



Fredholm transform and local rapid stabilization for a Kuramoto–Sivashinsky equation

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Abstract

This paper is devoted to the study of the local rapid exponential stabilization problem for a controlled Kuramoto–Sivashinsky equation on a bounded interval. We build a feedback control law to force the solution of the closed-loop system to decay exponentially to zero with arbitrarily prescribed decay rates, provided that the initial datum is small enough. Our approach uses a method we introduced for the rapid stabilization of a Korteweg–de Vries equation. It relies on the construction of a suitable integral transform and can be applied to many other equations.

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1. Introduction

Consider the following Kuramoto–Sivashinsky equation:

$$\begin{cases} v_t + v_{xxxx} + \lambda v_{xx} + vv_x = 0 & \text{in } (0, 1) \times (0, +\infty), \\ v(t, 0) = v(t, 1) = 0 & \text{on } (0, +\infty), \\ v_{xx}(t, 0) = f(t), v_{xx}(t, 1) = 0 & \text{on } (0, +\infty), \\ v(0, \cdot) = v^0(\cdot) & \text{in } (0, 1), \end{cases} \quad (1.1)$$

where $\lambda > 0$ and $v^0(\cdot) \in L^2(0, 1)$.

For $T > 0$, let us define

$$X_T \triangleq C^0([0, T]; L^2(0, 1)) \cap L^2(0, T; H^2(0, 1) \cap H_0^1(0, 1)),$$

which is endowed with the norm

$$|\cdot|_{X_T} = \left(|\cdot|_{C^0([0, T]; L^2(0, 1))}^2 + |\cdot|_{L^2(0, T; H^2(0, 1) \cap H_0^1(0, 1))}^2 \right)^{\frac{1}{2}}.$$

We first present the following locally well-posedness result, which is proved in [Appendix A](#) of this paper.

Theorem 1.1. *Let $F : L^2(0, 1) \rightarrow \mathbb{R}$ be a continuous linear map and let $T_0 \in (0, +\infty)$. Then for given $v^0 \in L^2(0, 1)$, there exists at most one solution $v \in X_{T_0}$ of (1.1) with $f(t) = F(v(t, \cdot))$.*

Moreover, there exist $r_0 > 0$ and $C_0 > 0$ such that, for every $v^0 \in L^2(0, 1)$ with

$$|v^0|_{L^2(0, 1)} \leq r_0, \quad (1.2)$$

there exists one solution $v \in X_{T_0}$ of (1.1) with $f(t) = F(v(t, \cdot))$ and this solution satisfies

$$|v|_{X_T} \leq C_0 |v^0|_{L^2(0, 1)}. \quad (1.3)$$

The Kuramoto–Sivashinsky (K–S for short) equation was first derived in [\[24\]](#) as a model for Belousov–Zhabotinskii reaction patterns in reaction–diffusion systems. It describes a lot of physical and chemical systems, such as unstable flame fronts (see [\[28\]](#) for example), falling liquid films (see [\[10\]](#) for example) and interfacial instabilities between two viscous fluids (see [\[20\]](#) for example). Since the pioneer works in [\[15, 26\]](#), the well-posedness and dynamical properties of the K–S equation were well studied (see [\[7, 12, 16, 17, 31\]](#) and the references therein).

We are interested in the stabilization problems of the K–S equation. There are many degrees of freedom to choose the feedback controls. For instance, one can choose the internal controls, the Dirichlet boundary control, the Neumann boundary control, etc., and the feedback law can be linear or nonlinear with respect to the output. Such kind of problems were studied extensively in the literature. Let us recall some of them.

In [\[1, 11\]](#), the authors studied the global stabilization for the K–S equation with periodic boundary conditions. The feedback controller acts on the whole domain. They first considered the ordinary differential equation approximations of the system, which accurately describe the

dominant dynamics, and then obtained a local stabilization result through nonlinear Galerkin’s method.

In the above two works, the control acts in the whole domain. In [21], the authors formulated and solved a robust boundary control problem for the K–S equation. In [25], with the assumption that $\lambda < 4\pi^2$, the authors studied the global stabilization problems of the K–S equation by a nonlinear boundary feedback control. The control acts on any two of the four variables u, u_x, u_{xx} and u_{xxx} at the boundary. It was derived using the combination of spectral analysis and Lyapunov techniques which guarantees L^2 -global exponential stability. It seems that their method does not work for $\lambda > 4\pi^2$.

In [27], with the assumption of the existence of the global solution to the K–S equation and λ is small, the authors studied the robust global stabilization of the equation subject to two nonlinear boundary feedback controls by Lyapunov techniques.

In [22], under the assumption of the existence of the global solution and $\lambda < 1$, the author studied the adaptive stabilization of the K–S equation by four nonlinear boundary feedback controls. The adaptive stabilizer was constructed by the concept of high-gain nonlinear output feedback and the estimation mechanism of the unknown parameters.

In this paper, we study the local rapid stabilization problem of (1.1). For this purpose, we first put a restriction on λ , i.e., we assume that

$$\lambda \notin \mathcal{N} \triangleq \left\{ j^2\pi^2 + k^2\pi^2 : j, k \in \mathbb{Z}^+, j \neq k \right\}. \tag{1.4}$$

In (1.4) and in the following, \mathbb{Z}^+ denotes the set of positive integers: $\mathbb{Z}^+ \triangleq \{1, 2, 3 \dots\}$.

Remark 1.1. The condition (1.4) seems strange. However, it is natural in the sense that if $\lambda \in \mathcal{N}$, then the linearized system of (1.1) at zero is not approximately controllable. See Theorem B.1 in Appendix B.

In [8], the same phenomenon of critical values of λ appears in the study of the boundary null controllability of the K–S equation with other boundary conditions. Moreover, a pole shifting method is applied in order to stabilize the noncritical cases. Furthermore, in [9], the authors show that when controlling all the boundary data at one point ($x = 0$ or $x = 1$), the equation is always null controllable.

Let us introduce an integral transform $K : L^2(0, 1) \rightarrow L^2(0, 1)$ as

$$(Kv)(x) = \int_0^1 k(x, y)v(y)dy, \quad \text{for } v \in L^2(0, 1).$$

Here k is the solution to

$$\begin{cases} k_{xxxx} + \lambda k_{xx} - k_{yyyy} - \lambda k_{yy} + ak = a\delta(x - y) & \text{in } (0, 1) \times (0, 1), \\ k(x, 0) = k(x, 1) = 0 & \text{on } (0, 1), \\ k_{yy}(x, 0) = k_{yy}(x, 1) = 0 & \text{on } (0, 1), \\ k_y(x, 0) = 0 & \text{on } (0, 1), \\ k(0, y) = k(1, y) = 0 & \text{on } (0, 1), \\ k_{xx}(1, y) = 0 & \text{on } (0, 1), \end{cases} \tag{1.5}$$

where $a \in \mathbb{R}$ and $\delta(x - y)$ denotes the Dirac measure on the diagonal of the square $[0, 1] \times [0, 1]$. The definition of a solution to (1.5) is given in Section 2.

We assume that

$$a \notin \mathcal{N}_1 \triangleq \left\{ -k^4\pi^4 + \lambda k^2\pi^2 + j^4\pi^4 - \lambda j^2\pi^2 : j, k \in \mathbb{Z}^+ \right\}. \tag{1.6}$$

Under this assumption, we are going to show the following results:

1. Eq. (1.5) has one and only one solution. The proof of this result is given in Section 2.
2. The operator $I - K$ is invertible. The proof of this result is given in Section 3.
3. Assume that

$$a > -j^4\pi^4 + \lambda j^2\pi^2, \quad \forall j \in \mathbb{Z}^+. \tag{1.7}$$

Let $\nu > 0$ be such that

$$0 < \nu < a + j^4\pi^4 - \lambda j^2\pi^2, \quad \forall j \in \mathbb{Z}^+. \tag{1.8}$$

Then, for every $v^0 \in L^2(0, 1)$ satisfying $|v^0|_{L^2(0,1)} \leq r$, if we define the feedback law $F(\cdot)$ by

$$F(v) = f(t) \triangleq \int_0^1 k_{xx}(0, y)v(t, y)dy, \tag{1.9}$$

then, for the solution v of (1.1), one has

$$|(I - K)v(t)|_{L^2(0,1)} \leq e^{-\nu t} |(I - K)v^0|_{L^2(0,1)}, \quad \forall t \in [0, +\infty). \tag{1.10}$$

This result is proved in Section 4.

From the above results, we get the following theorem.

Theorem 1.2. *Let us assume that (1.4) hold. Let $a > 0$ be such that (1.6) and (1.7) hold and let $\nu > 0$ be such that (1.8) holds. Then, there exist $r > 0$ and $C > 0$ such that, for every $v^0 \in L^2(0, 1)$ satisfying $|v^0|_{L^2(0,1)} \leq r$, the solution v to (1.1) with $f(t) \triangleq F(v(t, \cdot))$ and $v(0) = v^0$ is defined on $[0, +\infty)$ and satisfies*

$$|v(t, \cdot)|_{L^2(0,1)} \leq Ce^{-\nu t} |v(0, \cdot)|_{L^2(0,1)}, \quad \text{for every } t \geq 0. \tag{1.11}$$

Remark 1.2. Clearly, for every $\nu > 0$, there exists $a > 0$ such that (1.6) and (1.7). Hence we got the rapid stabilization of our K–S control system (1.1), i.e., for every $\nu > 0$, there exists a (linear) feedback law such that $0 \in L^2(0, 1)$ is exponential stable for the closed loop system with an exponential decay rate at least equal to ν .

Remark 1.3. There are three differences between system (1.1) and systems studied in [25,27]. The first one is that we only employ one control. The second one is the feedback control law is linear. The third one is we do not assume that λ is small.

Our method can also be applied to deal with the rapid stabilization problem of the K–S equation with other type boundary conditions and controls, such as

$$y(t, 0) = y(t, 1) = 0, \quad y_x(t, 0) = f(t), \quad y_x(t, 1) = 0 \text{ for } t \in (0, +\infty),$$

$$y_{xx}(t, 0) = y_{xx}(t, 1) = 0, \quad y_{xxx}(t, 0) = f(t), \quad y_{xxx}(t, 1) = 0 \text{ for } t \in (0, +\infty).$$

The control can also be put on the right end point of the boundary. In these cases, one has to modify the set \mathcal{N} given in (1.4) according to the boundary conditions. The proofs of the stabilization result for these cases are quite similar to that given in this paper and so are omitted.

The way of constructing the feedback control in the form of (1.9) was first introduced in [14] for obtaining the local rapid stabilization result for KdV equations. It was motivated by the backstepping method (see [23] for a systematic introduction of this method). Some ingredients of the proofs given here for the K–S equation are also inspired from [14].

Compared our way with the classical backstepping method, we use a Fredholm type transform rather than Volterra type transform. Fredholm type transforms were already used to study the stabilization of control system by some authors. In [18], the authors derived an explicit Volterra/Fredholm type transformation (and its inverse) of the closed-loop system into an exponentially stable target system. Based on this, they presented a novel control design for the Euler–Bernoulli beam which achieves rapid stabilization of the closed-loop system. In [2], the authors employed a novel Fredholm type transformation, which was called forwarding–backstepping transformation in that paper, to compensate the input and sensor delays in a controlled wave equation and to stabilize the overall system. Fredholm type transforms were also used in [3,4] to compensate the input and sensor delays and to stabilize the control system. In [6], a Fredholm type transform is introduced which allows the construction of boundary controllers and observers for a class of first-order hyperbolic PDEs. Some more results for the applying the Fredholm transforms in stabilization problems can also be found in [5,19,29] and the rich references therein. Compared to the above references, the interest of our approach is that it allows to prove the existence of the Fredholm transform and its invertibility precisely under the assumption of controllability of the linearized system, a condition which, in finite dimension, is necessary for rapid stabilization by means of smooth feedback laws.

Let us also point out that our proof shows the uniqueness of the kernel k (see Section 2). This uniqueness also holds for the control systems considered in [23] where the backstepping method is working. Hence, in these cases, our kernel k will be the same as the ones given in [23]. In particular, it will have its support included in one of the triangles $0 \leq x \leq y \leq 1$ and $0 \leq y \leq x \leq 1$, and the Fredholm transforms will be the Volterra transforms already presented in [23].

For the numerical side, in the cases presented in [23] where the backstepping method is working, the kernel k , when not available analytically, is obtained by rapidly convergent symbolic or numerical iterative schemes. It is unlikely that these iterative schemes work for (1.5). A possible strategy to solve numerically (1.5) is to use finite difference methods. Another approach might be to use a continuation method: one starts with $a = 0$, for which the solution is $k = 0$, and then one increases progressively the parameter a until the desired value. Note that, for every value of the parameter a , there exists a unique solution k to (1.5). Hence there will be no turning points.

Let us briefly explain the idea for introducing the transform K . For simplicity, let us forget the nonlinearity in (1.1) and therefore consider the following linear equation:

$$\begin{cases} u_t + u_{xxxx} + \lambda u_{xx} = 0 & \text{in } (0, 1) \times (0, +\infty), \\ u(t, 0) = u(t, 1) = 0 & \text{on } (0, +\infty), \\ u_{xx}(t, 0) = g(t), u_{xx}(t, 1) = 0 & \text{on } (0, +\infty), \\ u(0, \cdot) = u^0(\cdot) & \text{in } (0, 1). \end{cases} \tag{1.12}$$

Let

$$\tilde{u}(t, x) = u(t, x) - \int_0^1 k(x, y)u(t, y)dy, \quad g(t) = \int_0^1 k_{xx}(0, y)u(t, y)dy,$$

where $k(\cdot, \cdot)$ is a solution to (1.5). If $k(\cdot, \cdot)$ is smooth enough, one can easily see that $\tilde{u}(\cdot, \cdot)$ is the solution to

$$\begin{cases} \tilde{u}_t + \tilde{u}_{xxxx} + \lambda \tilde{u}_{xx} + a\tilde{u} = 0 & \text{in } (0, 1) \times (0, +\infty), \\ \tilde{u}(t, 0) = \tilde{u}(t, 1) = 0 & \text{on } (0, +\infty), \\ \tilde{u}_{xx}(t, 0) = 0, \tilde{u}_{xx}(t, 1) = 0 & \text{on } (0, +\infty), \\ \tilde{u}(0, \cdot) = u^0(\cdot) - \int_0^1 k(\cdot, y)u^0(y)dy & \text{in } (0, 1). \end{cases} \tag{1.13}$$

Let us point out that $\|\tilde{u}\|_{L^2(0,1)}$ decays exponentially if we take $a > 0$ large enough and that the exponential decay rate goes to $+\infty$ as $a \rightarrow +\infty$. To deal with the nonlinear term vu_x requires further arguments which will be given in Section 4.

It seems that it is very difficult to apply the backstepping method to solve our problem. Indeed, let us consider the linear system (1.12) again. If we follow the backstepping method and choose the feedback control as

$$\hat{u}(t, x) = u(t, x) - \int_x^1 \hat{k}(x, y)u(t, y)dy, \quad g(t) = \int_0^1 \hat{k}_{xx}(0, y)u(t, y)dy, \tag{1.14}$$

then, to guarantee that $\hat{u}(\cdot, \cdot)$ is a solution to (1.13), \hat{k} should solve

$$\begin{cases} \hat{k}_{xxxx} + \lambda \hat{k}_{xx} - \hat{k}_{yyyy} - \lambda \hat{k}_{yy} + a\hat{k} = 0 & \text{in } (0, 1) \times (0, 1), \\ \hat{k}(x, 1) = \hat{k}_{yy}(x, 1) = 0 & \text{on } (0, 1), \\ \hat{k}(x, x) = 0 & \text{on } (0, 1), \\ \hat{k}_{xx}(x, x) + \hat{k}_{xy}(x, x) = 0 & \text{on } (0, 1), \\ 4\hat{k}_{xxx}(x, x) + 2\hat{k}_{yyy}(x, x) + 6\hat{k}_{xxy}(x, x) + 4\hat{k}_{xyy}(x, x) + (\lambda - 1)\hat{k}_y(x, x) = 0 & \text{on } (0, 1). \end{cases} \tag{1.15}$$

With these five boundary restrictions, the fourth order equation (1.15) becomes overdetermined. Therefore, it is not clear whether such a function $\hat{k}(\cdot, \cdot)$ exists.

The rest of this paper is organized as follows. In Section 2, we establish the well-posedness of Eq. (1.5). In Section 3, we show that $I - K$ is an invertible operator. At last, in Section 4, we prove Theorem 1.2.

2. Well-posedness of (1.5)

This section is devoted to the study of the well-posedness of equation (1.5). We first introduce the definition of the solution to (1.5). Let

$$\mathcal{E} \triangleq \{ \rho \in C^\infty([0, 1] \times [0, 1]) : \rho(0, y) = \rho(1, y) = \rho(x, 0) = \rho(x, 1) = 0, \rho_x(0, y) = \rho_{xx}(0, y) = \rho_{xx}(1, y) = \rho_{yy}(x, 1) = 0 \} \tag{2.1}$$

and let \mathcal{G} be the set of $k \in H^2((0, 1) \times (0, 1)) \cap H_0^1((0, 1) \times (0, 1))$ such that

$$\left(x \in (0, 1) \mapsto k_{xx}(x, \cdot) \in L^2(0, 1) \right) \in C^0([0, 1]; L^2(0, 1)), \tag{2.2}$$

$$\left(y \in (0, 1) \mapsto k_{yy}(\cdot, y) \in L^2(0, 1) \right) \in C^0([0, 1]; L^2(0, 1)), \tag{2.3}$$

$$k_{yy}(\cdot, 0) = k_{yy}(\cdot, 1) = 0 \text{ in } L^2(0, 1). \tag{2.4}$$

We call $k(\cdot, \cdot) \in \mathcal{G}$ a solution to (1.5) if

$$\int_0^1 \int_0^1 [\rho_{xxxx}(x, y) + \lambda \rho_{xx}(x, y) - \rho_{yyyy}(x, y) - \lambda \rho_{yy}(x, y) + a \rho(x, y)] k(x, y) dx dy - \int_0^1 a \rho(x, x) dx = 0, \quad \text{for every } \rho \in \mathcal{E}. \tag{2.5}$$

We have the following well-posedness result for (1.5).

Theorem 2.1. *Suppose that (1.4) and (1.6) hold. Eq. (1.5) has a unique solution in \mathcal{G} .*

Remark 2.1. Theorem 2.1 concludes the existence and uniqueness of the solution to Eq. (1.5). However, to construct the feedback control, one needs further information about the solution. For example, can we find an explicit formula $k(\cdot, \cdot)$ as people do for the integral kernel introduced in the backstepping method? Can we solve it numerically or approximately? The proof of the existence of the solution give us a way to construct the approximate solution of Eq. (1.5). Nevertheless, since it is not very easy to compute ϕ_j (see (2.53) for the definition of ϕ_j), it is hard for us to compute $k(\cdot, \cdot)$. On the other hand, since the domain $(0, 1) \times (0, 1)$ is very regular, one can use finite difference method to get a numerical solution in an efficient way. The detailed analysis of this is out of the scorpion of this paper and will be appeared in our forthcoming work.

Before proceeding our proof, we recall the following two results.

Lemma 2.1. (See [30, p. 45, Theorem 15].) *Let H be a separable Hilbert space and let $\{e_j\}_{j \in \mathbb{Z}^+}$ be an orthonormal basis for H . If $\{f_j\}_{j \in \mathbb{Z}^+}$ is an ω -independent sequence such that*

$$\sum_{j \in \mathbb{Z}^+} \|f_j - e_j\|_H^2 < +\infty,$$

then $\{f_j\}_{j \in \mathbb{Z}^+}$ is a Riesz basis for H .

Lemma 2.2. (See [30, p. 40, Theorem 12].) *Let $\{e_j\}_{j \in \mathbb{Z}^+}$ be a basis of a Banach space X and let $\{f_j\}_{j \in \mathbb{Z}^+}$ be the associated sequence of coefficient functionals. If $\{b_j\}_{j \in \mathbb{Z}^+}$ is complete in X and if*

$$\sum_{j \in \mathbb{Z}^+} \|e_j - b_j\|_X \|f_j\|_{X'} < +\infty,$$

then $\{b_j\}_{j \in \mathbb{Z}^+}$ is a basis for X which is equivalent to $\{e_j\}_{j \in \mathbb{Z}^+}$.

Proof of Theorem 2.1. We divide the proof into two steps:

- Step 1: proof of the uniqueness of the solution to (1.5).
- Step 2: proof of the existence of a solution to (1.5).

Step 1: proof of the uniqueness of the solution to (1.5)

Assume that $k_1(\cdot, \cdot)$ and $k_2(\cdot, \cdot)$ are two solutions to (1.5). Let $k_3(\cdot, \cdot) = k_1(\cdot, \cdot) - k_2(\cdot, \cdot)$. Then $k_3(\cdot, \cdot)$ is such that (2.5) holds without the last integral term, i.e., $k_3(\cdot, \cdot)$ is a solution to

$$\begin{cases} k_{3,xxxx} + \lambda k_{3,xx} - k_{3,yyyy} - \lambda k_{3,yy} + ak_3 = 0 & \text{in } (0, 1) \times (0, 1), \\ k_3(x, 0) = k_3(x, 1) = 0 & \text{on } (0, 1), \\ k_{3,yy}(x, 0) = k_{3,yy}(x, 1) = 0 & \text{on } (0, 1), \\ k_{3,y}(x, 0) = 0 & \text{on } (0, 1), \\ k_3(0, y) = k_3(1, y) = 0 & \text{on } (0, 1), \\ k_{3,xx}(1, y) = 0 & \text{on } (0, 1). \end{cases} \tag{2.6}$$

Let us define an unbounded linear operator $A : D(A) \subset L^2(0, 1) \rightarrow L^2(0, 1)$ as follows:

$$\begin{cases} D(A) = \{w \in H^4(0, 1) : w(0) = w(1) = w_{xx}(0) = w_{xx}(1) = 0\}, \\ Aw = -w_{xxxx} - \lambda w_{xx}, \quad \forall w \in D(A). \end{cases} \tag{2.7}$$

A is a self-adjoint operator with compact resolvent. Simple computations give that the eigenvalues of A are

$$\mu_j = -j^4 \pi^4 + \lambda j^2 \pi^2, \quad j \in \mathbb{Z}^+ \tag{2.8}$$

and that the eigenfunction corresponding to the eigenvalue μ_j is

$$\varphi_j(x) = \sqrt{2} \sin(j\pi x), \quad j \in \mathbb{Z}^+. \tag{2.9}$$

Note that, by (1.4), one has

$$(j \in \mathbb{Z}^+, \quad k \in \mathbb{Z}^+ \text{ and } j \neq k) \Rightarrow (\mu_j \neq \mu_k). \tag{2.10}$$

Let us write

$$k_3(x, y) = \sum_{j \in \mathbb{Z}^+} \psi_j(x) \varphi_j(y) \tag{2.11}$$

for the solution to (2.6). Then, ψ_j solves

$$\begin{cases} \psi_j'''' + \lambda \psi_j'' + a \psi_j + \mu_j \psi_j = 0 & \text{in } (0, 1), \\ \psi_j(0) = \psi_j(1) = \psi_j''(1) = 0. \end{cases} \tag{2.12}$$

Let $c_j \triangleq \psi_j''(0)$ ($j \in \mathbb{Z}^+$). We consider the following equation:

$$\begin{cases} \check{\psi}_j'''' + \lambda \check{\psi}_j'' + a \check{\psi}_j + \mu_j \check{\psi}_j = 0 & \text{in } (0, 1), \\ \check{\psi}_j(0) = \check{\psi}_j(1) = \check{\psi}_j''(1) = 0, \\ \check{\psi}_j''(0) = 1. \end{cases} \tag{2.13}$$

Since, by (1.6) and (2.8), $a + \mu_j$ is not an eigenvalue of A , (2.13) has a unique solution. Moreover, $\psi_j = c_j \check{\psi}_j$ for every $j \in \mathbb{Z}^+$. The four roots of

$$r^4 + \lambda r^2 + a + \mu_j = 0$$

are

$$\begin{aligned} r_j^{(1)} &= \left(\frac{-\lambda + \sqrt{\lambda^2 - 4(a + \mu_j)}}{2} \right)^{\frac{1}{2}}, \quad r_j^{(2)} = -\left(\frac{-\lambda + \sqrt{\lambda^2 - 4(a + \mu_j)}}{2} \right)^{\frac{1}{2}} = -r_j^{(1)}, \\ r_j^{(3)} &= \left(\frac{-\lambda - \sqrt{\lambda^2 - 4(a + \mu_j)}}{2} \right)^{\frac{1}{2}}, \quad r_j^{(4)} = -\left(\frac{-\lambda - \sqrt{\lambda^2 - 4(a + \mu_j)}}{2} \right)^{\frac{1}{2}} = -r_j^{(3)}. \end{aligned} \tag{2.14}$$

Easy computations show that there exists $C > 0$ such that, with $i \triangleq \sqrt{-1}$,

$$\left| r_j^{(1)} - j\pi + \frac{\lambda}{2j\pi} \right| \leq \frac{C}{j^3}, \quad \left| i r_j^{(4)} - \left(j\pi - \frac{a}{4j^3\pi^3} \right) \right| \leq \frac{C}{j^5}, \quad \forall j \in \mathbb{Z}^+. \tag{2.15}$$

Let us assume that

$$a + \mu_j \neq 0 \text{ and } \lambda^2 \neq 4(a + \mu_j). \tag{2.16}$$

The cases where (2.16) does not hold require slight modifications in the arguments given below. We omit them. Note that (2.16) holds for j large enough. From (2.14) and (2.16), one gets that the four roots $r_j^{(1)}, r_j^{(2)}, r_j^{(3)}$ and $r_j^{(4)}$ are distinct. Hence there exists four complex numbers $\alpha_j^{(1)}, \alpha_j^{(2)}, \alpha_j^{(3)}$ and $\alpha_j^{(4)}$ such that

$$\check{\psi}_j = \alpha_j^{(1)} \cosh(r_j^{(1)} x) + \alpha_j^{(2)} \sinh(r_j^{(1)} x) + \alpha_j^{(3)} \cos(ir_j^{(4)} x) + \alpha_j^{(4)} \sin(ir_j^{(4)} x). \tag{2.17}$$

Let us emphasize that throughout this section, the functions as well as the sequences of numbers are complex valued. However, at the end of this section we will check that k is real valued. From the boundary conditions of (2.13) and (2.17), we get that

$$\begin{cases} \alpha_j^{(1)} + \alpha_j^{(3)} = 0, \\ \alpha_j^{(1)} \cosh(r_j^{(1)}) + \alpha_j^{(2)} \sinh(r_j^{(1)}) + \alpha_j^{(3)} \cos(ir_j^{(4)}) + \alpha_j^{(4)} \sin(ir_j^{(4)}) = 0, \\ \alpha_j^{(1)} (r_j^{(1)})^2 + \alpha_j^{(3)} (r_j^{(4)})^2 = 1, \\ \alpha_j^{(1)} (r_j^{(1)})^2 \cosh(r_j^{(1)}) + \alpha_j^{(2)} (r_j^{(1)})^2 \sinh(r_j^{(1)}) - \alpha_j^{(3)} (r_j^{(4)})^2 \cos(ir_j^{(4)}) - \alpha_j^{(4)} (r_j^{(4)})^2 \sin(ir_j^{(4)}) = 0. \end{cases} \tag{2.18}$$

By means of the first equation of (2.18), we find that $\alpha_j^{(1)} = -\alpha_j^{(3)}$. This, together with (2.14) and the third equation of (2.18), implies that

$$\alpha_j^{(1)} = -\alpha_j^{(3)} = \left(\sqrt{\lambda^2 - 4(a + \mu_j)} \right)^{-1}. \tag{2.19}$$

(Note that by (2.16), the right hand side of (2.19) is well defined.) From (2.16) and (2.16), one gets

$$(r_j^{(1)})^2 + (r_j^{(4)})^2 \neq 0. \tag{2.20}$$

According to (2.14), (2.19), the second and fourth equations in (2.18) and (2.20), we find that

$$\alpha_j^{(2)} = -\alpha_j^{(1)} \coth(r_j^{(1)}), \quad \alpha_j^{(4)} = -\alpha_j^{(3)} \cot(ir_j^{(4)}). \tag{2.21}$$

Thus, we obtain that

$$\begin{aligned} \check{\psi}_j &= \alpha_j^{(1)} \cosh(r_j^{(1)} x) - \alpha_j^{(1)} \coth(r_j^{(1)}) \sinh(r_j^{(1)} x) \\ &\quad - \alpha_j^{(1)} \cos(ir_j^{(4)} x) + \alpha_j^{(1)} \cot(ir_j^{(4)}) \sin(ir_j^{(4)} x). \end{aligned} \tag{2.22}$$

Note that, by (1.6),

$$a \neq 0. \tag{2.23}$$

Due to (2.15) and (2.23), we have

$$\cot(ir_j^{(4)}) = O(j^3) \text{ as } j \rightarrow \infty. \tag{2.24}$$

Let

$$\hat{\psi}_j = \frac{\sqrt{2}}{\alpha_j^{(1)} \cot(ir_j^{(4)})} \check{\psi}_j \text{ for } j \in \mathbb{Z}^+. \tag{2.25}$$

Let us now prove the following lemma.

Lemma 2.3. *The family $\{\hat{\psi}_j\}_{j \in \mathbb{Z}^+}$ is a Riesz basis of $L^2(0, 1)$.*

Proof of Lemma 2.3. First, we claim that

$$\sum_{j \in \mathbb{Z}^+} |\hat{\psi}_j - \varphi_j|_{L^2(0,1)}^2 < \infty. \tag{2.26}$$

Indeed, from (2.15) and (2.22) to (2.25), we see that there exists $C > 0$ such that, for every $j \in \mathbb{Z}^+$,

$$\begin{aligned} & \int_0^1 |\hat{\psi}_j - \varphi_j|^2 dx \\ & \leq 2 \left(\int_0^1 |\sqrt{2} \sin(ir_j^{(4)} x) - \sqrt{2} \sin(j\pi x)|^2 dx \right. \\ & \quad \left. + \int_0^1 \frac{2}{|\cot(ir_j^{(4)})|^2} |\cosh(r_j^{(1)} x) - \coth(r_j^{(1)}) \sinh(r_j^{(1)} x) - \cos(ir_j^{(4)} x)|^2 dx \right) \\ & \leq \frac{C}{j^6}. \end{aligned} \tag{2.27}$$

Since $\sum_{j=1}^{+\infty} j^{-6} < +\infty$, we have (2.26).

Let $T > 0$. Consider the following control system:

$$\begin{cases} \vartheta_t + \vartheta_{xxxx} + \lambda \vartheta_{xx} = 0 & \text{in } (0, T) \times (0, 1), \\ \vartheta(t, 0) = \vartheta(t, 1) = 0 & \text{in } (0, T), \\ \vartheta_{xx}(t, 0) = \eta(t), \vartheta_{xx}(t, 1) = 0 & \text{in } (0, T), \end{cases} \tag{2.28}$$

where $\eta(\cdot) \in L^2(0, T)$ is the control. Let $\tilde{\vartheta} = A^{-1} \vartheta$ (recall (2.7) for the definition of A). Then, $\tilde{\vartheta}$ solves

$$\partial_t \tilde{\vartheta} = A \tilde{\vartheta} - \eta(t)b, \tag{2.29}$$

here $b(\cdot)$ is the solution to

$$\begin{cases} b_{xxxx} + \lambda b_{xx} = 0 & \text{in } (0, 1), \\ b(0) = b(1) = 0, b_{xx}(0) = 1, b_{xx}(1) = 0. \end{cases} \tag{2.30}$$

Clearly, $b \in L^2(0, 1)$. Let $b = \sum_{j \in \mathbb{Z}^+} b_j \varphi_j$. Since system (2.28) is approximately controllable in $L^2(0, 1)$ (see Theorem B.1), we get that (2.28) is also approximately controllable in $A^{-1}(L^2(0, 1))$. In particular,

$$b_j \neq 0, \quad j \in \mathbb{Z}^+. \tag{2.31}$$

For $j \in \mathbb{Z}^+$, since, as already used above, $a + \mu_j$ is not an eigenvalue of A (recall once more (1.6) and (2.8)), there exists a $\tilde{\psi}_j \in D(A)$ such that

$$-A\tilde{\psi}_j + (a + \mu_j)\tilde{\psi}_j = -\frac{\sqrt{2}}{\alpha_j^{(1)} \cot(ir_j^{(4)})} (a + \mu_j)b. \tag{2.32}$$

Thanks to (2.13), (2.25) and (2.32), for every $j \in \mathbb{Z}^+$, we have

$$A^{-1}\tilde{\psi}_j = (a + \mu_j)^{-1}\tilde{\psi}_j - \frac{\sqrt{2}}{\alpha_j^{(1)} \cot(ir_j^{(4)})} A^{-1}b, \tag{2.33}$$

$$\hat{\psi}_j = \tilde{\psi}_j + \frac{\sqrt{2}}{\alpha_j^{(1)} \cot(ir_j^{(4)})} b. \tag{2.34}$$

Let us assume that there exists $\{a_j\}_{j \in \mathbb{Z}^+} \in \ell^2(\mathbb{Z}^+)$ such that $\sum_{j \in \mathbb{Z}^+} a_j \hat{\psi}_j = 0$. Then, from (2.34), we obtain that

$$\sum_{j \in \mathbb{Z}^+} a_j \left(\tilde{\psi}_j + \frac{\sqrt{2}}{\alpha_j^{(1)} \cot(ir_j^{(4)})} b \right) = 0. \tag{2.35}$$

Applying A^{-1} to (2.35), and using (2.33), one gets that

$$\sum_{j \in \mathbb{Z}^+} a_j (a + \mu_j)^{-1} \tilde{\psi}_j = 0, \tag{2.36}$$

which, together with (2.34), deduces that

$$\left[\sum_{j \in \mathbb{Z}^+} a_j \frac{\sqrt{2}}{\alpha_j^{(1)} \cot(ir_j^{(4)})} (a + \mu_j)^{-1} \right] b - \sum_{j \in \mathbb{Z}^+} a_j (a + \mu_j)^{-1} \hat{\psi}_j = 0. \tag{2.37}$$

Applying A^{-1} to (2.36) and using (2.33) again, we find that

$$\left[\sum_{j \in \mathbb{Z}^+} a_j \frac{\sqrt{2}}{\alpha_j^{(1)} \cot(ir_j^{(4)})} (a + \mu_j)^{-1} \right] A^{-1}b - \sum_{j \in \mathbb{Z}^+} a_j (a + \mu_j)^{-2} \tilde{\psi}_j = 0, \tag{2.38}$$

which, together with (2.34), gives

$$\begin{aligned} & \left[\sum_{j \in \mathbb{Z}^+} a_j \frac{\sqrt{2}}{\alpha_j^{(1)} \cot(ir_j^{(4)})} (a + \mu_j)^{-1} \right] A^{-1}b + \left[\sum_{j \in \mathbb{Z}^+} a_j \frac{\sqrt{2}}{\alpha_j^{(1)} \cot(ir_j^{(4)})} (a + \mu_j)^{-2} \right] b \\ & - \sum_{j \in \mathbb{Z}^+} a_j (a + \mu_j)^{-2} \hat{\psi}_j = 0. \end{aligned} \tag{2.39}$$

By induction, one gets that, for every positive integer p ,

$$\begin{aligned} & \left(\sum_{j \in \mathbb{Z}^+} a_j \frac{\sqrt{2}}{\alpha_j^{(1)} \cot(ir_j^{(4)})} (a + \mu_j)^{-1} \right) A^{-p}b \\ & + \sum_{k=2}^p \left(\sum_{j \in \mathbb{Z}^+} a_j \frac{\sqrt{2}}{\alpha_j^{(1)} \cot(ir_j^{(4)})} (a + \mu_j)^{-k} \right) A^{k-p-1}b \\ & + \sum_{j \in \mathbb{Z}^+} a_j \frac{\sqrt{2}}{\alpha_j^{(1)} \cot(ir_j^{(4)})} (a + \mu_j)^{-p-1}b - \sum_{j \in \mathbb{Z}^+} a_j (a + \mu_j)^{-p-1} \hat{\psi}_j = 0. \end{aligned} \tag{2.40}$$

If

$$\sum_{j \in \mathbb{Z}^+} a_j \frac{\sqrt{2}}{\alpha_j^{(1)} \cot(ir_j^{(4)})} (a + \mu_j)^{-1} \neq 0, \tag{2.41}$$

we get from (2.40) that

$$\{A^{-p}b\}_{p \in (\{0\} \cup \mathbb{Z}^+)} \subset \text{span}\{\hat{\psi}_j\}_{j \in \mathbb{Z}^+}. \tag{2.42}$$

If $\overline{\text{span}\{\hat{\psi}_j\}_{j \in \mathbb{Z}^+}} \neq L^2(0, 1)$, then we can find a nonzero function $d = \sum_{j \in \mathbb{Z}^+} d_j \varphi_j \in L^2(0, 1)$ such that

$$(h, d)_{L^2(0,1)} = 0, \text{ for every } h \in \text{span}\{\hat{\psi}_j\}_{j \in \mathbb{Z}^+}. \tag{2.43}$$

From (2.42) and (2.43), we obtain that $(A^{-p}b, d)_{L^2(0,1)} = 0$ for every $p \in (\{0\} \cup \mathbb{Z}^+)$. Therefore, we get

$$\sum_{j \in \mathbb{Z}^+} b_j \mu_j^{-p} d_j = 0 \text{ for all } p \in (\{0\} \cup \mathbb{Z}^+). \tag{2.44}$$

Let us define a complex variable function $G(\cdot) : \mathbb{C} \rightarrow \mathbb{C}$ as

$$G(z) = \sum_{j \in \mathbb{Z}^+} d_j b_j e^{\mu_j^{-1} z}, \quad z \in \mathbb{C}.$$

Then, it is clear that $G(\cdot)$ is a holomorphic function. From (2.44), we see that $G^{(p)}(0) = 0$ for every $p \in (\{0\} \cup \mathbb{Z}^+)$. Thus, we obtain that

$$G(\cdot) = 0. \tag{2.45}$$

Using (2.45) and (2.10), and looking at the asymptotic behavior of $G(z)$ as $z \in \mathbb{R}$ tends to $+\infty$ – if some μ_j are positive – and to $-\infty$ one gets that $d_j b_j = 0$ for all $j \in \mathbb{Z}^+$. Since $b_j \neq 0$, we get that $d_j = 0$ for all $j \in \mathbb{Z}^+$. Therefore, we get $d = 0$, which leads to a contradiction. Hence, (2.41) implies that

$$\overline{\text{span}\{\hat{\psi}_j\}_{j \in \mathbb{Z}^+}} = L^2(0, 1). \tag{2.46}$$

If (2.41) does not hold but

$$\sum_{j \in \mathbb{Z}^+} a_j \frac{\sqrt{2}}{\alpha_j^{(1)} \cot(ir_j^{(4)})} (a + \mu_j)^{-2} \neq 0,$$

then by using (2.40) again, we obtain that

$$\{A^{-p}b\}_{p \in (\{0\} \cup \mathbb{Z}^+)} \subset \text{span}\{\hat{\psi}_j\}_{j \in \mathbb{Z}^+}.$$

By a similar argument, we find that (2.46) again holds. Similarly, we can get that, if there is a $p \in \mathbb{Z}^+$ such that

$$\sum_{j \in \mathbb{Z}^+} a_j \frac{\sqrt{2}}{\alpha_j^{(1)} \cot(ir_j^{(4)})} (a + \mu_j)^{-p} \neq 0, \tag{2.47}$$

then (2.46) holds.

On the other hand, if

$$\sum_{j \in \mathbb{Z}^+} a_j \frac{\sqrt{2}}{\alpha_j^{(1)} \cot(ir_j^{(4)})} (a + \mu_j)^{-p} = 0 \text{ for every } p \in (\{0\} \cup \mathbb{Z}^+),$$

we define a function

$$\tilde{G}(z) \triangleq \sum_{j \in \mathbb{Z}^+} a_j \frac{\sqrt{2}}{\alpha_j^{(1)} \cot(ir_j^{(4)})} (a + \mu_j)^{-1} e^{(a+\mu_j)^{-1}z} \text{ for every } z \in \mathbb{C},$$

and it is clear that $\tilde{G}(\cdot)$ is a holomorphic function and $\tilde{G}^{(p)}(0) = 0$ for every $p \in \{0\} \cup \mathbb{Z}^+$, which implies that $\tilde{G}(\cdot) = 0$. Therefore, as above, we conclude that $a_j = 0$ for every $j \in \mathbb{Z}^+$.

By the above argument, we know that either $\{\hat{\psi}_j\}_{j \in \mathbb{Z}^+}$ is ω -independent or it is complete in $L^2(0, 1)$.

We first deal with the case that $\{\hat{\psi}_j\}_{j \in \mathbb{Z}^+}$ is ω -independent. Let us take $H = L^2(0, 1)$ and put $e_j = \varphi_j$, $f_j = \hat{\psi}_j$ for $j \in \mathbb{Z}^+$ in Lemma 2.1. Then, by (2.27), the conditions of Lemma 2.1 are fulfilled. Thus, $\{\hat{\psi}_j\}_{j \in \mathbb{Z}^+}$ is a Riesz basis of $L^2(0, 1)$. This concludes the proof of Lemma 2.3. \square

Next, we consider the case that $\{\hat{\psi}_j\}_{j \in \mathbb{Z}^+}$ is complete in $L^2(0, 1)$. In Lemma 2.2, let us set $X \triangleq L^2(0, 1)$, $e_j \triangleq \varphi_j$, $f_j \triangleq \varphi_j$, $b_j \triangleq \hat{\psi}_j$ for $j \in \mathbb{Z}^+$. Then, by (2.27), it is easy to see that the conditions of Lemma 2.2 are fulfilled. Therefore, $\{\hat{\psi}_j\}_{j \in \mathbb{Z}^+}$ is a Riesz basis of $L^2(0, 1)$.

Now we estimate $\{c_j\}_{j \in \mathbb{Z}^+}$. From (2.9), we get that $\int_0^1 |\varphi_{j,yy}(y)|^2 dy = j^4 \pi^4$. This, together with the fact that $\{\hat{\psi}_j\}_{j \in \mathbb{Z}^+}$ is a Riesz basis of $L^2(0, 1)$ and $k_3(\cdot, \cdot) \in H^2((0, 1) \times (0, 1))$, implies that

$$\begin{aligned} +\infty &> \int_0^1 \int_0^1 |k_{3,yy}(x, y)|^2 dx dy \geq \int_0^1 \int_0^1 \left| \sum_{j \in \mathbb{Z}^+} c_j \frac{\sqrt{2}}{\alpha_j^{(1)} \cot(ir_j^{(4)})} \hat{\psi}_j(x) \varphi_j''(y) \right|^2 dx dy \\ &\geq C \int_0^1 \sum_{j \in \mathbb{Z}^+} \left| c_j \frac{\sqrt{2}}{\alpha_j^{(1)} \cot(ir_j^{(4)})} \varphi_j''(y) \right|^2 dy \geq C \sum_{j \in \mathbb{Z}^+} \left| c_j \frac{\sqrt{2}}{\alpha_j^{(1)} \cot(ir_j^{(4)})} j^2 \right|^2. \end{aligned}$$

Hence, we find that

$$\left\{ c_j \frac{\sqrt{2}}{\alpha_j^{(1)} \cot(ir_j^{(4)})} j^2 \right\}_{j \in \mathbb{Z}^+} \in \ell^2(\mathbb{Z}^+). \tag{2.48}$$

From (2.9), we get that

$$\varphi_j'(0) = \sqrt{2} j \pi, \quad \forall j \in \mathbb{Z}^+. \tag{2.49}$$

From (2.48) and (2.49), we obtain that

$$\left\{ c_j \frac{\sqrt{2}}{\alpha_j^{(1)} \cot(ir_j^{(4)})} \varphi_j'(0) \right\}_{j \in \mathbb{Z}^+} \in \ell^2(\mathbb{Z}^+). \tag{2.50}$$

Using (2.11), $k_{3,y}(x, 0) = 0$ and the second inequality of (2.49), we find that

$$\sum_{j \in \mathbb{Z}^+} \psi_j(\cdot) \varphi_j'(0) = \sum_{j \in \mathbb{Z}^+} c_j \frac{\sqrt{2}}{\alpha_j^{(1)} \cot(ir_j^{(4)})} \varphi_j'(0) \hat{\psi}_j(\cdot) = 0 \quad \text{in } L^2(0, 1).$$

Since $\{\hat{\psi}_j\}_{j \in \mathbb{Z}}$ is a Riesz basis of $L^2(0, 1)$, we get that

$$c_j \frac{\sqrt{2}}{\alpha_j^{(1)} \cot(ir_j^{(4)})} \varphi_j'(0) = 0 \text{ for every } j \in \mathbb{Z}^+. \tag{2.51}$$

From (2.49), we have $\varphi_j'(0) \neq 0$, which together with (2.51), gives that $c_j = 0$ for every $j \in \mathbb{Z}^+$. This implies that Eq. (2.6) admits a unique solution $k_3 = 0$. Therefore, we obtain that Eq. (1.5) admits at most one solution. This concludes Step 1.

Step 2: proof of the existence of a solution to (1.5)

Denote by $D(A)'$ the dual space of $D(A)$ with respect to the pivot space $L^2(0, 1)$. Let $h(\cdot) = \sum_{j \in \mathbb{Z}^+} h_j \varphi_j(\cdot) \in D(A)$, i.e., $|h(\cdot)|_{D(A)}^2 = \sum_{j \in \mathbb{Z}^+} |h_j \mu_j|^2 < +\infty$. We have that

$$\sum_{j \in \mathbb{Z}^+} |h_j \varphi'_j(0)| \leq \left(\sum_{j \in \mathbb{Z}^+} |h_j \mu_j|^2 \right)^{\frac{1}{2}} \left(\sum_{j \in \mathbb{Z}^+} |\varphi'_j(0) \mu_j^{-1}|^2 \right)^{\frac{1}{2}} < +\infty.$$

Hence

$$\sum_{j=1}^n \varphi'_j(0) \varphi_j(\cdot) \text{ is convergent in } D(A)' \text{ as } n \text{ tends to } +\infty, \tag{2.52}$$

which allows us to define

$$\hat{b}(\cdot) \triangleq \sum_{j \in \mathbb{Z}^+} \varphi'_j(0) \varphi_j(\cdot) \in D(A)'. \tag{2.53}$$

Furthermore, it is clear that

$$(h, \hat{b})_{D(A), D(A)'} = \sum_{j \in \mathbb{Z}^+} h_j \varphi'_j(0) = h'(0) = -\delta'_0(h),$$

which implies that

$$\hat{b} = -\delta'_0 \quad \text{in } D(A)'. \tag{2.54}$$

Let

$$a_j \triangleq \frac{a}{\varphi'_j(0)}, \quad \phi_j \triangleq \varphi_j + (-A + \mu_j + a)^{-1} (a_j \hat{b} - a \varphi_j) \in L^2(0, 1) \text{ for } j \in \mathbb{Z}^+. \tag{2.55}$$

From (2.9) and (2.55), we have that

$$\begin{aligned} & \int_0^1 |\varphi_j(x) - \phi_j(x)|^2 dx \\ &= \int_0^1 |(-A + \mu_j + a)^{-1} (a_j \hat{b} - a \varphi_j)|^2 dx \\ &= \int_0^1 \left| \sum_{k \in \mathbb{Z}^+ \setminus \{j\}} (-\mu_k + \mu_j + a)^{-1} \frac{a \varphi'_k(0)}{\varphi'_j(0)} \varphi_k(x) \right|^2 dx \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k \in \mathbb{Z}^+ \setminus \{j\}} \left| (-\mu_k + \mu_j + a)^{-1} \frac{a\varphi'_k(0)}{\varphi'_j(0)} \right|^2 \\
 &\leq C \sum_{k \in \mathbb{Z}^+ \setminus \{j\}} \left| k^4 - j^4 \right|^{-2} j^{-2} k^2 \quad \text{for all } j \in \mathbb{Z}^+.
 \end{aligned}
 \tag{2.56}$$

From (2.56), for $j > 0$,

$$\begin{aligned}
 &\int_0^1 \left| \varphi_j(x) - \phi_j(x) \right|^2 dx \\
 &\leq C \left(\sum_{k > 2j} \frac{k^2}{|k^4 - j^4|^2 j^2} + \sum_{j < k \leq 2j} \frac{k^2}{|k^4 - j^4|^2 j^2} + \sum_{0 < k < j} \frac{k^2}{|k^4 - j^4|^2 j^2} \right).
 \end{aligned}
 \tag{2.57}$$

Now we estimate the terms in the right hand side of (2.57). First, we have that

$$\sum_{k > 2j} \frac{k^2}{|k^4 - j^4|^2 j^2} \leq \sum_{k > 2j} \frac{16}{15k^6 j^2} \leq \frac{16}{15j^6} \sum_{k > 2j} \frac{1}{k^2} \leq \frac{1}{j^7}.
 \tag{2.58}$$

Next,

$$\begin{aligned}
 \sum_{j < k \leq 2j} \frac{k^2}{|k^4 - j^4|^2 j^2} &\leq 4 \sum_{j < k \leq 2j} \frac{1}{|k^4 - j^4|^2} = 4 \sum_{1 \leq l \leq j} \frac{1}{|(j+l)^4 - j^4|^2} \\
 &\leq 4 \sum_{1 \leq l \leq j} \frac{1}{|4j^3l + 6j^2l^2|^2} \leq \frac{1}{4} \sum_{-j \leq l \leq -1} \frac{1}{j^4 (j+l)^2 l^2} \\
 &= \frac{1}{4} \sum_{1 \leq l \leq j} \frac{1}{j^4} \left[\frac{1}{l} - \frac{1}{j+l} \right]^2 \\
 &\leq \frac{1}{2} \sum_{1 \leq l \leq j} \frac{1}{j^6} \left(\frac{1}{l^2} + \frac{1}{(j-l)^2} \right) \leq \frac{\pi^2}{6} \frac{1}{j^6}.
 \end{aligned}
 \tag{2.59}$$

Similarly, we can obtain that

$$\sum_{0 < k < j} \frac{k^2}{|k^4 - j^4|^2 j^2} \leq \frac{\pi^2}{6} \frac{1}{j^6}.
 \tag{2.60}$$

From (2.57) to (2.60), we know that there is a constant $C > 0$ such that, for all positive integer j ,

$$\int_0^1 \left| \varphi_j(x) - \phi_j(x) \right|^2 dx \leq \frac{C}{j^6}.
 \tag{2.61}$$

Furthermore, by similar arguments, we can obtain that $\phi_j \in H^2(0, 1)$ and that

$$\int_0^1 \left| \phi'_j(x) - \phi'_j(x) \right|^2 dx \leq \frac{C}{j^4} \quad \text{for all } j \in \mathbb{Z}^+ \tag{2.62}$$

and

$$\int_0^1 \left| \phi''_j(x) - \phi''_j(x) \right|^2 dx \leq \frac{C}{j^2} \quad \text{for all } j \in \mathbb{Z}^+. \tag{2.63}$$

Let us check that

$$\phi_j \in H^6(0, L) \text{ and } \phi''_j(1) = 0, \quad \forall j \in \mathbb{Z}^+. \tag{2.64}$$

Simple computations show that

$$\sum_{k=1}^{+\infty} \frac{1}{k} \sin(k\pi x) = \frac{\pi}{2}(1-x) \text{ in } L^2(0, 1). \tag{2.65}$$

Let $j \in \mathbb{Z}^+$. From (2.8), there exists $C > 0$ (depending on j) such that

$$\left| \frac{1}{-\mu_k + \mu_j + a} - \frac{1}{\pi^4 k^4} \right| \leq \frac{C}{k^8}, \quad \forall k \in \mathbb{Z}^+. \tag{2.66}$$

The two statements of (2.64) follow from (2.9), (2.53), (2.55), (2.65) and (2.66).

With the same strategy to prove that $\{\hat{\psi}_j\}_{j \in \mathbb{Z}^+}$ is a Riesz basis of $L^2(0, 1)$, we can also show that

$$\left\{ -(a + \mu_j)A^{-1}\phi_j \right\}_{j \in \mathbb{Z}^+} \text{ is a Riesz basis of } L^2(0, 1). \tag{2.67}$$

From (2.8) and (2.49), we get that $\sum_{j \in \mathbb{Z}^+} \phi'_j(0)\mu_j^{-1}\phi_j$ is convergent in $L^2(0, 1)$. From (2.67), there is $\{\hat{c}_j\}_{j \in \mathbb{Z}^+} \in l^2(\mathbb{Z}^+)$ such that

$$-\sum_{j \in \mathbb{Z}^+} \hat{c}_j(a + \mu_j)A^{-1}\phi_j = \sum_{j \in \mathbb{Z}^+} \phi'_j(0)\mu_j^{-1}\phi_j \quad \text{in } L^2(0, 1),$$

which, together with (2.53) and (2.54), implies that

$$-\sum_{j \in \mathbb{Z}^+} \hat{c}_j(a + \mu_j)\phi_j = \sum_{j \in \mathbb{Z}^+} \phi'_j(0)\phi_j = -\delta'_0 \quad \text{in } D(A)'. \tag{2.68}$$

Since $\varphi_j \in D(A)$, from (2.68), we find that

$$-\hat{c}_j(a + \mu_j) \int_0^1 \varphi_j(x)\phi_j(x)dx - \sum_{k \in \mathbb{Z}^+ \setminus \{j\}} \hat{c}_k(a + \mu_k) \int_0^1 \varphi_j(x)\phi_k(x)dx = \varphi'_j(0). \tag{2.69}$$

For $j \neq k$, by (2.9), (2.53) and (2.55), we have that

$$\begin{aligned} & \hat{c}_k(a + \mu_k) \int_0^1 \varphi_j(x)\phi_k(x)dx \\ &= \hat{c}_k(a + \mu_k) \int_0^1 \varphi_j(x) \{ \varphi_k(x) + (-A + \mu_k + a)^{-1} [a_k \hat{b}(x) - a\varphi_k(x)] \} dx \\ &= \hat{c}_k(a + \mu_k) \int_0^1 \varphi_j(x) (-A + \mu_k + a)^{-1} a_k \varphi'_j(0) \varphi_j(x) dx \\ &= \hat{c}_k(a + \mu_k) (-\mu_j + \mu_k + a)^{-1} a_k \varphi'_j(0) \\ &= \hat{c}_k a \frac{a + \mu_k}{-\mu_j + \mu_k + a} \frac{\varphi'_j(0)}{\varphi'_k(0)}. \end{aligned}$$

This, together with (2.69), implies that

$$-\hat{c}_j(a + \mu_j) - \sum_{k \in \mathbb{Z}^+ \setminus \{j\}} \hat{c}_k a \frac{a + \mu_k}{-\mu_j + \mu_k + a} \frac{\varphi'_j(0)}{\varphi'_k(0)} = \varphi'_j(0) \text{ for all } j \in \mathbb{Z}^+. \tag{2.70}$$

For $j, k \in \mathbb{Z}^+$, let

$$c_j \triangleq -\frac{\hat{c}_j(a + \mu_j)}{\varphi'_j(0)}, \quad a_{jk} \triangleq \frac{1}{-\mu_j + \mu_k + a}. \tag{2.71}$$

From (2.70) and (2.71), we have that

$$c_j + a \sum_{k \in \mathbb{Z}^+ \setminus \{j\}} a_{jk} c_k = 1. \tag{2.72}$$

Let us now estimate c_j . From (2.8), (2.71) and (2.72), for every $j \in \mathbb{Z}^+$, we have

$$\begin{aligned} \left| \sum_{k \in \mathbb{Z}^+ \setminus \{j\}} a_{jk} c_k \right| &= \left| \sum_{k \in \mathbb{Z}^+ \setminus \{j\}} \hat{c}_k \frac{a + \mu_k}{-\mu_j + \mu_k + a} \frac{1}{\varphi'_k(0)} \right| \\ &\leq C \sum_{k \in \mathbb{Z}^+ \setminus \{j\}} \left| \hat{c}_k \frac{k^3}{j^4 - k^4} \right| \end{aligned}$$

$$\leq C \left(\sum_{k \in \mathbb{Z}^+ \setminus \{j\}} |\hat{c}_k|^2 \right)^{\frac{1}{2}} \left(\sum_{k \in \mathbb{Z}^+ \setminus \{j\}} \left| \frac{k^3}{j^4 - k^4} \right|^2 \right)^{\frac{1}{2}}. \tag{2.73}$$

For any $j \in \mathbb{Z}^+$,

$$\sum_{k \in \mathbb{Z}^+ \setminus \{j\}} \left| \frac{k^3}{j^4 - k^4} \right|^2 \leq \sum_{k > 2j} \left| \frac{k^3}{j^4 - k^4} \right|^2 + \sum_{j < k \leq 2j} \left| \frac{k^3}{j^4 - k^4} \right|^2 + \sum_{0 < k < j} \left| \frac{k^3}{j^4 - k^4} \right|^2. \tag{2.74}$$

We now estimate the three terms in the right hand side of (2.74). Firstly, we have that

$$\sum_{k > 2j} \left| \frac{k^3}{j^4 - k^4} \right|^2 \leq \frac{16}{15} \sum_{k > 2j} \frac{1}{k^2} \leq \frac{C}{j}. \tag{2.75}$$

Secondly,

$$\begin{aligned} \left| \frac{k^3}{j^4 - k^4} \right|^2 &\leq C \sum_{1 \leq l \leq j} \left| \frac{8j^3}{j^4 - (j+l)^4} \right|^2 \leq C \sum_{1 \leq l \leq j} \left| \frac{8j^3}{4j^3l + 4j^2l^2 + 4jl^3 + l^4} \right|^2 \\ &\leq C \sum_{1 \leq l \leq j} \left| \frac{j}{jl + l^2} \right|^2 \leq C \sum_{1 \leq l \leq j} \left| \left(\frac{1}{l} - \frac{1}{j+l} \right) \right|^2 \leq C \sum_{1 \leq l \leq j} \left(\frac{1}{l^2} - \frac{1}{(j+l)^2} \right) \leq C. \end{aligned} \tag{2.76}$$

Thirdly, with similar arguments as for (2.76), we can obtain that

$$\sum_{0 < k < j} \left| \frac{k^3}{j^4 - k^4} \right|^2 \leq C. \tag{2.77}$$

From (2.73) to (2.77), we find that there is a constant $C > 0$ such that, for all positive integer j ,

$$\left| \sum_{k \in \mathbb{Z}^+ \setminus \{j\}} a_{jk} c_k \right| \leq C. \tag{2.78}$$

Combining (2.72) and (2.78), we know that there is a constant $C > 0$ such that, for all $j \in \mathbb{Z}^+$,

$$|c_j| \leq C. \tag{2.79}$$

We now estimate $|c_j|$ for large j . From (2.71), we get that

$$\begin{aligned} \sum_{k \in \mathbb{Z}^+ \setminus \{j\}} |a_{jk}| &= \sum_{k \in \mathbb{Z}^+ \setminus \{j\}} \frac{1}{|-\mu_j + \mu_k + a|} \leq \sum_{k \in \mathbb{Z}^+ \setminus \{j\}} \frac{C}{|j^4 - k^4|} \\ &\leq \sum_{k > 2j} \frac{C}{|j^4 - k^4|} + \sum_{j < k \leq 2j} \frac{C}{|j^4 - k^4|} + \sum_{0 < k < j} \frac{C}{|j^4 - k^4|}. \end{aligned} \tag{2.80}$$

Let us estimate the terms in the last line of (2.80) one by one. First,

$$\sum_{k>2j} \frac{1}{|j^4 - k^4|} \leq \frac{16}{15} \sum_{k>2j} \frac{1}{k^4} \leq \frac{16}{15} \sum_{k>2j} \frac{1}{k^2} \frac{1}{k(k-1)} \leq \frac{16}{15} \frac{1}{(2j+1)4j^2} \leq \frac{2}{15j^3}. \tag{2.81}$$

Second,

$$\begin{aligned} \sum_{j<k\leq 2j} \frac{1}{|j^4 - k^4|} &\leq \sum_{1<l\leq j} \frac{1}{|j^4 - (j+l)^4|} \leq \sum_{1<l\leq j} \frac{1}{4j^3l + 6j^2l^2 + 4jl^3 + l^4} \\ &\leq \frac{1}{4} \sum_{1<l\leq j} \frac{1}{j^3l + j^2l^2} \leq \frac{1}{4j^3} \sum_{1<l\leq j} \left(\frac{1}{j+l} + \frac{1}{l}\right) \leq \frac{\ln j}{2j^3}. \end{aligned} \tag{2.82}$$

Similarly, we can obtain that

$$\sum_{0<k<j} \frac{1}{|j^4 - k^4|} \leq \frac{\ln j}{2j^3}. \tag{2.83}$$

From (2.80) to (2.83), we know that there is a constant $C > 0$ such that, for all positive integer j ,

$$\sum_{k \in \mathbb{Z}^+ \setminus \{j\}} |a_{jk}| \leq \frac{C(1 + \ln j)}{j^3}. \tag{2.84}$$

Combining (2.72), (2.79) and (2.84), we obtain that

$$|c_j - 1| \leq \frac{C(1 + \ln j)}{j^3}. \tag{2.85}$$

Let us now turn to the construction of $k(\cdot, \cdot)$. From (2.9), (2.61) and (2.85), one has that

$$\begin{aligned} &\sum_{j \in \mathbb{Z}^+} \int_0^1 |\varphi_j(x) - c_j \phi_j(x)|^2 dx \\ &\leq 2 \sum_{j \in \mathbb{Z}^+} \int_0^1 |(1 - c_j) \varphi_j(x)|^2 dx + 2 \sum_{j \in \mathbb{Z}^+} \int_0^1 |c_j|^2 |\varphi_j(x) - \phi_j(x)|^2 dx \\ &\leq C \sum_{j \in \mathbb{Z}^+} \left(\frac{\ln^2 j}{j^6} + \frac{1}{j^6}\right) < +\infty. \end{aligned} \tag{2.86}$$

Inequality (2.86) allows us to define a $k(\cdot, \cdot) \in L^2((0, 1) \times (0, 1))$ by

$$k(x, y) \triangleq \sum_{j \in \mathbb{Z}^+} [\varphi_j(x) - c_j \phi_j(x)] \varphi_j(y), \quad 0 \leq x, y \leq 1. \tag{2.87}$$

Let us prove that $k(\cdot, \cdot) \in H^2((0, 1) \times (0, 1))$. Thanks to (2.9), (2.62) and (2.85), we find that

$$\begin{aligned}
 & \int_0^1 \int_0^1 \left| \sum_{j \in \mathbb{Z}^+} [\varphi'_j(x) - c_j \phi'_j(x)] \varphi_j(y) \right|^2 dx dy \\
 &= \sum_{j \in \mathbb{Z}^+} \int_0^1 |\varphi'_j(x) - c_j \phi'_j(x)|^2 dx \\
 &\leq \sum_{j \in \mathbb{Z}^+} \int_0^1 |(1 - c_j) \varphi'_j(x)|^2 dx + \sum_{j \in \mathbb{Z}^+} \int_0^1 |c_j|^2 |\varphi'_j(x) - \phi'_j(x)|^2 dx \\
 &\leq C \sum_{j \in \mathbb{Z}^+} \left(\frac{\ln^2 j}{j^4} + \frac{1}{j^4} \right) < +\infty,
 \end{aligned} \tag{2.88}$$

which implies that

$$k_x(\cdot, \cdot) \in L^2((0, 1) \times (0, 1)). \tag{2.89}$$

Utilizing (2.9), (2.55) and (2.79), we get that

$$\begin{aligned}
 & \int_0^1 \int_0^1 \left| \sum_{j \in \mathbb{Z}^+} c_j [\varphi_j(x) - \phi_j(x)] \varphi'_j(y) \right|^2 dx dy \\
 &= \int_0^1 \sum_{j \in \mathbb{Z}^+} j^2 |c_j|^2 |\varphi_j(x) - \phi_j(x)|^2 dx \\
 &= \int_0^1 \sum_{j \in \mathbb{Z}^+} j^2 |c_j|^2 |(-A + \mu_j + a)^{-1} (a_j \hat{b} - a \varphi_j)(x)|^2 dx \\
 &= \int_0^1 \sum_{j \in \mathbb{Z}^+} j^2 |c_j|^2 \left| \sum_{l \in \mathbb{Z}^+ \setminus \{j\}} (-\mu_l + \mu_j + a)^{-1} \frac{a \varphi'_l(0)}{\varphi'_j(0)} \varphi_l(x) \right|^2 dx \\
 &= \sum_{l \in \mathbb{Z}^+} l^2 \sum_{l \in \mathbb{Z}^+ \setminus \{l\}} |c_j|^2 |(-\mu_l + \mu_j + a)^{-1}|^2.
 \end{aligned} \tag{2.90}$$

Similarly to the proof of (2.84), we can obtain that

$$\sum_{j \in \mathbb{Z}^+ \setminus \{l\}} |(-\mu_l + \mu_j + a)^{-1}|^2 \leq \frac{C}{l^6}. \tag{2.91}$$

Combining (2.79), (2.90) and (2.91), one has that

$$\int_0^1 \int_0^1 \left| \sum_{j \in \mathbb{Z}^+} c_j [\varphi_j(x) - \phi_j(x)] \varphi'_j(y) \right|^2 dx dy \leq C \sum_{l \in \mathbb{Z}^+} \frac{1}{l^4} < +\infty. \tag{2.92}$$

From (2.85) and (2.92), it follows that

$$\begin{aligned} & \int_0^1 \int_0^1 \left| \sum_{j \in \mathbb{Z}^+} [\varphi_j(x) - c_j \phi_j(x)] \varphi'_j(y) \right|^2 dx dy \\ & \leq 2 \int_0^1 \int_0^1 \left| \sum_{j \in \mathbb{Z}^+} (1 - c_j) \varphi_j(x) \varphi'_j(y) \right|^2 dx dy + 2 \int_0^1 \int_0^1 \left| \sum_{j \in \mathbb{Z}^+} c_j [\varphi_j(x) - \phi_j(x)] \varphi'_j(y) \right|^2 dx dy \\ & \leq 2 \sum_{j \in \mathbb{Z}^+} \int_0^1 |(1 - c_j) \varphi'_j(y)|^2 dy + 2 \int_0^1 \int_0^1 \left| \sum_{j \in \mathbb{Z}^+} c_j [\varphi_j(x) - \phi_j(x)] \varphi'_j(y) \right|^2 dx dy \\ & \leq C \left(\sum_{j \in \mathbb{Z}^+} \frac{1 + \ln^2 j}{j^4} \right) < +\infty. \end{aligned} \tag{2.93}$$

This, together with (2.86) and (2.88), deduces that

$$k_y(\cdot, \cdot) \in L^2((0, 1) \times (0, 1)), \tag{2.94}$$

which, together with (2.89), shows that $k(\cdot, \cdot) \in H^1((0, 1) \times (0, 1))$. Clearly, $k(\cdot, \cdot) = 0$ on the boundary of $(0, 1) \times (0, 1)$. Thus, we conclude that

$$k(\cdot, \cdot) \in H_0^1((0, 1) \times (0, 1)). \tag{2.95}$$

Furthermore, by (2.9), (2.63) and (2.85), we find that

$$\begin{aligned} & \int_0^1 \int_0^1 \left| \sum_{j \in \mathbb{Z}^+} [\varphi''_j(x) - c_j \phi''_j(x)] \varphi_j(y) \right|^2 dx dy \\ & = \sum_{j \in \mathbb{Z}^+} \int_0^1 |\varphi''_j(x) - c_j \phi''_j(x)|^2 dx \\ & \leq \sum_{j \in \mathbb{Z}^+} \int_0^1 |(1 - c_j) \varphi''_j(x)|^2 dx + \sum_{j \in \mathbb{Z}^+} \int_0^1 |c_j|^2 |\varphi''_j(x) - \phi''_j(x)|^2 dx \\ & \leq C \sum_{j \in \mathbb{Z}^+} \left(\frac{\ln^2 j}{j^2} + \frac{1}{j^2} \right) < +\infty, \end{aligned} \tag{2.96}$$

which shows that

$$k_{xx}(\cdot, \cdot) \in L^2((0, 1) \times (0, 1)). \tag{2.97}$$

Utilizing (2.9), (2.55) and (2.79), we get that

$$\begin{aligned} & \int_0^1 \int_0^1 \left| \sum_{j \in \mathbb{Z}^+} c_j [\varphi_j(x) - \phi_j(x)] \varphi_j''(y) \right|^2 dx dy \\ &= \int_0^1 \sum_{j \in \mathbb{Z}^+} j^4 |c_j|^2 |\varphi_j(x) - \phi_j(x)|^2 dx \\ &= \int_0^1 \sum_{j \in \mathbb{Z}^+} j^4 |c_j|^2 \left| (-A + \mu_j + a)^{-1} (a_j \hat{b} - a \varphi_j)(x) \right|^2 dx \\ &= \int_0^1 \sum_{l \in \mathbb{Z}^+} j^4 |c_j|^2 \left| \sum_{l \in \mathbb{Z}^+ \setminus \{j\}} (-\mu_l + \mu_j + a)^{-1} \frac{a \varphi_l'(0)}{\varphi_j'(0)} \varphi_l(x) \right|^2 dx \\ &= a^2 \sum_{j \in \mathbb{Z}^+} j^2 |c_j|^2 \sum_{l \in \mathbb{Z}^+ \setminus \{j\}} l^2 |(-\mu_l + \mu_j + a)^{-1}|^2 \\ &= a^2 \sum_{k \in \mathbb{Z}^+} l^2 \sum_{j \in \mathbb{Z}^+ \setminus \{k\}} j^2 |c_j|^2 |(-\mu_l + \mu_j + a)^{-1}|^2. \end{aligned} \tag{2.98}$$

Similar to the proof of (2.84), we can obtain that

$$\sum_{j \in \mathbb{Z}^+ \setminus \{l\}} j^2 |(-\mu_l + \mu_j + a)^{-1}|^2 \leq \frac{C}{l^4}. \tag{2.99}$$

Combining (2.79), (2.98) and (2.99), one has

$$\int_0^1 \int_0^1 \left| \sum_{j \in \mathbb{Z}^+} c_j [\varphi_j(x) - \phi_j(x)] \varphi_j''(y) \right|^2 dx dy \leq C \sum_{l \in \mathbb{Z}^+} \frac{1}{l^2} < +\infty. \tag{2.100}$$

From (2.9), (2.85) and (2.100), we get that

$$\begin{aligned} & \int_0^1 \int_0^1 \left| \sum_{j \in \mathbb{Z}^+} [\varphi_j(x) - c_j \phi_j(x)] \varphi_j''(y) \right|^2 dx dy \\ & \leq 2 \int_0^1 \int_0^1 \left| \sum_{j \in \mathbb{Z}^+} (1 - c_j) \varphi_j(x) \varphi_j''(y) \right|^2 dx dy \end{aligned}$$

$$\begin{aligned}
 &+ 2 \int_0^1 \int_0^1 \left| \sum_{j \in \mathbb{Z}^+} c_j [\varphi_j(x) - \phi_j(x)] \varphi_j''(y) \right|^2 dx dy \\
 &\leq 2 \sum_{j \in \mathbb{Z}^+} \int_0^1 |(1 - c_j) \varphi_j''(y)|^2 dy + 2 \int_0^1 \int_0^1 \left| \sum_{j \in \mathbb{Z}^+} c_j [\varphi_j(x) - \phi_j(x)] \varphi_j''(y) \right|^2 dx dy \\
 &\leq C \sum_{j \in \mathbb{Z}^+} \frac{1 + (\ln j)^2}{j^2} < +\infty.
 \end{aligned} \tag{2.101}$$

According to (2.96) and (2.101), we see that

$$k_{yy}(\cdot, \cdot) \in L^2((0, 1) \times (0, 1)), \tag{2.102}$$

which, together with (2.95), deduces that $k(\cdot, \cdot) \in H^2((0, 1) \times (0, 1))$.

Define

$$k^{(n)}(x, y) \triangleq \sum_{0 < j \leq n} [\varphi_j(x) - c_j \phi_j(x)] \varphi_j(y) \text{ in } (0, 1) \times (0, 1). \tag{2.103}$$

From (2.9), (2.64) and (2.103), one has

$$\left(x \in (0, 1) \mapsto k_{xx}^{(n)}(x, \cdot) \in L^2(0, 1) \right) \text{ is in } C^0([0, 1]; L^2(0, 1)). \tag{2.104}$$

For any $m, n \in \mathbb{Z}^+, m < n$, we have that

$$\begin{aligned}
 &\int_0^1 \left| \sum_{m < j \leq n} [\varphi_j''(x) - c_j \phi_j''(x)] \varphi_j(y) \right|^2 dy \\
 &= \sum_{m < j \leq n} |\varphi_j''(x) - c_j \phi_j''(x)|^2 \\
 &\leq 2 \sum_{m < j \leq n} |(1 - c_j) \varphi_j''(x)|^2 + 2 \sum_{m < j \leq n} |c_j|^2 |\varphi_j''(x) - \phi_j''(x)|^2.
 \end{aligned} \tag{2.105}$$

By means of (2.85), we find that

$$\max_{x \in [0, 1]} \sum_{m < |j| \leq n} |(1 - c_j) \varphi_j''(x)|^2 \leq C \sum_{m < |j| \leq n} \frac{1 + (\ln j)^2}{j^2}. \tag{2.106}$$

From (2.55), similarly to the proof of (2.84), we obtain that

$$\begin{aligned}
 & \max_{x \in [0,1]} \sum_{m < j \leq n} |c_j|^2 |\varphi_j''(x) - \phi_j''(x)|^2 \\
 &= \max_{x \in [0,1]} \sum_{m < j \leq n} |c_j|^2 \left| \sum_{l \in \mathbb{Z}^+ \setminus \{j\}} (-\mu_l + \mu_j + a)^{-1} \frac{a\varphi_l'(0)}{\phi_l'(0)} \varphi_l''(x) \right|^2 \\
 &\leq C \sum_{m < j \leq n} \sum_{l \in \mathbb{Z}^+ \setminus \{j\}} \left| \frac{l^3}{(j^4 - l^4)j} \right|^2 \\
 &\leq C \sum_{m < j \leq n} \frac{(\ln j)^2}{j^2}. \tag{2.107}
 \end{aligned}$$

Combining (2.106) and (2.107), we get that

$$\left\{ x \in (0, 1) \mapsto k_{xx}^{(n)}(x, \cdot) \in L^2(0, 1) \right\}_{n=1}^{+\infty} \text{ is a Cauchy sequence in } C^0([0, 1]; L^2(0, 1)), \tag{2.108}$$

which deduces that (2.2) holds. Proceeding as in the proofs of (2.102) and of (2.108), one gets that

$$\left\{ y \in (0, 1) \mapsto k_{yy}^{(n)}(\cdot, y) \in L^2(0, 1) \right\}_{n=1}^{+\infty} \text{ is a Cauchy sequence in } C^0([0, 1]; L^2(0, 1)), \tag{2.109}$$

which also gives (2.3). Moreover, (2.9), (2.87), (2.103) and (2.109) imply that

$$k_{yy}(\cdot, 0) = \lim_{n \rightarrow +\infty} k_{yy}^{(n)}(\cdot, 0) = 0 \text{ in } L^2(0, 1). \tag{2.110}$$

Similarly, one can show that

$$k_{yy}(\cdot, 1) = \lim_{n \rightarrow +\infty} k_{yy}^{(n)}(\cdot, 1) = 0 \text{ in } L^2(0, 1). \tag{2.111}$$

Let us point that

$$k_y(\cdot, 0) = 0 \text{ in } L^2(0, L). \tag{2.112}$$

Indeed, from (2.53), (2.55), (2.71), (2.72), (2.79), (2.84) and (2.87), one has, in $D(A)'$,

$$\begin{aligned}
 k_y(x, 0) &= \sum_{j \in \mathbb{Z}^+} [\varphi_j(x) - c_j \phi_j(x)] \varphi_j'(0) \\
 &= \sum_{j \in \mathbb{Z}^+} (1 - c_j) \varphi_j'(0) \varphi_j(x) - \sum_{j \in \mathbb{Z}^+} \sum_{k \in \mathbb{Z}^+ \setminus \{j\}} c_j a_{kj} a \varphi_k'(0) \varphi_k(x)
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{j \in \mathbb{Z}^+} (1 - c_j) \varphi'_j(0) \varphi_j(x) - \sum_{k \in \mathbb{Z}^+} (1 - c_k) \varphi'_k(0) \varphi_k(x) \\
 &= 0.
 \end{aligned}$$

Let us now prove that $k(\cdot, \cdot)$ satisfies (2.5). From (2.8) and (2.9) one has

$$\begin{aligned}
 &(\partial_{xxxx} + \lambda \partial_{xx} - \partial_{yyyy} - \lambda \partial_{yy} + a)(1 - c_j) \varphi_j(x) \varphi_j(y) \\
 &= (-\mu_j + \mu_j + a)(1 - c_j) \varphi_j(x) \varphi_j(y) = a(1 - c_j) \varphi_j(x) \varphi_j(y). \tag{2.113}
 \end{aligned}$$

From (2.53), (2.54) and (2.55), one has

$$\begin{aligned}
 &(\partial_{xxxx} + \lambda \partial_{xx} - \partial_{yyyy} - \lambda \partial_{yy} + a) c_j [\varphi_j(x) - \phi_j(x)] \varphi_j(y) \\
 &= -(\partial_{xxxx} + \lambda \partial_{xx} - \partial_{yyyy} - \lambda \partial_{yy} + a) \sum_{k \in \mathbb{Z}^+ \setminus \{j\}} c_j \frac{a \varphi'_k(0) \varphi_k(x)}{\varphi'_j(0) (-\mu_k + \mu_j + a)} \varphi_j(y) \\
 &= -c_j a \sum_{k \in \mathbb{Z}^+ \setminus \{j\}} \frac{\varphi'_k(0)}{\varphi'_j(0)} \varphi_k(x) \varphi_j(y) \\
 &= c_j a \varphi_j(x) \varphi_j(y) - \frac{c_j a}{\varphi'_j(0)} \sum_{k \in \mathbb{Z}^+} \varphi'_k(0) \varphi_k(x) \varphi_j(y) \\
 &= c_j a \varphi_j(x) \varphi_j(y) + \frac{c_j a}{\varphi'_j(0)} \delta'_{x=0} \otimes \varphi_j(y) \quad \text{in } D(A)' \otimes L^2(0, 1). \tag{2.114}
 \end{aligned}$$

From (2.103), (2.113) and (2.114), we get, in $D(A)' \otimes L^2(0, 1)$,

$$\begin{aligned}
 &k_{xxxx}^{(n)}(x, y) + \lambda k_{xx}^{(n)}(x, y) - k_{yyyy}^{(n)}(x, y) - \lambda k_{yy}^{(n)}(x, y) + a k^{(n)}(x, y) \\
 &- \delta'_{x=0} \otimes \sum_{0 < j \leq n} \frac{c_j a}{\varphi'_j(0)} \varphi_j(y) = a \sum_{0 < j \leq n} \varphi_j(x) \varphi_j(y). \tag{2.115}
 \end{aligned}$$

From (2.1), (2.9), (2.103), (2.115), one sees, using integrations by parts that, for any $\rho \in \mathcal{E} \subset D(A) \otimes L^2(0, 1)$,

$$\begin{aligned}
 0 &= \int_0^1 \int_0^1 [\rho_{xxxx}(x, y) + \lambda \rho_{xx}(x, y) - \rho_{yyyy}(x, y) - \lambda \rho_{yy}(x, y) + a \rho(x, y)] k^{(n)}(x, y) dx dy \\
 &+ \int_0^1 k_y^{(n)}(x, 0) \rho_{yy}(x, 0) dx - a \sum_{0 < j \leq n} \int_0^1 \int_0^1 \left(\sum_{0 < j \leq n} \rho(x, y) \varphi_j(x) \varphi_j(y) \right) dx dy. \tag{2.116}
 \end{aligned}$$

By (2.9), (2.110), (2.111) and letting n tends to $+\infty$ in (2.116), we obtain that

$$\int_0^1 \int_0^1 [\rho_{xxxx}(x, y) + \lambda\rho_{xx}(x, y) - \rho_{yyyy}(x, y) - \lambda\rho_{yy}(x, y) + a\rho(x, y)]k(x, y)dx dy + \int_0^1 k_y(x, 0)\rho_{yy}(x, 0)dx - a \int_0^1 \rho(y, y)dy = 0, \tag{2.117}$$

which, together with (2.112), gives

$$\int_0^1 \int_0^1 [\rho_{xxxx}(x, y) + \lambda\rho_{xx}(x, y) - \rho_{yyyy}(x, y) - \lambda\rho_{yy}(x, y) + a\rho(x, y)]k(x, y)dx dy - a \int_0^1 \rho(y, y)dy = 0. \tag{2.118}$$

Hence k is a solution of (1.5). It only remains to check that the function k is real valued. This follows from the fact \bar{k} is also a solution of (1.5). Hence by the uniqueness result of Step 1, we must have $\bar{k} = k$, which shows that the function k is real valued. This concludes Step 2 and therefore the proof of Theorem 2.1. \square

3. Invertibility of $I - K$

The goal of this section is to prove the following theorem.

Theorem 3.1. *$I - K$ is an invertible operator from $L^2(0, 1)$ to $L^2(0, 1)$.*

Proof of Theorem 3.1. Since $k(\cdot, \cdot) \in L^2((0, 1) \times (0, 1))$, we get that K is a compact operator. Furthermore, thanks to $k \in H^2((0, 1) \times (0, 1)) \cap H_0^1((0, 1) \times (0, 1))$, we know that K is a continuous linear map from $L^2(0, 1)$ into $H^2(0, 1) \cap H_0^1(0, 1)$. Denote by K^* the adjoint operator of K . Then, it is easy to see that

$$K^*(v)(x) = \int_0^1 k^*(x, y)v(y)dy \quad \text{for any } v \in L^2(0, 1),$$

where k^* is defined by

$$k^*(x, y) \triangleq k(y, x), \quad (x, y) \in (0, 1) \times (0, 1). \tag{3.1}$$

From (1.5) and (3.1), we know that $k^*(\cdot, \cdot)$ solves (in a sense which is naturally adapted from (2.5))

$$\begin{cases} k_{yyyy}^* + \lambda k_{yy}^* - k_{xxxx}^* - \lambda k_{xx}^* + ak^* = a\delta(y - x) & \text{in } (0, 1) \times (0, 1), \\ k^*(0, y) = k^*(1, y) = 0 & \text{on } (0, 1), \\ k_{xx}^*(0, y) = k_{xx}^*(1, y) = 0 & \text{on } (0, 1), \\ k_x^*(0, y) = 0 & \text{on } (0, 1), \\ k^*(x, 0) = k^*(x, 1) = 0 & \text{on } (0, 1), \\ k_{yy}^*(x, 1) = 0 & \text{on } (0, 1). \end{cases} \tag{3.2}$$

Furthermore, as a result of (2.2), (2.3) and (3.1), we have the following regularity for $k^*(\cdot, \cdot)$:

$$y \in (0, 1) \mapsto k_{yy}^*(\cdot, y) \in L^2(0, 1) \text{ is in } C^0([0, 1]; L^2(0, 1)), \tag{3.3}$$

$$x \in (0, 1) \mapsto k_{xx}^*(x, \cdot) \in L^2(0, 1) \text{ is in } C^0([0, 1]; L^2(0, 1)). \tag{3.4}$$

From (3.1), (3.2) and (3.4), we know that

$$v \in C^2([0, 1]) \text{ and } v(0) = v_x(0) = v_{xx}(0) = v(1) = v_{xx}(1) = 0, \quad \forall v \in K^*(L^2(0, 1)), \tag{3.5}$$

$$K^* \text{ is a continuous linear map from } L^2(0, 1) \text{ to } H^2(0, 1). \tag{3.6}$$

We claim that the spectral radius $r(K^*)$ of K^* is equal to 0. Otherwise, since $K^* : L^2(0, 1) \rightarrow L^2(0, 1)$ is, as K , a compact linear operator, it has a nonzero eigenvalue α . Then, there exists a positive integer n_0 such that

$$\text{Ker}(K^* - \alpha I)^{n_0+1} = \text{Ker}(K^* - \alpha I)^{n_0}. \tag{3.7}$$

Let $\mathcal{F} \triangleq \text{Ker}(K^* - \alpha I)^{n_0}$. Since K^* is a compact operator and $\alpha \neq 0$, \mathcal{F} is a finite dimensional vector space. Moreover, since $\alpha \neq 0$, one has $\mathcal{F} \subset K^*(L^2(0, 1))$, which, together with (3.5), implies that

$$\mathcal{F} \subset \{v \in C^2([0, 1]) : v(0) = v_x(0) = v_{xx}(0) = v(1) = v_{xx}(1) = 0\}. \tag{3.8}$$

Using the fact that, as $k, k^* \in H_0^1((0, 1) \times (0, 1))$, we get that

$$K^* \text{ can be extended to be a continuous linear map from } H^{-1}(0, 1) \text{ into } L^2(0, 1). \tag{3.9}$$

Denote by \tilde{K}^* this extension and remark that, if $u \in H^{-1}(0, 1)$ is such that $\tilde{K}^*u = \alpha u$, then $u \in L^2(0, 1)$. Thus, $\text{Ker}(\tilde{K}^* - \alpha I) \subset L^2(0, 1)$ and $\text{Ker}(\tilde{K}^* - \alpha I) = \text{Ker}(K^* - \alpha I)$. Similarly, we have

$$\begin{cases} \text{Ker}(\tilde{K}^* - \alpha I)^{n_0} \subset L^2(0, 1), \\ \text{Ker}(\tilde{K}^* - \alpha I)^{n_0} = \mathcal{F}, \\ \text{Ker}(\tilde{K}^* - \alpha I)^{n_0} = \text{Ker}(\tilde{K}^* - \alpha I)^{n_0+1}. \end{cases} \tag{3.10}$$

By (3.2),

$$K^*(\partial_{xxxx} + \lambda \partial_{xx})v = (\partial_{xxxx} + \lambda \partial_{xx})K^*v - aK^*v + av, \quad \forall v \in C_0^\infty(0, 1). \tag{3.11}$$

From (3.11) we get that K^* can be extended to be a continuous linear map from $H^{-4}(0, 1)$ into $H^{-4}(0, 1)$. Hence, by interpolation and (3.6), we also get that K^* can be extended as a continuous linear map from $H^{-2}(0, 1)$ into $H^{-1}(0, 1)$. We denote by \widehat{K}^* this extension. Then, from (3.10), one has

$$\begin{cases} \text{Ker}(\widehat{K}^* - \alpha I)^{n_0} = \text{Ker}(K^* - \alpha I)^{n_0} \subset L^2(0, 1), \\ \text{Ker}(\widehat{K}^* - \alpha I)^{n_0} = \mathcal{F}, \\ \text{Ker}(\widehat{K}^* - \alpha I)^{n_0} = \text{Ker}(\widehat{K}^* - \alpha I)^{n_0+1}. \end{cases} \tag{3.12}$$

Using a density argument, (3.8) and (3.11), we get, for every $v \in \mathcal{F}$,

$$\widehat{K}^*(\partial_{xxxx} + \lambda \partial_{xx})v = (\partial_{xxxx} + \lambda \partial_{xx})\widehat{K}^*v - a\widehat{K}^*v + av \text{ in } H^{-2}(0, 1). \tag{3.13}$$

From (3.8), (3.13), and induction on n , one has, for every $v \in \mathcal{F}$,

$$(\widehat{K}^*)^n(\partial_{xxxx} + \lambda \partial_{xx})v = (\partial_{xxxx} + \lambda \partial_{xx})(\widehat{K}^*)^n v - na(\widehat{K}^*)^n v + na(\widehat{K}^*)^{n-1} v \text{ in } H^{-2}(0, 1),$$

and therefore, for every polynomial P and for every $v \in \mathcal{F}$,

$$P(\widehat{K}^*)(\partial_{xxxx} + \lambda \partial_{xx})v = (\partial_{xxxx} + \lambda \partial_{xx})P(\widehat{K}^*)v - a\widehat{K}^*P'(\widehat{K}^*)v + aP'(\widehat{K}^*)v \text{ in } H^{-2}(0, 1). \tag{3.14}$$

By virtue of (3.8), (3.12) and (3.14) with $P(X) \triangleq (X - \alpha)^{n_0+1}$, we see that $(\partial_{xxxx} + \lambda \partial_{xx})\mathcal{F} \subset \mathcal{F}$. Since \mathcal{F} is finite dimensional, we know that $\partial_{xxxx} + \lambda \partial_{xx}$ has an eigenfunction in \mathcal{F} , that is, there exist $\mu \in \mathbb{C}$ and $\xi \in \mathcal{F} \setminus \{0\}$ such that

$$\begin{cases} (\partial_{xxxx} + \lambda \partial_{xx})\xi = \mu\xi & \text{in } (0, 1), \\ \xi(0) = \xi_x(0) = \xi_{xx}(0) = \xi(1) = \xi_{xx}(1) = 0. \end{cases}$$

This leads to a contradiction with Corollary B.1. Then, we know $r(K^*) = 0$, so $r(K) = 0$. Hence, the real number 1 belongs to the resolvent set of K , which completes the proof of Theorem 3.1. \square

4. Proof of Theorem 1.2

This section is addressed to a proof of Theorem 1.2.

Proof of Theorem 1.2. Let $T > 0$, which will be specified later on. Consider the following equation

$$\begin{cases} v_{1,t} + v_{1,xxxx} + \lambda v_{1,xx} + v_1 v_{1,x} = 0 & \text{in } [0, T] \times (0, 1), \\ v_1(t, 0) = v_1(t, 1) = 0 & \text{on } [0, T], \\ v_{1,xx}(t, 0) = \int_0^1 k_{xx}(0, y)v_1(t, y)dy, \quad v_{1,xx}(t, 1) = 0 & \text{on } [0, T], \\ v_1(0) = v^0 & \text{in } (0, 1). \end{cases} \tag{4.1}$$

By **Theorem 1.1**, we know that there exist $r_T > 0$ and $C_T > 0$ such that, for all $v^0 \in L^2(0, 1)$ with $|v^0|_{L^2(0,1)} \leq r_T$, Eq. (4.1) admits a unique solution $v_1 \in X_T$ and this solution satisfies

$$|v_1|_{X_T} \leq C_T |v^0|_{L^2(0,1)}. \tag{4.2}$$

Let $w_1 = (I - K)v_1$. Then, from the boundary conditions of (1.5) and (4.1), we have

$$\begin{aligned} w_{1,t}(t, x) &= v_{1,t}(t, x) - \int_0^1 k(x, y)v_{1,t}(t, y)dy \\ &= v_{1,t}(t, x) + \int_0^1 k(x, y)[v_{1,yyyy}(t, y) + \lambda v_{1,yy}(t, y) + v_1(t, y)v_{1,y}(t, y)]dy \\ &= v_{1,t}(t, x) + \int_0^1 [k_{yyyy}(x, y) + \lambda k_{yy}(x, y)]v_1(t, y)dy - \frac{1}{2} \int_0^1 k_y(x, y)v_1(t, y)^2dy, \end{aligned} \tag{4.3}$$

$$\lambda w_{1,xx}(t, x) = \lambda v_{1,xx}(t, x) - \lambda \int_0^1 k_{xx}(x, y)v_1(t, y)dy \tag{4.4}$$

and

$$w_{1,xxxx}(t, x) = v_{1,xxxx}(t, x) - \int_0^1 k_{xxxx}(x, y)v_1(t, y)dy. \tag{4.5}$$

From (1.5), (4.1), (4.3), (4.4) and (4.5), we obtain

$$\begin{aligned} &w_{1,t}(t, x) + w_{1,xxxx}(t, x) + \lambda w_{1,xx}(t, x) + aw_1(t, x) \\ &\quad + \frac{1}{2} \int_0^1 k_y(x, y)v_1(t, y)^2dy + v_1(t, x)v_{1,x}(t, x) \\ &= av_1(t, x) - \int_0^1 v_1(t, y)[k_{xxxx}(x, y) + \lambda k_{xx}(x, y) - k_{yyyy}(x, y) - \lambda k_{yy}(x, y) + ak(x, y)]dy \\ &= - \int_0^1 v_1(t, y)[k_{xxxx}(x, y) + \lambda k_{xx}(x, y) - k_{yyyy}(x, y) - \lambda k_{yy}(x, y) \\ &\quad + ak(x, y) - a\delta(x - y)]dy \\ &= 0. \end{aligned} \tag{4.6}$$

Hence, if we take the feedback control $F(\cdot)$ defined by (1.9), we get that w_1 solves

$$\begin{cases} w_{1,t} + w_{1,xxxx} + \lambda w_{1,xx} + w_1 w_{1,x} + a w_1 \\ = -v_1 v_{1,x} - \frac{1}{2} \int_0^1 k_y(x, y) v_1(t, y)^2 dy & \text{in } [0, T] \times (0, 1), \\ w_1(t, 0) = w_1(t, 1) = 0 & \text{on } [0, T], \\ w_{1,xx}(t, 0) = w_{1,xx}(t, 1) = 0 & \text{on } [0, T]. \end{cases} \tag{4.7}$$

Multiplying the first equation of (4.7) with w_1 and integrating on $(0, 1)$, we get, using integrations by parts,

$$\begin{aligned} \frac{d}{dt} \int_0^1 |w_1(t, x)|^2 dx &= -2Q(w_1(t, \cdot)) - \int_0^1 w_1(t, x) v_1(t, x) v_{1,x}(t, x) dx \\ &\quad - \frac{1}{2} \int_0^1 w_1(t, x) \left[\int_0^1 k_y(x, y) v_1(t, y)^2 dy \right] dx, \end{aligned} \tag{4.8}$$

with

$$Q(\varphi) \triangleq \int_0^1 |\varphi''(x)|^2 dx - \lambda \int_0^1 |\varphi'(x)|^2 dx + a \int_0^1 |\varphi(x)|^2 dx, \quad \forall \varphi \in H^2(0, 1) \cap H_0^1(0, 1). \tag{4.9}$$

Now we estimate the second and third terms in the right hand side of (4.8). First,

$$\begin{aligned} &\left| \int_0^1 w_1(t, x) v_1(t, x) v_{1,x}(t, x) dx \right| \\ &= \left| \int_0^1 \left[v_1(t, x) - \int_0^1 k(x, y) v_1(t, y) dy \right] v_1(t, x) v_{1,x}(t, x) dx \right| \\ &= \left| \int_0^1 \left(\int_0^1 k(x, y) v_1(t, y) dy \right) v_1(t, x) v_{1,x}(t, x) dx \right| \\ &= \frac{1}{2} \left| \int_0^1 \left(\int_0^1 k_x(x, y) v_1(t, y) dy \right) v_1(t, x)^2 dx \right| \\ &\leq \frac{1}{2} \left| \int_0^1 |k_x(\cdot, y)|^2 dy \right|_{L^\infty(0,1)} \left(\int_0^1 v_1(t, x)^2 dx \right)^{\frac{3}{2}} \end{aligned}$$

$$\leq |(I - K)^{-1}|^3_{\mathcal{L}(L^2(0,1))} \frac{1}{2} \left| \int_0^1 |k_x(\cdot, y)|^2 dy \right|_{L^\infty(0,1)} \left(\int_0^1 w_1^2(t, x) dx \right)^{\frac{3}{2}}. \tag{4.10}$$

Next,

$$\begin{aligned} & \left| \int_0^1 w_1(t, x) \int_0^1 k_y(x, y) v_1(t, y)^2 dy dx \right| \\ &= \left| \int_0^1 \left[v_1(t, x) - \int_0^1 k(x, y) v_1(t, y) dy \right] \left(\int_0^1 k_y(x, y) v_1(t, y)^2 dy \right) dx \right| \\ &\leq \left| \int_0^1 v_1(t, x) \left(\int_0^1 k_y(x, y) v_1(t, y)^2 dy \right) dx \right| \\ &\quad + \left| \int_0^1 \left(\int_0^1 k(x, y) v_1(t, y) dy \right) \int_0^1 k_y(x, y) v_1(t, y)^2 dy dx \right|. \end{aligned} \tag{4.11}$$

The first term in the right hand side of (4.11) satisfies that

$$\begin{aligned} & \left| \int_0^1 v_1(t, x) \left(\int_0^1 k_y(x, y) v_1(t, y)^2 dy \right) dx \right| \\ &= \left| \int_0^1 v_1(t, y)^2 \left(\int_0^1 k_y(x, y) v_1(t, x) dx \right) dy \right| \\ &\leq \int_0^1 v_1(t, y)^2 \left(\int_0^1 |k_y(x, y)|^2 dx \right)^{\frac{1}{2}} \left(\int_0^1 v_1(t, x)^2 dx \right)^{\frac{1}{2}} dy \\ &\leq \left| \left(\int_0^1 |k_y(x, \cdot)|^2 dx \right)^{\frac{1}{2}} \right|_{L^\infty(0,1)} \left(\int_0^1 v_1(t, x)^2 dx \right)^{\frac{3}{2}} \\ &\leq |(I - K)^{-1}|^3_{\mathcal{L}(L^2(0,1))} \left| \left(\int_0^1 |k_y(x, \cdot)|^2 dx \right)^{\frac{1}{2}} \right|_{L^\infty(0,1)} \left(\int_0^1 w_1(t, x)^2 dx \right)^{\frac{3}{2}}. \end{aligned} \tag{4.12}$$

The second term in the right hand side of (4.11) satisfies that

$$\left| \int_0^1 \left(\int_0^1 k(x, z) v_1(t, z) dz \right) \int_0^1 k_y(x, y) v_1(t, y)^2 dy dx \right|$$

$$\begin{aligned}
 &= \left| \int_0^1 v_1(t, y)^2 \int_0^1 k_y(x, y) \left(\int_0^1 k(x, z) v_1(t, z) dz \right) dx dy \right| \\
 &\leq \int_0^1 v_1(t, y)^2 \left(\int_0^1 |k_y(x, y)|^2 dx \right)^{\frac{1}{2}} \left[\int_0^1 \left(\int_0^1 k(x, z) v_1(t, z) dz \right)^2 dx \right]^{\frac{1}{2}} dy \\
 &\leq \left| \left(\int_0^1 |k_y(x, \cdot)|^2 dx \right)^{\frac{1}{2}} \right|_{L^\infty(0,1)} \left(\int_0^1 \int_0^1 |k(x, y)|^2 dx dy \right)^{\frac{1}{2}} \left(\int_0^1 v_1(t, x)^2 dx \right)^{\frac{3}{2}} \\
 &\leq |(I - K)^{-1}|_{\mathcal{L}(L^2(0,1))}^3 \left| \left(\int_0^1 |k_y(x, \cdot)|^2 dx \right)^{\frac{1}{2}} \right|_{L^\infty(0,1)} \\
 &\quad \times \left(\int_0^1 \int_0^1 |k(x, y)|^2 dx dy \right)^{\frac{1}{2}} \left(\int_0^1 w_1(t, x)^2 dx \right)^{\frac{3}{2}}. \tag{4.13}
 \end{aligned}$$

Let

$$\begin{aligned}
 \widehat{C} &= \frac{1}{2} |(I - K)^{-1}|_{\mathcal{L}(L^2(0,1))}^3 \left| \int_0^1 |k_x(\cdot, y)|^2 dy \right|_{L^\infty(0,1)} \\
 &\quad + |(I - K)^{-1}|_{\mathcal{L}(L^2(0,1))}^3 \left| \left(\int_0^1 |k_y(x, \cdot)|^2 dx \right)^{\frac{1}{2}} \right|_{L^\infty(0,1)} \\
 &\quad + |(I - K)^{-1}|_{\mathcal{L}(L^2(0,1))}^3 \left| \left(\int_0^1 |k_y(x, \cdot)|^2 dx \right)^{\frac{1}{2}} \right|_{L^\infty(0,1)} \left(\int_0^1 \int_0^1 |k(x, y)|^2 dx dy \right)^{\frac{1}{2}}.
 \end{aligned}$$

From (4.8) to (4.13), we get that

$$\frac{d}{dt} \int_0^1 |w_1(t, x)|^2 dx \leq -2Q(w_1(t, \cdot)) + \widehat{C} \left(\int_0^1 w_1(t, x)^2 dx \right)^{\frac{3}{2}}. \tag{4.14}$$

By (1.8), there exists $v' \in \mathbb{R}$ such that

$$v < v' \leq a + j^4 \pi^4 - \lambda j^2 \pi^2, \quad \forall j \in \mathbb{Z}^+. \tag{4.15}$$

From (4.9) and the second inequality of (4.15), we have

$$Q(\varphi) \geq v' \int_0^1 |\varphi(x)|^2 dx, \quad \forall \varphi \in H^2(0, 1) \cap H_0^1(0, 1). \tag{4.16}$$

From (4.14) and (4.16), one has

$$\frac{d}{dt} \int_0^1 |w_1(t, x)|^2 dx \leq -2v' \int_0^1 |w_1(t, x)|^2 dx + \widehat{C} \left(\int_0^1 w_1(t, x)^2 dx \right)^{\frac{3}{2}}. \tag{4.17}$$

By Theorem 1.1 and the first inequality of (4.15), there exists $\delta_1 > 0$ be such that, if $|w_1(0)|_{L^2(0,1)} \leq \delta_1$, then

$$\widehat{C} \left(\int_0^1 w_1(t, x)^2 dx \right)^{\frac{1}{2}} \leq 2(v' - v), \quad \forall t \in [0, T].$$

This, together with (4.14) and (4.17), implies that

$$\frac{d}{dt} \int_0^1 |w_1(t, x)|^2 dx \leq -2v \int_0^1 |w_1(t, x)|^2 dx, \quad \forall t \in [0, T],$$

which gives us that

$$|w_1(t, \cdot)|_{L^2(0,1)} \leq e^{-vt} |w_1(0, \cdot)|_{L^2(0,1)}, \quad \forall t \in [0, T]. \tag{4.18}$$

Then, from Theorem 3.1 and (4.18), we know that, if

$$|v_1(0, \cdot)|_{L^2(0,1)} \leq \min\{|I - K|_{\mathcal{L}(L^2(0,1))}^{-1} \delta_1, r_T\},$$

then

$$|v_1(t, \cdot)|_{L^2(0,1)} \leq |(I - K)^{-1}|_{\mathcal{L}(L^2(0,1))} |I - K|_{\mathcal{L}(L^2(0,1))} e^{-vt} |v_1(0, \cdot)|_{L^2(0,1)}, \quad \forall t \in [0, T]. \tag{4.19}$$

Now we choose $T > 0$ such that $e^{-vT} \leq |(I - K)^{-1}|_{\mathcal{L}(L^2(0,1))} |I - K|_{\mathcal{L}(L^2(0,1))}$. From (4.19), we find that $|v_1(T)|_{L^2(0,1)} \leq |v_1(0)|_{L^2(0,1)} \leq r_T$. Thus, by Theorem 1.1, we know that

$$\begin{cases} v_{2,t} + v_{2,xxxx} + \lambda v_{2,xx} + v_2 v_{2,x} = 0 & \text{in } [0, T] \times (0, 1), \\ v_2(t, 0) = v_2(t, 1) = 0 & \text{on } [0, T], \\ v_{2,xx}(t, 0) = \int_0^1 k_{xx}(0, y) v_2(t, y) dy, \quad v_{2,xx}(t, 1) = 0 & \text{on } [0, T], \\ v_2(0) = v_1(T) & \text{in } (0, 1), \end{cases} \tag{4.20}$$

is well-posed. Furthermore, as for v_1 and w_1 , one can prove that $w_2 \triangleq (I - K)v_2$ satisfies

$$|w_2(t, \cdot)|_{L^2(0,1)} \leq e^{-vt} |w_2(0, \cdot)|_{L^2(0,1)}, \quad \forall t \in [0, T]$$

and

$$|v_2(T, \cdot)|_{L^2(0,1)} \leq |v_2(0, \cdot)|_{L^2(0,1)}.$$

Then, we can define v_3 and w_3 in a similar manner. By induction, we can find $v_n \in X_T$ ($n > 1$), which solves

$$\begin{cases} v_{n,t} + v_{n,xxxx} + \lambda v_{n,xx} + v_n v_{n,x} = 0 & \text{in } [0, T] \times (0, 1), \\ v_n(t, 0) = v_n(t, 1) = 0 & \text{on } [0, T], \\ v_{n,xx}(t, 0) = \int_0^1 k_{xx}(0, y)v_n(t, y)dy, \quad v_{n,xx}(t, 1) = 0 & \text{on } [0, T], \\ v_n(0) = v_{n-1}(T) & \text{in } (0, 1). \end{cases} \tag{4.21}$$

Moreover, we have that $w_n = (I - K)v_n$ satisfies

$$|w_n(t, \cdot)|_{L^2(0,1)} \leq e^{-\nu t} |w_n(0, \cdot)|_{L^2(0,1)} = e^{-\nu t} |w_{n-1}(T, \cdot)|_{L^2(0,1)} \tag{4.22}$$

and

$$|v_n(T, \cdot)|_{L^2(0,1)} \leq |v_n(0, \cdot)|_{L^2(0,1)} = |v_{n-1}(T, \cdot)|_{L^2(0,1)}.$$

Now we put

$$v(t + (n - 1)T, x) = v_n(t, x), \quad w(t + (n - 1)T, x) = w_n(t, x) \quad \text{for } (t, x) \in [0, T] \times [0, 1].$$

Then, it is an easy matter to see that v solves (1.1) and $w = (I - K)v$. From (4.22), we get that

$$|w(t, \cdot)|_{L^2(0,1)} \leq e^{-\nu t} |w(0, \cdot)|_{L^2(0,1)}, \quad \forall t \geq 0. \tag{4.23}$$

This, together with $w = (I - K)v$, implies that for all $t \geq 0$,

$$\begin{aligned} |v(t, \cdot)|_{L^2(0,1)} &\leq e^{-\nu t} |(I - K)^{-1}|_{\mathcal{L}(L^2(0,1))} |I - K|_{\mathcal{L}(L^2(0,1))} |v(0, \cdot)|_{L^2(0,1)} \\ &\leq C e^{-\nu t} |v(0, \cdot)|_{L^2(0,1)}. \end{aligned}$$

Let $\delta_0 = \min\{|(I - K)^{-1}|_{\mathcal{L}(L^2(0,1))}^{-1} \delta_1, r_T\}$. Then, we know that for any $v^0 \in L^2(0, 1)$ with $|v^0|_{L^2(0,1)} \leq \delta_0$, Eq. (1.1) admits a solution $v \in X_T$. Furthermore, we have

$$|v(t, \cdot)|_{L^2(0,1)} \leq C e^{-\nu t} |v(0, \cdot)|_{L^2(0,1)}, \quad \forall t \geq 0, \tag{4.24}$$

which concludes the proof of Theorem 1.2. \square

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Appendix A. Well-posedness of the Cauchy problem associated to the K–S control system

This section is devoted to a proof of [Theorem 1.1](#). Let $T > 0$. We consider the following linearized K–S equation with non-homogeneous boundary condition:

$$\begin{cases} u_t + u_{xxxx} + \lambda u_{xx} = \tilde{h} & \text{in } [0, T] \times (0, 1), \\ u(t, 0) = u(t, 1) = 0 & \text{on } [0, T], \\ u_{xx}(t, 0) = h(t), u_{xx}(t, 1) = 0 & \text{on } [0, T], \\ u(0) = u^0 & \text{in } (0, 1). \end{cases} \tag{A.1}$$

Here $u^0 \in L^2(0, 1)$, $h \in L^2(0, T)$ and $\tilde{h} \in L^2(0, T; L^2(0, 1))$.

We first prove the following result:

Lemma A.1. *Let $u^0 \in L^2(0, 1)$. There exists a unique solution $u \in C^0([0, T]; L^2(0, 1)) \cap L^2(0, T; H^2(0, 1))$ of (A.1) such that $u(0, \cdot) = u^0(\cdot)$. Moreover, there exists $C_1 \geq 1$, independent of $\tilde{h} \in L^1(0, T; L^2(0, 1))$ and $u^0 \in L^2(0, 1)$, such that*

$$\|u\|_{X_T} \leq C_1 (\|u^0\|_{L^2(0,1)} + \|h\|_{L^2(0,T)} + \|\tilde{h}\|_{L^2(0,T;L^2(0,1))}). \tag{A.2}$$

Proof of Lemma A.1. First, we assume $u^0 \in D(A)$ and $h \in H^1(0, T)$. Consider the following equation:

$$\begin{cases} \hat{u}_t + \hat{u}_{xxxx} + \lambda \hat{u}_{xx} = \hat{h} & \text{in } [0, T] \times (0, 1), \\ \hat{u}(t, 0) = \hat{u}(t, 1) = 0 & \text{on } [0, T], \\ \hat{u}_{xx}(t, 0) = \hat{u}_{xx}(t, 1) = 0 & \text{on } [0, T], \\ \hat{u}(0) = \hat{u}^0, & \end{cases} \tag{A.3}$$

where

$$\hat{u}^0 \triangleq u^0 + \frac{1}{6}(x^3 - 3x^2 + 2x)h(0), \tag{A.4}$$

$$\hat{h} = \tilde{h} + \lambda(x - 1)h(t) + \frac{1}{6}(x^3 - 3x^2 + 2x)h'(t). \tag{A.5}$$

By the classical semigroup theory, we know that (A.3) admits a unique solution \hat{u} in $C^0([0, T]; D(A))$. Let

$$u(t, x) \triangleq \hat{u}(t, x) - \frac{1}{6}(x^3 - 3x^2 + 2x)h(t). \tag{A.6}$$

Then it is easy to see that $u(\cdot, \cdot)$ solves (A.1).

Furthermore, multiplying both sides of (A.1) by u and integrating the product in $(0, t) \times (0, 1)$, we get that

$$\begin{aligned} & \int_0^1 u(t, x)^2 dx - \int_0^1 u(0, x)^2 dx + 2 \int_0^t h(s) u_x(s, 0) ds + 2 \int_0^t \int_0^1 u_{xx}(s, x)^2 dx ds \\ & + 2\lambda \int_0^t \int_0^1 u_{xx}(s, x) u(s, x) dx ds - 2\lambda \int_0^t \int_0^1 u_x(s, x)^2 dx ds = 2 \int_0^t \int_0^1 \tilde{h}(s, x) u(s, x) dx ds, \end{aligned} \tag{A.7}$$

which implies that, for every positive real number ε ,

$$\begin{aligned} & \int_0^1 u(t, x)^2 dx + 2 \int_0^t \int_0^1 u_{xx}(s, x)^2 dx ds \\ & \leq \int_0^1 u(0, x)^2 dx + 2 \int_0^t \int_0^1 \tilde{h}(s, x) u(s, x) dx ds - 2\lambda \int_0^t \int_0^1 u_{xx}(s, x) u(s, x) dx ds \\ & \quad + \frac{1}{\varepsilon} \int_0^t h(s)^2 ds + \varepsilon \int_0^t u_x(s, 0)^2 ds. \end{aligned} \tag{A.8}$$

We have

$$-2\lambda \int_0^t \int_0^1 u_{xx}(s, x) u(s, x) dx ds \leq \frac{1}{2} \int_0^t \int_0^1 u_{xx}(s, x)^2 dx ds + 2\lambda^2 \int_0^t \int_0^1 u(s, x)^2 dx ds. \tag{A.9}$$

From (A.9) and integrations by parts, we obtain that

$$\begin{aligned} & \int_0^t u_x(s, 0)^2 ds = \int_0^t \int_0^1 [(x - 1) u_x(s, x)^2]_x dx ds \\ & = 2 \int_0^t \int_0^1 (x - 1) u_x(s, x) u_{xx}(s, x) dx ds + \int_0^t \int_0^1 u_x(s, x)^2 dx ds \\ & \leq \int_0^t \int_0^1 u_{xx}(s, x)^2 dx ds + 2 \int_0^t \int_0^1 u_x(s, x)^2 dx ds \\ & \leq \int_0^t \int_0^1 u_{xx}(s, x)^2 dx ds - 2 \int_0^t \int_0^1 u(s, x) u_{xx}(s, x) dx ds \\ & \leq 2 \int_0^t \int_0^1 u_{xx}(s, x)^2 dx ds + \int_0^t \int_0^1 u(s, x)^2 dx ds. \end{aligned} \tag{A.10}$$

From (A.8) to (A.10), by choosing $\varepsilon = 1/4$, we find that

$$\begin{aligned} & \int_0^1 u(t, x)^2 dx + \int_0^t \int_0^1 u_{xx}(s, x)^2 dx ds \\ & \leq \int_0^1 u(0, x)^2 dx + 4 \int_0^t h(s)^2 ds + \int_0^t \int_0^1 \tilde{h}(s, x)^2 dx ds + 2(1 + \lambda^2) \int_0^t \int_0^1 u(s, x)^2 dx ds. \end{aligned} \tag{A.11}$$

This, together with Gronwall’s inequality, implies that

$$|u|_{C^0([0, T]; L^2(0, 1))} \leq C(|u^0|_{L^2(0, 1)} + |h|_{L^2(0, T)} + |\tilde{h}|_{L^2(0, T; L^2(0, 1))}). \tag{A.12}$$

According to (A.11) and (A.12), we get that

$$|u|_{X_T} \leq C(|u^0|_{L^2(0, 1)} + |h|_{L^2(0, T)} + |\tilde{h}|_{L^2(0, T; L^2(0, 1))}). \tag{A.13}$$

Now, by a standard density argument, we know that for any $u^0 \in L^2(0, 1)$ and $h \in L^2(0, T)$, (A.1) admits a solution $u \in X_T$ which satisfies (A.13). The uniqueness of the solution follows from (A.13), which holds for every solution of (A.1) in X_T . This concludes the proof of Lemma A.1. \square

Lemma A.2. *Let $z \in X_T$. Then $zz_x \in L^2(0, T; L^2(0, 1))$ and the map $z \in X_T \mapsto zz_x \in L^2(0, T; L^2(0, 1))$ is continuous.*

Proof of Lemma A.2. By the Sobolev embedding theorem, we know that there is a constant $\kappa > 0$ such that

$$|v|_{W^{1, \infty}(0, 1)} \leq \kappa |v|_{H^2(0, 1)}. \tag{A.14}$$

Then, we see that

$$\begin{aligned} \int_0^T \int_0^1 |z(t, x)z_x(t, x)|^2 dx dt & \leq \int_0^T \left(|z_x(t)|_{L^\infty(0, 1)}^2 \int_0^1 |z(t, x)|^2 dx \right) dt \\ & \leq \max_{t \in [0, T]} \int_0^1 |z(t, x)|^2 dx \int_0^T |z_x(t)|_{L^\infty(0, 1)}^2 dt \\ & = |z|_{C^0([0, T]; L^2(0, 1))}^2 |z_x|_{L^2(0, T; L^\infty(0, 1))}^2 \\ & \leq C |z|_{C^0([0, T]; L^2(0, 1))}^2 |z|_{L^2(0, T; H^2(0, 1))}^2. \quad \square \end{aligned} \tag{A.15}$$

Proof of Theorem 1.1: uniqueness of the solution. Assume that u and v are two solutions to (1.1). Let $w = u - v$. Then we know that w solves

$$\begin{cases} w_t + w_{xxxx} + \lambda w_{xx} + uw_x + vw_x = 0 & \text{in } [0, T] \times (0, 1), \\ w(t, 0) = w(t, 1) = 0 & \text{on } [0, T], \\ w_{xx}(t, 0) = w_{xx}(t, 1) = 0 & \text{on } [0, T], \\ w(0, x) = 0 & \text{in } (0, 1). \end{cases} \tag{A.16}$$

Multiplying (A.16) by w , integrating the product in $(0, t) \times (0, 1)$ and performing integration by parts, we get

$$\begin{aligned} & \int_0^1 w(t, x)^2 dx + 2 \int_0^t \int_0^1 w_{xx}(s, x)^2 dx ds \\ &= -2\lambda \int_0^t \int_0^1 w_{xx}(s, x)w(s, x) dx ds + 2 \int_0^t \int_0^1 u(s, x)w_x(s, x)w(s, x) dx ds \\ & \quad - 2 \int_0^t \int_0^1 v(s, x)w_x(s, x)w(s, x) dx ds. \end{aligned} \tag{A.17}$$

Since $w(s, \cdot) \in H^2(0, 1) \cap H_0^1(0, 1)$, we know that there is a constant $C_1 > 0$ such that $|w(s)|_{H^2(0,1)} \leq C_1 |w_{xx}(s)|_{L^2(0,1)}$ for any $s \in [0, T]$. Then, by Sobolev’s embedding theorem, we get that there is a constant $C_2 > 0$ such that $|w(s)|_{W^{1,\infty}(0,1)} \leq C_2 |w_{xx}(s)|_{L^2(0,1)}$ for all $s \in [0, T]$. Thus,

$$\begin{aligned} & \left| \int_0^t \int_0^1 u(s, x)w_x(s, x)w(s, x) dx ds \right| \\ & \leq |u|_{C^0([0,T];L^2(0,1))} \int_0^t \left(\int_0^1 |w_x(s, x)w(s, x)|^2 dx \right)^{\frac{1}{2}} ds \\ & \leq |u|_{C^0([0,T];L^2(0,1))} \int_0^t |w_x(s)|_{L^\infty(0,1)} \left(\int_0^1 |w(s, x)|^2 dx \right)^{\frac{1}{2}} ds \\ & \leq \frac{\varepsilon}{4} \int_0^t |w_x(s)|_{L^\infty(0,1)}^2 ds + \frac{|u|_{C^0([0,T];L^2(0,1))}^2}{\varepsilon} \int_0^t \int_0^1 |w(s, x)|^2 dx ds \\ & \leq C_2 \varepsilon \int_0^t \int_0^1 |w_{xx}(s, x)|^2 dx ds + \frac{|u|_{C^0([0,T];L^2(0,1))}^2}{\varepsilon} \int_0^t \int_0^1 |w(s, x)|^2 dx ds. \end{aligned} \tag{A.18}$$

Similarly, we have that

$$\begin{aligned} & \left| \int_0^t \int_0^1 v(s, x) w_x(s, x) w(s, x) dx ds \right| \\ & \leq C_2 \varepsilon \int_0^t \int_0^1 |w_{xx}(s, x)|^2 dx ds + \frac{|v|_{C^0([0, T]; L^2(0, 1))}^2}{\varepsilon} \int_0^t \int_0^1 |w(s, x)|^2 dx ds. \end{aligned} \tag{A.19}$$

Moreover

$$-2\lambda \int_0^t \int_0^1 w_{xx}(s, x) w(s, x) dx ds \leq \int_0^t \int_0^1 w_{xx}(s, x)^2 dx ds + \lambda^2 \int_0^t \int_0^1 w(s, x)^2 dx ds. \tag{A.20}$$

From (A.17) to (A.20) and taking $\varepsilon = C_2/4$, we get that, for any $t \in [0, T]$,

$$\int_0^1 |w(t, x)|^2 dx \leq \left(\lambda^2 + \frac{|u|_{C^0([0, T]; L^2(0, 1))}^2 + |v|_{C^0([0, T]; L^2(0, 1))}^2}{\varepsilon} \right) \int_0^t \int_0^1 |w(s, x)|^2 dx ds. \tag{A.21}$$

This, together with the Gronwall inequality, implies that $w = 0$ in $[0, T] \times (0, 1)$.

Existence of the solution.

Let us extend \tilde{h} and h to be functions on $(0, +\infty) \times (0, 1)$ and $(0, +\infty)$ by setting them to be zero on $(T, +\infty) \times (0, 1)$ and $(T, +\infty)$, respectively. Denote by $\|F\|$ the norm of the continuous linear map $F : L^2(0, 1) \mapsto \mathbb{R}$. Set

$$T_1 \triangleq \min \left\{ \frac{1}{2C_1^2 \|F\|^2}, T \right\}. \tag{A.22}$$

Let $u \in X_T$. We know that

$$h(\cdot) \triangleq F(u(\cdot)) \in L^2(0, T_1).$$

Hence, for v^0 given in $L^2(0, 1)$, we can define a map

$$\mathcal{J} : X_{T_1} \rightarrow X_{T_1}$$

by $\mathcal{J}(u) = v$, where $v \in X_{T_1}$ solves (A.1) with $h(\cdot) = F(u(\cdot))$ and $v(0, \cdot) = v^0(\cdot)$.

For $\tilde{u}, \hat{u} \in X_{T_1}$, from (A.2) and (A.22) one has that

$$\begin{aligned} & |\mathcal{J}(\tilde{u}) - \mathcal{J}(\hat{u})|_{X_{T_1}} \\ & \leq C_1 |F(\tilde{u}(t, \cdot)) - F(\hat{u}(t, \cdot))|_{L^2(0, T_1)} \end{aligned}$$

$$\begin{aligned} &\leq C_1 \|F\| \left(\int_0^{T_1} \int_0^1 |\tilde{u}(t, y) - \hat{u}(t, y)|^2 dy dt \right)^{\frac{1}{2}} \\ &\leq \sqrt{T_1} C_1 \|F\| |\tilde{u} - \hat{u}|_{C^0([0, T_1]; L^2(0, 1))} \\ &\leq \frac{1}{\sqrt{2}} |\tilde{u} - \hat{u}|_{X_{T_1}}. \end{aligned}$$

Hence, we get that $\mathcal{J}(\cdot)$ is a contractive map. By the Banach fixed point theorem, we know that $\mathcal{J}(\cdot)$ has a unique fixed point v_1 , which is the solution to the following equation

$$\begin{cases} v_{1,t} + v_{1,xxxx} + \lambda v_{1,xx} = \tilde{h} & \text{in } [0, T_1] \times (0, 1), \\ v_1(t, 0) = v_1(t, 1) = 0 & \text{on } [0, T_1], \\ v_{1,xx}(t, 0) = F(v_1(t, \cdot)), v_{1,xx}(t, 1) = 0 & \text{on } [0, T_1], \\ v_1(0, x) = v^0(x) & \text{in } (0, 1). \end{cases} \tag{A.23}$$

Using $C_1 \geq 1$, (A.2), (A.22) and (A.23), we find that

$$\begin{aligned} |v_1|_{C^0([0, T_1]; L^2(0, 1))} &\leq C_1 \left[|v^0|_{L^2(0, 1)} + \left(\int_0^{T_1} |F(v(t, \cdot))|^2 dt \right)^{1/2} + |\tilde{h}|_{L^2(0, T_1; L^2(0, 1))} \right] \\ &\leq C_1 \left(|v^0|_{L^2(0, 1)} + \sqrt{T_1} \|F\| |v_1|_{C^0([0, T_1]; L^2(0, 1))} + |\tilde{h}|_{L^2(0, T_1; L^2(0, 1))} \right) \\ &\leq C_1 \left(|v^0|_{L^2(0, 1)} + |\tilde{h}|_{L^2(0, T_1; L^2(0, 1))} \right) + \frac{1}{\sqrt{2}} |v_1|_{C^0([0, T_1]; L^2(0, 1))}, \end{aligned}$$

which implies that

$$|v_1|_{C^0([0, T_1]; L^2(0, 1))} \leq (2 + \sqrt{2}) C_1 \left(|v^0|_{L^2(0, 1)} + |\tilde{h}|_{L^2(0, T_1; L^2(0, 1))} \right). \tag{A.24}$$

Thanks to (A.2), (A.22), (A.23) and (A.24), we obtain that

$$\begin{aligned} |v_1|_{X_T} &\leq C_1 \left[|v^0|_{L^2(0, 1)} + \left(\int_0^{T_1} |F(v(t, \cdot))|^2 dt \right)^{1/2} + |\tilde{h}|_{L^2(0, T_1; L^2(0, 1))} \right] \\ &\leq C_1 \left(|v^0|_{L^2(0, 1)} + |\tilde{h}|_{L^2(0, T_1; L^2(0, 1))} \right) + C_1 T_1^{1/2} \|F\| |v_1|_{C^0([0, T_1]; L^2(0, 1))} \\ &\leq C_1 \left(|v^0|_{L^2(0, 1)} + |\tilde{h}|_{L^2(0, T_1; L^2(0, 1))} \right) + \frac{1}{\sqrt{2}} |v_1|_{C^0([0, T_1]; L^2(0, 1))} \\ &\leq 4C_1 \left(|v^0|_{L^2(0, 1)} + |\tilde{h}|_{L^2(0, T_1; L^2(0, 1))} \right). \end{aligned} \tag{A.25}$$

By a similar argument, we can prove that the following equation

$$\begin{cases} v_{2,t} + v_{2,xxxx} + \lambda v_{2,xx} = \tilde{h} & \text{in } [0, T_1] \times (0, 1), \\ v_2(t, 0) = v_2(t, 1) = 0 & \text{on } [0, T_1], \\ v_{2,xx}(t, 0) = F(v_2(t, \cdot)), v_{2,xx}(t, 1) = 0 & \text{on } [0, T_1], \\ v_2(T_1, x) = v_1(T_1, x) & \text{in } (0, 1) \end{cases} \tag{A.26}$$

admits a unique solution in $C^0([T_1, 2T_1]; L^2(0, 1)) \cap L^2(T_1, 2T_1; H^2(0, 1) \cap H_0^1(0, 1))$. Furthermore, this solution satisfies that

$$\begin{aligned} & \left(|v_2|_{C^0([T_1, 2T_1]; L^2(0,1))}^2 + |v_2|_{L^2(T_1, 2T_1; H^2(0,1) \cap H_0^1(0,1))}^2 \right)^{1/2} \\ & \leq 4C_1 \left(|v_1(T_1)|_{L^2(0,1)} + |\tilde{h}|_{L^2(T_1, 2T_1; L^2(0,1))} \right). \end{aligned} \tag{A.27}$$

By induction, we know that for an integer $n \geq 2$, the following equation

$$\begin{cases} v_{n,t} + v_{n,xxxx} + \lambda v_{n,xx} = \tilde{h} & \text{in } [(n-1)T_1, nT_1] \times (0, 1), \\ v_n(t, 0) = v_n(t, 1) = 0 & \text{on } [(n-1)T_1, nT_1], \\ v_{n,xx}(t, 0) = F(v_n(t, \cdot)), v_{n,xx}(t, 1) = 0 & \text{on } [(n-1)T_1, nT_1], \\ v_n((n-1)T_1, x) = v_{n-1}((n-1)T_1, x) & \text{in } (0, 1), \end{cases} \tag{A.28}$$

admits a unique solution. Moreover, one has

$$\begin{aligned} & \left(|v_n|_{C^0([(n-1)T_1, nT_1]; L^2(0,1))}^2 + |v_n|_{L^2((n-1)T_1, nT_1; H^2(0,1) \cap H_0^1(0,1))}^2 \right)^{1/2} \\ & \leq 4C_1 \left(|v_{n-1}((n-1)T_1)|_{L^2(0,1)} + |\tilde{h}|_{L^2((n-1)T_1, nT_1; L^2(0,1))} \right). \end{aligned} \tag{A.29}$$

Let $n_0 \in \mathbb{Z}^+$ be such that $(n_0 - 1)T_1 \leq T < n_0T_1$. Let

$$v(t, x) \triangleq v_n(t, x) \text{ for } t \in [(n-1)T_1, nT_1] \cap [0, T], \quad n = 1, 2, \dots, n_0, x \in (0, 1).$$

Then v is the solution to

$$\begin{cases} v_t + v_{xxxx} + \lambda v_{xx} = \tilde{h} & \text{in } [0, T] \times (0, 1), \\ v(t, 0) = v(t, 1) = 0 & \text{on } [0, T], \\ v_{xx}(t, 1) = F(v(t, y)), v_{xx}(t, 1) = 0 & \text{on } [0, T], \\ v(0, x) = v^0(x) & \text{in } (0, 1). \end{cases} \tag{A.30}$$

Moreover, there is a constant $C = C(T) > 0$ (independent of $v^0 \in L^2(0, 1)$ and of $\tilde{h} \in L^2(0, T; L^2(0, 1))$) such that

$$|v|_{X_T}^2 \leq C(T) \left(|v^0|_{L^2(0,1)}^2 + |\tilde{h}|_{L^2(0,T; L^2(0,1))}^2 \right). \tag{A.31}$$

From now on, we assume that $v^0 \in L^2(0, 1)$ satisfies

$$|v^0|_{L^2(0,1)}^2 \leq \frac{5}{36C(T)^2\kappa^2}, \tag{A.32}$$

where κ is the constant given in (A.14).

Let

$$\mathcal{B} \triangleq \left\{ u \in X_T : |u|_{X_T}^2 \leq \frac{1}{6C(T)\kappa^2} \right\}.$$

Then \mathcal{B} is a nonempty closed subset of the Banach space X_T . Let us define a map \mathcal{K} from \mathcal{B} to X_T as follows:

$$\mathcal{K}(z) = v, \text{ where } v \text{ is the solution to (A.30) with } \tilde{h} = zz_x.$$

From Lemma A.2 and the above well-posedness result for (A.30), we know that \mathcal{K} is well-defined. From (A.31) and (A.32), we have that

$$\begin{aligned} |\mathcal{K}(z)|_{X_T}^2 &\leq C(T) \left(|v^0|_{L^2(0,1)}^2 + |zz_x|_{L^2(0,T;L^2(0,1))}^2 \right) \\ &\leq C(T) \left(|v^0|_{L^2(0,1)}^2 + |z|_{C^0([0,T];L^2(0,1))}^2 |z_x|_{L^2(0,T;L^\infty(0,1))}^2 \right) \\ &\leq C(T) \left(|v^0|_{L^2(0,1)}^2 + \kappa^2 |z|_{X_T}^4 \right) \\ &\leq \frac{1}{6C(T)\kappa^2} \end{aligned}$$

and

$$\begin{aligned} &| \mathcal{K}(z_1) - \mathcal{K}(z_2) |_{X_T}^2 \\ &\leq C(T) |z_1 z_{1,x} - z_2 z_{2,x}|_{L^2(0,T;L^2(0,1))}^2 \\ &= C(T) |z_1 z_{1,x} - z_1 z_{2,x} + z_1 z_{2,x} - z_2 z_{2,x}|_{L^2(0,T;L^2(0,1))}^2 \\ &\leq 2C(T)\kappa^2 \left(|z_1|_{C^0([0,T];L^2(0,1))}^2 |z_1 - z_2|_{L^2(0,T;H^2(0,1))}^2 \right. \\ &\quad \left. + |z_1 - z_2|_{C^0([0,T];L^2(0,1))}^2 |z_2|_{L^2(0,T;H^2(0,1))}^2 \right) \\ &\leq 2C(T)\kappa^2 \left(|z_1|_{C^0([0,T];L^2(0,1))}^2 + |z_2|_{L^2(0,T;H^2(0,1))}^2 \right) |z_1 - z_2|_{X_T}^2 \\ &\leq \frac{2}{3} |z_1 - z_2|_{X_T}^2. \end{aligned}$$

Therefore, we know that \mathcal{K} is from \mathcal{B} to \mathcal{B} and is contractive. Then, by the Banach fixed point theorem, there is a (unique) fixed point v , which is the solution to (1.1) with the initial datum $v(0, \cdot) = v^0(\cdot)$. \square

Appendix B. On the approximate controllability of the linear K–S control system (2.28)

In this section we prove the following theorem.

Theorem B.1. *System (2.28) is approximately controllable if and only if $\lambda \notin \mathcal{N}$.*

Proof of Theorem B.1. Consider the following equation

$$\begin{cases} z_t + z_{xxxx} + \lambda z_{xx} = 0 & \text{in } (0, T) \times (0, 1), \\ z(t, 0) = z(t, 1) = 0 & \text{in } (0, T), \\ z_{xx}(t, 1) = 0, z_{xx}(t, 0) = 0 & \text{in } (0, T), \\ z_x(t, 0) = 0 & \text{in } (0, T), \\ z(0) = z_0 & \text{in } (0, 1), \end{cases} \tag{B.1}$$

where $z_0 \in L^2(0, 1)$. Let us recall that the approximate controllability of (2.28), is equivalent to the property that the only solution to (B.1) is zero (see, e.g. [13, Theorem 2.43]).

The “if” part: Let $z_{0,j} = \int_0^1 z_0(x) \sqrt{2} \sin(j\pi x) dx$ for $j \in \mathbb{Z}^+$. The solution to (B.1) reads

$$z(t, x) = \sum_{j \in \mathbb{Z}^+} z_{0,j} e^{(-j^4\pi^4 + \lambda j^2\pi^2)t} \sqrt{2} \sin(j\pi x).$$

Then,

$$z_x(t, 0) = \sum_{j \in \mathbb{Z}^+} z_{0,j} e^{(-j^4\pi^4 + \lambda j^2\pi^2)t} \sqrt{2} j\pi = 0.$$

Clearly, $z_x(t, 0)$ is analytic in $(0, T]$. Since $z_x(t, 0) = 0$ in $(0, T]$, we know that

$$z_x(t, 0) = \sum_{j \in \mathbb{Z}^+} z_{0,j} e^{(-j^4\pi^4 + \lambda j^2\pi^2)t} \sqrt{2} j\pi = 0 \text{ in } (0, +\infty). \tag{B.2}$$

Let $j_0 \in \mathbb{Z}^+$ such that $-j_0^4\pi^4 + \lambda j_0^2\pi^2 = \max_{j \in \mathbb{Z}^+} \{-j^4\pi^4 + \lambda j^2\pi^2\}$. From (1.4), we have that j_0 is unique. Let us multiply both sides of (B.2) by $e^{(j_0^4\pi^4 - \lambda j_0^2\pi^2)t}$ and let t tends to $+\infty$. Then, we find that $z_{0,j_0} = 0$. Now let $j_1 \in \mathbb{Z}^+$ be such that $-j_1^4\pi^4 + \lambda j_1^2\pi^2 = \max_{j \in (\mathbb{Z}^+ \setminus \{j_0\})} \{-j^4\pi^4 + \lambda j^2\pi^2\}$. From (1.4), we have that j_1 is unique. Let us multiply both sides of (B.2) by $e^{(j_1^4\pi^4 - \lambda j_1^2\pi^2)t}$ and let $t \rightarrow +\infty$. Then, we find that $z_{0,j_1} = 0$. By induction, we can conclude that $z_{0,j} = 0$ for all $j \in \mathbb{Z}^+$, which implies that $z = 0$ in $(0, T) \times (0, 1)$.

The “only if” part: If $\lambda \in \mathcal{N}$, then there are $j_0, k_0 \in \mathbb{Z}^+$ with $j_0 \neq k_0$ such that $\lambda = j_0^2\pi^2 + k_0^2\pi^2$. Thus, we get $e^{(-j_0^4\pi^4 + \lambda j_0^2\pi^2)t} = e^{j_0^2 k_0^2 \pi^4 t}$. Let us choose

$$z(t, x) = e^{(-j_0^4\pi^4 + \lambda j_0^2\pi^2)t} \sin(j_0\pi x) - e^{[-k_0^4\pi^4 + \lambda k_0^2\pi^2]t} \frac{j_0}{k_0} \sin(k_0\pi x).$$

Then, clearly, z solves (B.1) and z is not zero. This concludes the proof of Theorem B.1. \square

As a corollary of Theorem B.1, one has the following corollary (which can be also checked directly)

Corollary B.1. Assume that $\lambda \notin \mathcal{N}$ and that $\varphi \in C^4([0, 1]; \mathbb{C})$ is such that, for some $\mu \in \mathbb{C}$,

$$\begin{cases} \varphi'''' + \lambda\varphi'' = \mu\varphi \text{ in } (0, 1), \\ \varphi(0) = \varphi(1) = \varphi'(0) = \varphi''(0) = \varphi''(1) = 0, \end{cases} \tag{B.3}$$

then, $\varphi = 0$.

Proof of Corollary B.1. It suffices to remark that, if (B.3) holds, then $z(t, x) = e^{-\mu t} \varphi(x)$ is a solution of (B.1). \square

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