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# DISSIPATIVE BOUNDARY CONDITIONS FOR NONLINEAR 1-D HYPERBOLIC SYSTEMS: SHARP CONDITIONS THROUGH AN APPROACH VIA TIME-DELAY SYSTEMS*

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**Abstract.** We analyze dissipative boundary conditions for nonlinear hyperbolic systems in one space dimension. We show that a known sufficient condition for exponential stability with respect to the $H^2$-norm is not sufficient for the exponential stability with respect to the $C^1$-norm. Hence, due to the nonlinearity, even in the case of classical solutions, the exponential stability depends strongly on the norm considered. We also give a new sufficient condition for the exponential stability with respect to the $W^{2,p}$-norm. The methods used are inspired from the theory of the linear time-delay systems and incorporate the characteristic method.

**Key words.** hyperbolic systems, dissipative boundary conditions, exponential stability, time-delay systems, nonlinearities

**AMS subject classifications.** 35L50, 93D20

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## 1. Introduction.

Let $n$ be a positive integer. We are concerned with the following nonlinear hyperbolic system:

$$u_t + F(u)u_x = 0 \quad \text{for every } (t,x) \in [0,+\infty) \times [0,1], \tag{1.1}$$

where $u : [0,+\infty) \times [0,1] \to \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathcal{M}_{n,n}(\mathbb{R})$. Here, as usual, $\mathcal{M}_{n,n}(\mathbb{R})$ denotes the set of $n \times n$ real matrices. We assume that $F$ is of class $C^\infty$ and $F(0)$ has $n$ distinct real nonzero eigenvalues. Then, replacing, if necessary, $u$ by $Mu$ where $M \in \mathcal{M}_{n,n}(\mathbb{R})$ is a suitable invertible matrix, we may assume that

$$F(0) = \operatorname{diag}(\Lambda_1, \dots, \Lambda_n) \tag{1.2}$$

with

$$\Lambda_i \in \mathbb{R},\ \Lambda_i \neq \Lambda_j \text{ for } i \neq j,\ i \in \{1,\dots,n\},\ j \in \{1,\dots,n\}. \tag{1.3}$$

For simple presentation, we assume that

$$\Lambda_i > 0 \text{ for } i = 1,\dots,n. \tag{1.4}$$

The case where $\Lambda_i$ changes sign can be worked out similarly as in [3].

In this article, we consider the boundary condition

$$u(t,0) = G\big(u(t,1)\big) \quad \text{for every } t \in [0,+\infty), \tag{1.5}$$

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where the map $G : \mathbb{R}^n \to \mathbb{R}^n$ is of class $C^\infty$ and satisfies

$$G(0) = 0, \tag{1.6}$$

which implies that 0 is a solution of

$$\begin{cases} u_t + F(u)u_x = 0 & \text{for every } (t, x) \in [0, +\infty) \times [0, 1], \\ u(t, 0) = G\big(u(t, 1)\big) & \text{for every } t \in [0, +\infty). \end{cases} \tag{1.7}$$

We are concerned about conditions on $G$ for which the equilibrium solution 0 of (1.7) is exponentially stable for (1.7).

We first review known results in the linear case, i.e., when $F$ and $G$ are linear. In that case, (1.7) is equivalent to

$$\phi_i(t) = \sum_{j=1}^{n} K_{ij}\phi_j(t - r_j) \quad \text{for } i = 1, \dots, n, \tag{1.8}$$

where

$$K = G'(0) \in \mathcal{M}_{n\times n}(\mathbb{R}) \tag{1.9}$$

and

$$\phi_i(t) := u_i(t, 0), \quad r_i := 1/\Lambda_i \quad \text{for } i = 1, \dots, n. \tag{1.10}$$

Hence, (1.7) can be viewed as a linear time-delay system. It is known from the work of Hale and Verduyn Lunel [6, Theorem 3.5, p. 275] on delay equations that 0 is exponentially stable for (1.8) if and only if there exists $\delta > 0$ such that

$$\Big(\det\big(Id_n - \big(\text{diag}(e^{-r_1 z}, \dots, e^{-r_n z})\big)K\big) = 0, z \in \mathbb{C}\Big) \implies \Re(z) \le -\delta. \tag{1.11}$$

Let us point out that [6, Theorem 3.5, p. 275] is dealing with exponentially stability with respect to the $C^0$-norm; however, the proof given in this reference also works for the $L^p$-norm for every $p \in [1, +\infty]$ with the same condition (1.11).

For many applications it is interesting to have an exponential stability of (1.8) which is robust with respect to the small changes on the $\Lambda_i$'s (or, equivalently, on the $r_i$'s), i.e., the speeds of propagation. One says that the exponential stability of 0 for (1.8) is robust with respect to the small changes on the $r_i's$ if there exists $\varepsilon \in (0, \text{Min}\{r_1, r_2, \dots, r_n\})$ such that, for every $(\tilde r_1, \tilde r_2, \dots, \tilde r_n) \in \mathbb{R}^n$ such that

$$|\tilde r_i - r_i| \le \varepsilon \quad \text{for } i = 1, \dots, n, \tag{1.12}$$

0 is exponentially stable (in $L^2((0, 1); \mathbb{R}^n)$) for

$$\phi_i(t) = \sum_{j=1}^{n} K_{ij}\phi_j(t - \tilde r_j) \quad \text{for } i = 1, \dots, n. \tag{1.13}$$

Silkowski (see, e.g., [6, Theorem 6.1, p. 286]) proved that 0 is exponentially stable (in $L^2((0, 1); \mathbb{R}^n)$) for (1.8) with an exponential stability which is robust with respect to the small changes on the $r_i$'s if and only if

$$\rho_0\big(K\big) < 1. \tag{1.14}$$

Here,

$$\rho_0(K) := \max\Big\{\rho\big(\operatorname{diag}(e^{i\theta_1},\dots,e^{i\theta_n})K\big);\theta_i\in\mathbb{R}\Big\}, \tag{1.15}$$

where, for $M\in\mathcal{M}_{n\times n}(\mathbb{R})$, $\rho(M)$ denotes the spectral radius of $M$. In fact, Silkowski proved that if the $r_i$'s are rationally independent, i.e., if

$$\left(\sum_{i=1}^n q_i r_i = 0 \text{ and } q:=(q_1,\dots,q_n)^T\in\mathbb{Q}^n\right) \implies (q=0)\,, \tag{1.16}$$

then 0 is exponentially stable (in $L^2((0,1);\mathbb{R}^n)$) for (1.8) if and only if (1.14) holds. In (1.16) and in the following, $\mathbb{Q}$ denotes the set of rational numbers.

The nonlinear case has been considered in the literature for more than three decades. To our knowledge, the first results are due to Slemrod in [13] and Greenberg and Li in [5] in two dimensions, i.e., $n = 2$. These results were later generalized for the higher dimensions. All these results rely on a systematic use of direct estimates of the solutions and their derivatives along the characteristic curves. The weakest sufficient condition in this direction was obtained by Qin [11], Zhao [15], and Li [8, Theorem 1.3, p. 173]. In these references, it is proved that 0 is exponentially stable for system (1.7) with respect to the $C^1$-norm if

$$\rho_\infty\big(K\big) < 1. \tag{1.17}$$

Here and in the following, for $1\le p\le\infty$,

$$\rho_p(M) := \inf\left\{\|\Delta M\Delta^{-1}\|_p;\ \Delta\in\mathcal{D}_{n,+}\right\} \quad \text{for every } M\in\mathcal{M}_{n\times n}(\mathbb{R}), \tag{1.18}$$

where $\mathcal{D}_{n,+}$ denotes the set of all $n\times n$ real diagonal matrices whose entries on the diagonal are strictly positive. The following standard notation is used:

$$\|x\|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \quad \forall x:=(x_1,\dots,x_n)^T\in\mathbb{R}^n,\ \forall p\in[1,+\infty), \tag{1.19}$$

$$\|x\|_\infty := \max\left\{|x_i|;\ i\in\{1,\dots,n\}\right\} \quad \forall x:=(x_1,\dots,x_n)^T\in\mathbb{R}^n, \tag{1.20}$$

$$\|M\|_p := \max_{\|x\|_p=1}\|Mx\|_p \quad \forall M\in\mathcal{M}_{n\times n}(\mathbb{R}). \tag{1.21}$$

(In fact, in [8, 11, 15], $K$ is assumed to have a special structure; however, it was pointed out in [7] that the case of a general $K$ can be reduced to the case of this special structure.) We will see later that (1.17) is also a sufficient condition for the exponential stability with respect to the $W^{2,\infty}$-norm (see Theorem 1.5). Robustness issues of the exponential stability were studied by Prieur, Winkin, and Bastin in [10] using again direct estimates of the solutions and their derivatives along the characteristic curves.

Using a totally different approach, which is based on a Lyapunov stability analysis, a new criterion on the exponential stability is obtained in [3]: it is proved there that 0 is exponentially stable for system (1.7) with respect to the $H^2$-norm if

$$\rho_2\big(K\big) < 1. \tag{1.22}$$

This result extends a previous one obtained in [4], where the same result is established under the assumption that $n = 2$ and $F$ is diagonal. See also the prior works [12] by

Rauch and Taylor and [14] by Xu and Sallet in the case of linear hyperbolic systems. Let us also point out that, adapting [3], one can also prove that (1.22) implies that 0 is exponentially stable for system (1.7) with respect to the $H^k$-norm for every integer $k \geq 2$. (Proceed as in [2, section 4].)

It is known (see [3]) that

$$\rho_0(M) \leq \rho_2(M) \leq \rho_\infty(M)$$

and that the second inequality is strict in general if $n \geq 2$: for $n \geq 2$, there exists $M \in \mathcal{M}_{n,n}(\mathbb{R})$ such that

$$\rho_2(M) < \rho_\infty(M). \tag{1.23}$$

In fact, let $a > 0$ and define

$$M := \begin{pmatrix} a & a \\ -a & a \end{pmatrix}.$$

Then

$$\rho_2(M) = \sqrt{2}a$$

and

$$\rho_\infty(M) = 2a.$$

This implies (1.23) in the case $n = 2$. The case $n \geq 3$ follows similarly by considering the matrices

$$\begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_{n,n}(\mathbb{R}).$$

The Lyapunov approach introduced in [3] has been successfully used in [2] to rediscover the exponential stability with respect to the $C^1$-norm.

The result obtained in [3] is sharp for $n \leq 5$. In fact, in [3] the authors established the following result:

$$\rho_0 = \rho_2 \quad \text{for } n = 1,\ 2,\ 3,\ 4,\ 5.$$

For $n \geq 6$, they also showed that there exists $M \in \mathcal{M}_{n,n}(\mathbb{R})$ such that

$$\rho_0(M) < \rho_2(M).$$

Taking into account these results, a natural question is the following: does the condition $\rho_2(K) < 1$ imply that 0 is exponentially stable for (1.7) with respect to the $C^1$-norm? We give a negative answer to this question (Theorem 1.3). Hence, different norms require different criteria for the exponential stability with respect to them even in the framework of classical solutions. Let us emphasize that this phenomenon is due to the nonlinearities: it does not appear when $F$ is constant. We then show that the condition $\rho_p(K) < 1$ is sufficient to obtain the exponential stability with respect to the $W^{2,p}$-norm (Theorem 1.5). The method used in this paper is strongly inspired from the theory of the linear time-delay systems and incorporates the characteristic method.

In order to precisely state our first result, we need to recall the compatibility conditions in connection with the well-posedness for the Cauchy problem associated to (1.7). Let $m \in \mathbb{N}$. Let $\mathcal{H} : C^0([0,1];\mathbb{R}^n) \to C^0([0,1];\mathbb{R}^n)$ be a map of class $C^m$. For $k \in \{0,1,\dots,m\}$, we define, by induction on $k$, $D^k\mathcal{H} : C^k([0,1];\mathbb{R}^n) \to C^0([0,1];\mathbb{R}^n)$ by

$$(D^0\mathcal{H})(\varphi) := \mathcal{H}(\varphi) \quad \forall \varphi \in C^0([0,1];\mathbb{R}^n), \tag{1.24}$$

$$(D^k\mathcal{H})(\varphi) := \big((D^{k-1}\mathcal{H})'(\varphi)\big)F(\varphi)\varphi_x \quad \forall \varphi \in C^k([0,1];\mathbb{R}^n), \quad \forall k \in \{0,1,\dots,m\}. \tag{1.25}$$

For example, if $m = 2$,

$$(D^1\mathcal{H})(\varphi) = \mathcal{H}'(\varphi)F(\varphi)\varphi_x \quad \forall \varphi \in C^1([0,1];\mathbb{R}^n), \tag{1.26}$$

$$(D^2\mathcal{H})(\varphi) = \mathcal{H}''(\varphi)\big(F(\varphi)\varphi_x, F(\varphi)\varphi_x\big) + \mathcal{H}'(\varphi)\big(F'(\varphi)F(\varphi)\varphi_x\big)\varphi_x, + \mathcal{H}'(\varphi)F(\varphi)\big((F'(\varphi)\varphi_x)\varphi_x + F(\varphi)\varphi_{xx}\big) \quad \forall \varphi \in C^2([0,1];\mathbb{R}^n). \tag{1.27}$$

Let $\mathcal{I}$ be the identity map from $C^0([0,1];\mathbb{R}^n)$ into $C^0([0,1];\mathbb{R}^n)$ and let us define $\mathcal{G} : C^0([0,1];\mathbb{R}^n) \to C^0([0,1];\mathbb{R}^n)$ by

$$\big(\mathcal{G}(\varphi)\big)(x) = G\big(\varphi(x)\big) \quad \text{for every } \varphi \in C^0([0,1];\mathbb{R}^n) \text{ and for every } x \in [0,1]. \tag{1.28}$$

Let $u^0 \in C^m([0,1];\mathbb{R}^n)$. We say that $u^0$ satisfies the compatibility conditions of order $m$ if

$$((D^k\mathcal{I})(u^0))(0) = ((D^k\mathcal{G})(u^0))(1) \quad \text{for every } k \in \{0,1,\dots,m\}. \tag{1.29}$$

For example, for $m = 1$, $u^0 \in C^1([0,1];\mathbb{R}^n)$ satisfies the compatibility conditions of order 1 if and only if

$$u^0(0) = G\big(u(1)\big), \tag{1.30}$$

$$F\big(u^0(0)\big)u^0_x(0) = G'\big(u(1)\big)F\big(u^0(1)\big)u^0_x(1). \tag{1.31}$$

With this definition of the compatibility conditions of order $m$, we can recall the following classical theorem on the well-posedness of the Cauchy problem associated to (1.7).

THEOREM 1.1. *Let $m \in \mathbb{N} \setminus \{0\}$. Given $T > 0$, there exists $\varepsilon_0 > 0$ such that if $\|u^0\|_{C^m([0,1];\mathbb{R}^n)} \le \varepsilon$ and the compatibility conditions of order $m$ (1.29) holds, then there exists a unique solution $u \in C^m([0,T]\times[0,1];\mathbb{R}^n)$ of (1.7) satisfying the initial condition $u(0,\cdot) = u^0$. Moreover,*

$$\|u\|_{C^m([0,T]\times[0,1];\mathbb{R}^n)} \le C\|u^0\|_{C^m([0,1];\mathbb{R}^n)} \tag{1.32}$$

*for some positive constant $C$ independent of $u^0$.*

The case $m = 1$ is due to Li and Yu [9, Chapter 4]. The general case can be obtained by recurrence as follows. Assuming the result holds for $m$, we prove that the result holds for $m+1$. Set $v = u_x$. Then $v \in C^{m-1}$ is the unique broad solution (see [1, Chapter 3] for related situations) to the equation

$$v_t + A(t,x)v_x = h(t,x,v), \quad v(t,0) = B(t)v(t,1) \text{ and } v(0,x) = u^0_x, \tag{1.33}$$

where

$$A(t,x) := F(u(t,x)), \quad h(t,x,v) := -\left(\nabla F(u(t,x))v\right)v, \tag{1.34}$$

$$B(t) := (F(u(t,0)))^{-1}G'(u(t,1))F(u(t,1)). \tag{1.35}$$

Since $u$ satisfies the compatibility of order $m+1$, it follows that $v(0,\cdot)$ satisfies the compatibility condition of order $m$; this means that (1.29) holds where $F(u)$ is replaced by $A(t,x)$ in (1.25). Since $u \in C^m$, we derive that $A \in C^m$ and $h \in C^m$. It follows that $v \in C^m$ and hence $u \in C^{m+1}$. One also obtains an estimate for $\|v\|_{C^m}$, which in turn implies the estimate for $\|u\|_{C^{m+1}}$.

We can now define the notion of exponential stability with respect to the $C^m$-norm.

DEFINITION 1.2. *The equilibrium solution $u \equiv 0$ is exponentially stable for system* (1.7) *with respect to the $C^m$-norm if there exist $\varepsilon > 0$, $\nu > 0$, and $C > 0$ such that, for every $u^0 \in C^m([0,1];\mathbb{R}^n)$ satisfying the compatibility conditions of order $m$* (1.29) *and such that $\|u^0\|_{C^m([0,1];\mathbb{R}^n)} \leq \varepsilon$, there exists one and only one solution $u \in C^m([0,+\infty) \times [0,1];\mathbb{R}^n)$ of* (1.7) *satisfying the initial condition $u(0,\cdot) = u^0$ and this solution satisfies*

$$\|u(t,\cdot)\|_{C^m([0,1];\mathbb{R}^n)} \leq Ce^{-\nu t}\|u^0\|_{C^m([0,1];\mathbb{R}^n)} \quad \forall t > 0.$$

With this definition, let us return to the results which are already known concerning the exponential stability with respect to the $C^m$-norm.

(i) *For linear $F$ and $G$.* Let $m \in \mathbb{N}$. If $\rho_0\big(G'(0)\big) < 1$, then 0 is exponentially stable for system (1.7) with respect to the $C^m$-norm and the converse holds if the $r_i$'s are rationally independent. This result was proved for the $L^2$-norm. But the proof can be adapted to treat the case of the $C^m$-norm.

(ii) *For general $F$ and $G$.* Let $m \in \mathbb{N} \setminus \{0\}$. If $\rho_\infty\big(G'(0)\big) < 1$, then 0 is exponentially stable for system (1.7) with respect to the $C^m$-norm. This result was proved for the case $m = 1$ in [8, 11, 15]. However, the result provided in [8, Theorem 1.3, Chapter 5] for this case gives also the case $m \geq 2$ by an induction argument and by considering the quasi-linear hyperbolic system (with a source term which is quadratic in $u_x$) satisfied by $(u, u_x)$; note that [8, Theorem 1.3, Chapter 5] considers a quasi-linear hyperbolic system with a source term which is quadratic. For a different proof based on a Lyapunov approach, see [2].

(iii) *For general $F$ and $G$ and $n = 1$.* Let $m \in \mathbb{N} \setminus \{0\}$. Then 0 is exponentially stable for system (1.7) with respect to the $C^m$-norm if and only if $\rho_0\big(G'(0)\big) < 1$. Note that, for $n = 1$, the $\rho_p(G'(0))$'s do not depend on $p \in [1,+\infty]$: they are all equal to $|G'(0)|$.

The first result of this paper is the following one.

THEOREM 1.3. *Let $m \in \mathbb{N} \setminus \{0\}$, $n \geq 2$, and $\tau > 0$. There exist a linear map $G : \mathbb{R}^n \to \mathbb{R}^n$ and $F \in C^\infty(\mathbb{R}^n; \mathcal{M}_{n\times n}(\mathbb{R}))$ such that $F$ is diagonal, $F(0)$ has $n$ distinct positive real eigenvalues,*

$$\rho_\infty\big(G'(0)\big) < 1 + \tau,\ \rho_0\big(G'(0)\big) = \rho_2\big(G'(0)\big) < 1, \tag{1.36}$$

*and 0 is* not *exponentially stable for system* (1.7) *with respect to the $C^m$-norm.*

The second result of this paper is on a sufficient condition for the exponential stability with respect to the $W^{2,p}$-norm. In order to state it, we use the following definition, adapted from Definition 1.2.

DEFINITION 1.4. *Let $p \in [1,+\infty]$. The equilibrium solution $u \equiv 0$ is exponentially stable for* (1.7) *with respect to the $W^{2,p}$-norm if there exist $\varepsilon > 0$, $\nu > 0$, and $C > 0$ such that, for every $u^0 \in W^{2,p}((0,1);\mathbb{R}^n)$ satisfying the compatibility conditions of order* 1 (1.30)–(1.31) *and such that*

$$\|u^0\|_{W^{2,p}((0,1);\mathbb{R}^n)} \leq \varepsilon, \tag{1.37}$$

*there exists one and only one solution* $u \in C^1([0,+\infty) \times [0,1]; \mathbb{R}^n)$ *of* (1.7) *satisfying the initial condition* $u(0,\cdot) = u^0$ *and this solution satisfies*

$$\|u(t,\cdot)\|_{W^{2,p}((0,1);\mathbb{R}^n)} \le Ce^{-\nu t}\|u^0\|_{W^{2,p}((0,1);\mathbb{R}^n)} \quad \forall t > 0.$$

Again, for every $T > 0$, for every initial condition $u^0 \in W^{2,p}((0,1);\mathbb{R}^n)$ satisfying the compatibility conditions (1.30)–(1.31) and such that $\|u^0\|_{W^{2,p}((0,1);\mathbb{R}^n)}$ is small enough, there exist a unique $C^1$ solution $u \in L^\infty([0,T]; W^{2,p}((0,1);\mathbb{R}^n))$ of (1.7) satisfying the initial condition $u(0,\cdot) = u^0$ (and, if $p \in [1,+\infty)$, this solution is in $C^0([0,T]; W^{2,p}((0,1);\mathbb{R}^n))$). (See Lemma 3.1.) Our next result is the following theorem, where the assumptions on the regularity of $F$ and $G$ are weakened.

THEOREM 1.5. *Let* $p \in [1,+\infty]$. *Assume that* $F$ *and* $G$ *are of class* $C^2$. *Assume that* $F(0)$ *has* $n$ *real distinct positive eigenvalues,* $G(0) = 0$, *and*

$$\rho_p\big(G'(0)\big) < 1. \tag{1.38}$$

*Then, the equilibrium solution* $u \equiv 0$ *of the system* (1.7) *is exponentially stable with respect to the* $W^{2,p}$*-norm.*

Let us recall that the case $p = 2$ is proved in [3]. Let us emphasize that, even in this case, our proof is completely different from the one given in [3].

*Remark* 1. The notation on various conditions on exponential stability used in this paper is different from the ones in [3] but the same as the ones used in [2].

The paper is organized as follows. In sections 2 and 3, we establish Theorems 1.3 and 1.5, respectively.

**2. Proof of Theorem 1.3.** We give the proof in the case $n = 2$. The general case $n \ge 2$ follows immediately from the case considered here.

Let $F \in C^\infty(\mathbb{R}^2; \mathcal{M}_{2\times 2}(\mathbb{R}))$ be such that

$$F(u) = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \dfrac{1}{r_2 + u_2} \end{pmatrix} \quad \forall u = (u_1, u_2)^T \in \mathbb{R}^2 \text{ with } u_2 > -\frac{r_2}{2} \tag{2.1}$$

for some $0 < \Lambda_1 < \Lambda_2$. We recall that

$$r_1 = 1/\Lambda_1 \quad \text{and} \quad r_2 = 1/\Lambda_2.$$

We assume that $r_1$ and $r_2$ are independent in $\mathbb{Z}$, i.e.,

$$\big(k_1 r_1 + k_2 r_2 = 0 \text{ and } (k_1,k_2)^T \in \mathbb{Z}^2\big) \implies (k_1 = k_2 = 0)\,. \tag{2.2}$$

Define $G : \mathbb{R}^2 \to \mathbb{R}^2$ as the following linear map:

$$G(u) := a \begin{pmatrix} 1 & \xi \\ -1 & \eta \end{pmatrix} u \quad \text{for } u \in \mathbb{R}^2. \tag{2.3}$$

Here, $a > 0$ and $\xi, \eta$ are two positive numbers such that

$$\text{if } P_k(\xi,\eta) = 0, \quad \text{then} \quad P_k \equiv 0 \tag{2.4}$$

for every polynomial $P_k$ of degree $k$ ($k \ge 0$) with rational coefficients.

Note that if

$$a \text{ is close to } 1/2 \quad \text{and} \quad \xi, \eta \text{ are close to } 1, \tag{2.5}$$

then

$$\rho_\infty(G) \text{ is close to } 1 \tag{2.6}$$

and

$$\rho_0(G) = \rho_2(G) \text{ are close to } \frac{1}{\sqrt{2}} < 1. \tag{2.7}$$

Here, and in the following, for notational ease, we use the convention $G = K = G'(0)$.

Let $\tau_0 > 1$ (which will be defined below). We take $a \in \mathbb{Q}$, $a > 1/2$ but close to $1/2$ and choose $\xi, \eta > 1$ but close to 1 so that

$$\rho_\infty(G) < \tau_0, \tag{2.8}$$

$$a(1 + \xi + \eta) \leq 2, \tag{2.9}$$

and there exists $c > 0$ such that

$$\frac{\max\{\xi, \eta\}}{a(\xi + \eta)} < c < 1. \tag{2.10}$$

We also impose that $\xi$, $\eta$ satisfy (2.4).

We start with the case $m = 1$. We argue by contradiction. Assume that there exists $\tau_0 > 1$ such that for all $G$ with $\rho_\infty(G'(0)) < \tau_0$, there exist $\varepsilon_0$, $C_0$, $\nu$ positive numbers such that

$$\|u(t, \cdot)\|_{C^1([0,1];\mathbb{R}^2)} \leq Ce^{-\nu t}\|u^0\|_{C^1([0,1];\mathbb{R}^2)} \tag{2.11}$$

if $u^0 \in C^1([0, 1]; \mathbb{R}^2)$ satisfies the compatibility conditions (1.30)–(1.31) and is such that $\|u^0\|_{C^1([0,1];\mathbb{R}^2)} \leq \varepsilon_0$. Here, $u$ denotes the solution of (1.7) satisfying the initial condition $u(0, \cdot) = u^0$.

Let $u \in C^1([0, +\infty) \times [0, 1]; \mathbb{R}^2)$ be a solution to (1.7) and define

$$v(t) = u(t, 0).$$

Then

$$v\Big(t + r_2 + v_2(t)\Big) = v_1\Big(t + r_2 + v_2(t) - r_1\Big)G_1 + v_2(t)G_2, \tag{2.12}$$

where $G_1$ and $G_2$ are the first and the second column of $G$. Equation (2.12) motivates our construction below.

Fix $T > 0$ (arbitrarily large) such that

$$T - (kr_1 + lr_2) \neq 0 \quad \text{ for every } k, l \in \mathbb{N}.$$

Let $\varepsilon \in (0, 1)$ be (arbitrarily) small such that

$$\inf_{k,l\in\mathbb{N}} |T - (kr_1 + lr_2)| \geq \varepsilon. \tag{2.13}$$

(Note that the smallness of $\varepsilon$ in order to have (2.13) depends on $T$: It goes to 0 as $T \to +\infty$.) Let $n$ be the integer part of $T/r_2$ plus 1. In particular, $nr_2 > T$. Fix $n$ rational points $(s_i^0, t_i^0)^T \in \mathbb{Q}^2$, $i = 1, \ldots, n$, such that their coordinates are distinct, i.e., $s_i^0 \neq s_j^0$, $t_i^0 \neq t_j^0$ for $i \neq j$, and

$$\|(s_i^0, t_i^0)\|_\infty \leq \varepsilon^3/4^n \quad \text{ for every } i \in \{1, \ldots, n\}. \tag{2.14}$$

For $0 \le k \le n-1$, we define $(s_i^{k+1}, t_i^{k+1})^T$ for $i = 1, n-(k+1)$ by recurrence as follows:

$$(s_i^{k+1}, t_i^{k+1})^T = G(s_i^k, t_{i+1}^k)^T = a \begin{pmatrix} s_i^k + \xi t_{i+1}^k \\ -s_i^k + \eta t_{i+1}^k \end{pmatrix}. \tag{2.15}$$

Set

$$V(T) := (s_1^n, t_1^n), \quad dV(T) = \varepsilon(1,0)^T. \tag{2.16}$$

Define

$$T_1 := T - r_1, \qquad T_2 := T - r_2 - t_2^{n-1}, \tag{2.17}$$

$$V(T_1) = (s_1^{n-1}, t_1^{n-1}), \qquad V(T_2) = (s_2^{n-1}, t_2^{n-1}), \tag{2.18}$$

$$dV(T_1) = \varepsilon\left(\frac{\eta}{a(\xi+\eta)}, 0\right), \qquad dV(T_2) = \varepsilon\left(0, \frac{1}{a(\xi+\eta)}\right). \tag{2.19}$$

Assume that $T_{\gamma_1\ldots\gamma_k}$ is defined for $\gamma_i = 1, 2$. Set

$$T_{\gamma_1\ldots\gamma_k 1} = T_{\gamma_1\ldots\gamma_k} - r_1 \tag{2.20}$$

and

$$T_{\gamma_1\ldots\gamma_k 2} = T_{\gamma_1\ldots\gamma_k} - r_2 - t_{1+l}^{n-(k+1)}, \tag{2.21}$$

where[1]

$$l = \sum_{j=1}^{k} (\gamma_j - 1). \tag{2.22}$$

Note that, by (2.14), (2.15), (2.17), (2.20), (2.21), and (2.22),

$$\left| T_{\gamma_1\ldots\gamma_k} - kr_1 - (r_2 - r_1)\sum_{j=1}^{k}(\gamma_j - 1)\right| \le C\varepsilon^3 \quad \forall k \in \{1, \ldots, n\} \tag{2.23}$$

for some $C > 0$ which is independent of $T > r_1$ and $\varepsilon \in (0, +\infty)$.

We claim that

$$\text{the } T_{\gamma_1\ldots\gamma_k},\ k \in \{1, \ldots, n-1\}, \text{ are distinct.} \tag{2.24}$$

(See Figure 1.) We admit this fact, which will be proved later on, and continue the proof.

Define $V(T_{\gamma_1\ldots\gamma_k\gamma_{k+1}})$ and $dV(T_{\gamma_1\ldots\gamma_k\gamma_{k+1}})$ as

$$V(T_{\gamma_1\ldots\gamma_k\gamma_{k+1}}) = (s_{1+l}^{n-(k+1)}, t_{1+l}^{n-(k+1)})^T \tag{2.25}$$

and

$$dV(T_{\gamma_1\ldots\gamma_k 1}) = (x, 0)^T \qquad dV(T_{\gamma_1\ldots\gamma_k 2}) = (0, y)^T, \tag{2.26}$$

[1] Roughly speaking, $l$ describes the number of times which comes from $r_2$ in the construction of $\gamma_1 \ldots \gamma_k$.

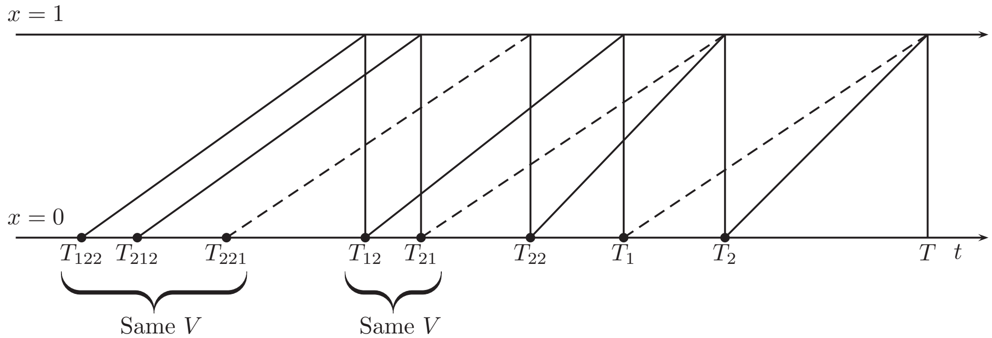


FIG. 1. $V(T_{122}) = V(T_{212}) = V(T_{221}) \neq V(T_{12}) = V(T_{21})$ *and the* $T_\gamma$*'s are different. The slope of the dashed lines is* $\Lambda_1 = r_1^{-1}$.

where $l$ is given by (2.22) and the real numbers $x, y$ are chosen such that

$$G(x,y)^T = dV(T_{\gamma_1\ldots\gamma_k}). \tag{2.27}$$

Let us also point out that, by (2.19) and (2.26),

(2.28) at least one of the two components of $dV(T_{\gamma_1\ldots\gamma_k})$ is 0.

From (2.3), we have

$$G^{-1} = \frac{1}{a(\eta+\xi)}\begin{pmatrix}\eta & -\xi \\ 1 & 1\end{pmatrix}. \tag{2.29}$$

It follows from (2.10), (2.26), (2.27), (2.28), and (2.29) that

$$\|dV(T_{\gamma_1\ldots\gamma_k\gamma_{k+1}})\|_\infty \leq c\|dV(T_{\gamma_1\ldots\gamma_k})\|_\infty. \tag{2.30}$$

Using (2.24), we may construct $\mathfrak{v} \in C^1([0,r_1];\mathbb{R}^2)$ such that

$$\mathfrak{v}'(T_{\alpha_1\ldots\alpha_k}) = dV(T_{\alpha_1\ldots\alpha_k}) \tag{2.31}$$

and

$$\mathfrak{v}(T_{\alpha_1\ldots\alpha_k}) = V(T_{\alpha_1\ldots\alpha_k}) \tag{2.32}$$

if

$$T_{\alpha_1\ldots\alpha_k} \in (0,r_1).$$

(Recall that $r_1 > r_2 > 0$ and $nr_2 > T$.) It follows from (2.9), (2.14), (2.15), (2.25), and (2.32), that

$$\|\mathfrak{v}(T_{\alpha_1\ldots\alpha_k})\|_\infty \leq \varepsilon^3 \quad \text{if } T_{\alpha_1\ldots\alpha_k} \in (0,r_1). \tag{2.33}$$

Let $T_{\alpha_1\ldots\alpha_k} \in (0,r_1)$ and $T_{\gamma_1\ldots\gamma_m} \in (0,r_1)$ be such that

$$\mathfrak{v}(T_{\alpha_1\ldots\alpha_k}) \neq \mathfrak{v}(T_{\gamma_1\ldots\gamma_m}). \tag{2.34}$$

From (2.15), (2.25), (2.32), and (2.34), we get that

$$k \neq m \text{ or } \operatorname{card}\{i \in \{1, \dots, k\};\ \alpha_i = 1\} \neq \operatorname{card}\{i \in \{1, \dots, m\};\ \gamma_i = 1\}. \tag{2.35}$$

See also Figure 1.

From (2.13), (2.17), (2.20), (2.21), and (2.35), we get that, at least if $\varepsilon > 0$ is small enough,

$$|T_{\alpha_1 \dots \alpha_k} - T_{\gamma_1 \dots \gamma_m}| \geq \varepsilon/2. \tag{2.36}$$

Using (2.13), (2.33), and (2.36), we may also impose that

$$\mathfrak{v} = 0 \text{ in a neighborhood of } 0 \text{ in } [0, r_1], \tag{2.37}$$

$$\mathfrak{v} = 0 \text{ in a neighborhood of } r_1 \text{ in } [0, r_1], \tag{2.38}$$

$$\mathfrak{v} = 0 \text{ in a neighborhood of } r_2, \tag{2.39}$$

$$\|\mathfrak{v}\|_{C^1([0,r_1])} \leq C \max\{\varepsilon^2, A\}, \tag{2.40}$$

where

$$A := \max\left\{\|dV(T_{\alpha_1 \dots \alpha_k})\|_\infty;\ T_{\alpha_1 \dots \alpha_k} \in (0, r_1)\right\}. \tag{2.41}$$

In (2.40), $C$ denotes a positive constant that does not depend on $T > r_1$ and on $\varepsilon > 0$ provided that $\varepsilon > 0$ is small enough, this smallness depending on $T$. We use this convention until the end of this section, and the constants $C$ may vary from one place to another.

Note that if $T_{\alpha_1 \dots \alpha_k} \in (0, r_1)$, then

$$k r_1 > T/2.$$

It follows that

$$k > T/(2r_1),$$

which, together with (2.16), (2.30), and $c \in (0, 1)$, implies that

$$\|dV(T_{\alpha_1 \dots \alpha_k})\|_\infty \leq \varepsilon c^{T/(2r_1)}. \tag{2.42}$$

From (2.40) and (2.42), one has

$$\|\mathfrak{v}\|_{C^1([0,r_1];\mathbb{R}^2)} \leq C \max\left\{\varepsilon^2, \varepsilon c^{T/(2r_1)}\right\} \leq C \varepsilon c^{T/(2r_1)}. \tag{2.43}$$

Let $\tilde{u} \in C^1([0, r_1] \times [0, 1]; \mathbb{R}^2)$ be the solution to the backward Cauchy problem

$$\begin{cases} \tilde{u}_t + F(\tilde{u})\tilde{u}_x = 0 & \text{for every } (t, x) \in [0, r_1] \times [0, 1], \\ \tilde{u}(t, 1) = G^{-1} v(t) & \text{for every } t \in [0, r_1], \\ \tilde{u}(r_1, x) = 0 & \text{for every } x \in [0, 1]. \end{cases} \tag{2.44}$$

Note that, by (2.38), the boundary condition at $x = 1$ for the backward Cauchy problem (2.44) vanishes in a neighborhood of $r_1$ in $[0, 1]$ and therefore the necessary compatibility conditions for the existence of $\tilde{u}$, namely,

$$G^{-1} v(t_1) = 0 \text{ and } G^{-1} v'(t_1) = 0, \tag{2.45}$$

are satisfied. Moreover, if $\varepsilon > 0$ is small enough, this solutions indeed exists by [9, pp. 96–107]. Let $u^0 \in C^1([0,1];\mathbb{R}^2)$ be defined by

$$u^0(x) := \tilde{u}(0,x) \quad \text{for every } x \in [0,1]. \tag{2.46}$$

Using (2.10), (2.43), and the definition of $u^0$, we have

$$\|u^0\|_{C^1([0,1];\mathbb{R}^2)} \leq C\|v\|_{C^1([0,r_1];\mathbb{R}^2)} \leq C \max\left\{\varepsilon^2, \varepsilon c^{T/(2r_1)}\right\} \leq C\varepsilon. \tag{2.47}$$

Note that $u^0$ satisfies the compatibility condition (1.30) and (1.31) since, by (2.38) and (2.39), $u^0$ vanishes in a neighborhood of 0 in $[0,1]$ and, by (2.37), $u^0$ vanishes in a neighborhood of 1 in $[0,1]$. Let $u \in C^1([0,+\infty) \times [0,1];\mathbb{R}^2)$ be the solution of (1.7) satisfying the initial condition

$$u(0,x) = u^0(x) \qquad \text{for every } x \in [0,1].$$

Since 0 is assumed to be exponentially stable for (1.7) with respect to the $C^1$-norm, $u$ exists for all positive time if $\varepsilon$ is small enough. Let us define $v \in C^1([0,+\infty);\mathbb{R}^2)$ by

$$v(t) := u(t,0) \qquad \text{for every } t \in [0,+\infty). \tag{2.48}$$

Then, by the constructions of $u$ and $\tilde{u}$, one has

$$v(t) = \mathfrak{v}(t) \quad \text{for every } t \in [0,r_1]. \tag{2.49}$$

Then, using (2.12) together with the definition of $T_{\gamma_1\ldots\gamma_k}$ and $V(T_{\gamma_1\ldots\gamma_k})$, one has

$$v(T_{\gamma_1\ldots\gamma_k}) = V(T_{\gamma_1\ldots\gamma_k}) \quad \text{if } T_{\gamma_1\ldots\gamma_k} \in [0,T] \tag{2.50}$$

with the convention that if $k=0$, $T_{\gamma_1\ldots\gamma_k} = T$.

Differentiating (2.12) with respect to $t$, we get

$$\left(1+v_2'(t)\right)v'\Big(t+r_2+v_2(t)\Big) = \left(1+v_2'(t)\right)v_1'\Big(t+r_2+v_2(t)-r_1\Big)G_1 + v_2'(t)G_2. \tag{2.51}$$

It follows that

$$v'\Big(t+r_2+v_2(t)\Big) = v_1'\Big(t+r_2+v_2(t)-r_1\Big)G_1 + v_2'(t)G_2 - \frac{v_2'(t)^2}{1+v_2'(t)}G_2. \tag{2.52}$$

From the definition of $dV$, (2.31), (2.42), (2.49), and (2.52), one gets, for every $T > r_1$, the existence of $C(T) > 0$ such that

$$|v'(T) - dV(T)| \leq C(T)\varepsilon^2, \tag{2.53}$$

provided that $\varepsilon$ is small enough (the smallness depending on $T$). In (2.53) and in the following, we use the notation

$$|x| := \|x\|_2 \quad \forall x \in \mathbb{R}^n. \tag{2.54}$$

From (1.7), (2.11), and (2.48),

$$|v'(t)| \leq 2\Lambda_2 C_0 e^{-\nu t}\|u^0\|_{C^1([0,1];\mathbb{R}^2)} \quad \text{for every } t \in [0,+\infty), \tag{2.55}$$

provided that $\|u^0\|_{C^1([0,1];\mathbb{R}^2)} \le \varepsilon_0$. Using (2.16), (2.47), (2.53), and (2.55), one gets the existence of $C_1 > 0$ such that, for every $T > 0$, there exist $C(T) > 0$ and $\varepsilon(T) > 0$ such that

$$1 \le C_1 e^{-\nu T} + C(T)\varepsilon \quad \text{for every } T > 0, \text{ for every } \epsilon \in (0, \varepsilon(T)]. \tag{2.56}$$

We choose $T > 0$ large enough so that $C_1 e^{-\nu T} \le (1/2)$. Then letting $\varepsilon \to 0^+$ in (2.56), we get a contradiction.

It remains to prove (2.24) in order to conclude the proof of Theorem 1.3 if $m = 1$. Let us assume

$$T_{\gamma_1 \dots \gamma_k} = T_{\alpha_1 \dots \alpha_m} \text{ with } k, m \in \{1, \dots, n-1\} \tag{2.57}$$

($\gamma_i, \alpha_i = 1, 2$). Using (2.2) and (2.23), we derive that

$$m = k, \ \mathrm{card}\big\{i; \gamma_i = 2\big\} = \mathrm{card}\big\{i; \alpha_i = 2\big\} =: \ell \tag{2.58}$$

for some $0 \le \ell \le m$. Let $k_1 < \dots < k_\ell$ and $m_1 < \dots < m_\ell$ be such that

$$\gamma_{k_l} = \alpha_{m_l} = 2 \quad \text{for } 1 \le l \le \ell.$$

Define

$$i_l := \sum_{i=1}^{k_l} (\gamma_i - 1) \quad \text{and} \quad j_l := \sum_{i=1}^{k_l} (\alpha_i - 1).$$

It follows from (2.21), (2.22), and (2.57) that

$$\sum_{l=1}^{\ell} t_{i_l}^{n-k_l} = \sum_{l=1}^{\ell} t_{j_l}^{n-m_l}. \tag{2.59}$$

Hence, the fact

$$\gamma_i = \alpha_i \quad \text{for } i = 1, \dots, k = m \tag{2.60}$$

is proved if one can verify that

$$i_l = j_l \quad \text{and} \quad k_l = m_l \qquad \forall l = 1, \dots, \ell. \tag{2.61}$$

By a recurrence argument on $\ell$, it suffices to prove that

$$i_\ell = j_\ell \quad \text{and } k_\ell = m_\ell. \tag{2.62}$$

Note that, by (2.15),

$$t_j^k = a^k \eta^k t_{j+k}^0 + P_{k-1}(\xi, \eta), \tag{2.63}$$

where $P_{k-1}$ is a polynomial of degree $k-1$ with rational coefficients. Since $\xi, \eta$ satisfy (2.4), it follows from (2.59) and (2.63) that

$$k_\ell = m_\ell$$

and

$$i_\ell = j_\ell.$$

Thus, claim (2.62) is proved and so are claims (2.61), (2.60), and (2.24). This concludes the proof of Theorem 1.3 if $m = 1$.

Let us show how to modify the above proof to treat the case $m \geq 2$. Instead of (2.14), one requires

$$\|(s_i^0, t_i^0)\|_\infty \leq \varepsilon^{2+m}/4^n \quad \text{for every } i,\, j \in \{1, \dots, n\}. \tag{2.64}$$

Then, instead of (2.33), one gets

$$\|\mathfrak{v}(T_{\alpha_1 \dots \alpha_k})\|_\infty \leq \varepsilon^{2+m} \quad \text{if } T_{\alpha_1 \dots \alpha_k} \in (0, r_1). \tag{2.65}$$

Instead of (2.31), one requires

$$\mathfrak{v}^{(m)}(T_{\alpha_1 \dots \alpha_k}) = dV(T_{\alpha_1 \dots \alpha_k}), \tag{2.66}$$

and instead of (2.40), one has

$$\|\mathfrak{v}\|_{C^m([0,r_1])} \leq C \max\{\varepsilon^2, A\}, \tag{2.67}$$

where $A$ is still given by (2.41). Then (2.47) is now

$$\|u^0\|_{C^m([0,1];\mathbb{R}^2)} \leq C\|v\|_{C^m([0,r_1];\mathbb{R}^2)} \leq C\varepsilon c^{T/(2r_1)}. \tag{2.68}$$

In the case $m = 1$, we differentiated once (2.12) with respect to $t$ in order to get (2.52). Now we differentiate (2.12) $m$ times with respect to $t$ in order to get

$$\left| v^{(m)}\Big(t + r_2 + v_2(t)\Big) - v_1^{(m)}\Big(t + r_2 + v_2(t) - r_1\Big)G_1 + v_2^{(m)}(t)G_2 \right| \leq C \sum_{i=0}^{m} v^{(i)}(t)^2,$$

which allows us to get, instead of (2.53),

$$|v^{(m)}(T) - dV(T)| \leq C(T)\varepsilon^2. \tag{2.69}$$

We then get a contradiction as in the case $m = 1$. This concludes the proof of Theorem 1.3. ∎

*Remark* 2. Property (2.24) is a key point. It explains why the condition $\rho_0(K) < 1$ is not sufficient for exponential stability in the case of *nonlinear* systems. Indeed, $\rho_0(K) < 1$ gives an exponential stability that is robust with respect to perturbations on the delays that are *constant*: these perturbations are not allowed to depend on time. However, with these type of perturbations, (2.24) does not hold: with constant perturbations on the delays, one has

$$T_{12} = T_{21},\, T_{122} = T_{212} = T_{221}$$

and, more generally,

$$T_{\gamma_1 \dots \gamma_k} = T_{\alpha_1 \dots \alpha_k} \text{ if } \operatorname{card}\{i \in \{1, \dots, k\};\, \gamma_i = 1\} = \operatorname{card}\{i \in \{1, \dots, k\};\, \alpha_i = 1\}.$$

**3. Proof of Theorem 1.5.** This section containing two subsections is devoted to the proof of Theorem 1.5. In the first subsection, we present some lemmas which will be used in the proof. In the second subsection, we give the proof of Theorem 1.5.

**3.1. Some useful lemmas.** The first lemma is a standard one on the well-posedness of (1.1) and (1.5).

LEMMA 3.1. *Let* $p \in [1, +\infty]$. *Given* $T > 0$, *there exists* $\varepsilon_0 > 0$ *such that if* $\|u_0\|_{W^{2,p}((0,1);\mathbb{R}^n)} < \varepsilon_0$ *and the compatibility conditions* (1.30)–(1.31) *holds, then there exists a unique solution* $u \in C^1([0,T] \times [0,1]; \mathbb{R}^n)$ *of* (1.7) *satisfying the initial condition* $u(0,\cdot) = u^0$. *Moreover,*

$$\|u(t,\cdot)\|_{W^{2,p}((0,1);\mathbb{R}^n)} \le C\|u^0\|_{W^{2,p}((0,1);\mathbb{R}^n)} \tag{3.1}$$

*for some positive constant* $C$ *independent of* $u^0$.

*Proof of Lemma* 3.1. The case $p = 2$ was established in [3]. The (sketches of) proof given there can be adapted to treat the other cases. The proof presented here is based on the characteristic method. The existence and uniqueness in $C^1$ follows from Theorem 1.1 with $m = 1$. To obtain (3.1), we proceed as follows. Set $v = u_x$. Then $v \in C^0$ is the unique broad solution to (1.33), where $A$, $h$, and $B$ are still defined by (1.34) and (1.35). Since $A \in C^1$, $h \in C^1$, and $B \in C^1$, estimate (3.1) now follows from the characteristic methods. (See also (3.48) in the proof of Theorem 1.5.) The details are left to the reader. ◻

We next present two lemmas dealing with the system

$$v_t + A(t,x)v_x = 0$$

and its perturbation where $A$ is diagonal. The first lemma is the following one.

LEMMA 3.2. *Let* $p \in [1, +\infty]$, $m$ *be a positive integer,* $\lambda_1 \ge \cdots \ge \lambda_m > 0$, *and* $\hat{K} \in (0,1)$. *There exist three constants* $\varepsilon_0 > 0$, $\gamma > 0$, *and* $C > 0$ *such that, for every* $T > 0$, *every* $A \in C^1([0,T]\times[0,1]; \mathcal{D}_{m,+})$, *every* $K \in C^1([0,T]; \mathcal{M}_{m,m}(\mathbb{R}))$, *and every* $v \in W^{1,p}([0,T] \times [0,1]; \mathbb{R}^m)$ *such that*

$$v_t + A(t,x)v_x = 0 \text{ for } (t,x) \in (0,T) \times (0,1), \tag{3.2}$$

$$v(t,0) = K(t)v(t,1) \text{ for } t \in [0,T], \tag{3.3}$$

$$\sup_{t\in[0,T]} \|K(t)\|_p \le \hat{K} < 1, \tag{3.4}$$

$$\|A - \operatorname{diag}(\lambda_1, \dots, \lambda_m)\|_{C^1([0,T]\times[0,1];\mathcal{M}_{m,m}(\mathbb{R}))} + \sup_{t\in[0,T]} \|K'(t)\|_p \le \varepsilon_0, \tag{3.5}$$

*one has*

$$\|v(t,\cdot)\|_{W^{1,p}((0,1);\mathbb{R}^m)} \le Ce^{-\gamma t}\|v(0,\cdot)\|_{W^{1,p}((0,1);\mathbb{R}^m)} \text{ for } t \in [0,T].$$

*Proof of Lemma* 3.2. We only consider the case $1 \le p < +\infty$, the case $p = +\infty$ follows similarly (the proof is even easier) and is left to the reader. Note that, from (3.5), one has

$$0 < \frac{\lambda_m}{2} \le A_{ii}(s, \varphi_i(t,s)) \le 2\lambda_1, \tag{3.6}$$

at least if $\varepsilon_0 > 0$ is small enough (for example, if $\varepsilon_0 < \lambda_m/C$ for some large constant $C$ depending only on $n$), a property which is always assumed in this proof. For $t \ge 0$, let $\varphi_i(t,s)$ be such that

$$\partial_s \varphi_i(t,s) = A_{ii}(s, \varphi_i(t,s)) \quad \text{and} \quad \varphi_i(t,t) = 0.$$

Then

$$v_i(s, \varphi_i(t, s)) = v_i(t, 0). \tag{3.7}$$

Note that, by (3.6), for every $t \in [0, T - (2/\lambda_m)]$, there exists a unique $s_i(t) \in (t, T]$ such that

$$\varphi_i(t, s_i(t)) = 1 \text{ and } \varphi_i(t, s) < 1 \quad \forall s \in [t, s_i(t)). \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$v_i(s_i(t), 1) = v_i(t, 0). \tag{3.9}$$

Using classical results on the dependence of solutions of ordinary differential equations on the initial conditions together with the inverse mapping theorem, one gets

$$|s_i'(t) - 1| \leq C\varepsilon_0. \tag{3.10}$$

Here and in what follows in this proof, $'$ denotes the derivative with respect to $t$, e.g., $s_i'(t) = ds_i/dt$ and $v'(t, x) = \partial_t v(t, x)$, and $C$ denotes a positive constant which changes from one place to another and may depend on $p$, $m$, $\lambda_1 \geq \cdots \geq \lambda_m > 0$, and $\hat{K} \in (0, 1)$ but is independent of $\varepsilon_0 > 0$, which is always assumed to be small enough, $T > 0$, $A$, and $v$, which are always assumed to satisfy (3.2) to (3.5).

Define, for $t \geq 2\lambda_1$,

$$\hat{r}_i(t) := t - s_i^{-1}(t). \tag{3.11}$$

From (3.10), we have

$$\sup_{t \in [2\lambda_1, T]} |\hat{r}_i'| \leq C\varepsilon_0. \tag{3.12}$$

Set

$$V(t) = v(t, 0).$$

We derive from (3.3), (3.9), and (3.11) that

$$V(t) = K(t)\Big(V_1\big(t - \hat{r}_1(t)\big), \ldots, V_i\big(t - \hat{r}_i(t)\big), \ldots, V_m\big(t - \hat{r}_m(t)\big)\Big)^T \quad \text{for } t \geq 2r_m. \tag{3.13}$$

In (3.13) and in the following, $r_i := 1/\lambda_i$ for every $i \in \{1, \ldots, m\}$. From (3.4) and (3.13), we obtain

$$\int_{2r_m}^T \|V(t)\|_p^p \, dt \leq \hat{K}^p \sum_{i=1}^n \int_{2r_m}^T |V_i\big(t - \hat{r}_i(t)\big)|^p \, dt. \tag{3.14}$$

Since

$$\int_{2r_m}^T |V_i\big(t - \hat{r}_i(t)\big)|^p \, dt = \int_{2r_m - \hat{r}_i(2r_m)}^{T - \hat{\lambda}_i(T)} |V_i(t)|^p s_i'(t) \, dt,$$

it follows from (3.10) that

$$\int_{2r_m}^T |V_i(t - \hat{r}_i)|^p \leq \int_0^T (1 + C\varepsilon_0)|V_i(t)|^p \, dt. \tag{3.15}$$

A combination of (3.14) and (3.15) yields

$$\int_{2r_m}^{T} \|V(t)\|_p^p \, dt \le \int_0^T \hat{K}^p (1 + C\varepsilon_0) \|V(t)\|_p^p \, dt.$$

By taking $\varepsilon_0$ small enough so that $\hat{K}^p(1 + C\varepsilon_0) \le [(1 + \hat{K})/2]^p$, we have

$$\int_0^T \|V(t)\|_p^p \, dt \le C \int_0^{2r_m} \|V(t)\|_p^p \, dt. \tag{3.16}$$

We next establish similar estimates for the derivatives of $V$. Let us define

$$W(t) := (W_1(t), \dots, W_m(t))^T := V'(t). \tag{3.17}$$

Differentiating (3.13) with respect to $t$, we have

$$W(t) = K(t)\Big(W_1\big(t - \hat{r}_1(t)\big), \dots, W_i\big(t - \hat{r}_i(t)\big), \dots, W_m\big(t - \hat{r}_m(t)\big)\Big)^T + g_1(t) + f_1(t), \tag{3.18}$$

where

$$g_1(t) := -K(t)\Big(W_1\big(t - \hat{r}_1(t)\big)\hat{r}_1'(t), \dots, W_i\big(t - \hat{r}_i(t)\big)\hat{r}_i'(t), \dots, W_m\big(t - \hat{r}_m(t)\big)\hat{r}_m'(t)\Big)^T \tag{3.19}$$

and

$$f_1(t) := K'(t)\Big(V_1\big(t - \hat{r}_1(t)\big), \dots, V_i\big(t - \hat{r}_i(t)\big), \dots, V_m\big(t - \hat{r}_m(t)\big)\Big)^T. \tag{3.20}$$

From (3.18), we have

$$|W(t)|_p^p \le [(\hat{K} + 1)/2]^p \sum_{i=1}^m |W_i\big(t - \hat{r}_i(t)\big)|^p + C\Big(|f_1(t)|_p^p + |g_1(t)|_p^p\Big). \tag{3.21}$$

Using (3.5) and (3.12), we derive from (3.19) and (3.20), as in (3.15), that

$$\int_{2r_m}^T \big(\|g_1(t)\|_p^p + \|f_1(t)\|_p^p\big) \, dt \le C\varepsilon_0^p \int_0^T \big(\|W\|_p^p + \|V(t)\|_p^p\big) \, dt. \tag{3.22}$$

It follows from (3.21), as in (3.16), that

$$\int_0^T \|V'(t)\|_p^p \, dt \le C \int_0^{2r_m} \big(\|V(t)\|_p^p + \|V'(t)\|_p^p\big) \, dt. \tag{3.23}$$

Combining (3.16) and (3.23), we reach the conclusion. ∎

As a consequence of Lemma 3.2, we obtain the following lemma, where $\mathcal{B}(\mathbb{R}^m)$ denotes the set of bilinear forms on $\mathbb{R}^m$.

LEMMA 3.3. *Let $p \ge 1$, $m$ be a positive integer, $\lambda_1 \ge \cdots \ge \lambda_m > 0$, $\hat{K} \in (0, 1)$, and $M \in (0, +\infty)$. Then there exist three constants $\varepsilon_0 > 0$, $\gamma > 0$, and $C > 0$ such that, for every $T > 0$, every $A \in C^1([0,T] \times [0,1]; \mathcal{D}_{m,+})$, every $K \in C^1([0,T]; \mathcal{M}_{m,m}(\mathbb{R}))$, every $Q \in C^1([0,T] \times [0,1]; \mathcal{B}(\mathbb{R}^m))$, and every $v \in W^{1,p}([0,T] \times$*

$[0,1];\mathbb{R}^m)$ *such that*

$$v_t + A(t,x)v_x = Q(t,x)(v,v) \text{ for } (t,x) \in (0,T)\times(0,1), \tag{3.24}$$

$$v(t,0) = K(t)v(t,1) \text{ for } t \in (0,T), \tag{3.25}$$

$$\sup_{t\in[0,T]} \|K(t)\|_p \leq \hat{K} < 1, \tag{3.26}$$

$$\|A - \operatorname{diag}(\lambda_1,\ldots,\lambda_m)\|_{C^1([0,T]\times[0,1])} + \sup_{t\in[0,T]} \|K'(t)\|_p \leq \varepsilon_0, \tag{3.27}$$

$$\|Q\|_{C^1([0,T]\times[0,1];\mathcal{B}(\mathbb{R}^m))} \leq M, \tag{3.28}$$

$$\|v(0,\cdot)\|_{W^{1,p}((0,1);\mathbb{R}^m)} \leq \varepsilon_0, \tag{3.29}$$

*one has*

$$\|v(t,\cdot)\|_{W^{1,p}((0,1);\mathbb{R}^m)} \leq Ce^{-\gamma t}\|v(0,\cdot)\|_{W^{1,p}((0,1);\mathbb{R}^m)} \text{ for } t \in (0,T).$$

*Proof of Lemma* 3.3. Let $\tilde{v} \in W^{1,p}([0,T]\times[0,1];\mathbb{R}^m)$ be the solution of the linear Cauchy problem

$$\tilde{v}_t + A(t,x)\tilde{v}_x = 0 \text{ for } (t,x) \in (0,T)\times(0,1), \tag{3.30}$$

$$\tilde{v}(t,0) = K(t)\tilde{v}(t,1) \text{ for } t \in (0,T), \tag{3.31}$$

$$\tilde{v}(0,x) = v(0,x) \text{ for } x \in (0,1). \tag{3.32}$$

(Note that $v(0,0) = K(0)v(0,1)$; hence, such a $\tilde{v}$ exists.) From Lemma 3.2, (3.30), (3.31), and (3.32), one has

$$\|\tilde{v}(t,\cdot)\|_{W^{1,p}((0,1);\mathbb{R}^m)} \leq Ce^{-\gamma t}\|v(0,\cdot)\|_{W^{1,p}((0,1);\mathbb{R}^m)} \text{ for } t \in [0,T]. \tag{3.33}$$

Let

$$\bar{v} := v - \tilde{v}. \tag{3.34}$$

From (3.24), (3.25), (3.30), (3.31), (3.32), and (3.34), one has

$$\bar{v}_t + A(t,x)\bar{v}_x = Q(t,x)(\tilde{v}+\bar{v},\tilde{v}+\bar{v}) \text{ for } (t,x) \in (0,T)\times(0,1), \tag{3.35}$$

$$\bar{v}(t,0) = K(t)\bar{v}(t,1) \text{ for } t \in (0,T), \tag{3.36}$$

$$\bar{v}(0,x) = 0 \text{ for } x \in (0,1). \tag{3.37}$$

Let, for $t \in [0,T]$,

$$e(t) := \|\bar{v}(t,\cdot)\|_{L^\infty((0,1);\mathbb{R}^m)}. \tag{3.38}$$

Following the characteristics and using (3.33), (3.35), (3.36), and the Sobolev imbedding $W^{1,p}((0,1);\mathbb{R}^m) \subset L^\infty((0,1);\mathbb{R}^m)$, one gets, in the sense of distribution in $(0,T)$,

$$e'(t) \leqslant C(\|v(0,\cdot)\|^2_{W^{1,p}((0,1);\mathbb{R}^m)} + e(t) + e(t)^2). \tag{3.39}$$

In (3.39), $C$ is as in the proof of Lemma 3.2 except that it may now depend on $M$. From (3.37), (3.38), and (3.39), one gets the existence of $\varepsilon_0$, of an increasing function $T \in [0,+\infty) \mapsto C(T) \in (0,+\infty)$ and of a decreasing function $T \in [0,+\infty) \mapsto \varepsilon(T) \in (0,+\infty)$, such that, for every $T \in [0,+\infty)$, every $A \in C^1([0,T]\times[0,1];\mathcal{D}_{m,+})$, every $K \in C^1([0,T];\mathcal{M}_{m,m}(\mathbb{R}))$, every $Q \in C^1([0,T]\times[0,1];\mathcal{B}(\mathbb{R}^m))$, and every $v \in W^{1,p}([0,T]\times[0,1];\mathbb{R}^m)$ satisfying (3.24) to (3.29),

$$(3.40) \quad \left(\|v(0,\cdot)\|_{W^{1,p}((0,1);\mathbb{R}^m)} \le \varepsilon(T)\right) \implies \left(\|\bar v(t,\cdot)\|_{L^\infty((0,1);\mathbb{R}^m)} \le C(T)\|v(0,\cdot)\|^2_{W^{1,p}((0,1);\mathbb{R}^m)} \text{ for } t \in (0,T)\right).$$

Let $\bar w := \bar v_x$. Differentiating (3.35) with respect to $x$, we get

$$(3.41) \quad \begin{aligned} \bar w_t + A(t,x)\bar w_x + A_x(t,x)\bar w = {}& Q_x(t,x)(\tilde v + \bar v, \tilde v + \bar v) \\ &+ Q(t,x)(\tilde v_x + \bar w, \tilde v + \bar v) + Q(t,x)(\tilde v + \bar v, \tilde v_x + \bar w) \\ &\text{for } (t,x) \in (0,T)\times(0,1). \end{aligned}$$

Differentiating (3.36) with respect to $t$ and using (3.35), we get, for $t \in [0,T]$,

$$(3.42) \quad \begin{aligned} &A(t,0)\bar w(t,0) - Q(t,0)(\tilde v(t,0) + \bar v(t,0), \tilde v(t,0) + \bar v(t,0)) \\ &\quad = K(t)\big(A(t,1)\bar w(t,1) - Q(t,1)(\tilde v(t,1) + \bar v(t,1), \tilde v(t,1) + \bar v(t,1))\big) - K'(t)\bar v(t,1). \end{aligned}$$

Differentiating (3.37) with respect to $x$, one gets

$$(3.43) \quad \bar w(0,x) = 0 \text{ for } x \in (0,1).$$

We consider (3.41), (3.42), and (3.43) as a nonhomogeneous linear hyperbolic system where the unknown is $w$ and the data are $A$, $K$, $Q$, $\tilde v$, and $\bar v$. Then, from straightforward estimates on the solutions of linear hyperbolic equations, one gets that, for every $t \in [0,T]$,

$$(3.44) \quad \begin{aligned} \|\bar w(t,\cdot)\|_{L^p((0,1);\mathbb{R}^m)} \le {}& e^{CT\left(1 + \|\tilde v\|_{L^\infty((0,T)\times(0,1);\mathbb{R}^m)} + \|\bar v\|_{L^\infty((0,T)\times(0,1);\mathbb{R}^m)}\right)} \\ &\times \left(\|\tilde v\|^2_{L^\infty((0,T);W^{1,p}((0,1);\mathbb{R}^m))} + \|\bar v\|^2_{L^\infty((0,T)\times(0,1);\mathbb{R}^m)}\right). \end{aligned}$$

From (3.33), (3.40), and (3.44), one gets the existence of $\varepsilon_0$, of an increasing function $T \in [0,+\infty) \mapsto C(T) \in (0,+\infty)$ and of a decreasing function $T \in [0,+\infty) \mapsto \varepsilon(T) \in (0,+\infty)$, such that, for every $T \in [0,+\infty)$, every $A \in C^1([0,T]\times[0,1];\mathcal{D}_{m,+})$, every $K \in C^1([0,T];\mathcal{M}_{m,m}(\mathbb{R}))$, every $Q \in C^1([0,T]\times[0,1];\mathcal{B}(\mathbb{R}^m))$, and every $v \in W^{1,p}([0,T]\times[0,1];\mathbb{R}^m)$ satisfying (3.24) to (3.29),

$$(3.45) \quad \left(\|v(0,\cdot)\|_{W^{1,p}((0,1);\mathbb{R}^m)} \le \varepsilon(T)\right) \implies \left(\|\bar v(t,\cdot)\|_{W^{1,p}((0,1);\mathbb{R}^m)} \le C(T)\|v(0,\cdot)\|^2_{W^{1,p}((0,1);\mathbb{R}^m)} \text{ for } t \in (0,T)\right),$$

which, together with (3.33) and (3.34), concludes the proof of Lemma 3.3. ∎

**3.2. Proof of Theorem 1.5.** Replacing, if necessary, $u$ by $Du$ where $D$ (depending only on $K$) is a diagonal matrix with positive entries, we may assume that

$$(3.46) \quad \|G'(0)\|_p < 1.$$

For $a \in \mathbb{R}^n$, let $\lambda_i(a)$ be the $i$th eigenvalue of $F(a)$ and $l_i(a)$ be a left eigenvector of $F(a)$ for this eigenvalue. The functions $\lambda_i$ are of class $C^\infty$ in a neighborhood of

$0 \in \mathbb{R}^n$. We may also impose on the $l_i$ to be of class $C^\infty$ in a neighborhood of $0 \in \mathbb{R}^n$ and impose that $l_i(0)^T$ is the $i$th vector of the canonical basis of $\mathbb{R}^n$. Set

$$\begin{cases} v_i = l_i(u)u \\ w_i = l_i(u)\partial_t u \end{cases} \quad \text{for } i = 1, \dots, n.$$

From [8, (3.5) and (3.6), p. 187], we have, for $i = 1, \dots, n$,

$$\begin{cases} u_i = v_i + \sum_{j,k}^n b_{ijk}(v) v_j v_k, \\ \partial_t u_i = w_i + \sum_{ijk} \bar{b}_{ijk}(v) v_j w_k, \end{cases} \tag{3.47}$$

where $b_{ijk}$ and $\bar{b}_{ijk}$ are of class $C^\infty$. From [8, (3.7) and (3.8)], we obtain, for $i = 1, \dots, n$,

$$\begin{cases} \partial_t v_i + \lambda_i(u)\partial_x v_i = \sum_{ijk}^n c_{ijk}(u) v_j v_k + \sum_{ijk}^n d_{ijk}(u) v_j w_k, \\ \partial_t w_i + \lambda_i(u)\partial_x w_i = \sum_{ijk}^n \bar{c}_{ijk}(u) w_j w_k + \sum_{ijk}^n \bar{d}_{ijk}(u) v_j w_k, \end{cases} \tag{3.48}$$

where $c_{ijk}, \bar{c}_{ijk}, d_{ijk}, \bar{d}_{ijk}$ are of class $C^\infty$ in a neighborhood of $0 \in \mathbb{R}^n$. We also have, for some $\hat{G} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ of class $C^\infty$ in a neighborhood of $0 \in \mathbb{R}^{2n}$,

$$\begin{pmatrix} v(t,0) \\ w(t,0) \end{pmatrix} = \hat{G} \begin{pmatrix} v(t,1) \\ w(t,1) \end{pmatrix}$$

and, by (1.5),

$$\hat{G}' \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} G'(0) & 0 \\ 0 & G'(0) \end{pmatrix},$$

which, together with (3.46), implies that

$$\|\hat{G}'(0)\|_p < 1.$$

Applying Lemma 3.3 for (3.48), we obtain the exponential stability for $(v, w)$ with respect to the $W^{1,p}$-norm, from which, noticing that $u_x = -F(u)^{-1}u_t$, Theorem 1.5 readily follows. ▯

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