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Brief paper

Stability of linear density-flow hyperbolic systems under PI boundary control ☆

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ABSTRACT

We consider a class of density-flow systems, described by linear hyperbolic conservation laws, which can be monitored and controlled at the boundaries. These control systems are open-loop unstable and subject to unmeasured flow disturbances. We address the issue of feedback stabilization and disturbance rejection under PI boundary control. Explicit necessary and sufficient stability conditions in the frequency domain are provided.

1. Introduction

We are concerned with hyperbolic systems of two linear conservation laws over a finite interval in one spatial dimension of the general form:

$$\begin{aligned}&\partial_t H + \partial_x Q = 0,\\ &\partial_t Q + \lambda_1\lambda_2\partial_x H + (\lambda_1 - \lambda_2)\partial_x Q = 0,\end{aligned} \tag{1}$$

where $t \in [0, +\infty)$, $x \in [0, L]$, λ_1 and λ_2 are two real positive constants. In these equations $H(t, x)$ is the density and $Q(t, x)$ is the flow density of some extensive quantity of interest. Therefore, this system is called a "density-flow" system.

The model (1) may be used to represent many physical systems. A very simple and relevant engineering example is given by distribution networks of liquid fluids which are made of pipes interconnected by pumps as shown in Fig. 1. Under the assumptions that

- the flow is normal to the cross section,
- the pressure and the fluid velocity are uniform in a cross-section,
- the sound velocity in the fluid is much larger than the flow velocity,
- the friction is negligible,

the dynamics of the fluid in a pipe of the network are described by the following hyperbolic system:

$$\begin{aligned}&\partial_t H + \partial_x Q = 0,\\ &\partial_t Q + c^2\partial_x H = 0,\end{aligned}$$

where H is the piezometric head and Q is the flow rate, while $\lambda_1 = \lambda_2 = c$ is the sound velocity. A detailed justification of this model is nicely presented by Nicolet in his Ph.D. thesis (see Chapter 2 of Nicolet, 2007). In such networks, it may be relevant to provide the system with feedback controllers that regulate the piezometric head at certain places in order, for instance, to prevent water hammer phenomena.

The system (1) may also be used as a valid approximate linearized model for many other engineering applications where dissipation is neglected, such as for example gas pipelines where H is the gas density and Q is the gas flow rate (see e.g. Banda, Herty, & Klar, 2006), open channels where H is the water depth and Q is the water flow rate (see e.g. Bastin, Coron, & d'Andréa-Novel, 2009) or electrical transmission lines where H is the charge density and Q is the current density.

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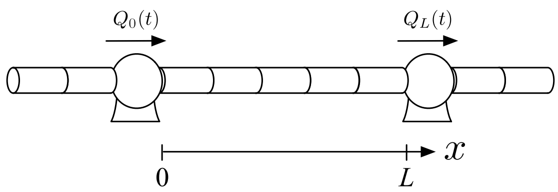

Fig. 1. A density-flow system.

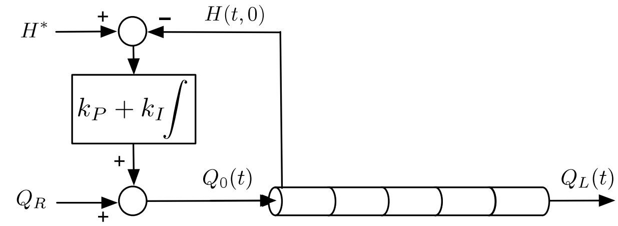

Fig. 2. Block diagram of the closed-loop system with a Proportional–Integral control.

In this paper, we are concerned with the solutions of the Cauchy problem for the system (1) under an initial condition:

$$H(0,x), Q(0,x), \quad x \in [0,L],$$

and two boundary conditions of the form:

$$Q(t,0) = Q_0(t), \qquad Q(t,L) = Q_L(t), \quad t \in [0,+\infty). \tag{2}$$

Since any pair of constant states H^*, Q^* can be a steady-state, it is clear that the system has a continuum of non-isolated equilibria which are not asymptotically stable.

It is therefore relevant to study the boundary feedback stabilization of the control system (1)–(2). Our main concern in this paper is to give explicit stabilizability conditions in the particular case where *Proportional–Integral* (PI) boundary control is used for stabilization and disturbance attenuation. The analysis is in the continuation of previous contributions on PI control of hyperbolic systems by Dos Santos, Bastin, Coron, and d'Andréa-Novel (2008), Dos Santos Martins and Rodrigues (2011) and Xu and Sallet (1999) where conservative sufficient stability conditions are given using respectively spectral, Lyapunov and LMI approaches. In the present paper, our main contribution is to give complete and explicit necessary and sufficient conditions.

The motivation for using a PI control structure is addressed in Section 2. In Section 3, the necessary and sufficient stability conditions in the frequency domain are provided (Theorem 1 and Corollary 1). Finally in Section 4 we show how the stability analysis can be extended to (acyclic) networks of density-flow systems.

2. The PI control structure

We consider the situation where there is only one boundary control input, say $Q_0(t)$, available for feedback stabilization. The other boundary flow $Q_L(t)$ perturbs the system in an unpredictable manner. It is assumed that this so-called "load disturbance" cannot be measured and cannot therefore be directly compensated in the control.

We assume that, in addition to stabilization, the control objective is to regulate $H(t,x)$ at the "set point" H^* and to attenuate the incidence of the load disturbance $Q_L(t)$ by using on-line feedback measurements of $H(t,0)$.

In such case, it is well known that it is useful to implement an "integral" action in addition to the proportional action. The PI control law may be of the following form:

$$Q_0(t) \triangleq Q_R + k_P(H^* - H(t,0)) + k_I \int_0^t (H^* - H(\sigma,0))d\sigma. \tag{3}$$

The first term Q_R is a constant reference value for the flow which is arbitrary and freely chosen by the designer. The second term is the proportional correction action with the tuning parameter k_P. The last term is the integral action with the tuning parameter k_I. The control structure is illustrated in Fig. 2. In the specific case of a constant disturbance $Q_L(t) = Q^*$, it is readily seen that the closed-loop system has a unique steady-state (H^*, Q^*).

As it is explained in detail in Chapter 11 of the textbook *Feedback Systems* by Astrom and Murray (2009), PI control is by far the most popular way of using feedback in engineering systems because it is the simplest way to cancel offset errors and to attenuate load disturbances in a robust way. The integral gain k_I is a measure of the disturbance attenuation but a too large value of k_I may lead to instability in some instances. It is therefore of interest to characterize the range of values of k_I for which the closed loop system is guaranteed to be stable.

3. Stability conditions

The Riemann coordinates are defined around a steady-state by the following change of coordinates:

$$R_1 = Q - Q^* + \lambda_2(H - H^*),$$
$$R_2 = Q - Q^* - \lambda_1(H - H^*).$$

The inverse change of coordinates is:

$$H = H^* + \frac{R_1 - R_2}{\lambda_1 + \lambda_2},$$
$$Q = Q^* + \frac{\lambda_1 R_1 + \lambda_2 R_2}{\lambda_1 + \lambda_2}.$$

With Riemann coordinates, the system (1) is written in characteristic form:

$$\partial_t R_1 + \lambda_1 \partial_x R_1 = 0, \qquad \partial_t R_2 - \lambda_2 \partial_x R_2 = 0. \tag{4}$$

In these coordinates, the control law (3) provides a first boundary condition at $x = 0$:

$$R_1(t,0) = k_1 R_2(t,0) + k_0(\lambda_1 + \lambda_2)Z(t), \tag{5}$$

with $k_1 \triangleq \dfrac{k_P - \lambda_2}{k_P + \lambda_1}$, $\quad k_0 \triangleq \dfrac{k_I}{k_P + \lambda_1}$ and

$$Z(t) \triangleq \frac{Q_R - Q^*}{k_I} + \frac{1}{\lambda_1 + \lambda_2}\int_0^t (R_2(\sigma,0) - R_1(\sigma,0))d\sigma.$$

The specific case of a constant disturbance $Q_L(t) = Q^*$ gives a second boundary condition at $x = L$:

$$R_2(t,L) = k_2 R_1(t,L) \quad \text{with } k_2 = -\frac{\lambda_1}{\lambda_2}. \tag{6}$$

From (6), since $R_1(t,x)$ and $R_2(t,x)$ are constant along their respective characteristic lines, we have that

$$R_2(t+\tau,0) = k_2 R_1(t,0) \quad \text{with } \tau \triangleq \frac{L}{\lambda_1} + \frac{L}{\lambda_2} \tag{7}$$

and therefore that

$$\frac{dR_2(t+\tau,0)}{dt} = k_2 \frac{dR_1(t,0)}{dt}. \tag{8}$$

Moreover, by differentiating (5) with respect to time, the first boundary condition is rewritten as:

$$\frac{dR_1(t,0)}{dt} = k_1 \frac{dR_2(t,0)}{dt} + k_0\big(R_2(t,0) - R_1(t,0)\big). \tag{9}$$

Then, by eliminating $R_1(t,0)$ and $dR_1(t,0)/dt$ between (7)–(9), we get that $R_2(t,0)$ is the solution of the following delay-differential equation of neutral type:

$$\frac{dR_2(t+\tau,0)}{dt} - k_1 k_2 \frac{dR_2(t,0)}{dt} + k_0\Big(R_2(t+\tau,0) - k_2 R_2(t,0)\Big) = 0. \tag{10}$$

The Laplace transform of this equation is:

$$\Big[(e^{s\tau} - k_1k_2)s + k_0(e^{s\tau} - k_2)\Big]R_2(s,0) = 0. \tag{11}$$

This is a so-called *neutral delay-differential* equation. The roots of the characteristic equation

$$(e^{s\tau} - k_1k_2)s + k_0(e^{s\tau} - k_2) = 0 \tag{12}$$

are called *the poles* of the system (4), (5), (6).

In the next theorem, we give necessary and sufficient conditions to have stable poles, i.e. poles located in the left-half complex plane and bounded away from the imaginary axis. The stability of the poles implies, in the time domain, the exponential stability of the equilibrium (when the disturbance $Q_L(t) = Q^*$ is constant) and therefore the input-to-state stability (when the disturbance $Q_L(t)$ is bounded time-varying) for the L^∞-norms (see e.g. Hale & Verduyn-Lunel, 2002, Section 3 and Michiels & Niculescu, 2007, Section 1.2).

In the proof of the theorem we use a variant of the Walton–Marshall procedure (see Silva, Datta, & Bhattacharyva, 2005, Section 5.6 and Walton & Marshall, 1987).

Theorem 1. *There exists $\delta > 0$ such that the poles of the system (4), (5), (6) are located in the half plane $(-\infty, -\delta] \times \mathbb{R}$ if and only if*

- *when $\lambda_1 \leqslant \lambda_2$ (i.e. $-1 \leqslant k_2 < 0$),*

  $|k_1k_2| < 1$ *and* $0 < k_0$;
- *when $\lambda_1 > \lambda_2$ (i.e. $k_2 < -1$),*

  $$|k_1k_2| < 1 \quad \text{and} \quad 0 < k_0 < \omega_0 \frac{k_2(1+k_1)}{1-k_2^2}\sin(\omega_0\tau)$$

  where ω_0 is the smallest positive ω such that $\cos(\omega\tau) = \frac{1+k_1k_2^2}{k_2(1+k_1)}$.

Proof. A fundamental property in the stability analysis of this neutral delay-differential equation is that $|k_1k_2| < 1$ is a necessary condition to have stable poles i.e. $\Re(s) < -\delta$ for some $\delta > 0$ (see e.g. Hale & Verduyn-Lunel, 2002 and Michiels & Vyhlidal, 2005). From now on, we therefore assume that $|k_1k_2| < 1$. Then for every k_1 and k_2, for every $\eta > \ln(|k_1k_2|)/\tau$ and for every $C_0 > 0$, there exists $C_1 > 0$ such that

$$\Big\{|k_0| \leqslant C_0,\ |s| \geqslant C_1 \text{ and } (12)\Big\} \Rightarrow \Big\{\Re(s) \leqslant \eta\Big\}. \tag{13}$$

Indeed the existence of C_1 results from rewriting (12) under the form

$$e^{s\tau} = \frac{k_1k_2s + k_0k_2}{s + k_0} \tag{14}$$

which implies

$$\tau\Re(s) = \ln\left|\frac{k_1k_2s + k_0k_2}{s+k_0}\right| \xrightarrow{|s|\to\infty} \ln|k_1k_2|$$

where the convergence is uniform for $|k_0| \leqslant C_0$.

With the notation $s \triangleq \sigma + i\omega$, the poles satisfy the following equation:

$$\begin{aligned} k_0 &= -\frac{(e^{s\tau} - k_1k_2)s}{e^{s\tau} - k_2} \\ &= \frac{[\omega a(\sigma,\omega) - \sigma b(\sigma,\omega)] - i[\sigma a(\sigma,\omega) + \omega b(\sigma,\omega)]}{e^{2\sigma\tau} + k_2^2 - 2k_2e^{\sigma\tau}\cos(\omega\tau)} \end{aligned} \tag{15}$$

with

$$a(\sigma,\omega) \triangleq k_2e^{\sigma\tau}(k_1 - 1)\sin\omega\tau \quad \text{and} \tag{16}$$

$$b(\sigma,\omega) \triangleq e^{2\sigma\tau} - k_2(1+k_1)e^{\sigma\tau}\cos\omega\tau + k_1k_2^2. \tag{17}$$

Since the left-hand side of Eq. (15) is real, it follows that the imaginary part of the right-hand side must be zero. Therefore we are looking for the values of σ and ω such that

$$\sigma a(\sigma,\omega) + \omega b(\sigma,\omega) = 0. \tag{18}$$

Let us now consider the poles with non-positive real parts, i.e. $\sigma \leqslant 0$.

(1) If $k_0 = 0$, we see that the poles are roots of $(e^{s\tau} - k_1k_2)s = 0$. This means that there is a pole $s = 0$ at the origin and the other poles are stable if and only if $|k_1k_2| < 1$. Now for small non-zero k_0, we have:

$$(1 - k_1k_2)s + k_0(1 - k_2) \approx 0,$$

that is

$$s \approx -k_0\frac{1-k_2}{1-k_1k_2}.$$

This approximation is justified by using the implicit function theorem applied to the map

$$(s, k_0) \in \mathbb{C}\times\mathbb{R} \longrightarrow F(s,k_0) = (e^{s\tau} - k_1k_2)s + k_0(e^{s\tau} - k_2)$$

since $\partial F/\partial s(0,0) = 1 - k_1k_2 \neq 0$. Then, since $|k_1k_2| < 1$ and $k_2 = -\lambda_1/\lambda_2 < 0$, it follows that for small $k_0 > 0$ the pole at zero moves inside the negative half-plane while the other poles stay inside the negative half-plane. Moreover, for small $k_0 < 0$, the pole at zero moves inside the right half plane. As k_0 decreases, this simple pole cannot come back on the imaginary axis (since $k_2 \neq 0$) and therefore it remains in the right half plane for all $k_0 < 0$.

(2) Now, in order to analyze what happens when $k_0 > 0$ becomes larger, we consider the conditions for having poles on the imaginary axis, i.e. $\sigma = 0$. Since $k_0 \neq 0$, the case $\sigma = 0, \omega = 0$ is excluded. Therefore $\sigma = 0$ implies $b = 0$ from (18), which together with (17) gives:

$$\cos(\omega\tau) = \frac{1+k_1k_2^2}{k_2(1+k_1)}. \tag{19}$$

In this case, it can be readily verified that, since $|k_1k_2| < 1$,

$$\begin{aligned} \lambda_1 < \lambda_2 &\Leftrightarrow |k_2| < 1 \\ &\Leftrightarrow 1 - k_2^2 > 0 \\ &\Leftrightarrow (1-k_2^2)(1-k_1^2k_2^2) > 0 \\ &\Leftrightarrow 1 + k_1^2k_2^4 + 2k_1k_2^2 > k_2^2(1+k_1^2) + 2k_1k_2^2 \\ &\Leftrightarrow \left|\frac{1+k_1k_2^2}{k_2(1+k_1)}\right| > 1. \end{aligned}$$

This implies that, if $\lambda_1 < \lambda_2$, there are no eigenvalues on the imaginary axis (since $|\cos\omega\tau| \leqslant 1$ obviously).

Then, using also (13), we can conclude, using a standard deformation argument on k_0, that, when $|k_2| < 1$ and $|k_1k_2| < 1$, the poles remain stable for every $k_0 > 0$.

(3) Let us now consider the case where $\lambda_1 > \lambda_2$ i.e. $k_2 < -1$ (the case $\lambda_1 = \lambda_2$ is discussed later). In this case, it can be readily verified that

$$\left|\frac{1+k_1k_2^2}{k_2(1+k_1)}\right| < 1.$$

Therefore, from (15) and (17) with $\sigma = 0$, there is a pair of poles $\pm i\omega$ on the imaginary axis for any positive value of ω such that:

$$\cos(\pm\omega\tau) = \frac{1+k_1k_2^2}{k_2(1+k_1)}$$

$$\text{and} \quad \omega\sin(\omega\tau) = -\frac{k_0(k_2^2-1)}{k_2(1+k_1)}. \tag{20}$$

Let ω_0 be the smallest value of ω such that (20) is satisfied. Now, if $i\omega_0$ is a pole on the imaginary axis, the corresponding value of k_0 computed from (20) with $\omega = \omega_0$ is:

$$k_0^* = \omega_0 \frac{k_2(1+k_1)}{1-k_2^2} \sin(\omega_0 \tau) > 0.$$

Then, using again (13), we can conclude, using a standard deformation argument on k_0, that the poles are stable for any k_0 such that $0 < k_0 < k_0^*$. In order to determine the motion of the pole on the imaginary axis for small variations of k_0 around k_0^*, we consider the root s of the characteristic equation as an explicit function of k_0. Then, by differentiating the characteristic equation (12), we have the following expression for the derivative of s with respect to k_0:

$$s' \triangleq \frac{ds}{dk_0} = \frac{k_2 - e^{s\tau}}{e^{s\tau}(1+\tau(s+k_0)) - k_1 k_2}. \tag{21}$$

We now evaluate this expression at $i\omega$:

$$s' = \frac{k_2 - e^{i\omega\tau}}{e^{i\omega\tau}(1+\tau(i\omega+k_0)) - k_1 k_2}.$$

Using (20), after some calculations, we obtain that the real part of s' at $i\omega$ is given by:

$$\Re(s') = \frac{\tau k_0 (k_2^2 - 1)}{\left| e^{i\omega\tau}(1+\tau(i\omega+k_0)) - k_1 k_2 \right|^2}.$$

Hence, since $k_0 > 0$ and $k_2^2 > 1$ by assumption, $\Re(s')$ is a positive number. It follows that any pole reaching the imaginary axis from the left when k_0 is increasing will cross the imaginary axis from left to right. This readily implies that, as soon as $k_0 > k_0^*$, there is necessarily at least one pole in the right half plane.

(4) Let us finally consider the case where $\lambda_1 = \lambda_2$ (i.e. $k_2 = -1$). In that case, it follows directly from (19) that $\cos(\omega\tau) = -1$ and $\sin(\omega\tau) = 0$ for any pole $i\omega$ on the imaginary axis. Therefore the characteristic equation (12) reduces to

$$(k_1 - 1) i\omega = 0$$

which is impossible if $\omega \neq 0$ because the conditions $k_2 = -1$ and $|k_1 k_2| < 1$ imply that $|k_1| < 1$. Hence there is no imaginary pole when $\lambda_1 = \lambda_2$.
This completes the proof of Theorem 1. □

In the previous theorem, for the clarity of the proof, we have carried out the analysis in terms of the parameters k_0, k_1 and k_2. However, from a practical viewpoint, it is clearly more relevant and more interesting to express the stability conditions in terms of the control tuning parameters k_P and k_I. Replacing k_0, k_1 and k_2 by their expressions in function of k_P, k_I, λ_1 and λ_2 as given in (5)–(6), the conditions of Theorem 1 are translated as follows.

Corollary 1. *There exists $\delta > 0$ such that the poles of the system (4), (5), (6) are in the half plane $(-\infty, -\delta] \times \mathbb{R}$ if and only if the control tuning parameters k_P, k_I are selected such that:*

- *when $\lambda_1 < \lambda_2$,*

$$k_P > 0 \text{ and } k_I > 0 \quad \text{or} \quad k_P < -\frac{2\lambda_1\lambda_2}{\lambda_2 - \lambda_1} \text{ and } k_I < 0;$$

- *when $\lambda_1 = \lambda_2$, $k_P > 0$ and $k_I > 0$;*
- *when $\lambda_1 > \lambda_2$,*

$$k_P > 0 \quad \text{and} \quad 0 < k_I < \omega_0 \frac{(2k_P + \lambda_1 - \lambda_2)\lambda_1\lambda_2}{(\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2)} \sin(\omega_0 \tau)$$

where ω_0 is the smallest positive ω such that

$$\cos(\omega\tau) = \frac{\lambda_2^2(k_P + \lambda_1) + \lambda_1^2(k_P - \lambda_2)}{\lambda_1\lambda_2(\lambda_2 - \lambda_1 - 2k_P)}.$$

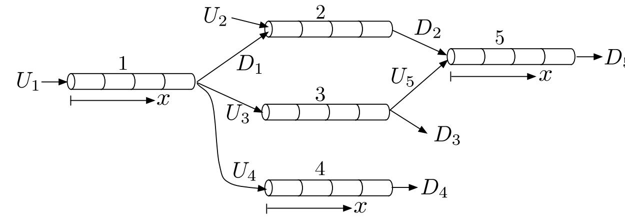

Fig. 3. Physical network of density-flow systems.

4. Networks of density-flow systems

In this section, we examine how the previous stability analysis can be extended to acyclic networks of density-flow systems. Depending on the concerned application, there are different ways of designing such networks. Here, as a matter of example, we consider a specific structure which leads to a natural generalization of the control problem addressed in the previous section. But other structures could be dealt with in a similar way (see e.g. Engel, Fijavz, Nagel, & Sikolya, 2008 and Marigo, 2007 for relevant related references).

The network has a compartmental structure illustrated in Fig. 3. The nodes of the network are n storage compartments having the dynamics of density-flow systems (e.g. the pipes of an hydraulic network):

$$\begin{aligned} &\partial_t H_j + \partial_x Q_j = 0, \\ &\partial_t Q_j + \lambda_j \lambda_{n+j} \partial_x H_j + (\lambda_j - \lambda_{n+j}) \partial_x Q_j = 0, \\ &\quad j = 1, \ldots, n. \end{aligned} \tag{22}$$

Without loss of generality and for simplicity, it can always be assumed that, by an appropriate scaling, all the pipes have exactly the same length L.

The directed arcs $i \to j$ of the network represent instantaneous transfer flows between the compartments. Additional input and output arcs represent interactions with the surroundings: either inflows injected from the outside into some compartments or outflows from some compartments to the outside. We assume that there is exactly one and only one control flow, denoted as U_i, at the input of each compartment. All the other flows are assumed to be disturbances and denoted by D_k ($k = 1, \ldots, m$). The set of $2n$ PDEs (22) is therefore subject to $2n$ boundary flow balance conditions of the following form for $i = 1, \ldots, n$:

$$\begin{aligned} Q_i(t,0) &= U_i(t) + \sum_{k=1}^{m} \beta_{ik} D_k(t), \\ Q_i(t,L) &= \sum_{j=1}^{n} \alpha_{ij} U_j(t) + \sum_{k=1}^{m} \gamma_{ik} D_k(t). \end{aligned} \tag{23}$$

In the summations, the coefficients α_{ij}, β_{ik} and γ_{ik} are equal to 1 for the existing links between adjacent compartments of the network and 0 for the others (see Fig. 3 for illustration).

With the matrix notations

$$\mathbf{H} \triangleq \begin{pmatrix} H_1 \\ \vdots \\ H_n \end{pmatrix} \quad \mathbf{Q} \triangleq \begin{pmatrix} Q_1 \\ \vdots \\ Q_n \end{pmatrix} \quad \mathbf{U} \triangleq \begin{pmatrix} U_1 \\ \vdots \\ U_n \end{pmatrix} \quad \mathbf{D} \triangleq \begin{pmatrix} D_1 \\ \vdots \\ D_m \end{pmatrix},$$

$$\Lambda^+ = \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}, \qquad \Lambda^- = \operatorname{diag}\{\lambda_{n+1}, \ldots, \lambda_{2n}\},$$

the system (22) is written

$$\begin{aligned} &\partial_t \mathbf{H} + \partial_x \mathbf{Q} = 0, \\ &\partial_t \mathbf{Q} + \Lambda^+ \Lambda^- \partial_x \mathbf{H} + (\Lambda^+ - \Lambda^-) \partial_x \mathbf{Q} = 0. \end{aligned} \tag{24}$$

The boundary conditions (23) are written

$$\mathbf{Q}(t,0) = \mathbf{U}(t) + B_0\mathbf{D}(t),$$
$$\mathbf{Q}(t,L) = A_L\mathbf{U}(t) + B_L\mathbf{D}(t), \tag{25}$$

with appropriate matrices A_L, B_0, B_L. Since the network is acyclic, the nodes of the network can be numbered such that the square matrix A_L is strictly upper triangular. Therefore the matrix A_L has the property that

$$A_L^p = 0, \tag{26}$$

where p is the length of the longest path in the network.

A steady state for the system (24)–(25) is a quadruple

$$\{\mathbf{H}^*, \mathbf{Q}^*, \mathbf{U}^*, \mathbf{D}^*\}$$

which satisfies the boundary conditions:

$$\mathbf{Q}^* = \mathbf{U}^* + B_0\mathbf{D}^*,$$
$$\mathbf{Q}^* = A_L\mathbf{U}^* + B_L\mathbf{D}^*.$$

The network has an infinity of non-isolated steady states which are not asymptotically stable. In order to stabilize the network, each control input is endowed with a PI control law of the form:

$$U_i(t) \triangleq U_{Ri} + k_{Pi}(H_i^* - H_i(t,0)) + k_{Ii}\int_0^t (H_i^* - H_i(\sigma,0))d\sigma, \tag{27}$$

where U_{Ri} is an arbitrary scaling constant, H_i^* is the set point for the ith compartment, k_{Pi} and k_{Ii} are the control tuning parameters. In matrix form, the set of control laws (27) is written

$$\mathbf{U} = \mathbf{U}_R + K_P\Big(\mathbf{H}^* - \mathbf{H}(t,0)\Big) + K_I\int_0^t \Big(\mathbf{H}^* - \mathbf{H}(\sigma,0)\Big)d\sigma, \tag{28}$$

with $K_P \triangleq \mathrm{diag}\{k_{P1},\dots,k_{Pn}\}$ and $K_I \triangleq \mathrm{diag}\{k_{I1},\dots,k_{In}\}$.

We shall now examine how the stability analysis of Section 4 for the "single pipe" case can be generalized to the closed-loop network (24)–(25)–(28) for constant unknown disturbances $\mathbf{D}^*$. The Riemann coordinates are defined as follows:

$$\begin{matrix} R_i \triangleq Q_i - Q_i^* + \lambda_{n+i}(H_i - H_i^*) \\ R_{n+i} \triangleq Q_i - Q_i^* - \lambda_i(H_i - H_i^*) \end{matrix} \quad i = 1,\dots,n,.$$

Using this definition, the following equalities hold at the boundaries:

$$\begin{aligned}
&(\lambda_i + \lambda_{n+i})(Q_i(t,0) - Q_i^*) \\
&\quad = (\lambda_i + \lambda_{n+i})\Big[k_{P_i}(H_i^* - H_i(t,0)) + k_{Ii}Z_i(t)\Big] \\
&\quad = k_{P_i}(R_{n+i}(t,0) - R_i(t,0)) + (\lambda_i + \lambda_{n+i})k_{Ii}Z_i(t), \\
&(\lambda_i + \lambda_{n+i})(Q_i(t,L) - Q_i^*) = \lambda_i R_i(t,L) + \lambda_{n+i}R_{n+i}(t,L),
\end{aligned}$$

with $Z_i(t)$ such that

$$\frac{dZ_i}{dt} = H_i^* - H_i(t,0) = \frac{R_{n+i}(t,0) - R_i(t,0)}{\lambda_i + \lambda_{n+i}}.$$

Since $R_i(t,x)$ and $R_{n+i}(t,x)$ are constant along their respective characteristic lines, we have that

$$R_i\left(t + \frac{L}{\lambda_i}, L\right) = R_i(t,0)$$
$$\text{and} \quad R_{n+i}\left(t + \frac{L}{\lambda_{n+i}}, 0\right) = R_{n+i}(t,L). \tag{29}$$

Then, combining appropriately these equalities, it can be shown after some computations that, in the frequency domain, the transfer function between $(Q_i(t,L) - Q_i^*)$ and $(Q_i(t,0) - Q_i^*)$ is given by:

$$\begin{aligned}
G_i(s) &\triangleq \frac{Q_i(s,0) - Q_i^*}{Q_i(s,L) - Q_i^*} \\
&= \frac{1}{\lambda_{n+i}}\frac{s(\lambda_i k_i - \lambda_{n+i}) + c_i(\lambda_i - \lambda_{n+i})}{(e^{s\tau_i} - k_i k_{n+i})s + c_i(e^{s\tau_i} - k_{n+i})}e^{\frac{sL}{\lambda_i}},
\end{aligned} \tag{30}$$

with the following notations:

$$k_i \triangleq \frac{k_{Pi} - \lambda_{n+i}}{k_{Pi} + \lambda_i}, \qquad k_{n+i} \triangleq -\frac{\lambda_i}{\lambda_{n+i}},$$
$$c_i \triangleq \frac{k_{Ii}}{k_{Pi} + \lambda_i}, \qquad \tau_i \triangleq \frac{L}{\lambda_i} + \frac{L}{\lambda_{n+i}}.$$

It follows that the poles of the transfer function $G_i(s)$ are the roots of the characteristic equation

$$(e^{s\tau_i} - k_i k_{n+i})s + c_i(e^{s\tau_i} - k_{n+i}) = 0$$

which is, as expected, identical to the characteristic equation of the simple case of Section 3.

Let us now consider the closed-loop system (24)–(25)–(28) as an input–output dynamical system with input $\mathbf{D}$ and output $\mathbf{U}$. Then, by iterating Eq. (25) p-times and using property (26), it can be shown that the transfer matrix of the system is as follows:

$$H(s) \triangleq \sum_{i=0}^{p-1}(G(s)A_L)^i(G(s)B_L - B_0),$$

with $G(s) \triangleq \mathrm{diag}\{G_1(s),\dots,G_n(s)\}$. It follows readily that the poles of $H(s)$ are given by the collection of the poles of the individual scalar transfer function $G_i(s)$. Consequently, the system is stable if and only if the conditions of Corollary 1 hold for each PI controller of the network.

5. Conclusion

In this paper we have addressed the issue of feedback stabilization and load disturbance attenuation for hyperbolic density flow systems under PI boundary control. Explicit necessary and sufficient stability conditions in the frequency domain have been provided. It has also been shown how the stability analysis can be extended to acyclic networks of density-flow systems. Finally, let us also point out that the control system (1)–(2) subject to *Proportional–Integral–Derivative* (PID) boundary controls is known to be always unstable (see e.g. Coron & Tamasoiu, in press).

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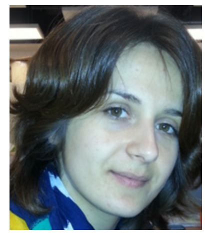

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