

## CONTROL OF THREE HEAT EQUATIONS COUPLED WITH TWO CUBIC NONLINEARITIES\*

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**Abstract.** We study the null controllability of three parabolic equations. The control is acting only on one of the three equations. The three equations are coupled by means of two cubic nonlinearities. The linearized control system around 0 is not null controllable. However, using the cubic nonlinearities, we prove the (global) null controllability of the control system. The proof relies on the return method, an algebraic solvability, and smoothing properties of the parabolic equations.

**Key words.** null controllability, parabolic system, nonlinear coupling, return method, algebraic solvability

**AMS subject classifications.** 35K40, 93B05

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**1. Introduction.** Let  $N$  be a positive integer, and let  $\Omega$  be a nonempty connected bounded subset of  $\mathbb{R}^N$  of class  $C^2$ . Let  $\omega$  be a nonempty open subset of  $\Omega$ . We denote by  $\chi_\omega : \Omega \rightarrow \mathbb{R}$  the characteristic function of  $\omega$  and let  $T \in (0, +\infty)$ . We are interested in the control system

$$(1.1) \quad \begin{cases} \alpha_t - \Delta\alpha = \beta^3 & \text{in } (0, T) \times \Omega, \\ \beta_t - \Delta\beta = \gamma^3 & \text{in } (0, T) \times \Omega, \\ \gamma_t - \Delta\gamma = u\chi_\omega & \text{in } (0, T) \times \Omega, \\ \alpha = \beta = \gamma = 0 & \text{in } (0, T) \times \partial\Omega. \end{cases}$$

It is a control system where, at time  $t \in [0, T]$ , the state is  $(\alpha(t, \cdot), \beta(t, \cdot), \gamma(t, \cdot))^{\text{tr}} : \Omega \rightarrow \mathbb{R}^3$  and the control is  $u(t, \cdot) : \Omega \rightarrow \mathbb{R}$ . Let us point out that, due to the recursive structure of (1.1) (one first solves the last parabolic equation of (1.1), then the second one, and finally the first one), it follows from classical results on linear parabolic equations that the Cauchy problem associated to (1.1) is globally well-posed in the  $L^\infty$  setting, i.e., with bounded measurable initial data, controls, and solutions.

The main goal of this paper is to prove the following global null controllability result on control system (1.1).

**THEOREM 1.1.** *For every  $(\alpha^0, \beta^0, \gamma^0)^{\text{tr}} \in L^\infty(\Omega)^3$ , there exists a control  $u \in L^\infty((0, T) \times \Omega)$  such that the solution  $(\alpha, \beta, \gamma)^{\text{tr}} \in L^\infty((0, T) \times \Omega)^3$  to the Cauchy problem*

$$(1.2) \quad \begin{cases} \alpha_t - \Delta\alpha = \beta^3 & \text{in } (0, T) \times \Omega, \\ \beta_t - \Delta\beta = \gamma^3 & \text{in } (0, T) \times \Omega, \\ \gamma_t - \Delta\gamma = u\chi_\omega & \text{in } (0, T) \times \Omega, \\ \alpha = \beta = \gamma = 0 & \text{in } (0, T) \times \partial\Omega, \\ \alpha(0, \cdot) = \alpha^0(\cdot), \beta(0, \cdot) = \beta^0(\cdot), \gamma(0, \cdot) = \gamma^0(\cdot) & \text{in } \Omega \end{cases}$$

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satisfies

$$(1.3) \quad \alpha(T, \cdot) = \beta(T, \cdot) = \gamma(T, \cdot) = 0 \quad \text{in } \Omega.$$

The controllability of systems of partial differential equations with a small number of controls is an important subject which has been recently investigated in a large number of papers. For the case of linear systems, let us mention, in particular, the following:

- For systems of parabolic equations in dimension 1 or larger: [21, 20, 26, 29]. A key step in these papers is to establish suitable Carleman estimates. In dimension 1, the method of moments can lead to very precise (and sometimes unexpected) results; see, in particular, [8, 7, 10, 11]. See also the survey paper [6] and the reference therein.
- For systems of Schrödinger equations: [2], which uses transmutation, together with a controllability result for systems of wave equations proved in the same paper. See also [30] for the controllability of a cascade system of conservative equations.
- For Stokes equations of incompressible fluids: [23, 28, 17, 12]. Again, Carleman estimates are key ingredients here.
- For hyperbolic equations: [1, 2], which rely on multiplier methods, and [4], which uses microlocal analysis.

Let us assume that 0 is a trajectory (i.e., a solution) of the system of partial differential equations. If the linearized control system is controllable, one can expect to get the local null controllability. For systems of partial differential equations with a small number of controls, it has been proven to be the case, for example, for the Navier–Stokes equations in [12].

Note that the linearized control system of (1.1) around 0 is clearly not controllable. When the linearized control system around 0 is not controllable, one may still expect that the nonlinearities can give the controllability. A method for treating this case is the return method. It consists of looking for (nonzero) trajectories of the control system going from 0 to 0 such that the linearized control system is controllable. This method was introduced in [13] for a stabilization issue and used for the first time in [14] to get the controllability of a partial differential equation (the Euler equation of incompressible fluids). This method can also be used to get controllability of systems of partial differential equations with a small number of controls. See, for example,

- [15] for a water tank control system modeled by means of the Saint-Venant equations,
- [17, 19] for the Navier–Stokes equations,
- [18] for a system of two nonlinear heat equations.

Let us give more details about [18], since it deals with a control system related to our system (1.1). The control system considered in [18] is

$$(1.4) \quad \begin{cases} \beta_t - \Delta\beta = \gamma^3 & \text{in } (0, T) \times \Omega, \\ \gamma_t - \Delta\gamma = u\chi_\omega & \text{in } (0, T) \times \Omega, \\ \beta = \gamma = 0 & \text{in } (0, T) \times \partial\Omega, \end{cases}$$

where, at time  $t \in [0, T]$ , the state is  $(\beta(t, \cdot), \gamma(t, \cdot))^{\text{tr}} : \Omega \rightarrow \mathbb{R}^2$  and the control is  $u(t, \cdot) : \Omega \rightarrow \mathbb{R}$ . (In fact, slightly more general control systems of two coupled parabolic equations are considered in [18].) Using the return method, it is proved in [18] that the control system (1.4) is locally null controllable. We use the same method here. However, the construction of trajectories of the control system going

from 0 to 0 such that the linearized control system is (null) controllable is much more complicated for the control system (1.1) than for the control system (1.4).

The construction of trajectories of the control system (1.1) going from 0 to 0 such that the linearized control system is (null) controllable follows from simple scaling arguments (see (4.2)–(4.5) below) and the following theorem.

**THEOREM 1.2.** *There exists  $(a, b, c) \in C_0^\infty(\mathbb{R} \times \mathbb{R})^3$  such that*

- (1.5) *the supports of  $a, b,$  and  $c$  are included in  $[-1, 1] \times [-1, 1],$*
- (1.6)  *$\{(t, r); r > 0, b(t, r) \neq 0 \text{ and } c(t, r) \neq 0\} \neq \emptyset,$*
- (1.7)  *$a(t, r) = a(t, -r), b(t, r) = b(t, -r), c(t, r) = c(t, -r) \quad \forall (t, r) \in \mathbb{R} \times \mathbb{R},$*
- (1.8)  *$a_t - a_{rr} - \frac{N-1}{r}a_r = b^3 \quad \text{in } \mathbb{R} \times \mathbb{R}^*,$*
- (1.9)  *$b_t - b_{rr} - \frac{N-1}{r}b_r = c^3 \quad \text{in } \mathbb{R} \times \mathbb{R}^*.$*

An important ingredient of the proof of Theorem 1.2 is the following proposition which is related to Theorem 1.2 in the stationary case.

**PROPOSITION 1.3.** *There exist  $(A, B, C) \in C^\infty(\mathbb{R})^3$  and  $\delta_A \in (0, 1/2)$  such that*

- (1.10) *the supports of  $A, B,$  and  $C$  are included in  $[-1, 1],$*
- (1.11)  *$\{z; z > 0, B(z) \neq 0 \text{ and } C(z) \neq 0\} \neq \emptyset,$*
- (1.12)  *$A(z) = A(-z), B(z) = B(-z), C(z) = C(-z) \quad \forall z \in \mathbb{R},$*
- (1.13)  *$A(z) = e^{-1/(1-z^2)} \quad \text{if } 1 - \delta_A < z < 1,$*
- (1.14)  *$-A'' - \frac{N-1}{z}A' = B^3 \quad \text{in } \mathbb{R}^*,$*
- (1.15)  *$-B'' - \frac{N-1}{z}B' = C^3 \quad \text{in } \mathbb{R}^*,$*
- (1.16)  *$(B(z) = 0 \text{ and } z \in [0, 1]) \Leftrightarrow \left(z = \frac{1}{2}\right),$*
- (1.17)  *$B'\left(\frac{1}{2}\right) < 0,$*
- (1.18)  *$C\left(\frac{1}{2}\right) > 0,$*
- (1.19)  *$(C(z) = 0 \text{ and } z \in [0, 1]) \Rightarrow (z \in (0, 1) \text{ and } C'(z) \neq 0).$*

This proposition is proved in section 2. In section 3 we show how to use Proposition 1.3 in order to prove Theorem 1.2. Finally, in section 4, we deduce Theorem 1.1 from Theorem 1.2.

*Remark 1.4.* Looking to our proof of Theorem 1.1, it is natural to conjecture that this theorem still holds if, in (1.2),  $\beta^3$  and  $\gamma^3$  are replaced by  $\beta^{2p+1}$  and  $\gamma^{2q+1}$ , respectively, where  $p$  and  $q$  are arbitrary nonnegative integers.

**2. Proof of Proposition 1.3 (stationary case).** In order to construct  $A,$  one uses the following lemma.

**LEMMA 2.1.** *There exists  $\delta_0 \in (0, 1)$  such that, for every  $\delta \in (0, \delta_0),$  there exists*

a function  $G \in C^\infty([0, +\infty))$  such that

$$(2.1) \quad G(z) = z^3 \left(z - \frac{1}{2}\right)^3 \quad \text{for } \frac{1}{2} - \delta < z < \frac{1}{2} + \delta,$$

$$(2.2) \quad \left(z - \frac{1}{2}\right)G(z) > 0 \quad \text{for } 0 < z < 1, \quad z \neq \frac{1}{2},$$

$$(2.3) \quad \left\{ z \in (0, 1); (G^{1/3})''(z) + \frac{N-1}{z}(G^{1/3})'(z) = 0 \right\} \quad \text{is finite,}$$

and such that the solution  $A : (0, +\infty) \rightarrow \mathbb{R}$  to the Cauchy problem

$$(2.4) \quad A(1) = A'(1) = 0, \quad A''(z) + \frac{N-1}{z}A'(z) = G(z), \quad z > 0,$$

satisfies

$$(2.5) \quad \text{there exists } c_0 \in \mathbb{R} \text{ such that } A(z) = c_0 - z^8 \text{ if } 0 < z < \delta,$$

$$(2.6) \quad A(z) = e^{-1/(1-z^2)} \quad \text{if } 1 - \delta < z < 1,$$

$$(2.7) \quad A(z) = 0 \quad \text{if } z \in [1, +\infty).$$

*Proof of Lemma 2.1.* Let us first emphasize that it follows from (2.1) and (2.2) that  $G^{1/3}$  is of class  $C^\infty$  on  $(0, 1)$ ; hence (2.3) makes sense. Let  $\delta \in (0, 1/4)$ . Let  $\bar{G} \in C^\infty([0, +\infty))$  be such that (2.1) and (2.2) hold for  $G = \bar{G}$  and

$$(2.8) \quad \bar{G}(z) = -8(6+N)z^6 \quad \forall z \in (0, \delta),$$

$$(2.9) \quad \bar{G}(z) = \left( \frac{-2+6z^4}{(1-z^2)^4} - \frac{2(N-1)}{(1-z^2)^2} \right) e^{-1/(1-z^2)} \quad \forall z \in ((1-\delta), 1),$$

$$(2.10) \quad \bar{G}(z) = 0 \quad \forall z \in (1, +\infty),$$

$$(2.11) \quad \bar{G} \text{ is analytic on } (0, 1) \setminus \{\delta, (1/2) - \delta, (1/2) + \delta, 1 - \delta\}.$$

One easily sees that such a  $\bar{G}$  exists if  $\delta \in (0, 1/4)$  is small enough, with the smallness depending on  $N$ . From now on,  $\delta$  is always assumed to be small enough. Let  $\kappa \in \mathbb{R}$ . Let us define  $G \in C^\infty([0, +\infty))$  by

$$(2.12) \quad G := \bar{G} \quad \text{in } [0, \delta] \cup [(1/2) - \delta, (1/2) + \delta] \cup [1 - \delta, +\infty),$$

$$(2.13) \quad G(z) := \bar{G}(z) + \min\{\kappa, 0\}e^{-1/(z-\delta)}e^{-1/(1-2\delta-2z)} \quad \forall z \in (\delta, (1/2) - \delta),$$

$$(2.14) \quad G(z) := \bar{G}(z) + \max\{\kappa, 0\}e^{-1/(2z-1-2\delta)}e^{-1/(1-\delta-z)} \quad \forall z \in ((1/2) + \delta, 1 - \delta).$$

Let  $A$  be the solution of the Cauchy problem (2.4). From (2.12), one has (2.1) and (2.2). From (2.11), (2.13), and (2.14), one gets that

$$(2.15) \quad G \text{ is analytic on } (0, 1) \setminus \{\delta, (1/2) - \delta, (1/2) + \delta, 1 - \delta\},$$

which implies (2.3), since  $(G^{1/3})''$  cannot be identically equal to 0 on one of the five intervals  $(0, \delta)$ ,  $(\delta, (1/2) - \delta)$ ,  $((1/2) - \delta, (1/2) + \delta)$ ,  $((1/2) + \delta, 1 - \delta)$ , and  $(1 - \delta, 1)$ .

*Remark 2.2.* We require (2.15) only to get (2.3). However, (2.3) can also be obtained without requiring (2.15) by using genericity arguments.

From (2.4), (2.9), and (2.12), one gets (2.6). From (2.4), (2.10), and (2.12), one gets (2.7).

It remains to prove that, for some  $\kappa \in \mathbb{R}$ , one has (2.5). Let us first point out that, for every  $y \in C^2((0, \delta))$ ,

$$(2.16) \quad \left( y'' + \frac{N-1}{z}y' = 0 \right) \\ \Rightarrow (\exists (c_0, c_1) \in \mathbb{R}^2 \text{ such that } y(z) = c_0 + c_1E(z) \forall z \in (0, \delta)),$$

where

$$(2.17) \quad \text{if } N \neq 2, \quad E(z) := \frac{1}{(2-N)z^{N-2}} \quad \forall z \in (0, +\infty),$$

$$(2.18) \quad \text{if } N = 2, \quad E(z) := -\ln(z) \quad \forall z \in (0, +\infty).$$

From (2.4), (2.8), and (2.12), one gets that  $y := A + z^8$  satisfies the assumption of the implication (2.16). Hence, by (2.16) one gets the existence of  $(c_0, c_1) \in \mathbb{R}^2$  such that

$$(2.19) \quad A(z) = c_0 - z^8 + c_1E(z) \quad \forall z \in (0, \delta).$$

It suffices to check that for some  $\kappa \in \mathbb{R}$ ,

$$(2.20) \quad c_1 = 0.$$

From (2.4), one has

$$(2.21) \quad \text{if } N \neq 2, \quad A(z) = -\frac{1}{(N-2)z^{N-2}} \int_1^z s^{N-1}G(s)ds + \frac{1}{N-2} \int_1^z sG(s)ds \quad \forall z \in (0, 1],$$

$$(2.22) \quad \text{if } N = 2, \quad A(z) = \ln(z) \int_1^z sG(s)ds - \int_1^z s \ln(s)G(s)ds \quad \forall z \in (0, 1],$$

which, together with (2.17), (2.18), (2.19), with  $z \rightarrow 0$ , give

$$(2.23) \quad c_1 = \int_0^1 s^{N-1}G(s)ds.$$

From (2.12), (2.13), and (2.14), one has

$$(2.24) \quad \lim_{\kappa \rightarrow +\infty} \int_0^1 s^{N-1}G(s)ds = +\infty \text{ and } \lim_{\kappa \rightarrow -\infty} \int_0^1 s^{N-1}G(s)ds = -\infty.$$

In particular, with the intermediate value theorem, there exists  $\kappa \in \mathbb{R}$  such that

$$(2.25) \quad \int_0^1 s^{N-1}G(s)ds = 0,$$

which, together with (2.23), concludes the proof of Lemma 2.1. □

We return to the proof of Proposition 1.3. We extend  $A$  to all of  $\mathbb{R}$  by requiring

$$(2.26) \quad A(0) = c_0,$$

$$(2.27) \quad A(z) = A(-z) \quad \forall z \in (-\infty, 0).$$

By (2.5), (2.26), and (2.27),  $A \in C^\infty(\mathbb{R})$ . Let  $B \in C^0(\mathbb{R}^*)$  be defined by

$$(2.28) \quad B := - \left( A'' + \frac{N-1}{z} A' \right)^{1/3}.$$

From (2.27) and (2.28), one gets that

$$(2.29) \quad B(z) = B(-z) \quad \forall z \in \mathbb{R}^*.$$

From (2.28), one sees that

$$(2.30) \quad B \text{ is of class } C^\infty \text{ on the set } \{z \in \mathbb{R}^*; B(z) \neq 0\}.$$

From (2.5), (2.27), and (2.28), one has

$$(2.31) \quad B(z) = 2(6+N)^{1/3} z^2 \quad \forall z \in (-\delta, \delta) \setminus \{0\},$$

which allows us to extend  $B$  to all of  $\mathbb{R}$  by continuity by requiring

$$(2.32) \quad B(0) = 0.$$

From (2.31) and (2.32), we get that

$$(2.33) \quad B \text{ is of class } C^\infty \text{ in } (-\delta, \delta).$$

From (2.2), (2.4), and (2.28), one gets that

$$(2.34) \quad B \neq 0 \quad \text{in } (0, 1) \setminus \{1/2\},$$

which, with (2.30), implies that

$$(2.35) \quad B \text{ is of class } C^\infty \text{ in } (0, 1) \setminus \{1/2\}.$$

From (2.1), (2.4), and (2.28), one has

$$(2.36) \quad B(z) = -z \left( z - \frac{1}{2} \right) \quad \forall z \in \left( \frac{1}{2} - \delta, \frac{1}{2} + \delta \right).$$

In particular, (1.17) holds. From (2.6) and (2.28), one gets

$$(2.37) \quad B(z) = - \left( \frac{-2 + 6z^4}{(1-z^2)^4} - \frac{2(N-1)}{(1-z^2)^2} \right)^{1/3} e^{-1/(3-3z^2)} \quad \forall z \in (1-\delta, 1),$$

which implies the existence of  $\delta_0 > 0$  such that, for every  $\delta \in (0, \delta_0]$ ,

$$(2.38) \quad B < 0 \quad \text{in } (1-\delta, 1).$$

From (2.7) and (2.28), one gets

$$(2.39) \quad B(z) = 0 \quad \forall z \in (1, +\infty),$$

which, together with (2.37), implies that

$$(2.40) \quad B \text{ is of class } C^\infty \text{ in } (1-\delta, +\infty).$$

From (2.29), (2.33), (2.35), (2.36), and (2.40), one gets that

$$(2.41) \quad B \text{ is of class } C^\infty \text{ in } \mathbb{R}.$$

Let us now define  $C \in C^0(\mathbb{R}^*)$  by

$$(2.42) \quad C(z) := - \left( B'' + \frac{N-1}{z} B' \right)^{1/3} \quad \forall z \in \mathbb{R}^*.$$

From (2.29) and (2.42), one has

$$(2.43) \quad C(z) = C(-z) \quad \forall z \in \mathbb{R}^*.$$

From (2.41) and (2.42), one gets that

$$(2.44) \quad C \text{ is of class } C^\infty \text{ on the set } \{z \in \mathbb{R}^*; C(z) \neq 0\}.$$

From (2.31) and (2.42), one has

$$(2.45) \quad C(z) = -(4N)^{\frac{1}{3}}(6+N)^{\frac{1}{3}} < 0 \quad \forall z \in [-\delta, \delta].$$

From (2.36) and (2.42), one has

$$(2.46) \quad C(z) = \left( 2N - \frac{N-1}{2z} \right)^{1/3} \quad \forall z \in \left[ \frac{1}{2} - \delta, \frac{1}{2} + \delta \right].$$

In particular, since  $\delta > 0$  is small enough,

$$(2.47) \quad C \text{ is positive and of class } C^\infty \text{ on } \left[ \frac{1}{2} - \delta, \frac{1}{2} + \delta \right].$$

From (2.37), (2.39), and (2.42), one gets that

$$(2.48) \quad C > 0 \text{ in } [1 - \delta, 1) \text{ and } C \text{ is of class } C^\infty \text{ in } [1 - \delta, +\infty).$$

From (2.43), (2.44), (2.45), (2.47), and (2.48), one sees that

$$(2.49) \quad C \in C^\infty(\mathbb{R})$$

if

$$(2.50) \quad C \text{ is of class } C^\infty \text{ in } (\delta, (1/2) - \delta) \cup ((1/2) + \delta, 1 - \delta).$$

Let us first point out that by (2.3), (2.4), (2.28), and (2.42),

$$(2.51) \quad \text{the set of } z_0 \in (\delta, (1/2) - \delta) \cup ((1/2) + \delta, 1 - \delta) \text{ such that } C(z_0) = 0 \text{ is finite.}$$

We are going to prove that (2.50) indeed holds provided that one no longer requires (2.15) and that one modifies  $G$  in a neighborhood of every  $z_0 \in (\delta, (1/2) - \delta) \cup ((1/2) + \delta, 1 - \delta)$  such that  $C(z_0) = 0$ . Since  $G = -B^3$ , this comes from the following lemma.

LEMMA 2.3. *Let  $\nu > 0$ ,  $\zeta > 0$ , and  $\eta > 0$  be such that  $[\zeta - \eta, \zeta + \eta] \subset (0, +\infty)$ . Let  $B \in C^\infty([\zeta - \eta, \zeta + \eta])$  be such that*

$$(2.52) \quad B''(z) + \frac{N-1}{z} B'(z) \neq 0 \quad \forall z \in [\zeta - \eta, \zeta + \eta] \setminus \{\zeta\}.$$

Then there exists  $\bar{B} \in C^\infty([\zeta - \eta, \zeta + \eta])$  satisfying

$$(2.53) \quad |\bar{B}(z) - B(z)| \leq \nu \quad \forall z \in [\zeta - \eta, \zeta + \eta],$$

$$(2.54) \quad \text{the support of } \bar{B} - B \text{ is included in } (\zeta - \eta, \zeta + \eta),$$

$$(2.55) \quad \left( \bar{B}'' + \frac{N-1}{z} \bar{B}' \right)^{1/3} \in C^\infty([\zeta - \eta, \zeta + \eta]),$$

and such that if  $\bar{A} \in C^\infty([\zeta - \eta, \zeta + \eta])$  is the solution of

$$(2.56) \quad \bar{A}'' + \frac{N-1}{z} \bar{A}' = -\bar{B}^3,$$

$$(2.57) \quad \bar{A}(\zeta - \eta) = A(\zeta - \eta), \quad \bar{A}'(\zeta - \eta) = A'(\zeta - \eta),$$

then

$$(2.58) \quad \bar{A}(\zeta + \eta) = A(\zeta + \eta), \quad \bar{A}'(\zeta + \eta) = A'(\zeta + \eta).$$

*Proof of Lemma 2.3.* Let us first consider the case where

$$(2.59) \quad \left( B''(\zeta - \eta) + \frac{N-1}{\zeta - \eta} B'(\zeta - \eta) \right) \left( B''(\zeta + \eta) + \frac{N-1}{\zeta + \eta} B'(\zeta + \eta) \right) < 0.$$

Then replacing, if necessary,  $B$  by  $-B$  and using (2.52), we may assume that

$$(2.60) \quad B''(z) + \frac{N-1}{z} B'(z) < 0 \quad \forall z \in [\zeta - \eta, \zeta],$$

$$(2.61) \quad B''(z) + \frac{N-1}{z} B'(z) > 0 \quad \forall z \in (\zeta, \zeta + \eta).$$

Let  $\varphi \in C^\infty(-\infty, +\infty)$  be such that

$$(2.62) \quad \varphi = 1 \text{ in } [-1/2, 1/2],$$

$$(2.63) \quad \varphi = 0 \text{ in } (-\infty, -1] \cup [1, +\infty),$$

$$(2.64) \quad \varphi(z) \in [0, 1] \quad \forall z \in (-\infty, \infty).$$

Let

$$(2.65) \quad \mathcal{E} := \{ \xi \in C^\infty([\zeta - \eta, \zeta + \eta]); \text{ the support of } \xi \text{ is included in } (\zeta - \eta, \zeta + \eta) \setminus \{ \zeta \} \}.$$

The vector space  $\mathcal{E}$  is equipped with the norm

$$(2.66) \quad |\xi| := \max\{ |\xi(x)|; x \in [\zeta - \eta, \zeta + \eta] \}.$$

For  $\varepsilon \in \mathbb{R}$  and  $\xi \in \mathcal{E}$ , one now defines  $H_{\varepsilon, \xi} \in C^\infty([\zeta - \eta, \zeta + \eta])$ , if  $\varepsilon \neq 0$ , by

$$(2.67) \quad H_{\varepsilon, \xi}(z) := \varepsilon^2 (z - \zeta)^3 \varphi \left( \frac{z - \zeta}{|\varepsilon|} \right) + \left( 1 - \varphi \left( \frac{z - \zeta}{|\varepsilon|} \right) \right) \left( B''(z) + \frac{N-1}{z} B'(z) + \xi(z) \right),$$

for every  $z \in [\zeta - \eta, \zeta + \eta]$  and

$$(2.68) \quad H_{0, \xi}(z) := B''(z) + \frac{N-1}{z} B'(z) + \xi(z) \quad \forall z \in [\zeta - \eta, \zeta + \eta].$$



We then define  $\bar{B} := B_{\varepsilon, \xi} \in C^\infty([\zeta - \eta, \zeta + \eta])$  by requiring

$$(2.69) \quad B''_{\varepsilon, \xi}(z) + \frac{N-1}{z} B'_{\varepsilon, \xi}(z) = H_{\varepsilon, \xi}(z),$$

$$(2.70) \quad B_{\varepsilon, \xi}(\zeta - \eta) = B(\zeta - \eta), \quad B'_{\varepsilon, \xi}(\zeta - \eta) = B'(\zeta - \eta).$$

Let  $C_{\varepsilon, \xi} \in C^0([\zeta - \eta, \zeta + \eta])$  be defined by

$$(2.71) \quad C_{\varepsilon, \xi}(z) := - \left( B''_{\varepsilon, \xi}(z) + \frac{N-1}{z} B'_{\varepsilon, \xi}(z) \right)^{1/3} = -H_{\varepsilon, \xi}(z)^{1/3}.$$

Note that by (2.62), (2.67), and (2.71), if  $\varepsilon \neq 0$ ,

$$(2.72) \quad C'_{\varepsilon, \xi}(\xi) = -|\varepsilon|^{2/3} \neq 0.$$

Using (2.63), (2.65), (2.67), (2.68), and (2.69), one sees that if  $\varepsilon < \eta$  (which is assumed from now on),  $B_{\varepsilon, \xi}$  and  $B$  are both solutions to the second order differential equation

$$(2.73) \quad Y''(z) + \frac{N-1}{z} Y'(z) = B''(z) + \frac{N-1}{z} B'(z)$$

in a neighborhood of  $\{\zeta - \eta, \zeta + \eta\}$  in  $[\zeta - \eta, \zeta + \eta]$ . In particular, by (2.70),  $B_{\varepsilon, \xi}$  and  $B$  are equal in a neighborhood of  $\zeta - \eta$  in  $[\zeta - \eta, \zeta + \eta]$ , and (2.54) is equivalent to

$$(2.74) \quad B_{\varepsilon, \xi}(\zeta + \eta) = B(\zeta + \eta), \quad B'_{\varepsilon, \xi}(\zeta + \eta) = B'(\zeta + \eta).$$

Let  $A_{\varepsilon, \xi} \in C^\infty([\zeta - \eta, \zeta + \eta])$  be the solution of

$$(2.75) \quad A''_{\varepsilon, \xi} + \frac{N-1}{z} A'_{\varepsilon, \xi} = -B_{\varepsilon, \xi}^3,$$

$$(2.76) \quad A_{\varepsilon, \xi}(\zeta - \eta) = A(\zeta - \eta), \quad A'_{\varepsilon, \xi}(\zeta - \eta) = A'(\zeta - \eta).$$

Let  $\mathcal{F} : (-\eta, \eta) \times \mathcal{E} \rightarrow \mathbb{R}^4$  be defined by

$$(2.77) \quad \mathcal{F}(\varepsilon, \xi) := (B_{\varepsilon, \xi}(\zeta + \eta) - B(\zeta + \eta), B'_{\varepsilon, \xi}(\zeta + \eta) - B'(\zeta + \eta), \\ A_{\varepsilon, \xi}(\zeta + \eta) - A(\zeta + \eta), A'_{\varepsilon, \xi}(\zeta + \eta) - A'(\zeta + \eta))^{\text{tr}}.$$

One easily checks that

$$(2.78) \quad \mathcal{F} \text{ is of class } C^1,$$

$$(2.79) \quad \mathcal{F}(0, 0) = 0.$$

Let us assume, for the moment, that

$$(2.80) \quad \frac{\partial \mathcal{F}}{\partial \xi}(0, 0) \text{ is onto.}$$

By (2.80), there exists a 4-dimensional subspace  $\mathcal{E}_0$  of  $\mathcal{E}$  such that

$$(2.81) \quad \frac{\partial \mathcal{F}}{\partial \xi}(0, 0) \mathcal{E}_0 = \mathbb{R}^4.$$

By (2.81) and the implicit function theorem, there exists  $\varepsilon_0 \in (0, \eta)$  and a map  $\xi : (-\varepsilon_0, \varepsilon_0) \rightarrow \mathcal{E}_0$  such that

$$(2.82) \quad \xi(0) = 0,$$

$$(2.83) \quad \mathcal{F}(\varepsilon, \xi(\varepsilon)) = 0 \quad \forall \varepsilon \in (-\varepsilon_0, \varepsilon_0).$$

From (2.60), (2.61), (2.65), (2.66), (2.67), (2.68), and (2.69), one gets the existence of  $\varepsilon_1 > 0$  such that for every  $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$  and for every  $\xi \in \mathcal{E}_0$  such that  $|\xi| \leq \varepsilon_1$ ,

$$(2.84) \quad B''_{\varepsilon, \xi}(z) + \frac{N-1}{z} B'_{\varepsilon, \xi}(z) < 0 \quad \forall z \in [\zeta - \eta, \zeta),$$

$$(2.85) \quad B''_{\varepsilon, \xi}(z) + \frac{N-1}{z} B'_{\varepsilon, \xi}(z) > 0 \quad \forall z \in (\zeta, \zeta + \eta].$$

From (2.62), (2.67), and (2.69), one gets that for every  $\varepsilon \in (0, +\infty)$  and for every  $\xi \in \mathcal{E}_0$ , one has

$$(2.86) \quad B''_{\varepsilon, \xi}(z) + \frac{N-1}{z} B'_{\varepsilon, \xi}(z) = \varepsilon^2(z - \zeta)^3 \quad \text{if } |z - \zeta| \leq \frac{\varepsilon}{2}.$$

From (2.71), (2.84), (2.85), and (2.86), one gets that for every  $\varepsilon \in [-\varepsilon_1, \varepsilon_1] \setminus \{0\}$  and for every  $\xi \in \mathcal{E}_0$  such that  $|\xi| \leq \varepsilon_1$ ,

$$(2.87) \quad C_{\varepsilon, \xi} \in C^\infty([\zeta - \eta, \zeta + \eta]),$$

$$(2.88) \quad (C_{\varepsilon, \xi}(z) = 0) \Leftrightarrow (z = \zeta),$$

which, together with (2.81) as above, (2.72), (2.82), and (2.83), conclude the proof of Lemma 2.3 when (2.59) holds.

It remains to prove (2.80). Simple computations show that

$$(2.89) \quad \frac{\partial \mathcal{F}}{\partial \xi}(0, 0)\xi = (x_1(\zeta + \eta), x_2(\zeta + \eta), x_3(\zeta + \eta), x_4(\zeta + \eta))^{\text{tr}},$$

where  $x : [\zeta - \eta, \zeta + \eta] \rightarrow \mathbb{R}^4$  is the solution of

$$(2.90) \quad \dot{x} = K(t)x + \xi(t)e,$$

with

$$(2.91) \quad K(t) := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{N-1}{t} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3B^2(t) & 0 & 0 & -\frac{N-1}{t} \end{pmatrix}, \quad e := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

which satisfies

$$(2.92) \quad x(\zeta - \eta) = 0.$$

Hence, using a standard density argument, (2.81) comes from the following lemma.

LEMMA 2.4. *Let  $\nu > 0$ ,  $\zeta > 0$ ,  $\eta > 0$  be such that  $[\zeta - \eta, \zeta + \eta] \subset (0, +\infty)$ . Let  $B \in C^\infty([\zeta - \eta, \zeta + \eta])$  be such that*

$$(2.93) \quad B \not\equiv 0.$$

*Then the control system (2.90), where the state is  $x \in \mathbb{R}^4$  and the control is  $\xi \in \mathbb{R}$ , is controllable on  $[\zeta - \eta, \zeta + \eta]$ ; i.e., for every  $X$  in  $\mathbb{R}^4$  there exists  $\xi \in L^\infty(\zeta - \eta, \zeta + \eta)$  such that the solution of (2.90) and (2.92) satisfies  $x(\zeta + \eta) = X$ .*

*Proof of Lemma 2.4.* We use a classical result on the controllability of time-varying linear finite-dimensional control systems (see, e.g., [16, Theorem 1.18]). One defines, by induction on  $i \in \mathbb{N}$ ,  $e_i \in C^\infty([\zeta - \eta, \zeta + \eta])$  by requiring

$$(2.94) \quad e_0(t) := e \quad \forall t \in [\zeta - \eta, \zeta + \eta],$$

$$(2.95) \quad e_i(t) := \dot{e}_{i-1}(t) - K(t)e_{i-1}(t) \quad \forall t \in [\zeta - \eta, \zeta + \eta], \quad \forall i \in \mathbb{N} \setminus \{0\}.$$

Let  $\theta \in C^\infty([\zeta - \eta, \zeta + \eta])$  be defined by

$$(2.96) \quad \theta(t) := -\frac{N-1}{t} \quad \forall t \in [\zeta - \eta, \zeta + \eta].$$

Straightforward computations lead to

$$(2.97) \quad e_1 = \begin{pmatrix} -1 \\ -\theta \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} \theta \\ -\dot{\theta} + \theta^2 \\ 0 \\ -3B^2 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 2\dot{\theta} - \theta^2 \\ -\ddot{\theta} + 3\theta\dot{\theta} - \theta^3 \\ 3B^2 \\ 6B^2\theta - 6B\dot{B} \end{pmatrix}.$$

From (2.91), (2.94), and (2.97), one gets

$$(2.98) \quad \det(e_0, e_1, e_2, e_3) = 9B^4,$$

which, with (2.93) and [16, Theorem 1.18], concludes the proof of Lemma 2.4.  $\square$

We now turn to the case where (2.59) does not hold. Then replacing, if necessary,  $B$  by  $-B$  and using (2.52), we may assume that

$$(2.99) \quad B''(z) + \frac{N-1}{z}B'(z) > 0 \quad \forall z \in [\zeta - \eta, \zeta + \eta] \setminus \{\zeta\}.$$

In the definition of  $H_{\varepsilon, \xi}$  one replaces (2.67) by

$$(2.100) \quad H_{\varepsilon, \xi}(z) := \varepsilon^2 \varphi\left(\frac{z-\zeta}{|\varepsilon|}\right) + \left(1 - \varphi\left(\frac{z-\zeta}{|\varepsilon|}\right)\right) \left(B''(z) + \frac{N-1}{z}B'(z) + \xi(z)\right)$$

and keeps (2.68). Now (2.84) and (2.85) are replaced by

$$(2.101) \quad C_{\varepsilon, \xi}(z) > 0 \quad \forall z \in [\zeta - \eta, \zeta + \eta], \quad \forall \varepsilon \in [-\varepsilon_1, \varepsilon_1] \setminus \{0\}, \quad \forall \xi \in \mathcal{E}_0 \text{ such that } |\xi| \leq \varepsilon_1.$$

Therefore (compare with (2.88)), provided that  $\varepsilon \neq 0$ , one can see that  $C_{\varepsilon, \xi}(z) \neq 0$  for every  $z \in [\zeta - \eta, \zeta + \eta]$ , and consequently (1.19) is satisfied. Moreover,

$$(2.102) \quad C_{\varepsilon, \xi} \in C^\infty([\zeta - \eta, \zeta + \eta]) \quad \forall \varepsilon \in [-\varepsilon_1, \varepsilon_1] \setminus \{0\}, \quad \forall \xi \in \mathcal{E}_0 \text{ such that } |\xi| \leq \varepsilon_1,$$

which, together with (2.102), (2.82), (2.83), and (2.101), concludes the proof of Proposition 1.3.  $\square$

**3. Proof of Theorem 1.2 (time-varying case).** In this section, we prove Theorem 1.2. We define  $\lambda \in C^\infty([-1, 1])$  and  $f_0 \in C^\infty([-1, 1])$  by

$$(3.1) \quad \lambda(t) := (1 - t^2)^2 \quad \forall t \in [-1, 1]$$

and

$$(3.2) \quad f_0(t) := \begin{cases} e^{-\frac{1}{1-t^2}} & \text{if } |t| < 1, \\ 0 & \text{if } t = 0. \end{cases}$$

Let  $\varepsilon \in (0, 1]$ . For  $r \in \mathbb{R}$  and  $t \in (-1, 1)$ , we set

$$(3.3) \quad z := \frac{r}{\varepsilon\lambda(t)} \in [0, +\infty).$$

Let  $A, B$ , and  $C$  be as in Proposition 1.3. By (2.43), (2.45), (2.47), (2.48), and (2.51), there exist  $p \in \mathbb{N}$  and  $\rho_1, \rho_2, \dots, \rho_p$  in  $(-1, 1) \setminus \{0\}$  such that

$$(3.4) \quad \{z \in (-1, 1); C(z) = 0\} = \{\rho_l; l \in \{1, 2, \dots, p\}\}.$$

Let

$$(3.5) \quad \rho_0 := \frac{1}{2}, \quad \rho_{-1} := -\frac{1}{2}.$$

Let  $\delta > 0$  be such that

$$(3.6) \quad [\rho_l - \delta, \rho_l + \delta] \subset (-1, 1) \setminus \{0\} \quad \forall l \in \{-1, 0, 1, \dots, p\},$$

$$(3.7) \quad [\rho_l - \delta, \rho_l + \delta] \cap [\rho_{l'} - \delta, \rho_{l'} + \delta] = \emptyset \quad \forall (l, l') \in \{-1, 0, 1, \dots, p\}^2 \text{ such that } l \neq l'.$$

Let  $\mathbb{D} := \{(t, r) \in (-1, 1) \times \mathbb{R}; |r| < \varepsilon\lambda(t)\}$ . We look for  $a : (t, r) \in \mathbb{D} \mapsto a(t, r) \in \mathbb{R}$  in the following form:

$$(3.8) \quad a(t, r) = f_0(t)A(z) + \sum_{l=-1}^p \sum_{i=1}^3 f_{il}(t)g_{il}(z),$$

where the functions  $f_{il}, g_{il}$  are to be determined with the requirement that

$$(3.9) \quad \text{the support of } g_{il} \text{ is included in } (\rho_l - \delta, \rho_l + \delta) \quad \forall i \in \{1, 2, 3\}, \quad \forall l \in \{-1, 0, 1, \dots, p\}.$$

Then  $b : (t, r) \in \mathbb{D} \mapsto b(t, r) \in \mathbb{R}$  is defined by

$$(3.10) \quad b := \left( a_t - a_{rr} - \frac{N-1}{r} a_r \right)^{1/3},$$

and, on every open subset of  $\mathbb{D}$  on which  $b$  is of class  $C^2$  and  $b_r/r$  is bounded,  $c$  is defined by

$$(3.11) \quad c := \left( b_t - b_{rr} - \frac{N-1}{r} b_r \right)^{1/3}.$$

For  $l \in \{-1, 0, 1, \dots, p\}$ , let  $\Sigma_l \subset \mathbb{R} \times \mathbb{R}$  be defined by

$$(3.12) \quad \Sigma_l := \{(t, r) \in (-1, 1) \times \mathbb{R}; z \in (\rho_l - \delta, \rho_l + \delta)\}.$$

Let us first study the case where, for some

$$(3.13) \quad \bar{l} \in \{1, 2, \dots, p\},$$

$(t, r) \in \Sigma_{\bar{l}}$ . By symmetry, we may only study the case where  $\rho_{\bar{l}} > 0$ . Note that (3.13), together with (1.18) and (3.4), implies that

$$(3.14) \quad \rho_{\bar{l}} \neq \frac{1}{2}.$$

From (3.7), (3.8), (3.9), and (3.12), we have

$$(3.15) \quad a(t, r) = f_0(t)A(z) + \sum_{i=1}^3 f_{i\bar{l}}(t)g_{i\bar{l}}(z).$$

In order to simplify the notation, we omit the index  $\bar{l}$  and define  $g_0$  by

$$(3.16) \quad g_0 := A.$$

(This definition is used throughout this section.) Then, (3.15) now reads

$$(3.17) \quad a(t, r) = \sum_{i=0}^3 f_i(t)g_i(z).$$

Note that (1.16), (3.14), and (3.16) imply that

$$(3.18) \quad B(\rho) \neq 0.$$

Moreover, by (1.15), (1.19), (3.4), (3.13), and (3.16),

$$(3.19) \quad \left( B^{(2)} + \frac{N-1}{z} B^{(1)} \right) (\rho) = 0,$$

$$(3.20) \quad \left( B^{(2)} + \frac{N-1}{z} B^{(1)} \right)_z (\rho) = 0,$$

$$(3.21) \quad \left( B^{(2)} + \frac{N-1}{z} B^{(1)} \right)_{zz} (\rho) = 0,$$

$$(3.22) \quad \left( B^{(2)} + \frac{N-1}{z} B^{(1)} \right)_{zzz} (\rho) \neq 0.$$

To simplify the notation we assume that, for example,

$$(3.23) \quad B(\rho) < 0,$$

$$(3.24) \quad \left( B^{(2)} + \frac{N-1}{z} B^{(1)} \right)_{zzz} (\rho) < 0.$$

From (3.20), (3.21), (3.23), and (3.24), if  $\delta \in (0, \rho)$  is small enough, there exists  $\mu > 0$  such that

$$(3.25) \quad B(z) \leq -\mu \quad \forall z \in [\rho - \delta, \rho + \delta],$$

$$(3.26) \quad \left( B^{(2)} + \frac{N-1}{z} B^{(1)} \right)_{zzz} (z) \leq -\mu \quad \forall z \in [\rho - \delta, \rho + \delta].$$

We now fix such a  $\delta$ .

From (3.10) and (3.17),

$$(3.27) \quad b = -\frac{1}{\varepsilon^{2/3}\lambda^{2/3}} \left( \sum_{i=0}^3 \left( f_i g_i^{(2)} + \frac{N-1}{z} f_i g_i^{(1)} + z\varepsilon^2 \lambda \dot{\lambda} f_i g_i^{(1)} - \varepsilon^2 \lambda^2 \dot{f}_i g_i \right) \right)^{1/3}.$$

Let us denote by  $M : \mathbb{R} \times \mathbb{R}^* \rightarrow \mathbb{R}$ ,  $(t, z) \mapsto M(t, z) \in \mathbb{R}$ , the function defined by

$$(3.28) \quad M(t, z) := \sum_{i=0}^3 \left( f_i(t) g_i^{(2)}(z) + \frac{N-1}{z} f_i(t) g_i^{(1)}(z) + z\varepsilon^2 \lambda(t) \dot{\lambda}(t) f_i(t) g_i^{(1)}(z) - \varepsilon^2 \lambda^2(t) \dot{f}_i(t) g_i(z) \right).$$

For the moment, let us assume that

$$(3.29) \quad M(t, z) \neq 0 \quad \forall (t, z) \in (-1, 1) \times (\rho - \delta, \rho + \delta).$$

Using (3.3), (3.11), (3.27), (3.28), and straightforward computations, one gets, on the open set of the  $(t, r) \in \Sigma$  such that  $M(t, z) \neq 0$ ,

$$(3.30) \quad 9\varepsilon^{8/3} \lambda^{8/3} c^3 = \nu,$$

with

$$(3.31) \quad \nu := \frac{1}{M^{2/3}} \left( 3M_{zz} - 2\frac{M_z^2}{M} + \frac{3(N-1)}{z} M_z + 6\varepsilon^2 \lambda \dot{\lambda} M - 3\varepsilon^2 \lambda^2 M_t + 3z\varepsilon^2 \lambda \dot{\lambda} M_z \right).$$

The idea is to construct the  $f_i$ 's and the  $g_i$ 's in order to have a precise knowledge of the places where  $\nu$  vanishes and of the order of the vanishing so that  $\nu$  is the cube of a  $C^\infty$  function. More precisely, we are going to check that one can construct the  $f_i$ 's and the  $g_i$ 's so that, at least if  $\varepsilon \in (0, 1]$  is small enough,

$$(3.32) \quad \nu(t, \rho) = 0 \quad \forall t \in (-1, 1),$$

$$(3.33) \quad \nu_z(t, \rho) = 0 \quad \forall t \in (-1, 1),$$

$$(3.34) \quad \nu_{zz}(t, \rho) = 0 \quad \forall t \in (-1, 1),$$

$$(3.35) \quad \nu_{zzz}(t, \rho) > 0 \quad \forall t \in (-1, 1).$$

From (3.28), one has

$$(3.36) \quad M_z = \sum_{i=0}^3 \left( f_i g_i^{(3)} + \varepsilon^2 (\dot{\lambda} f_i - \lambda \dot{f}_i) \lambda g_i^{(1)} + \varepsilon^2 z \lambda \dot{\lambda} f_i g_i^{(2)} + \frac{N-1}{z} f_i g_i^{(2)} - \frac{N-1}{z^2} f_i g_i^{(1)} \right),$$

$$(3.37) \quad M_{zz} = \sum_{i=0}^3 \left( f_i g_i^{(4)} + \varepsilon^2 (2\dot{\lambda} f_i - \lambda \dot{f}_i) \lambda g_i^{(2)} + \varepsilon^2 z \lambda \dot{\lambda} f_i g_i^{(3)} + \frac{N-1}{z} f_i g_i^{(3)} - \frac{2(N-1)}{z^2} f_i g_i^{(2)} + \frac{2(N-1)}{z^3} f_i g_i^{(1)} \right).$$

We impose that

$$(3.38) \quad g_i^{(j)}(\rho) = \begin{cases} 1 & \text{if } i = 1 \text{ and } j = 4, \\ 0 & \text{if } 1 \leq i \leq 3, 0 \leq j \leq 4, \text{ and } (i, j) \neq (1, 4). \end{cases}$$

From (3.28), (3.36), (3.37), and (3.38), we have

$$(3.39) \quad M(\cdot, \rho) = f_0 g_0^{(2)}(\rho) + \frac{N-1}{\rho} f_0 g_0^{(1)}(\rho) + \varepsilon^2 \rho \lambda \dot{\lambda} f_0 g_0^{(1)}(\rho) - \varepsilon^2 \lambda^2 \dot{f}_0 g_0(\rho),$$

$$(3.40) \quad \begin{aligned} M_z(\cdot, \rho) &= f_0 g_0^{(3)}(\rho) + \frac{N-1}{\rho} f_0 g_0^{(2)}(\rho) - \frac{N-1}{\rho^2} f_0 g_0^{(1)}(\rho) \\ &\quad + \varepsilon^2 (\dot{\lambda} f_0 - \lambda \dot{f}_0) \lambda g_0^{(1)}(\rho) + \varepsilon^2 \rho \lambda \dot{\lambda} f_0 g_0^{(2)}(\rho), \end{aligned}$$

$$(3.41) \quad \begin{aligned} M_{zz}(\cdot, \rho) &= f_0 g_0^{(4)}(\rho) + \frac{N-1}{\rho} f_0 g_0^{(3)}(\rho) - \frac{2(N-1)}{\rho^2} f_0 g_0^{(2)}(\rho) \\ &\quad + \frac{2(N-1)}{\rho^3} f_0 g_0^{(1)}(\rho) + f_1 \\ &\quad + \varepsilon^2 (2\dot{\lambda} f_0 - \lambda \dot{f}_0) \lambda g_0^{(2)}(\rho) + \varepsilon^2 \rho \lambda \dot{\lambda} f_0 g_0^{(3)}(\rho). \end{aligned}$$

From (1.14), (3.1), (3.2), (3.16), (3.23), (3.39), and (3.40), one has, at least if  $\varepsilon > 0$  is small enough, which is from now on assumed,

$$(3.42) \quad \forall t \in (-1, 1), \quad M(t, \rho) > 0.$$

Then, for  $z = \rho$ , one has

$$(3.43) \quad \begin{aligned} \nu(\cdot, \rho) &= \frac{1}{M^{2/3}(\cdot, \rho)} \left( 3M_{zz}(\cdot, \rho) - 2\frac{M_z^2(\cdot, \rho)}{M(\cdot, \rho)} + \frac{3(N-1)}{\rho} M_z(\cdot, \rho) \right. \\ &\quad \left. + 6\varepsilon^2 \lambda \dot{\lambda} M_z(\cdot, \rho) - 3\varepsilon^2 \lambda^2 M_t(\cdot, \rho) + 3\rho \varepsilon^2 \lambda \dot{\lambda} M_z(\cdot, \rho) \right). \end{aligned}$$

We then choose to define  $f_1 : t \in (-1, 1) \mapsto f_1(t) \in \mathbb{R}$  by

$$(3.44) \quad \begin{aligned} f_1 &:= -f_0 g_0^{(4)}(\rho) - \frac{N-1}{\rho} f_0 g_0^{(3)}(\rho) + \frac{2(N-1)}{\rho^2} f_0 g_0^{(2)}(\rho) - \frac{2(N-1)}{\rho^3} f_0 g_0^{(1)}(\rho) \\ &\quad - \varepsilon^2 (2\dot{\lambda} f_0 - \lambda \dot{f}_0) \lambda g_0^{(2)}(\rho) - \varepsilon^2 \rho \lambda \dot{\lambda} f_0 g_0^{(3)}(\rho) \\ &\quad + \frac{1}{3} \left( 2\frac{M_z^2(\cdot, \rho)}{M(\cdot, \rho)} - \frac{3(N-1)}{\rho} M_z(\cdot, \rho) - 6\varepsilon^2 \lambda \dot{\lambda} M(\cdot, \rho) \right. \\ &\quad \left. + 3\varepsilon^2 \lambda^2 M_t(\cdot, \rho) - 3\rho \varepsilon^2 \lambda \dot{\lambda} M_z(\cdot, \rho) \right). \end{aligned}$$

Note that, even if  $M$  depends on  $f_1$ ,  $f_2$ , and  $f_3$ , the right-hand side of (3.44) does not depend on  $f_1$ ,  $f_2$ , and  $f_3$ , and  $f_1$  is indeed well defined by (3.44). This definition of  $f_1$ , together with (3.41) and (3.43), implies that (3.32) holds. (In fact,  $f_1$  is defined by (3.44) precisely in order to have (3.32).) From (3.1), (3.2), (3.19), (3.23), (3.39), (3.40), and (3.44), we obtain the existence of two polynomials  $p_1(\varepsilon^2, t)$  and  $q_1(\varepsilon^2, t)$  in the variables  $\varepsilon^2$  and  $t$  such that

$$(3.45) \quad f_1(t) = \varepsilon^2 \frac{p_1(\varepsilon^2, t)}{1 + q_1(\varepsilon^2, t)} f_0(t) \quad \forall t \in (-1, 1).$$

In order to simplify the notation, we set

$$(3.46) \quad \begin{aligned} K(t, z) &:= -\frac{2M_z(t, z)^2}{M(t, z)} + \frac{3(N-1)}{z} M_z(t, z) + 6\varepsilon^2 \lambda \dot{\lambda} M(t, z) \\ &\quad - 3\varepsilon^2 \lambda^2 M_t(t, z) + 3z\varepsilon^2 \lambda \dot{\lambda} M_z(t, z). \end{aligned}$$

We then have

$$(3.47) \quad \nu = M^{-\frac{2}{3}}(3M_{zz} + K).$$

Differentiating this equality with respect to  $z$ , we obtain

$$(3.48) \quad \nu_z = M^{-\frac{5}{3}} \left( 3MM_{zzz} + MK_z - 2M_zM_{zz} - \frac{2}{3}M_zK \right).$$

Differentiating (3.46) with respect to  $z$ , we get

$$(3.49) \quad \begin{aligned} K_z &= -\frac{4M_zM_{zz}}{M^2} + \frac{2M_z^3}{M^3} + \frac{3(N-1)}{z}M_{zz} - \frac{3(N-1)}{z^2}M_z \\ &\quad + 9\varepsilon^2\lambda\dot{\lambda}M_z - 3\varepsilon^2\lambda^2M_{tz} + 3\varepsilon^2z\lambda\dot{\lambda}M_{zz}. \end{aligned}$$

Then, differentiating (3.37) with respect to  $z$ , we have

$$(3.50) \quad \begin{aligned} M_{zzz} &= \sum_{i=0}^3 \left( f_i g_i^{(5)} + \frac{N-1}{z} f_i g_i^{(4)} - \frac{3(N-1)}{z^2} f_i g_i^{(3)} + \frac{6(N-1)}{z^3} f_i g_i^{(2)} \right. \\ &\quad \left. - \frac{6(N-1)}{z^4} f_i g_i^{(1)} + \varepsilon^2 z \lambda \dot{\lambda} f_i g_i^{(4)} + \varepsilon^2 (3\dot{\lambda} f_i - \lambda \dot{f}_i) \lambda g_i^{(3)} \right). \end{aligned}$$

We impose that

$$(3.51) \quad g_i^{(5)}(\rho) = \begin{cases} 1 & \text{if } i = 2, \\ 0 & \text{if } i \in \{1, 3\}. \end{cases}$$

From (3.38), (3.50), and (3.51), we have

$$(3.52) \quad \begin{aligned} M_{zzz}(\cdot, \rho) &= f_0 g_0^{(5)}(\rho) + \frac{N-1}{\rho} f_0 g_0^{(4)}(\rho) - \frac{3(N-1)}{\rho^2} f_0 g_0^{(3)}(\rho) \\ &\quad + \frac{6(N-1)}{\rho^3} f_0 g_0^{(2)}(\rho) - \frac{6(N-1)}{\rho^4} f_0 g_0^{(1)}(\rho) + f_2 + \frac{N-1}{\rho} f_1 \\ &\quad + \varepsilon^2 (3\dot{\lambda} f_0 - \lambda \dot{f}_0) \lambda g_0^{(3)}(\rho) + \varepsilon^2 \rho \lambda \dot{\lambda} f_0 g_0^{(4)}(\rho) + \varepsilon^2 \rho \lambda \dot{\lambda} f_1. \end{aligned}$$

We then define  $f_2 : t \in (-1, 1) \mapsto f_2(t) \in \mathbb{R}$  by

$$(3.53) \quad \begin{aligned} f_2 &:= -f_0 g_0^{(5)}(\rho) - \frac{N-1}{\rho} f_0 g_0^{(4)}(\rho) + \frac{3(N-1)}{\rho^2} f_0 g_0^{(3)}(\rho) \\ &\quad - \frac{6(N-1)}{\rho^3} f_0 g_0^{(2)}(\rho) + \frac{6(N-1)}{\rho^4} f_0 g_0^{(1)}(\rho) - \frac{N-1}{\rho} f_1 \\ &\quad + \frac{1}{3M(\cdot, \rho)} \left( -M(\cdot, \rho) K_z(\cdot, \rho) + 2M_z(\cdot, \rho) M_{zz}(\cdot, \rho) + \frac{2}{3} M_z(\cdot, \rho) K(\cdot, \rho) \right) \\ &\quad - \varepsilon^2 \rho \lambda \dot{\lambda} f_0 g_0^{(4)}(\rho) - \varepsilon^2 \rho \lambda \dot{\lambda} f_1 - \varepsilon^2 (3\dot{\lambda} f_0 - \lambda \dot{f}_0) \lambda g_0^{(3)}(\rho). \end{aligned}$$

Note that, again, even if  $M$  depends on  $f_2$  and  $f_3$ , the right-hand side of (3.53) does not depend on  $f_2$  and  $f_3$  (it depends on  $f_1$ ; however,  $f_1$  is already defined in (3.44)),



and  $f_2$  is indeed well defined by (3.53). This definition of  $f_2$ , together with (3.48) and (3.52), implies (3.33). From (3.1), (3.2), (3.20), (3.39), (3.40), (3.41), (3.45), (3.46), (3.49), and (3.53), we obtain the existence of two polynomials  $p_2(\varepsilon^2, t)$  and  $q_2(\varepsilon^2, t)$  in the variables  $\varepsilon^2$  and  $t$  such that

$$(3.54) \quad f_2(t) = \varepsilon^2 \frac{p_2(\varepsilon^2, t)}{1 + q_2(\varepsilon^2, t)} f_0(t) \quad \forall t \in (-1, 1).$$

Differentiating (3.48) with respect to  $z$ , we obtain

$$(3.55) \quad \begin{aligned} \nu_{zz} = M^{-\frac{8}{3}} & \left( -4MM_zM_{zzz} - \frac{7}{3}MM_zK_z + \frac{10}{3}M_z^2M_{zz} + \frac{10}{9}M_z^2K \right. \\ & \left. + 3M^2M_{zzzz} + MM_zK_z + M^2K_{zz} - 2MM_z^2 - \frac{2}{3}MM_{zz}K \right). \end{aligned}$$

Differentiating (3.49) with respect to  $z$ , we obtain

$$(3.56) \quad \begin{aligned} K_{zz} = & -\frac{4M_{zz}^2}{M} - \frac{4M_zM_{zzz}}{M} + \frac{10M_z^2M_{zz}}{M^2} - \frac{4M_z^4}{M^3} + \frac{6(N-1)}{z^3}M_z \\ & - \frac{6(N-1)}{z^2}M_{zz} + \frac{3(N-1)}{z}M_{zzz} \\ & + 12\varepsilon^2\lambda\dot{\lambda}M_{zz} - 3\varepsilon^2\lambda^2M_{tzz} + 3z\varepsilon^2\lambda\dot{\lambda}M_{zzz}. \end{aligned}$$

Differentiating (3.50) with respect to  $z$ , one has

$$(3.57) \quad \begin{aligned} M_{zzzz} = \sum_{i=0}^3 & \left( f_i g_i^{(6)} + \frac{N-1}{z} f_i g_i^{(5)} - \frac{4(N-1)}{z^2} f_i g_i^{(4)} \right. \\ & + \frac{12(N-1)}{z^3} f_i g_i^{(3)} - \frac{24(N-1)}{z^4} f_i g_i^{(2)} + \frac{24(N-1)}{z^5} f_i g_i^{(1)} \\ & \left. + \varepsilon^2(4\dot{\lambda}f_i - \lambda\dot{f}_i)\lambda g_i^{(4)} + \varepsilon^2 z \lambda \dot{\lambda} f_i g_i^{(5)} \right). \end{aligned}$$

We then impose

$$(3.58) \quad g_i^{(6)}(\rho) = \begin{cases} 1 & \text{if } i = 3, \\ 0 & \text{if } i \in \{1, 2\}. \end{cases}$$

Evaluating  $M_{zzzz}$  at  $z = \rho$  in (3.57) gives

$$(3.59) \quad \begin{aligned} M_{zzzz}(\cdot, \rho) = & f_0 g_0^{(6)}(\rho) + f_3 + \frac{N-1}{\rho} f_0 g_0^{(5)}(\rho) - \frac{4(N-1)}{\rho^2} f_0 g_0^{(4)}(\rho) \\ & + \frac{12(N-1)}{\rho^3} f_0 g_0^{(3)}(\rho) - \frac{24(N-1)}{\rho^4} f_0 g_0^{(2)}(\rho) \\ & + \frac{24(N-1)}{\rho^5} f_0 g_0^{(1)}(\rho) + \frac{N-1}{\rho} f_2 - \frac{4(N-1)}{\rho^2} f_1 \\ & + \varepsilon^2(4\dot{\lambda}f_0 - \lambda\dot{f}_0)\lambda g_0^{(4)}(\rho) + \varepsilon^2(4\dot{\lambda}f_1 - \lambda\dot{f}_1)\lambda \\ & + \varepsilon^2 \rho \lambda \dot{\lambda} f_0 g_0^{(5)}(\rho) + \varepsilon^2 \rho \lambda \dot{\lambda} f_2. \end{aligned}$$

Then we define  $f_3 : t \in (-1, 1) \mapsto f_3(t) \in \mathbb{R}$  by  
 (3.60)

$$\begin{aligned} f_3 := & -f_0g_0^{(6)}(\rho) - \frac{N-1}{\rho}f_0g_0^{(5)}(\rho) + \frac{4(N-1)}{\rho^2}f_0g_0^{(4)}(\rho) - \frac{12(N-1)}{\rho^3}f_0g_0^{(3)}(\rho) \\ & + \frac{24(N-1)}{\rho^4}f_0g_0^{(2)}(\rho) - \frac{24(N-1)}{\rho^5}f_0g_0^{(1)}(\rho) - \frac{N-1}{\rho}f_2 + \frac{4(N-1)}{\rho^2}f_1 \\ & - \varepsilon^2\rho\lambda\dot{\lambda}f_0g_0^{(5)}(\rho) - \varepsilon^2(4\dot{\lambda}f_0 - \lambda\dot{f}_0)\lambda g_0^{(4)}(\rho) - \varepsilon^2\rho\lambda\dot{\lambda}f_2 - \varepsilon^2(4\dot{\lambda}f_1 - \lambda\dot{f}_1)\lambda \\ & + \frac{1}{3M^2(\cdot, \rho)} \left( 4M(\cdot, \rho)M_z(\cdot, \rho)M_{zzz}(\cdot, \rho) + \frac{7}{3}M(\cdot, \rho)M_z(\cdot, \rho)K_z(\cdot, \rho) \right. \\ & \quad - \frac{10}{3}M_z^2(\cdot, \rho)M_{zz}(\cdot, \rho) - \frac{10}{9}M_z^2(\cdot, \rho)K(\cdot, \rho) - M(\cdot, \rho)M_z(\cdot, \rho)K_z(\cdot, \rho) \\ & \quad \left. - M^2(\cdot, \rho)K_{zz}(\cdot, \rho) + 2M(\cdot, \rho)M_{zz}^2(\cdot, \rho) + \frac{2}{3}M(\cdot, \rho)M_{zz}(\cdot, \rho)K(\cdot, \rho) \right). \end{aligned}$$

Once more, even if  $M$  depends on  $f_3$ , the right-hand side of (3.60) does not depend on  $f_3$ , and  $f_3$  is indeed well defined by (3.60). This definition of  $f_3$ , together with (3.55) and (3.59), implies that (3.34) holds. From (3.1), (3.2), (3.21), (3.39), (3.40), (3.41), (3.45), (3.46), (3.49), (3.52), (3.54), (3.56), and (3.60), we obtain the existence of two polynomials  $p_3(\varepsilon^2, t)$  and  $q_3(\varepsilon^2, t)$  in the variables  $\varepsilon^2$  and  $t$ , such that

$$(3.61) \quad f_3(t) = \varepsilon^2 \frac{p_3(\varepsilon^2, t)}{1 + q_3(\varepsilon^2, t)} f_0(t) \quad \forall t \in (-1, 1).$$

We are now in a position to analyze the regularity of  $a$ ,  $b$ , and  $c$  on  $\Sigma$ . Let us first point out that, by (3.1), (3.2), (3.8), (3.45), (3.54), and (3.61), there exists  $\psi^a : (t, z) \in [-1, 1] \times [\rho - \delta, \rho + \delta] \mapsto \psi^a(t, z) \in \mathbb{R}$  of class  $C^\infty$  such that

$$(3.62) \quad a(t, r) = f_0(t)\psi^a(t, z) \quad \forall (t, r) \in \Sigma.$$

In particular,  $a$  is of class  $C^\infty$  in  $\Sigma$ . From (1.14), (3.1), (3.2), (3.16), (3.27), (3.25), (3.45), (3.54), and (3.61), we get that, at least if  $\varepsilon > 0$  is small enough, there exists  $\psi^b : (t, z) \in [-1, 1] \times [\rho - \delta, \rho + \delta] \mapsto \psi^b(t, z) \in \mathbb{R}$  of class  $C^\infty$  such that

$$(3.63) \quad b < 0 \text{ in } \Sigma,$$

$$(3.64) \quad b(t, r) = \lambda^{-2/3}f_0(t)^{1/3}\psi^b(t, z) \quad \forall (t, r) \in \Sigma.$$

In particular,  $b$  is of class  $C^\infty$  in  $\Sigma$ .

Let us now study  $c$ . Differentiating (3.55) with respect to  $z$ , one gets

$$(3.65) \quad \begin{aligned} \nu_{zzz} = & M^{-\frac{1}{3}} \left( 10MM_z^2M_{zzz} + \frac{10}{3}MM_z^2K_z - \frac{80}{9}M_z^3M_{zz} - \frac{80}{27}M_z^3K \right. \\ & - 6M^2M_zM_{zzzz} - 2M^2M_zK_{zz} + 10MM_zM_z^2 + \frac{10}{3}MM_zM_{zz}K \\ & - 8M^2M_{zz}M_{zzz} - 2M^2M_{zz}K_z + 3M^3M_{zzzzz} \\ & \left. + M^3K_{zzz} - \frac{2}{3}M^2M_{zzz}K \right). \end{aligned}$$

Differentiating (3.57) with respect to  $z$ , we get  
(3.66)

$$M_{zzzzz} = \sum_{i=0}^3 \left( f_i g_i^{(7)} + \frac{N-1}{z} f_i g_i^{(6)} - \frac{5(N-1)}{z^2} f_i g_i^{(5)} + \frac{20(N-1)}{z^3} f_i g_i^{(4)} - \frac{60(N-1)}{z^4} f_i g_i^{(3)} + \frac{120(N-1)}{z^5} f_i g_i^{(2)} - \frac{120(N-1)}{z^6} f_i g_i^{(1)} + \varepsilon^2 (5\dot{\lambda} f_i - \lambda \dot{f}_i) \lambda g_i^{(5)} + \varepsilon^2 z \lambda \dot{\lambda} f_i g_i^{(6)} \right).$$

Differentiating (3.56) with respect to  $z$ , we get  
(3.67)

$$K_{zzz} = \frac{6M_z^2 M_{zz}}{M} - \frac{4M_z M_{zzzz}}{M} + \frac{24M_z M_{zz}^2}{M^2} + \frac{4M_{zz} M_{zzz}}{M} - \frac{36M_z^3 M_{zz}}{M^3} + \frac{12M_z^5}{M^4} - \frac{18(N-1)}{z^4} M_z + \frac{18(N-1)}{z^3} M_{zz} - \frac{9(N-1)}{z^2} M_{zzz} + \frac{3(N-1)}{z} M_{zzzz} + 15\varepsilon^2 \lambda \dot{\lambda} M_{zzz} - 3\varepsilon^2 \lambda^2 M_{tzzz} + 3z\varepsilon^2 \lambda \dot{\lambda} M_{zzzz}.$$

From (3.1), (3.2), (3.26), (3.28), (3.30), (3.31), (3.32), (3.33), (3.34), (3.36), (3.37), (3.45), (3.46), (3.49), (3.50), (3.54), (3.56), (3.61), (3.65), (3.66), and (3.67), one gets the existence of  $\phi : (t, z) \in [-1, 1] \times [\rho - \delta, \rho + \delta] \mapsto \phi(t, z) \in \mathbb{R}$  of class  $C^\infty$  such that

(3.68)  $c^3(t, r) = \lambda^{8/3} f_0(t)^{1/3} \phi(t, z) \quad \forall (t, r) \in \Sigma,$

(3.69)  $\phi(t, \rho) = 0 \quad \forall t \in [-1, 1],$

(3.70)  $\partial_z \phi(t, \rho) = 0 \quad \forall t \in [-1, 1],$

(3.71)  $\partial_{zz}^2 \phi(t, \rho) = 0 \quad \forall t \in [-1, 1],$

(3.72)  $\partial_{zzz}^3 \phi(t, z) > 0 \quad \forall (t, z) \in [-1, 1] \times [\rho - \delta, \rho + \delta].$

Let  $\tilde{\phi} : (t, z) \in [-1, 1] \times [\rho - \delta, \rho + \delta] \mapsto \tilde{\phi}(t, z) \in \mathbb{R}$  be defined by

(3.73) 
$$\tilde{\phi}(t, z) := \frac{1}{2} \int_0^1 (1-s)^2 \partial_{zzz}^3 \phi(t, \rho + s(z-\rho)) ds \quad \forall (t, z) \in [-1, 1] \times [\rho - \delta, \rho + \delta].$$

Then,  $\tilde{\phi}$  is of class  $C^\infty$  on  $[-1, 1] \times [\rho - \delta, \rho + \delta]$  and, using (3.69), (3.70), (3.71), (3.72), we get

(3.74)  $\phi(t, z) = (z - \rho)^3 \tilde{\phi}(t, z) \quad \forall (t, z) \in [-1, 1] \times [\rho - \delta, \rho + \delta],$

(3.75)  $\tilde{\phi}(t, z) > 0 \quad \forall (t, z) \in [-1, 1] \times [\rho - \delta, \rho + \delta].$

Let  $\psi^c : (t, z) \in [-1, 1] \times [\rho - \delta, \rho + \delta] \mapsto \psi^c(t, z) \in \mathbb{R}$  be defined by

(3.76)  $\psi^c(t, z) := (z - \rho) \tilde{\phi}(t, z)^{1/3} \quad \forall (t, z) \in [-1, 1] \times [\rho - \delta, \rho + \delta].$

By (3.68), (3.74), (3.75), and (3.76), one gets that

(3.77)  $\psi^c \in C^\infty([-1, 1] \times [\rho - \delta, \rho + \delta]),$

(3.78)  $c(t, r) = \lambda^{-8/9} f_0(t)^{1/9} \psi^c(t, z) \quad \forall (t, r) \in \Sigma.$

In particular,  $c$  is of class  $C^\infty$  in  $\Sigma$ .

Let us now study the case  $l \in \{-1, 0\}$ , i.e.,  $\rho_l = 1/2$  or  $\rho_l = -1/2$ . By symmetry, we may assume that  $l = 0$  so that  $\rho_l = 1/2$ . This case is simpler than the previous one. It is already treated in [18], except that we now have to take care of  $c$ . So, we will only briefly sketch the arguments. By (1.18) we may impose that  $\delta$  be small enough so that

$$(3.79) \quad C(z) > 0 \quad \forall z \in [(1/2) - \delta, (1/2) + \delta].$$

We now define (see (3.27) and compare with (3.31))

$$(3.80) \quad \nu := \sum_{i=0}^3 \left( f_i g_i^{(2)} + \frac{N-1}{z} f_i g_i^{(1)} + z \varepsilon^2 \lambda \dot{f}_i g_i^{(1)} - \varepsilon^2 \lambda^2 \dot{f}_i g_i \right).$$

We still want to ensure that (3.32)–(3.34) hold. This is achieved by now imposing

$$(3.81) \quad f_1 := -\frac{1}{2} \varepsilon^2 \lambda \dot{f}_0 g_0^{(1)} \left( \frac{1}{2} \right) + \varepsilon^2 \lambda^2 \dot{f}_0 g_0 \left( \frac{1}{2} \right),$$

$$(3.82) \quad f_2 := - \left[ \left( 2(N-1) f_1 + \frac{1}{2} \varepsilon^2 \lambda \dot{\lambda} \right) + \frac{1}{2} \varepsilon^2 \lambda \dot{f}_0 g_0^{(2)} \left( \frac{1}{2} \right) + (\varepsilon^2 \lambda \dot{f}_0 - \varepsilon^2 \lambda^2 \dot{f}_0) g_0^{(1)} \left( \frac{1}{2} \right) \right],$$

$$(3.83) \quad \begin{aligned} f_3 := & - \left[ \left( 2(N-1) + \frac{1}{2} \varepsilon^2 \lambda \dot{\lambda} \right) f_2 + (2\varepsilon^2 \lambda \dot{\lambda} - 8(N-1)) f_1 - \varepsilon^2 \lambda^2 \dot{f}_1 \right. \\ & \left. + \frac{1}{2} \varepsilon^2 \lambda \dot{f}_0 g_0^{(3)} \left( \frac{1}{2} \right) + (2\varepsilon^2 \lambda \dot{f}_0 - \varepsilon^2 \lambda^2 \dot{f}_0) g_0^{(2)} \left( \frac{1}{2} \right) \right], \end{aligned}$$

where the  $g_i$ 's now satisfy

$$(3.84) \quad g_1^{(2)} \left( \frac{1}{2} \right) = g_2^{(3)} \left( \frac{1}{2} \right) = g_3^{(4)} \left( \frac{1}{2} \right) = 1,$$

$$(3.85) \quad g_i^{(j)} \left( \frac{1}{2} \right) = 0 \quad \forall (i, j) \in \{1, 2, 3\} \times \{0, 1, 2, 3, 4\} \setminus \{(1, 2), (2, 3), (3, 4)\}.$$

Then  $a$  still satisfies (3.62) for some function  $\psi^a$  of class  $C^\infty$  on  $[-1, 1] \times [\rho - \delta, \rho + \delta]$ . Proceeding as we did to prove (3.78), we get the existence of  $\psi^b$  of class  $C^\infty$  on  $[-1, 1] \times [\rho - \delta, \rho + \delta]$  such that (3.64) holds. Now the case of the function  $c$  is simpler than before, since, at least for  $\varepsilon > 0$  small enough, we get from (3.79) that  $c > 0$  in  $\Sigma$  and the existence  $\psi^c$  of class  $C^\infty$  on  $[-1, 1] \times [\rho - \delta, \rho + \delta]$  such that (3.78) holds.

The case where

$$(3.86) \quad (t, r) \in \Sigma' := \left\{ (t, r) \in (-1, 1) \times \mathbb{R}; z \in (-1, 1) \setminus \left( \cup_{l=-1}^p [\rho_l - (\delta/2), \rho_l + (\delta/2)] \right) \right\}$$

is even simpler than the two previous ones, since, by (3.9),

$$(3.87) \quad g_1 = g_2 = g_3 = 0.$$

One gets that (3.62), (3.64), and (3.78) hold on  $\Sigma'$  where

$$(3.88) \quad \psi^a, \psi^b, \psi^c \in C^\infty \left( [-1, 1] \times \left( [-1, 1] \setminus \left( \cup_{l=-1}^p (\rho_l - (\delta/2), \rho_l + (\delta/2)) \right) \right) \right).$$

In conclusion, from these three cases we get the existence of three functions  $\psi^a$ ,  $\psi^b$ , and  $\psi^c$  such that

$$(3.89) \quad \psi^a, \psi^b, \psi^c \in C^\infty([-1, 1] \times [-1, 1]),$$

$$(3.90) \quad a(t, r) = f_0(t)\psi^a(t, z) \quad \forall (t, r) \in \mathbb{D},$$

$$(3.91) \quad b(t, r) = \lambda(t)^{-2/3}f_0(t)\psi^b(t, z) \quad \forall (t, r) \in \mathbb{D},$$

$$(3.92) \quad c(t, r) = \lambda(t)^{-8/9}f_0(t)\psi^c(t, z) \quad \forall (t, r) \in \mathbb{D},$$

which, together with (3.1) and (3.2), imply that, if  $a$ ,  $b$ , and  $c$  are extended to all of  $\mathbb{R} \times \mathbb{R}$  by 0 outside  $\mathbb{D}$ , then  $a$ ,  $b$ , and  $c$  are of class  $C^\infty$  on  $\mathbb{R} \times \mathbb{R}$ . This concludes the proof of Theorem 1.2.  $\square$

**4. Proof of Theorem 1.1.** In this section, we show how to deduce Theorem 1.1 from Theorem 1.2 by means of the return method, an algebraic solvability, and classical controllability results.

Let  $x_0 \in \omega$ . Let  $\bar{r} > 0$  be small enough so that

$$(4.1) \quad \left( \left| t - \frac{T}{2} \right| \leq \bar{r}^2 \text{ and } |x - \bar{x}_0| \leq \bar{r} \right) \Rightarrow (t \in (0, T) \text{ and } x \in \omega).$$

Let  $\bar{\alpha} : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ ,  $\bar{\beta} : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ ,  $\bar{\gamma} : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ , and  $\bar{u} : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$  be defined by, for every  $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ ,

$$(4.2) \quad \bar{\alpha}(t, x) := \bar{r}^8 a \left( \frac{t - (T/2)}{\bar{r}^2}, \frac{1}{\bar{r}} |x - x_0| \right),$$

$$(4.3) \quad \bar{\beta}(t, x) := \bar{r}^2 b \left( \frac{t - (T/2)}{\bar{r}^2}, \frac{1}{\bar{r}} |x - x_0| \right),$$

$$(4.4) \quad \bar{\gamma}(t, x) := c \left( \frac{t - (T/2)}{\bar{r}^2}, \frac{1}{\bar{r}} |x - x_0| \right),$$

$$(4.5) \quad \bar{u}(t, x) := \bar{\gamma}_t(t, x) - \Delta \bar{\gamma}(t, x).$$

From (1.5), (1.7), (1.8), (1.9), (4.2), (4.3), (4.4), and (4.5), the functions  $\bar{\alpha}$ ,  $\bar{\beta}$ ,  $\bar{\gamma}$ , and  $\bar{u}$  are of class  $C^\infty$  and satisfy

$$(4.6) \quad \bar{\alpha}_t - \Delta \bar{\alpha} = \bar{\beta}^3 \quad \text{in } \mathbb{R} \times \mathbb{R}^N,$$

$$(4.7) \quad \bar{\beta}_t - \Delta \bar{\beta} = \bar{\gamma}^3 \quad \text{in } \mathbb{R} \times \mathbb{R}^N,$$

$$(4.8) \quad \bar{\gamma}_t - \Delta \bar{\gamma} = \bar{u} \chi_\omega \quad \text{in } \mathbb{R} \times \mathbb{R}^N,$$

$$(4.9) \quad \text{the supports of } \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \text{ and } \bar{u} \text{ are included in } (0, T) \times \omega.$$

Let  $(\alpha^0, \beta^0, \gamma^0)^{\text{tr}} \in L^\infty(\Omega)^3$ . For  $(\alpha, \beta, \gamma)^{\text{tr}} \in L^\infty((0, T) \times \Omega)^3$  and  $u \in L^\infty((0, T) \times \Omega)$ , let us define  $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})^{\text{tr}} \in L^\infty((0, T) \times \Omega)^3$  and  $\hat{u} \in L^\infty((0, T) \times \Omega)$ , for every  $(t, x) \in (0, T) \times \Omega$ , by

$$(4.10) \quad \hat{\alpha}(t, x) := \alpha(t, x) - \bar{\alpha}(t, x),$$

$$(4.11) \quad \hat{\beta}(t, x) := \beta(t, x) - \bar{\beta}(t, x),$$

$$(4.12) \quad \hat{\gamma}(t, x) := \gamma(t, x) - \bar{\gamma}(t, x),$$

$$(4.13) \quad \hat{u}(t, x) := u(t, x) - \bar{u}(t, x).$$

From (4.6), (4.7), (4.8), and (4.9),  $(\alpha, \beta, \gamma)^{\text{tr}} \in L^\infty((0, T) \times \Omega)^3$  is the solution of the Cauchy problem (1.2) if and only if  $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})^{\text{tr}} \in L^\infty((0, T) \times \Omega)^3$  is the solution of the Cauchy problem

$$(4.14) \quad \begin{cases} \hat{\alpha}_t - \Delta \hat{\alpha} = 3\bar{\beta}^2 \hat{\beta} + 3\bar{\beta} \hat{\beta}^2 + \hat{\beta}^3 & \text{in } (0, T) \times \Omega, \\ \hat{\beta}_t - \Delta \hat{\beta} = 3\bar{\gamma}^2 \hat{\gamma} + 3\bar{\gamma} \hat{\gamma}^2 + \hat{\gamma}^3 & \text{in } (0, T) \times \Omega, \\ \hat{\gamma}_t - \Delta \hat{\gamma} = \hat{u} \chi_\omega & \text{in } (0, T) \times \Omega, \\ \hat{\alpha} = \hat{\beta} = \hat{\gamma} = 0 & \text{in } (0, T) \times \partial\Omega, \\ \hat{\alpha}(0, \cdot) = \alpha^0(\cdot), \hat{\beta}(0, \cdot) = \beta^0(\cdot), \hat{\gamma}(0, \cdot) = \gamma^0(\cdot) & \text{in } \Omega. \end{cases}$$

Moreover, by (4.9), (4.10), (4.11), and (4.12), one has

$$(4.15) \quad \alpha(T, \cdot) = \hat{\alpha}(T, \cdot), \quad \beta(T, \cdot) = \hat{\beta}(T, \cdot), \quad \gamma(T, \cdot) = \hat{\gamma}(T, \cdot) \quad \text{in } \Omega.$$

Let us consider the system

$$(4.16) \quad \begin{cases} \hat{\alpha}_t - \Delta \hat{\alpha} = 3\bar{\beta}^2 \hat{\beta} + 3\bar{\beta} \hat{\beta}^2 + \hat{\beta}^3 & \text{in } (0, T) \times \Omega, \\ \hat{\beta}_t - \Delta \hat{\beta} = 3\bar{\gamma}^2 \hat{\gamma} + 3\bar{\gamma} \hat{\gamma}^2 + \hat{\gamma}^3 & \text{in } (0, T) \times \Omega, \\ \hat{\gamma}_t - \Delta \hat{\gamma} = \hat{u} \chi_\omega & \text{in } (0, T) \times \Omega, \\ \hat{\alpha} = \hat{\beta} = \hat{\gamma} = 0 & \text{in } (0, T) \times \partial\Omega \end{cases}$$

as a control system where, at time  $t \in [0, T]$ , the state is  $(\hat{\alpha}(t, \cdot), \hat{\beta}(t, \cdot), \hat{\gamma}(t, \cdot))^{\text{tr}} \in L^\infty(\Omega)^3$ , and the control is  $\hat{u}(t, \cdot) \in L^\infty(\Omega)$ . Note that  $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})^{\text{tr}} = 0$  and  $\hat{u} = 0$  is a trajectory (i.e., a solution) of this control system. The linearized control system around this (null) trajectory is the linear control system

$$(4.17) \quad \begin{cases} \hat{\alpha}_t - \Delta \hat{\alpha} = 3\bar{\beta}^2 \hat{\beta} & \text{in } (0, T) \times \Omega, \\ \hat{\beta}_t - \Delta \hat{\beta} = 3\bar{\gamma}^2 \hat{\gamma} & \text{in } (0, T) \times \Omega, \\ \hat{\gamma}_t - \Delta \hat{\gamma} = \hat{u} \chi_\omega & \text{in } (0, T) \times \Omega, \\ \hat{\alpha} = \hat{\beta} = \hat{\gamma} = 0 & \text{in } (0, T) \times \partial\Omega, \end{cases}$$

where, at time  $t \in [0, T]$ , the state is  $(\hat{\alpha}(t, \cdot), \hat{\beta}(t, \cdot), \hat{\gamma}(t, \cdot))^{\text{tr}} \in L^\infty(\Omega)^3$ , and the control is  $\hat{u}(t, \cdot) \in L^\infty(\Omega)$ .

By (1.6), (4.3), and (4.4), there exists a nonempty open subset  $\omega_1$  of  $\omega$ ,  $t_1 \in (0, T)$ , and  $t_2 \in (0, T)$  such that

$$(4.18) \quad \overline{\omega_1} \subset \omega,$$

$$(4.19) \quad 0 < t_1 < t_2 < T,$$

$$(4.20) \quad \bar{\beta}(t, x) \neq 0 \quad \forall (t, x) \in [t_1, t_2] \times \overline{\omega_1},$$

$$(4.21) \quad \bar{\gamma}(t, x) \neq 0 \quad \forall (t, x) \in [t_1, t_2] \times \overline{\omega_1}.$$

Let  $\omega_2$  be a nonempty open subset of  $\omega_1$  such that

$$(4.22) \quad \overline{\omega_2} \subset \omega_1.$$

Let us recall, by (the proof of) [24, Theorem 2.4, Chapter 1], the linear control system

$$(4.23) \quad \begin{cases} \hat{\alpha}_t - \Delta \hat{\alpha} = 3\bar{\beta}^2 \hat{\beta} + v_1 \chi_{(t_1, t_2) \times \omega_2} & \text{in } (0, t_2) \times \Omega, \\ \hat{\beta}_t - \Delta \hat{\beta} = 3\bar{\gamma}^2 \hat{\gamma} + v_2 \chi_{(t_1, t_2) \times \omega_2} & \text{in } (0, t_2) \times \Omega, \\ \hat{\gamma}_t - \Delta \hat{\gamma} = v_3 \chi_{(t_1, t_2) \times \omega_2} & \text{in } (0, t_2) \times \Omega, \\ \hat{\alpha} = \hat{\beta} = \hat{\gamma} = 0 & \text{in } (0, t_2) \times \partial\Omega, \end{cases}$$

where, at time  $t \in [0, t_2]$ , the state is  $(\hat{\alpha}(t, \cdot), \hat{\beta}(t, \cdot), \hat{\gamma}(t, \cdot))^{\text{tr}} \in L^\infty(\Omega)^3$  and the control is  $(v_1(t, \cdot), v_2(t, \cdot), v_3(t, \cdot))^{\text{tr}} \in L^\infty(\Omega)^3$ , is null controllable. We next point out that with the terminology of [27, page 148] (see also [19]), the underdetermined system

$$(4.24) \quad \begin{cases} \tilde{\alpha}_t - \Delta \tilde{\alpha} = 3\bar{\beta}^2 \tilde{\beta} + v_1 & \text{in } (t_1, t_2) \times \omega_1, \\ \tilde{\beta}_t - \Delta \tilde{\beta} = 3\bar{\gamma}^2 \tilde{\gamma} + v_2 & \text{in } (t_1, t_2) \times \omega_1, \\ \tilde{\gamma}_t - \Delta \tilde{\gamma} = v_3 + \tilde{u} & \text{in } (t_1, t_2) \times \omega_1, \end{cases}$$

where the data is  $(v_1, v_2, v_3)^{\text{tr}} : (t_1, t_2) \times \omega_1 \rightarrow \mathbb{R}^3$  and the unknown is  $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{u})^{\text{tr}} : (t_1, t_2) \times \omega_1 \rightarrow \mathbb{R}^4$ , is algebraically solvable; i.e., there are solutions of (4.24) such that the unknown can be expressed in terms of the derivatives of the data. Indeed, for  $(v_1, v_2, v_3)^{\text{tr}} \in \mathcal{D}'((t_1, t_2) \times \omega_1)^3$ , if  $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{u})^{\text{tr}} \in \mathcal{D}'((t_1, t_2) \times \omega_1)^4$  is defined by

$$(4.25) \quad \tilde{\alpha} := 0,$$

$$(4.26) \quad \tilde{\beta} := -\frac{v_1}{3\bar{\beta}^2},$$

$$(4.27) \quad \tilde{\gamma} := \frac{1}{3\bar{\gamma}^2} \left( -\left(\frac{v_1}{3\bar{\beta}^2}\right)_t + \Delta \left(\frac{v_1}{3\bar{\beta}^2}\right) - v_2 \right),$$

$$(4.28) \quad \begin{aligned} \tilde{u} := & -v_3 + \left( \frac{1}{3\bar{\gamma}^2} \left( -\left(\frac{v_1}{3\bar{\beta}^2}\right)_t + \Delta \left(\frac{v_1}{3\bar{\beta}^2}\right) - v_2 \right) \right)_t \\ & - \Delta \left( \frac{1}{3\bar{\gamma}^2} \left( -\left(\frac{v_1}{3\bar{\beta}^2}\right)_t + \Delta \left(\frac{v_1}{3\bar{\beta}^2}\right) - v_2 \right) \right), \end{aligned}$$

then (4.24) holds. This algebraic solvability is a key ingredient for the following proposition.

**PROPOSITION 4.1.** *There exists  $\eta > 0$  such that for every  $(\alpha^0, \beta^0, \gamma^0)^{\text{tr}} \in L^\infty(\Omega)^3$  satisfying*

$$(4.29) \quad |\alpha^0|_{L^\infty(\Omega)} + |\beta^0|_{L^\infty(\Omega)} + |\gamma^0|_{L^\infty(\Omega)} < \eta,$$

*there exists  $\hat{u} \in L^\infty((0, t_2) \times \Omega)$  such that the solution  $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})^{\text{tr}} \in L^\infty((0, t_2) \times \Omega)^3$  of the Cauchy problem*

$$(4.30) \quad \begin{cases} \hat{\alpha}_t - \Delta \hat{\alpha} = 3\bar{\beta}^2 \hat{\beta} + 3\bar{\beta} \hat{\beta}^2 + \hat{\beta}^3 & \text{in } (0, t_2) \times \Omega, \\ \hat{\beta}_t - \Delta \hat{\beta} = 3\bar{\gamma}^2 \hat{\gamma} + 3\bar{\gamma} \hat{\gamma}^2 + \hat{\gamma}^3 & \text{in } (0, t_2) \times \Omega, \\ \hat{\gamma}_t - \Delta \hat{\gamma} = \hat{u} \chi_\omega & \text{in } (0, t_2) \times \Omega, \\ \hat{\alpha} = \hat{\beta} = \hat{\gamma} = 0 & \text{in } (0, t_2) \times \partial\Omega, \\ \hat{\alpha}(0, \cdot) = \alpha^0(\cdot), \hat{\beta}(0, \cdot) = \beta^0(\cdot), \hat{\gamma}(0, \cdot) = \gamma^0(\cdot) & \text{in } \Omega \end{cases}$$

*satisfies*

$$(4.31) \quad \hat{\alpha}(t_2, \cdot) = \hat{\beta}(t_2, \cdot) = \hat{\gamma}(t_2, \cdot) = 0 \quad \text{in } \Omega.$$

The proof of Proposition 4.1 is given in Appendix A. It is an adaptation of [19], which deals with Navier–Stokes equations, to our parabolic system. Besides a suitable inverse mapping theorem, it mainly consists of the following two steps.

1. Prove that the control system (4.30) with two “fictitious” controls added on the first two equations is null controllable by means of smooth controls. See Proposition A.2.

2. Remove the two “fictitious” controls by using the algebraic solvability, as in [13] and [19]. See (the proof of) Proposition A.5.

With the notation of Proposition 4.1, we extend  $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})^{\text{tr}}$  and  $\hat{u}$  to all of  $(0, T) \times \Omega$  by requiring

$$(4.32) \quad \hat{\alpha}(t, x) = \hat{\beta}(t, x) = \hat{\gamma}(t, x) = \hat{u}(t, x) = 0 \quad \forall (t, x) \in (t_2, T) \times \Omega.$$

Then, by (4.30) and (4.31), one has (4.14) and

$$(4.33) \quad \hat{\alpha}(T, \cdot) = \hat{\beta}(T, \cdot) = \hat{\gamma}(T, \cdot) = 0 \quad \text{in } \Omega.$$

Let us define  $(\alpha, \beta, \gamma)^{\text{tr}} \in L^\infty((0, T) \times \Omega)^3$  and  $u \in L^\infty((0, T) \times \Omega)$  by imposing (4.10), (4.11), (4.12), and (4.13). Then, from (4.14), one has (1.2) and, using (4.15) together with (4.33), one has (1.3). This concludes the proof of Theorem 1.1 if (4.29) holds.

However, assumption (4.29) can be removed by using the following simple homogeneity argument: If  $((\alpha, \beta, \gamma)^{\text{tr}}, u) \in L^\infty((0, T) \times \Omega)^3 \times L^\infty((0, T) \times \Omega)$  is a trajectory (i.e., a solution) of the control system (1.1), then for every  $s > 0$ ,  $((\alpha^s, \beta^s, \gamma^s)^{\text{tr}}, u^s) := ((s^9 \alpha, s^3 \beta, s \gamma)^{\text{tr}}, su)$  is a trajectory (i.e., a solution) of the control system (1.1). This concludes the proof of Theorem 1.1.  $\square$

**Appendix A. Proof of Proposition 4.1.** Let  $\hat{1}_{\omega_2} : \mathbb{R}^3 \rightarrow [0, 1]$  be a function of class  $C^\infty$  which is equal to 1 on  $\omega_2$  and whose support is included in  $\omega_1$ , and let  $\zeta : \mathbb{R} \rightarrow [0, 1]$  be such that  $\zeta$  is equal to 0 on  $(-\infty, (2t_1 + t_2)/3]$  and is equal to 1 on  $((t_1 + 2t_2)/3, +\infty)$ . Let  $\vartheta : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}$  be defined by

$$(A.1) \quad \vartheta(t, x) := \zeta(t) \hat{1}_{\omega_2}(x) \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^3.$$

From now on, we set  $Q := (t_1, t_2) \times \Omega$  and, for  $\eta \in (0, 1)$  and  $K > 0$ ,

$$(A.2) \quad \rho_\eta(t) := e^{\frac{-K}{\eta(t_2-t)}}, \quad \rho_1(t) := e^{\frac{-K}{(t_2-t)}} \quad \forall t \in [t_1, t_2].$$

We have the following Carleman estimates, which were proved in [24, Chapter 1].

LEMMA A.1. *Let  $\eta \in (0, 1)$ . There exist  $K := K(\eta) > 0$  and  $C := C(K) > 0$  such that, for every  $g = (g_1, g_2, g_3)^{\text{tr}} \in L^2((t_1, t_2) \times \Omega)^3$  and for every solution  $z = (\hat{\alpha}, \hat{\beta}, \hat{\gamma})^{\text{tr}} \in L^2((t_1, t_2), H^2(\Omega)^3) \cap H^1((t_1, t_2), L^2(\Omega)^3)$  of the parabolic system*

$$(A.3) \quad \begin{cases} -\hat{\alpha}_t - \Delta \hat{\alpha} = g_1 & \text{in } (t_1, t_2) \times \Omega, \\ -\hat{\beta}_t - \Delta \hat{\beta} - 3\hat{\beta}^2 \hat{\alpha} = g_2 & \text{in } (t_1, t_2) \times \Omega, \\ -\hat{\gamma}_t - \Delta \hat{\gamma} - 3\hat{\gamma}^2 \hat{\beta} = g_3 & \text{in } (t_1, t_2) \times \Omega, \\ \hat{\alpha} = \hat{\beta} = \hat{\gamma} = 0 & \text{in } (t_1, t_2) \times \partial\Omega, \end{cases}$$

which is the adjoint of (4.17), one has

$$(A.4) \quad |\sqrt{\rho_\eta} z|_{L^2(Q)^3}^2 + |z(t_1, \cdot)|_{L^2(\Omega)^3}^2 \leq C \left( \int_{(t_1, t_2) \times \Omega} \vartheta \rho_1 |z|^2 + \int_{(t_1, t_2) \times \Omega} \rho_1 |g|^2 \right).$$

Let us now derive from Lemma A.1 a proposition on the null controllability with controls which are smooth functions for the control system (4.17) with a right-hand side term.



PROPOSITION A.2. Let  $\eta \in (0, 1)$  be such that

$$(A.5) \quad \eta > \frac{2}{3},$$

and let  $K$  be as in Lemma A.1. Let  $k \in \mathbb{N}$ , and let  $p \in [2, +\infty)$ . Then for every  $f = (f_1, f_2, f_3)^{\text{tr}} \in L^p(Q)^3$  such that  $\rho_\eta^{-1/2} f \in L^p(Q)^3$  and for every  $(\alpha^0, \beta^0, \gamma^0)^{\text{tr}} \in W_0^{1,p}(\Omega)^3 \cap W^{2,p}(\Omega)^3$ , there exists  $u = (u_1, u_2, u_3) \in L^2(Q)^3$  satisfying

$$(A.6) \quad e^{\frac{K\eta^2}{2(t_2-t)}} \vartheta u \in L^2((t_1, t_2), H^{2k}(\Omega)^3) \cap H^k((t_1, t_2), L^2(\Omega)^3),$$

such that the solution  $\hat{y} := (\hat{\alpha}, \hat{\beta}, \hat{\gamma})^{\text{tr}}$  of

$$(A.7) \quad \begin{cases} \hat{\alpha}_t - \Delta \hat{\alpha} = 3\bar{\beta}^2 \hat{\beta} + f_1 + \vartheta u_1 & \text{in } (t_1, t_2) \times \Omega, \\ \hat{\beta}_t - \Delta \hat{\beta} = 3\bar{\gamma}^2 \hat{\gamma} + f_2 + \vartheta u_2 & \text{in } (t_1, t_2) \times \Omega, \\ \hat{\gamma}_t - \Delta \hat{\gamma} = f_3 + \vartheta u_3 & \text{in } (t_1, t_2) \times \Omega, \\ \hat{\alpha} = \hat{\beta} = \hat{\gamma} = 0 & \text{in } (t_1, t_2) \times \partial\Omega, \\ \hat{\alpha}(t_1, \cdot) = \alpha^0(\cdot), \hat{\beta}(t_1, \cdot) = \beta^0(\cdot), \hat{\gamma}(t_1, \cdot) = \gamma^0(\cdot) & \text{in } \Omega \end{cases}$$

satisfies

$$(A.8) \quad e^{\frac{K}{2(t_2-t)}} \hat{y} \in L^p((t_1, t_2), W^{2,p}(\Omega)^3) \cap W^{1,p}((t_1, t_2), L^p(\Omega)^3).$$

*Proof of Proposition A.2.* We adapt the proof of [19, Proposition 4] to our situation. Modifying  $f$  if necessary, we may assume without loss of generality that

$$(A.9) \quad (\alpha^0, \beta^0, \gamma^0)^{\text{tr}} = 0.$$

Let us define a linear operator  $S : \mathcal{D}'(Q)^3 \rightarrow \mathcal{D}'(Q)^3$  by

$$(A.10) \quad Sz := \begin{pmatrix} -\alpha_t - \Delta \alpha \\ -\beta_t - \Delta \beta - 3\bar{\beta}^2 \alpha \\ -\gamma_t - \Delta \gamma - 3\bar{\gamma}^2 \beta \end{pmatrix} \quad \forall z = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \in \mathcal{D}'(Q)^3.$$

We define a closed linear unbounded operator  $\mathcal{S} : \mathcal{D}(\mathcal{S}) \subset L^2(Q)^3 \rightarrow L^2(Q)^3$  by

$$(A.11) \quad \mathcal{D}(\mathcal{S}) := \{z = (\alpha, \beta, \gamma)^{\text{tr}} \in L^2((t_1, t_2), H_0^1 \cap H^2(\Omega)^3) \\ \cap H^1((t_1, t_2), L^2(\Omega)^3); z(t_2, \cdot) = 0\},$$

$$(A.12) \quad \mathcal{S}z = Sz.$$

Let

$$(A.13) \quad X_0 := L^2(Q).$$

For  $m \in \mathbb{N} \setminus \{0\}$ , we set

$$(A.14) \quad X_m := \mathcal{D}(\mathcal{S}^m).$$

Let us point out that

$$(A.15) \quad \langle z_1, z_2 \rangle_{X_m} := \langle \mathcal{S}^m z_1, \mathcal{S}^m z_2 \rangle_{L^2(Q)^3}$$

is a scalar product on  $X_m$ . From now on,  $X_m$  is equipped with this scalar product. Then  $X_m$  is a Hilbert space. For  $m \in \mathbb{Z} \cap (-\infty, 0)$ , let

$$(A.16) \quad X_m := X'_{-m},$$

where  $X'_{-m}$  denotes the dual space of  $X_{-m}$ . We choose the pivot space  $L^2(Q)^3 = X_0$ . In particular, (A.16) is an equality for  $m = 0$ . For every  $(k, l) \in \mathbb{Z}^2$  such that  $k \leq l$ , one has

$$(A.17) \quad X_l \subset X_k.$$

Note that since  $\Omega$  is only of class  $C^2$ , in general, for  $m \in \mathbb{N} \setminus \{0, 1\}$ ,

$$(A.18) \quad X_m \not\subset L^2((t_1, t_2), H^{2m}(\Omega)^3) \cap H^m((t_1, t_2), L^2(\Omega)^3).$$

However, even with  $\Omega$  only of class  $C^2$ , by classical results on the interior regularity of parabolic systems, for every  $m \in \mathbb{N}$ , for every open subset  $\Omega_0$  such that  $\bar{\Omega}_0 \subset \Omega$ , and for every  $z \in X_m$ ,

$$(A.19) \quad z|_{(t_1, t_2) \times \Omega_0} \in L^2((t_1, t_2), H^{2m}(\Omega_0)^3) \cap H^m((t_1, t_2), L^2(\Omega_0)^3).$$

(Note that this property is not known to hold for the linearized Navier–Stokes equations considered in [19] for  $\Omega$  only of class  $C^2$ ; this is why  $\Omega$  is assumed to be of class  $C^\infty$  in [19].)

For  $m \in \mathbb{N}$ , one can define  $\mathcal{S}^*$  as an operator from  $X_{-m}$  into  $X_{-m-1}$  by setting, for every  $z_1 \in X_{-m-1}$  and for every  $z_2 \in X_{m+1}$ ,

$$(A.20) \quad \langle \mathcal{S}^* z_1, z_2 \rangle_{X_{-m-1}, X_{m+1}} := \langle z_1, \mathcal{S} z_2 \rangle_{X_{-m}, X_m}.$$

(One easily checks that this definition is consistent: it gives the same image if  $z_1$  is also in  $X_{-m'}$  for some  $m' \in \mathbb{N}$ .) This implies, in particular, that for every  $z_1 \in L^2(Q)^3$  and for every  $z_2 \in X_m$ , one has, for every  $0 \leq j \leq l$ ,

$$(A.21) \quad \langle (\mathcal{S}^*)^l z_1, z_2 \rangle_{X_{-l}, X_l} = \langle (\mathcal{S}^*)^{l-j} z_1, (\mathcal{S})^j z_2 \rangle_{X_{j-l}, X_{l-j}}.$$

Let  $\mathcal{H}_0$  be the set of  $z \in H^1((t_1, t_2), L^2(\Omega)^3) \cap L^2((t_1, t_2), H^2(\Omega)^3)$  such that

$$(A.22) \quad \sqrt{\rho_1} S z \in X_k,$$

$$(A.23) \quad \sqrt{\vartheta \rho_1} z \in L^2(Q)^3.$$

Let  $q$  be the following bilinear form defined on  $\mathcal{H}_0$ :

$$(A.24) \quad q(z, w) := \langle \sqrt{\rho_1} S z, \sqrt{\rho_1} S w \rangle_{X_k} + \int_Q \vartheta \rho_1 z \cdot w.$$

(This is the analogue of the bilinear form denoted by  $a$  in [19].) From (A.4), we deduce that  $q$  is a scalar product on  $\mathcal{H}_0$ . Let  $\mathcal{H}$  be the completion of  $\mathcal{H}_0$  for this scalar product. Note that, still from (A.4) and also from the definition of  $\mathcal{H}$ ,  $\mathcal{H}$  is a subspace of  $L^2_{loc}((t_1, t_2), H^1_0(\Omega)^3)$ , and, for every  $z \in \mathcal{H}$ , one has (A.22), (A.23), and

$$(A.25) \quad |\rho_\eta|^{1/2} z|_{L^2(Q)^3} \leq C \sqrt{q(z, z)} \quad \forall z \in \mathcal{H}.$$

As in [19], using the Riesz representation theorem, together with (A.25), one gets that there exists a unique

$$(A.26) \quad \hat{z} \in \mathcal{H}$$

verifying, for every  $w \in \mathcal{H}$ ,

$$(A.27) \quad \langle \mathcal{S}^k(\sqrt{\rho_1}S\hat{z}), \mathcal{S}^k(\sqrt{\rho_1}Sw) \rangle_{L^2(Q)^3} - \int_Q u \cdot w = \int_Q f \cdot w,$$

with

$$(A.28) \quad u := -\rho_1 \hat{z}.$$

We then set

$$(A.29) \quad \tilde{y} := (\mathcal{S}^*)^k \mathcal{S}^k(\sqrt{\rho_1}S\hat{z}) \in X_{-k}.$$

We want to gain regularity on  $\tilde{y}$  by accepting a weaker exponential decay rate for  $\tilde{y}$  when  $t$  is close to  $t_2$  (in the spirit of [24, Theorem 2.4, Chapter 1] and [9]). Let  $\psi \in C^\infty([t_1, t_2])$  and  $y \in X_{-1}$ . One can define  $\psi y \in X_{-1}$  in the following way. Since  $\mathcal{S}^* : X_0 \rightarrow X_{-1}$  is onto, there exists  $h \in X_0$  such that  $\mathcal{S}^*h = y$ . We define  $\psi y$  by

$$(A.30) \quad \psi y = \psi \mathcal{S}^*h := -\psi' h + \mathcal{S}^*(\psi h).$$

This definition is compatible with the usual definition of  $\psi y$  if  $y \in X_0$ . We can then define by induction on  $m$   $\psi y \in X_{-m}$  for  $\psi \in C^\infty([t_1, t_2])$  and  $y \in X_{-m}$  in the same way. Using (A.29), this allows us to define

$$(A.31) \quad \hat{y} := \sqrt{\rho_1} \tilde{y} \in X_{-k}.$$

From (A.27), (A.28), (A.29), and (A.31), one gets

$$(A.32) \quad \mathcal{S}^* \hat{y} = f + \vartheta u \quad \text{in } X_{-k-1}.$$

Let

$$(A.33) \quad \tilde{K} \in (0, K) \quad \text{and} \quad \tilde{\rho}_1 := e^{-\tilde{K}/(t_2-t)}.$$

Using (A.28), (A.29), and (A.32), one has

$$(A.34) \quad \mathcal{S}^* \left( \left( \frac{\sqrt{\rho_1}}{\sqrt{\tilde{\rho}_1}} \right) \tilde{y} \right) = \left( \frac{1}{\sqrt{\tilde{\rho}_1}} \right)' \sqrt{\rho_1} \tilde{y} + \left( \frac{1}{\sqrt{\tilde{\rho}_1}} \right) (f + u) \quad \text{in } X_{-k}.$$

We want to deduce from (A.34) some information on the regularity of  $\tilde{y}$ . This can be achieved thanks to the following lemma, the proof of which is similar to the proof of [19, Lemma 4].

LEMMA A.3. *Let  $m \in \mathbb{N}$ . If  $y \in X_{-m}$  and  $\mathcal{S}^*y \in X_{-m}$ , then  $y \in X_{-m+1}$ .*

From (A.31), (A.34), and Lemma A.3, one gets that

$$\left( \frac{\sqrt{\rho_1}}{\sqrt{\tilde{\rho}_1}} \right) \tilde{y} \in X_{-k+1} \quad \forall \tilde{K} \in (0, K).$$

Using an easy induction argument together with Lemma A.3 (and the fact that one can choose  $\tilde{K} < K$  arbitrarily close to  $K$ ), we deduce that for every  $\tilde{K} \in (0, K)$ ,  $(\sqrt{\rho_1}/\sqrt{\tilde{\rho}_1})\tilde{y} \in X_0$ .

Let us now focus on  $u$ . Let us define

$$(A.35) \quad v := \rho_1 \hat{z}.$$

Using (A.25), one gets that

$$(A.36) \quad \rho_1^{-1} \rho_\eta^{1/2} v \in L^2(Q)^3.$$

Using (A.26) together with regularity results for  $S$  applied on  $\tilde{\rho}_1^{-1} \rho_\eta^{1/2} v \in L^2(Q)^3$  and, as above for the proof of (A.36), a bootstrap argument (together with the fact that one can choose  $\tilde{K} \in (0, K)$  arbitrarily close to  $K$ ), one obtains that

$$(A.37) \quad \tilde{\rho}_1^{-1} \rho_\eta^{1/2} v \in X_k \quad \forall \tilde{K} \in (0, K).$$

Let us point out that (A.5) implies that

$$(A.38) \quad \eta^2 - 2 + \frac{1}{\eta} < 0.$$

From (A.5), (A.19), (A.28), (A.35), (A.37), and (A.38), one gets (A.6).

Let us now deal with  $\hat{y}$ . Without loss of generality, we may assume that

$$(A.39) \quad 4k > 2 + N,$$

so that

$$(A.40) \quad L^2((t_1, t_2), H^{2k}(\Omega)^3) \cap H^k((t_1, t_2), L^2(\Omega)^3) \subset L^\infty(Q).$$

From (A.32), (A.40), and (A.37), we deduce (by looking at the parabolic system verified by  $(1/\sqrt{\tilde{\rho}_1})\hat{y}$  and using usual regularity results for linear parabolic systems) that

$$(A.41) \quad \left(1/\sqrt{\tilde{\rho}_1}\right) \hat{y} \in L^p((t_1, t_2), W^{2,p}(\Omega)^3) \cap W^{1,p}((t_1, t_2), L^p(\Omega)^3) \quad \forall \tilde{K} \in (0, K),$$

which, together with (A.32), concludes the proof of Proposition A.2. □

To end the proof of Proposition 4.1, we are going to apply the following inverse mapping theorem (see [5, Chapter 2, section 2.3]).

**PROPOSITION A.4.** *Let  $E$  and  $F$  be two Banach spaces. Let  $\mathcal{F} : E \rightarrow F$  be of class  $C^1$  in a neighborhood of 0. Let us assume that the operator  $\mathcal{F}'(0) \in \mathcal{L}(E, F)$  is onto. Then there exist  $\eta > 0$  and  $C > 0$  such that for every  $g \in F$  verifying  $|g - \mathcal{F}(0)| < \eta$ , there exists  $e \in E$  such that*

1.  $\mathcal{F}(e) = g$ ,
2.  $|e|_E \leq C|g - \mathcal{F}(0)|_F$ .

We now use the same technique as in [24, Theorem 4.2]. For  $y := (\alpha, \beta, \gamma)^{\text{tr}} \in \mathcal{D}'(Q)^3$  and for  $v \in \mathcal{D}'(Q)$ , one defines  $\mathcal{L}(y, v) \in \mathcal{D}'(Q)^3$  by

$$(A.42) \quad \mathcal{L}(y, v) := \begin{pmatrix} \alpha_t - \Delta\alpha - 3\tilde{\beta}^2\beta \\ \beta_t - \Delta\beta - 3\tilde{\gamma}^2\gamma \\ \gamma_t - \Delta\gamma - v \end{pmatrix}.$$

Let  $\eta \in (0, 1)$ , and let  $K = K(\eta) > 0$  be as in Lemma A.1. We apply Proposition A.4 with  $E$  and  $F$  defined in the following way. Let  $E$  be the space of the functions

$$(y, v) \in L^p(Q)^3 \times L^\infty(Q)$$

such that

1.  $e^{\frac{\eta^3 K}{2(t_2-t)}} y \in L^p((t_1, t_2), W^{2,p}(\Omega)^3) \cap W^{1,p}((t_1, t_2), L^p(\Omega)^3)$ ,
2.  $e^{\frac{\eta^3 K}{2(t_2-t)}} v \in L^\infty(Q)^3$  and the support of  $v$  is included in  $(t_1, t_2) \times \omega$ ,
3.  $e^{\frac{K}{2\eta(t_2-t)}} \mathcal{L}(y, v) \in L^p(Q)^3$ ,
4.  $y(t_1, \cdot) \in W_0^{1,p}(\Omega)^3 \cap W^{2,p}(\Omega)^3$ ,

equipped with the following norm which makes it a Banach space:

$$(A.43) \quad |(y, v)|_E := \left| e^{\frac{\eta^3 K}{2(t_2-t)}} y \right|_{L^p((t_1, t_2), W^{2,p}(\Omega)^3) \cap W^{1,p}((t_1, t_2), L^p(\Omega)^3)} \\ + \left| e^{\frac{\eta^3 K}{2(t_2-t)}} v \right|_{L^\infty(Q)} + \left| e^{\frac{K}{2\eta(t_2-t)}} \mathcal{L}(y, v) \right|_{L^p(Q)^3} + |y(t_1, \cdot)|_{W^{2,p}(\Omega)^3}.$$

Let  $F$  be the space of the functions  $(h, y^0) \in L^p(Q)^3 \times (W_0^{1,p}(\Omega)^3 \cap W^{2,p}(\Omega)^3)$  such that

$$(A.44) \quad e^{\frac{K}{2\eta(t_2-t)}} h \in L^p(Q)^3,$$

equipped with the following norm which makes it a Banach space:

$$(A.45) \quad |(h, y^0)|_F := \left| e^{\frac{\eta K_1}{2(t_2-t)}} h \right|_{L^p(Q)^3} + |y^0|_{W^{2,p}(\Omega)^3}.$$

We define  $\mathcal{F} : E \rightarrow F$  by

$$(A.46) \quad \mathcal{F}(y, v) = \left( \mathcal{L}(y, v) - \begin{pmatrix} 3\bar{\beta}\beta^2 + \beta^3 \\ 3\bar{\gamma}\gamma^2 + \gamma^3 \\ 0 \end{pmatrix}, y(t_1, \cdot) \right).$$

One easily sees that  $\mathcal{F}$  is of class  $C^1$  if

$$(A.47) \quad p > \frac{N+2}{2} \quad \text{and} \quad \eta > \frac{1}{2^{1/4}}.$$

From now on, we assume  $p > 2$  and  $\eta \in (0, 1)$  are chosen so that (A.47) holds. Note that the second inequality of (A.47) implies that (A.5) holds. Let us assume for the moment that the following proposition holds.

**PROPOSITION A.5.** *One has*

$$(A.48) \quad \mathcal{F}'(0, 0)(E) = F.$$

Then the assumptions of Proposition A.4 hold. Since Proposition 4.1 follows from the conclusion of Proposition A.4 by taking  $\hat{u} = 0$  in  $(0, t_1) \times \Omega$ , this concludes the proof of Proposition 4.1.

It only remains to prove Proposition A.5. Let  $f = (f_1, f_2, f_3)^{\text{tr}}$  and  $y^0 = (\alpha^0, \beta^0, \gamma^0)^{\text{tr}}$  be such that  $(f, y^0) \in F$ . Let us choose  $k$  large enough so that

$$(A.49) \quad N + 2 < 4(k - 2).$$

Using Proposition A.2, we get the existence of  $u = (u_1, u_2, u_3) \in L^2(Q)^3$  satisfying (A.6) such that the solution  $\hat{y} := (\hat{\alpha}, \hat{\beta}, \hat{\gamma})^{\text{tr}}$  of (A.7) satisfies (A.8). We now use the algebraic solvability of (4.24) (i.e., that (4.25), (4.26), (4.27), and (4.28) imply (4.24)) with

$$(A.50) \quad v := \vartheta u.$$

We get that, if  $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{u})^{\text{tr}} \in \mathcal{D}'((t_1, t_2) \times \omega_1)^4$  is defined by (4.25), (4.26), (4.27), and (4.28), then (4.24) holds. We extend  $\tilde{\alpha}$ ,  $\tilde{\beta}$ ,  $\tilde{\gamma}$ , and  $\tilde{u}$  to  $(t_1, t_2) \times \Omega$  by 0 outside  $(t_1, t_2) \times (\Omega \setminus \omega_1)$  and still denote these extensions by  $\tilde{\alpha}$ ,  $\tilde{\beta}$ ,  $\tilde{\gamma}$ , and  $\tilde{u}$ . Note that (4.24) still holds on  $(t_1, t_2) \times \Omega$  and that (see, in particular, (A.1))

$$(A.51) \quad \tilde{\alpha}(t_1, \cdot) = \tilde{\beta}(t_1, \cdot) = \tilde{\gamma}(t_1, \cdot) = 0.$$

Finally, we define  $y := (\alpha, \beta, \gamma)^{\text{tr}} \in \mathcal{D}'((t_1, t_2) \times \omega_1)^3$  and  $u \in \mathcal{D}'((t_1, t_2) \times \omega_1)$  by

$$(A.52) \quad \alpha := \hat{\alpha} - \tilde{\alpha}, \quad \beta := \hat{\beta} - \tilde{\beta}, \quad \gamma := \hat{\gamma} - \tilde{\gamma}, \quad u := -\tilde{u}.$$

From (4.25), (4.26), (4.27) (4.28), (A.6), (A.8), (A.49), (A.50), and (A.52), we get that  $(y, u) \in E$ . Then, from (4.24), (A.7), (A.50), (A.51), and (A.52), we get that  $\mathcal{F}'(0, 0)(y, u) = (y^0, f)$ . This concludes the proof of Proposition A.5 and therefore also the proof of Proposition 4.1.  $\square$

*Remark A.6.* 1. Instead of proceeding as in [19] in order to prove Proposition 4.1, one can also proceed as in [18]. For that, an important step is to prove that small (in a suitable sense) perturbations of the linear control system (4.17) are controllable by means of bounded controls (see [18, section 3.1.2]). This controllability property follows from [25, Theorem 4.1], and one can also get it by following [18, section 3.1.2] or [22].

2. Let us emphasize that the algebraic solvability of (4.24) leads to a loss of derivatives. This problem is managed in our situation thanks to hypoelliptic properties of parabolic equations. These properties do not hold, for example, for hyperbolic equations. However, for these last equations, the loss of derivatives problem can be solved thanks to a Nash–Moser inverse mapping theorem due to Gromov [27, section 2.3.2, Main Theorem]. See [3] for the first use of this inverse mapping theorem in the context of control of partial differential equations.

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