Stability of Feedback Equilibrium Solutions for Noncooperative Differential Games

Alberto Bressan

Department of Mathematics, Penn State University

Differential games: the PDE approach

- The search for equilibrium solutions to noncooperative differential games in feedback form leads to a nonlinear system of Hamilton-Jacobi PDEs for the values functions

- main focus: existence and stability of the solutions to these PDEs

- Any system can be locally approximated by a linear one:

$$\dot{x} = f(x, u_1, u_2), \qquad \dot{x} = Ax + B_1 u_1 + B_2 u_2$$

- Any cost functional can be locally approximated by a quadratic one.

Main issue: Assume that an equilibrium solution is found, to an approximating game with linear dynamics and quadratic costs.

Does the original nonlinear game also have an equilibrium solution, close to the linear-quadratic one?

Two cases

(i) finite time horizon

(ii) infinite time horizon

An example - approximating a Cauchy problem for a PDE

$$u_t = u_{xx}$$

$$u(0,x) = \varphi(x)$$

- Approximate the initial data: $\varphi(x) \approx ax^2 + bx + c$
- Conclude: $u(t,x) \approx ax^2 + bx + c + 2at$

$$\begin{cases} \textit{CORRECT} & \text{for } t \geq 0 \\ \textit{WRONG} & \text{for } t < 0 \end{cases}$$

Finite Horizon Games

$x \in \mathbb{R}^n$ state of the system

u_1, u_2 controls implemented by the players

Dynamics: $\dot{x}(t) = f(x, u_1, u_2) \qquad x(\tau) = y$

Goal of i-th player:

$$\text{maximize:} \quad J_i(\tau, y, u_1, u_2) \doteq \psi_i\big(x(T)\big) - \int_\tau^T L_i\big(x(t),\ u_1(t), u_2(t)\big)\, dt$$

$=$ [terminal payoff] - [integral of a running cost]

Seek: Nash equilibrium solutions in feedback form

$$u_i = u_i^*(t, x)$$

• Given the strategy $u_2 = u_2^*(t, x)$ adopted by the second player, for every initial data (τ, y), the assignment $u_1 = u_1^*(t, x)$ is a feedback solution to **optimal control problem for the first player :**

$$\max_{u_1(\cdot)} \left(\psi_1(x(T)) - \int_\tau^T L_1(x, u_1, u_2^*(t, x))\, dt \right)$$

subject to

$$\dot{x} = f(x, u_1, u_2^*(t, x)), \qquad x(\tau) = y$$

• Similarly $u_2 = u_2^*(t, x)$ should provide a solution to the optimal control problem for the second player, given that $u_1 = u_1^*(t, x)$

The system of PDEs for the value functions

$V_i(\tau, y)$ = value function for the i-th player
(= expected payoff, if game starts at τ, y)

Assume: $$\begin{cases} f(x, u_1, u_2) \ = \ f_1(x, u_1) + f_2(x, u_2) \\ L_i(x, u_1, u_2) \ = \ L_{i1}(x, u_1) + L_{i2}(x, u_2) \end{cases}$$

Optimal feedback controls:

$$u_i^* \ = \ u_i^*(t, x, \nabla V_i) \ = \ \underset{\omega}{\operatorname{argmax}} \ \Big\{ \nabla V_i(t, x) \cdot f_i(x, \omega) - L_{ii}(x, \omega) \Big\}$$

The value functions satisfy a system of PDE's

$$\partial_t V_i + \nabla V_i \cdot f(x, u_1^*, u_2^*) \ = \ L_i(x, u_1^*, u_2^*) \qquad\qquad i = 1, 2$$

with terminal condition: $V_i(T, x) = \psi_i(x)$

Finite horizon game

$$\begin{cases} \partial_t V_1 & = & H^{(1)}(x, \nabla V_1, \nabla V_2)\,, \\ \partial_t V_2 & = & H^{(2)}(x, \nabla V_1, \nabla V_2)\,, \end{cases} \qquad \begin{cases} V_1(T, x) & = & \psi_1(x) \\ V_2(T, x) & = & \psi_2(x) \end{cases}$$

(backward Cauchy problem, with terminal conditions)

Test well-posedness: by locally linearizing of the equations

$$\partial_t V_i \;=\; H^{(i)}(x, \nabla V_1, \nabla V_2) \qquad\qquad i = 1,2$$

perturbed solution: $$V_i^{(\varepsilon)} \;=\; V_i + \varepsilon\, Z_i + o(\varepsilon)$$

Differentiating $H^{(i)}(x, p_1, p_2)$, obtain a linear equation satisfied by Z_i

$$\partial_t Z_i \;=\; \frac{\partial H^{(i)}}{\partial p_1}(x, \nabla V_1, \nabla V_2)\cdot \nabla Z_1 + \frac{\partial H^{(i)}}{\partial p_2}(x, \nabla V_1, \nabla V_2)\cdot \nabla Z_2$$

Freezing the coefficients at a point $(\bar{x},\ \nabla V_1(\bar{x}),\ \nabla V_2(\bar{x}))$, one obtains a linear system with constant coefficients

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}_t + \sum_{j=1}^{n} A_j \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}_{x_j} \;=\; 0 \qquad (1)$$

Each A_j is a 2×2 matrix

For a given vector $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, consider the matrix

$$A(\xi) \doteq \sum_j \xi_j A_j$$

Definition 1. The system (1) is *hyperbolic* if there exists a constant C such that

$$\sup_{\xi \in \mathbb{R}^n} \Big\| \exp iA(\xi) \Big\| \leq C$$

Computing solutions in terms of Fourier transform, which is an isometry on $\mathbf{L}^2$, the above definition is motivated by

Theorem 1. *The system (1) is hyperbolic if and only if the corresponding Cauchy problem is well posed in* $\mathbf{L}^2(\mathbb{R}^n)$.

$$\begin{aligned} \big\|Z(t)\big\|_{\mathbf{L}^2} &= \big\|\widehat{Z}(t)\big\|_{\mathbf{L}^2} \leq \sup_{\xi\in\mathbb{R}^n} \big\|\exp(-iA(\xi))\big\| \cdot \big\|\widehat{Z}(0)\big\|_{\mathbf{L}^2} \\ &= \sup_{\xi\in\mathbb{R}^n} \big\|\exp\, iA(\xi)\big\| \cdot \big\|Z(0)\big\|_{\mathbf{L}^2} \end{aligned}$$

Lemma 1 (necessary condition). *If the system (1) is hyperbolic, then for every* $\xi\in\mathbb{R}^m$ *the matrix* $A(\xi)$ *has a basis of eigenvectors* $r_1,\ldots,r_n$, *with real eigenvalues* $\lambda_1,\ldots,\lambda_n$ *(not necessarily distinct).*

Lemma 2 (sufficient condition). *Assume that, for* $|\xi|=1$, *the matrices* $A(\xi)$ *can be diagonalized in terms of a real, invertible matrix* $R(\xi)$ *continuously depending on* ξ. *Then the system (1) is hyperbolic.*

A class of differential games

$$\text{dynamics:} \qquad \dot{x} = f_1(x, u_1) + f_2(x, u_2)$$

$$\text{payoffs:} \qquad J_i \;=\; \psi_i(x(T)) - \int_0^T \Big(L_{i1}(x, u_1) + L_{i2}(x, u_2) \Big)\, dt\,, \qquad\qquad i = 1, 2$$

Value functions satisfy

$$\begin{cases} \partial_t V_1 + \nabla V_1 \cdot f(x, u_1^\sharp, u_2^\sharp) &=& L_1(x, u_1^\sharp, u_2^\sharp) \\ \partial_t V_2 + \nabla V_2 \cdot f(x, u_1^\sharp, u_2^\sharp) &=& L_2(x, u_1^\sharp, u_2^\sharp) \end{cases}$$

$$u_i^\sharp \;=\; u_i^\sharp(x, \nabla V_i) \;=\; \operatorname*{argmax}_\omega \Big\{ \nabla V_i \cdot f_i(x, \omega) - L_{ii}(x, \omega) \Big\}, \qquad\qquad i = 1, 2$$

$$V_1(T, x) \;=\; \psi_1(x)\,, \qquad\qquad V_2(T, x) \;=\; \psi_2(x)$$

$$f = (f_1, \ldots, f_n), \qquad\qquad \nabla V_i \;=\; p_i \;=\; (p_{i1}, \ldots p_{in})$$

Evolution of a perturbation:

$$Z_{i,t} + \sum_{k=1}^{n} f_k \, Z_{i,x_k} + \sum_{k=1}^{n} \sum_{j=1}^{2} \left(\nabla V_i \cdot \frac{\partial f}{\partial u_j} - \frac{\partial L_i}{\partial u_j} \right) \left(\frac{\partial u_j^\sharp}{\partial p_{1k}} \, Z_{1,x_k} + \frac{\partial u_j^\sharp}{\partial p_{2k}} \, Z_{2,x_k} \right) \;=\; 0$$

Maximality conditions $\implies$

$$\nabla V_1 \cdot \frac{\partial f}{\partial u_1} - \frac{\partial L_1}{\partial u_1} \;=\; 0\,, \qquad\qquad \nabla V_2 \cdot \frac{\partial f}{\partial u_2} - \frac{\partial L_2}{\partial u_2} \;=\; 0$$

Evolution of a first order perturbation

$$\begin{pmatrix} Z_{1,t} \\ Z_{2,t} \end{pmatrix} + \sum_{k=1}^{n} A_k \begin{pmatrix} Z_{1,x_k} \\ Z_{2,x_k} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where the 2×2 matrices A_k are given by

$$A_k = \begin{pmatrix} f_k & \left(\nabla V_1 \cdot \frac{\partial f}{\partial u_2} - \frac{\partial L_1}{\partial u_2}\right) \frac{\partial u_2^\sharp}{\partial p_{2k}} \\ \left(\nabla V_2 \cdot \frac{\partial f}{\partial u_1} - \frac{\partial L_2}{\partial u_1}\right) \frac{\partial u_1^\sharp}{\partial p_{1k}} & f_k \end{pmatrix}$$

$$A(\xi) = \sum_{k=1}^{n} \begin{pmatrix} f_k \xi_k & \left(\nabla V_1 \cdot \frac{\partial f}{\partial u_2} - \frac{\partial L_1}{\partial u_2}\right) \frac{\partial u_2^\sharp}{\partial p_{2k}} \xi_k \\ \left(\nabla V_2 \cdot \frac{\partial f}{\partial u_1} - \frac{\partial L_2}{\partial u_1}\right) \frac{\partial u_1^\sharp}{\partial p_{1k}} \xi_k & f_k \xi_k \end{pmatrix}$$

HYPERBOLICITY $\Longrightarrow$ $A(\xi)$ has real eigenvalues for every ξ

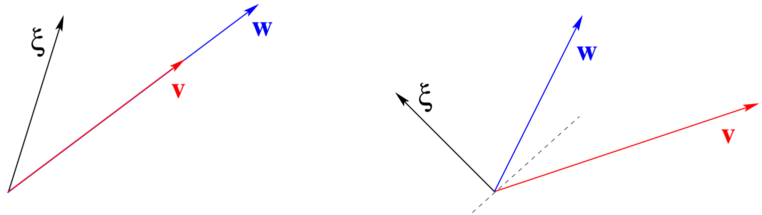
$$\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_n), \qquad \mathbf{v}_k \doteq \left(\nabla V_1 \cdot \frac{\partial f}{\partial u_2} - \frac{\partial L_1}{\partial u_2}\right) \frac{\partial u_2^\sharp}{\partial p_{2k}}$$

$$\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_n), \qquad \mathbf{w}_k \doteq \left(\nabla V_2 \cdot \frac{\partial f}{\partial u_1} - \frac{\partial L_2}{\partial u_1}\right) \frac{\partial u_1^\sharp}{\partial p_{1k}}$$

HYPERBOLICITY $\Longrightarrow$ $(\mathbf{v} \cdot \xi)(\mathbf{w} \cdot \xi) \geq 0$ for all $\xi \in \mathbb{R}^n$

HYPERBOLICITY $\implies$ $(\mathbf{v}\cdot\xi)(\mathbf{w}\cdot\xi) \geq 0$ for all $\xi \in \mathbb{R}^n$

- TRUE if $\mathbf{v}, \mathbf{w}$ are linearly dependent, with same orientation.
- FALSE if $\mathbf{v}, \mathbf{w}$ are linearly independent.



- In one space dimension, the Cauchy Problem can be well posed for a large set of data.
- In several space dimensions, generically the system is hyperbolic, and the Cauchy Problem is ill posed

A.B., W.Shen, Small BV solutions of hyperbolic non-cooperative differential games, SIAM J. Control Optim. **43** (2004), 194–215.

A.B., W.Shen, Semi-cooperative strategies for differential games, *Intern. J. Game Theory* **32** (2004), 561–593.

A.B., Noncooperative differential games. *Milan J. Math.*, **79** (2011), 357–427.

Differential games in infinite time horizon

$$\text{Dynamics:} \qquad \dot{x} = f(x, u_1, u_2), \qquad\qquad x(0) = x_0$$

$u_1, \; u_2$ controls implemented by the players

Goal of i-th player:

$$\text{maximize:} \quad J_i \doteq \int_0^{+\infty} e^{-\gamma t}\, \Psi_i\big(x(t),\; u_1(t),\; u_2(t)\big)\, dt$$

(running payoff, exponentially discounted in time)

A special case

Dynamics: $\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2$

Player i seeks to minimize: $J_i = \int_0^\infty e^{-\gamma t} \left(\phi_i(x(t)) + \frac{u_i^2(t)}{2} \right) dt$

A system of PDEs for the value functions

The value functions V_1, V_2 for the two players satisfy the system of H-J equations

$$\begin{cases} \gamma V_1 &= (f \cdot \nabla V_1) - \frac{1}{2}(g_1 \cdot \nabla V_1)^2 - (g_2 \cdot \nabla V_1)(g_2 \cdot \nabla V_2) + \phi_1 \\ \gamma V_2 &= (f \cdot \nabla V_2) - \frac{1}{2}(g_2 \cdot \nabla V_2)^2 - (g_1 \cdot \nabla V_1)(g_1 \cdot \nabla V_2) + \phi_2 \end{cases}$$

Optimal feedback controls: $\quad u_i^*(x) = -\nabla V_i(x) \cdot g_i(x) \qquad i = 1, 2$

nonlinear, implicit !

Linear - Quadratic games

Assume that the dynamics is linear:

$$\dot{x} \;=\; (Ax + \mathbf{b}_0) + \mathbf{b}_1 u_1 + \mathbf{b}_2 u_2\,, \qquad\qquad x(0) = y$$

and the cost functions are quadratic:

$$J_i \;=\; \int_0^{+\infty} e^{-\gamma t}\left(\mathbf{a}_i \cdot x + x^T P_i x + \frac{u_i^2}{2}\right) dt$$

Then the system of PDEs has a special solution of the form

$$V_i(x) \;=\; k_i + \beta_i \cdot x + x^T \Gamma_i x \qquad\qquad i = 1, 2 \qquad\qquad (*)$$

$$\text{optimal controls:}\quad u_i^*(x) \;=\; -\,(\beta_i + 2x^T \Gamma_i) \cdot \mathbf{b}_i$$

To find this solution, it suffices to
determine the coefficients k_i, β_i, Γ_i by solving a system of algebraic equations

Validity of linear-quadratic approximations ?

Assume the dynamics is almost linear

$$\dot{x} \;=\; f_0(x)+g_1(x)u_1+g_2(x)u_2 \;\approx\; (Ax+\mathbf{b}_0)+\mathbf{b}_1 u_1+\mathbf{b}_2 u_2\,, \qquad\qquad x(0)=y$$

and the cost functions are almost quadratic

$$J_i \;=\; \int_0^{+\infty} e^{-\gamma t}\Big(\phi_i(x)+\frac{u_i^2}{2}\Big)\,dt \;\approx\; \int_0^{+\infty} e^{-\gamma t}\Big(\mathbf{a}_i\cdot x+x^T P_i x+\frac{u_i^2}{2}\Big)\,dt$$

Is it true that the nonlinear game has a feedback solution close to the linear-quadratic game?

$$\dot{x} \;=\; (a_0 x + b_0) + b_1 u_1 + b_2 u_2$$

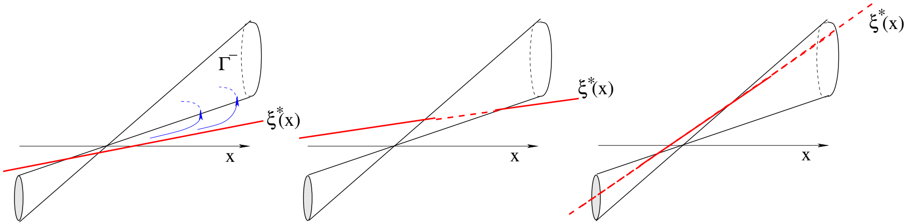
The ODE for the derivatives of the value functions $\xi_i = V_i'$ takes the form

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \xi_1' \\ \xi_2' \end{pmatrix} \;=\; \begin{pmatrix} \psi_1(x,\xi_1,\xi_2) \\ \psi_2(x,\xi_1,\xi_2) \end{pmatrix}, \qquad A_{ij} = A_{ij}(x,\xi_1,\xi_2)$$

- The map $(x,\xi_1,\xi_2) \;\mapsto\; \det A(x,\xi_1,\xi_2)$ is a homogeneous quadratic polynomial
- An affine solution exists: $\xi_1^*(x) \;=\; k_1 x + \beta_1 , \qquad \xi_2^*(x) \;=\; k_2 x + \beta_2$
- The map $x \mapsto \det A(x, \xi_1^*(x), \xi_2^*(x))$ is a quadratic polynomial

Stability w.r.t. perturbations (A.B. - Khai Nguyen, 2016)

- on a bounded interval $\Omega = [a, b]$
  (positively invariant for the feedback dynamics)
- on the whole real line



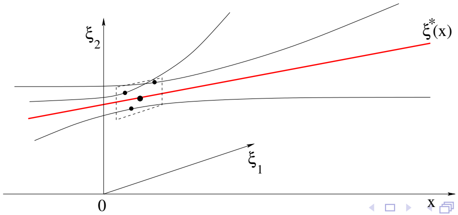
$$\Gamma^- = \Big\{(x, \xi_1, \xi_2)\,; \quad \det A(x, \xi_1, \xi_2) \leq 0\Big\}$$

Stability over a bounded interval

Easy case: $\det A(x, \xi_1^*(x), \xi_2^*(x)) \neq 0$ for all $x \in \mathbb{R}$.

$$\begin{pmatrix} \xi_1' \\ \\ \xi_2' \end{pmatrix} = A^{-1}(x, \xi_1, \xi_2) \begin{pmatrix} \psi_1(x, \xi_1, \xi_2) \\ \\ \psi_2(x, \xi_1, \xi_2) \end{pmatrix}$$

- The linear-quadratic game has a 2-parameter family of Nash equilibrium solutions in feedback form. One is affine, the other are nonlinear.
- All of the above solutions are stable w.r.t. small nonlinear perturbations of the dynamics and the cost functions.



Case 2: $\det A$ vanishes at two points $\bar{x}_1 < \bar{x}_2$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \xi_1' \\ \xi_2' \end{pmatrix} = \begin{pmatrix} \psi_1(x, \xi_1, \xi_2) \\ \psi_2(x, \xi_1, \xi_2) \end{pmatrix}$$

Equivalent system:

$$\begin{cases} A_{11}\, d\xi_1 + A_{12}\, d\xi_2 - \psi_1\, dx & = 0 \\ A_{21}\, d\xi_1 + A_{22}\, d\xi_2 - \psi_2\, dx & = 0 \end{cases}$$

Setting:

$$\mathbf{v} \doteq \begin{pmatrix} -\psi_1 \\ A_{11} \\ A_{12} \end{pmatrix}, \qquad \mathbf{w} \doteq \begin{pmatrix} -\psi_2 \\ A_{21} \\ A_{22} \end{pmatrix},$$

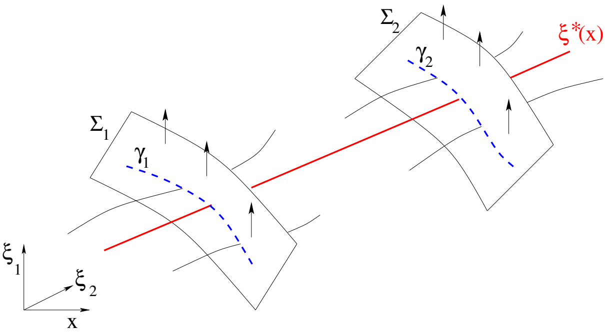
We seek continuously differentiable functions $x \mapsto (\xi_1(x), \xi_2(x))$ whose graph is obtained by concatenating trajectories of the system

$$\begin{pmatrix} \dot{x} \\ \dot{\xi}_1 \\ \dot{\xi}_2 \end{pmatrix} = \mathbf{v} \times \mathbf{w} = \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} \\ A_{22}\psi_1 - 12\psi_2 \\ A_{11}\psi_2 - A_{21}\psi_1 \end{pmatrix}.$$

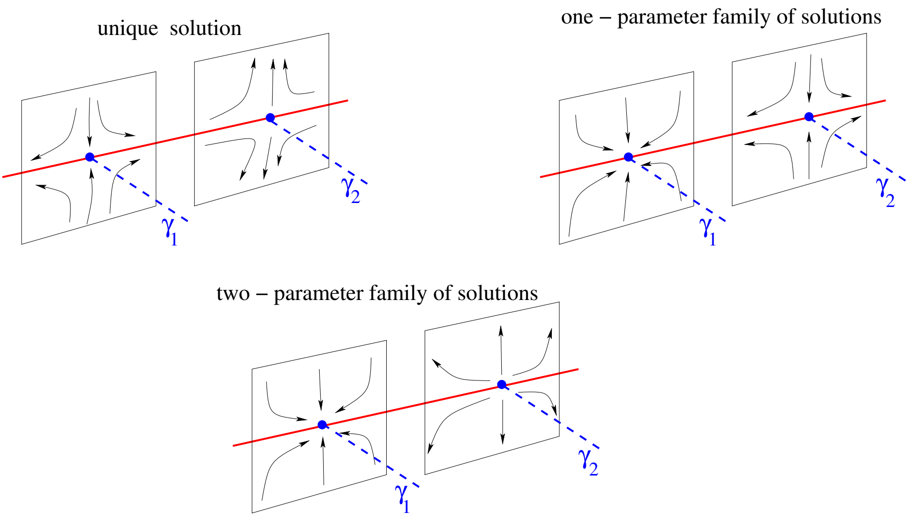
$$\det\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = 0 \quad \text{on} \quad \Sigma_1 \cup \Sigma_2$$

Under generic conditions on the coefficients of the linear-quadratic problem, there exists two curves $\gamma_1 \subset \Sigma_1$, $\gamma_2 \subset \Sigma_2$ such that

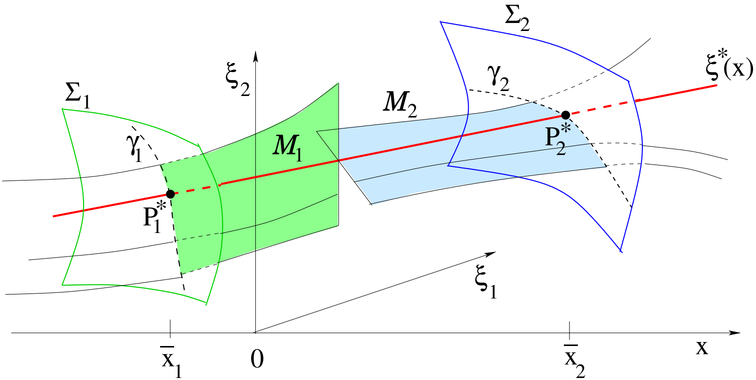
- $\mathbf{v} \times \mathbf{w} = 0$ on γ_1 and on γ_2
- $\mathbf{v} \times \mathbf{w}$ is vertical on $\Sigma_1 \setminus \gamma_1$ and on $\Sigma_2 \setminus \gamma_2$



Three generic cases



The saddle-saddle case



Under generic assumptions, a unique solution exists

Stability under perturbations, on the entire real line

(A.B., K.Nguyen, 2016)

$$\dot{x} \;=\; a_0 x + f_0(x) + (b_1 + h_1(x))u_1 + (b_2 + h_2(x))u_2$$

$$J_i \;=\; \int_0^{+\infty} e^{-\gamma t}\left(R_i x + S_i x^2 + \eta_i(x) + \frac{u_i^2}{2}\right) dt$$

Under generic assumptions on the coefficients $a_0, b_1, b_2, \ldots$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that the following holds. If the perturbations satisfy

$$\|f_0\|_{\mathcal{C}^2} + \|\eta_1\|_{\mathcal{C}^2} + \|\eta_2\|_{\mathcal{C}^2} + \|h_1\|_{\mathcal{C}^1} + \|h_2\|_{\mathcal{C}^1} \;\leq\; \delta,$$

then the perturbed equations for $\xi_1 = V_1'$, $\xi_2 = V_2'$ admit a solution such that

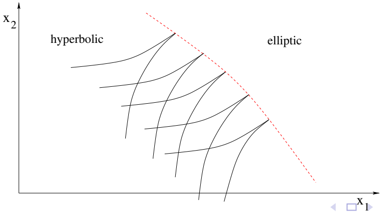
$$|\xi_1(x) - \xi_1^*(x)| + |\xi_2(x) - \xi_2^*(x)| \;\leq\; \varepsilon(1 + |x|) \qquad \text{for all } x \in \mathbb{R}$$

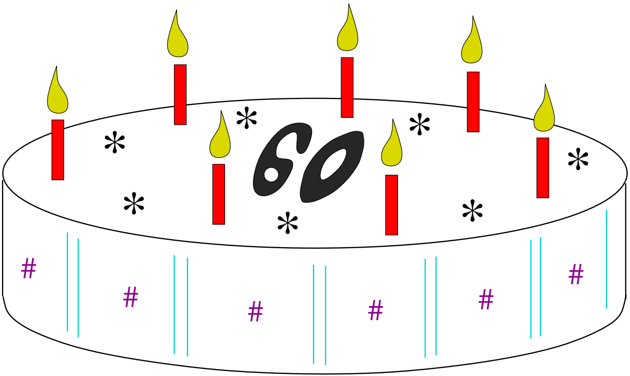
Future work: $x \in \mathbb{R}^2$

$$V_i = V_i(x_1, x_2)$$

$$\begin{cases} \gamma V_1 = (f \cdot \nabla V_1) + \frac{1}{2}(g_1 \cdot \nabla V_1)^2 + (g_2 \cdot \nabla V_1)(g_2 \cdot \nabla V_2) + \phi_1 \\ \gamma V_2 = (f \cdot \nabla V_2) + \frac{1}{2}(g_2 \cdot \nabla V_2)^2 + (g_1 \cdot \nabla V_1)(g_1 \cdot \nabla V_2) + \phi_2 \end{cases}$$

- Linearize around the affine solution of a L-Q game
- Determine if this linearized PDE is elliptic, hyperbolic, or mixed type
- Construct solutions to the perturbed nonlinear PDE





Happy Birthday

Jean Michel !!