MATHEMATICAL CONTROL AND
RELATED FIELDS
Volume **3**, Number **3**, September **2013**

doi:10.3934/mcrf.2013.3.303

pp. **303–322**

PHANTOM TRACKING METHOD, HOMOGENEITY AND RAPID STABILIZATION

Jean-Michel Coron

Institut universitaire de France and Laboratoire Jacques-Louis Lions
Université Pierre et Marie Curie
4, place Jussieu, 75005 Paris, France

Abstract. In this paper we explain on various examples the "phantom tracking" method, a method which can be used to stabilize nonlinear control systems modeled by ordinary differential equations or partial differential equations. We show how it can handle global controllability, homogeneity issues or fast stabilization.

1. Introduction.

It is a great pleasure for me to write this article in this special issue dedicated to Bernard Bonnard. I am very impressed by the number of deep and seminal papers written by Bernard Bonnard. I also want to express here my gratitude for his help when I started to work in control theory. This article is on a method to stabilize some control systems. Let us mention that, even if Bernard Bonnard had always more interest in controllability than in stabilization issues, it clearly follows from his works that these two problems have strong connexions. See, for example, [3] and [4].

The stabilization method is called here the "phantom tracking" method. It has been introduced in [7] in the context of the stabilization of the Euler equations of incompressible fluids. ([7] deals with simply connected domains in $\mathbb{R}^2$; for multiply connected domains in $\mathbb{R}^2$, see [14].) It has also been used in the context of finite dimensional quantum control systems in [1, 15]. Roughly speaking, this method can be described as follows. We consider the control system

$$\dot{x} = f(x, u), \tag{1.1}$$

where the state is $x \in \mathbb{R}^n$ and the control is $u \in \mathbb{R}^m$. We assume that $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$ is an equilibrium of the control system (1.1), i.e., we assume that

$$f(0, 0) = 0. \tag{1.2}$$

We want to construct a (continuous) feedback law $u : \mathbb{R}^n \to \mathbb{R}^m$, vanishing at $0 \in \mathbb{R}^n$ such that 0 is asymptotically stable for the closed loop system $\dot{x} = f(x, u(x))$. Let us assume that there exists a curve of equilibria $\gamma \in [0, \bar{\gamma}] \mapsto (x^\gamma, u^\gamma)$ of the control system $\dot{x} = f(x, u)$ such that $(x^0, u^0) = (0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$. We assume that for every $\gamma \in (0, \bar{\gamma}]$ there exist a feedback law u^γ which asymptotically stabilizes x^γ (with a large enough basin of attraction: in principle, it must contain 0). The idea is then to use for the control system $\dot{x} = f(x, u)$ the feedback law $u^{\tilde{\gamma}}$ where $\tilde{\gamma} : \mathbb{R}^n \to [0, \bar{\gamma}]$

2010 *Mathematics Subject Classification.* Primary: 93D15, 93D30; Secondary: 93C10.
Key words and phrases. Stabilization, feedback, homogeneity, backstepping, phantom tracking.
JMC was supported by the ERC advanced grant 266907 (CPDENL) of the 7th Research Framework Programme (FP7).

is a well chosen function. One steers the control system to the "phantom" x^γ with a γ which is moving to 0. This is summarized on Figure 1. On this figure one sees

- The curve $\gamma \in [0, \bar\gamma] \mapsto x^\gamma$ (dotted curve).
- A trajectory for a fixed γ when one applies the feedback laws u^γ (dashed curve). This trajectory converges to x^γ.
- A trajectory when one uses the feedback laws $u^{\tilde\gamma}$. This trajectory converges to 0.

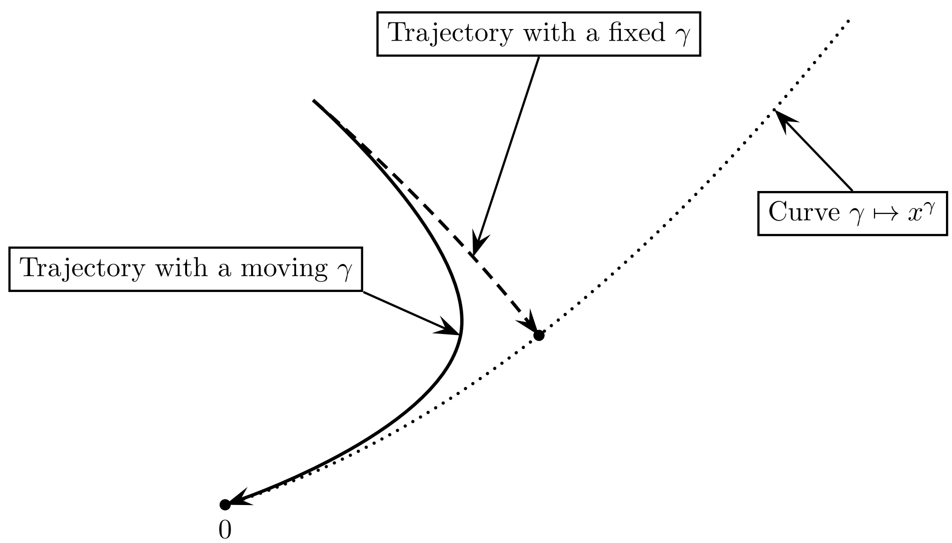

Figure 1. Phantom tracking.

2. **A first example.** Instead of giving a general result, let us explain how to use this method on the following control system:

$$\dot x_1 = x_2,\ \dot x_2 = -x_1 + u,\ \dot x_3 = x_4,\ \dot x_4 = -x_3 + 2x_1x_2, \tag{2.3}$$

where the state is $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ and the control is $u \in \mathbb{R}$. Note that this control system is not small-time locally controllable (see [9, Example 3.38, pages 148-19] but is controllable in large time (see [9, Example 6.4, pages 190-191]). In [9, Section 6.3] we have used this control system to explain our proof, given in [8], of the controllability of a water tank control system. The element $(x^\gamma, u^\gamma) := ((\gamma, 0, 0, 0), \gamma)$ is an equilibrium of the control system (2.3). The linearized control system of (2.3) at this equilibrium is the linear control system

$$\dot x_1 = x_2,\ \dot x_2 = -x_1 + u,\ \dot x_3 = x_4,\ \dot x_4 = -x_3 + 2\gamma x_2, \tag{2.4}$$

where the state is $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ and the control is $u \in \mathbb{R}$. This linear control system satisfies the Kalman rank condition (see, e.g., [9, Theorem 1.16]) if and only if $\gamma \neq 0$. This implies the existence of a feedback law which locally asymptotically stabilizes this equilibrium $((\gamma, 0, 0, 0), \gamma)$ if $\gamma \neq 0$ for the control system (2.4). In fact such a feedback law can be constructed by using the damping method (see, e.g., [9, Section 12.2] or [25, pages 240-242]). It suffices to consider the following control Lyapunov function $V^\gamma : \mathbb{R}^4 \to \mathbb{R}$ defined by

$$V^\gamma(x) := (x_1 - \gamma)^2 + x_2^2 + x_3^2 + x_4^2,\ \forall x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4. \tag{2.5}$$

Note that

$$V^\gamma(x) > V^\gamma(x^\gamma) = 0,\ \forall x \in \mathbb{R}^4 \setminus \{\gamma\}, \tag{2.6}$$

$$\lim_{|x| \to +\infty} V^\gamma(x) = +\infty. \tag{2.7}$$

The time derivative of V^γ along the trajectory of (2.3) is

$$\dot{V}^\gamma = 2x_2(u - \gamma + 2x_1x_4). \tag{2.8}$$

Hence, in order to asymptotically stabilize x^γ for the control system (2.4), it is natural to consider the feedback law $u^\gamma : \mathbb{R}^4 \to \mathbb{R}$ defined by

$$u^\gamma := \gamma - 2x_1x_4 - x_2. \tag{2.9}$$

From (2.8) and (2.9), one gets

$$\dot{V}^\gamma = -2x_2^2. \tag{2.10}$$

Let

$$\gamma \in \mathbb{R} \setminus \{0\}. \tag{2.11}$$

Let us check that

$(\mathcal{P}_\gamma)$ x^γ is globally asymptotically stable for the closed loop system

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + u^\gamma,\ \dot{x}_3 = x_4,\ \dot{x}_4 = -x_3 + 2x_1x_2. \tag{2.12}$$

By (2.6), (2.7), (2.10) and the LaSalle invariance principle (see, e.g., [19, Section 4.2] or [25, Lemma 5.7.8]), in order to prove $(\mathcal{P}_\gamma)$ it suffices to check that if $x = (x_1, x_2, x_3, x_4) \in C^1(\mathbb{R})^4$ is such that

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + \gamma - 2x_1x_4 - x_2,\ \dot{x}_3 = x_4,\ \dot{x}_4 = -x_3 + 2x_1x_2 \text{ in } \mathbb{R}, \tag{2.13}$$

$$x_2 = 0 \text{ in } \mathbb{R}, \tag{2.14}$$

then

$$\exists t_0 \in \mathbb{R} \text{ such that } x(t_0) = x^\gamma. \tag{2.15}$$

Taking the time derivative of the second equality of (2.13) and using the first and third equality of (2.13) as well as (2.14) one gets that

$$x_1x_3 = 0 \text{ in } \mathbb{R}. \tag{2.16}$$

Let us first deal with the case where there exists $t_0 \in \mathbb{R}$ such that

$$x_1(t_0) = 0. \tag{2.17}$$

Using the second equality of (2.13), (2.14) and (2.17), one gets that $\gamma = 0$, in contradiction with (2.11). Hence (2.17) cannot hold, which together with (2.16) implies that

$$x_3 = 0 \text{ in } \mathbb{R}. \tag{2.18}$$

From the three last equalities of (2.13), (2.14) and (2.18) one readily gets that

$$x = x^\gamma \text{ in } \mathbb{R},$$

which concludes the proof of $(\mathcal{P}_\gamma)$. Note that $(\mathcal{P}_0)$ does not hold: for every $r > 0$, $t \in \mathbb{R} \mapsto (x_1(t), x_2(t), x_3(t), x_4(t)) := (0, 0, r\sin t, r\cos t) \in \mathbb{R}^4$ is a solution of (2.12) for $\gamma = 0$.

Let us now follow the phantom tracking strategy. In fact, instead of using $u^{\tilde{\gamma}}$ with a suitable $\tilde{\gamma} : \mathbb{R}^4 \to \mathbb{R}$ it is better to use directly a control Lyapunov of the type $V^{\tilde{\gamma}}$. The best way, at least theoretically, to choose $\tilde{\gamma}$ is to define it implicitly by proceeding in the following way. There exit an open neighborhood Ω of $0 \in \mathbb{R}^4$ and $V \in C^\infty(\Omega; \mathbb{R})$ such that

$$V(0) = 0, \tag{2.19}$$

$$V(x) = (x_1 - V(x))^2 + x_2^2 + x_3^2 + x_4^2,\ \forall x = (x_1, x_2, x_3, x_4) \in \Omega. \tag{2.20}$$

In other words $\tilde{\gamma}$ is defined implicitly by requiring that $\tilde{\gamma}(x) = V^{\tilde{\gamma}(x)}$ and, of course, that $\tilde{\gamma}$ is smooth and satisfies $\tilde{\gamma}(0) = 0$. The existence of V follows from the implicit function theorem applied to the function

$$\begin{array}{rcl} \mathbb{R} \times \mathbb{R}^4 & \to & \mathbb{R} \\ (V, (x_1, x_2, x_3, x_4)) & \mapsto & (x_1 - V)^2 + x_2^2 + x_3^2 + x_4^2 - V, \end{array}$$

at the point $(0, 0) \in \mathbb{R} \times \mathbb{R}^4$. In fact, in this simple case, Ω and V can be given explicitly. For example, the properties required for V hold for

$$\Omega := \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4;\ |x| < 1/4\}, \tag{2.21}$$

$$V(x) := \frac{1 + 2x_1 - \sqrt{(1 + 2x_1)^2 - 4|x|^2}}{2},\ \forall x = (x_1, x_2, x_3, x_4) \in \Omega. \tag{2.22}$$

It is not hard to check that, diminishing Ω if necessary,

$$V(x) > V(0) = 0,\ \forall x \in \Omega \setminus \{0\}. \tag{2.23}$$

Differentiating (2.20) with respect to time, we have, along the trajectories of (2.3),

$$\dot{V} = 2(x_1 - V)(x_2 - \dot{V}) + 2x_2(-x_1 + u) + 2x_3x_4 + 2x_4(-x_3 + 2x_1x_2),$$

i.e.,

$$(1 + 2x_1 - 2V)\dot{V} = 2x_2(u - V + 2x_1x_4). \tag{2.24}$$

We define a feedback law $u : \Omega \to \mathbb{R}$ by

$$u := V - 2x_1x_4 - x_2, \tag{2.25}$$

so that, with (2.24),

$$(1 + 2x_1 - 2V)\dot{V} = -2x_2^2. \tag{2.26}$$

Let $\Omega' \subset \Omega$ be an open neighborhood of 0 such that

$$1 + 2x_1 - 2V > 0 \text{ in } \Omega'. \tag{2.27}$$

From (2.26) and (2.27), one gets

$$\dot{V} \leqslant 0 \text{ in } \Omega', \tag{2.28}$$

$$(\dot{V}(x) = 0 \text{ and } x \in \Omega') \Rightarrow (x_2 = 0). \tag{2.29}$$

From (2.28), by the LaSalle invariance principle, in order to prove that $0 \in \mathbb{R}^4$ is locally asymptotically stable for the closed loop system

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + u(x),\ \dot{x}_3 = x_4,\ \dot{x}_4 = -x_3 + 2x_1x_2, \tag{2.30}$$

with u defined by (2.25), it suffices to check that, if $x = (x_1, x_2, x_3, x_4) : \mathbb{R} \to \Omega'$ is a solution of (2.30) such that

$$\dot{V}(x(t)) = 0, \forall t \in \mathbb{R}, \tag{2.31}$$

then

$$\exists t_0 \in \mathbb{R} \text{ such that } x(t_0) = 0. \tag{2.32}$$

Let $x : \mathbb{R} \to \Omega'$ be a solution of (2.30) such that (2.31) holds. From (2.29) and (2.31), one has

$$x_2 = 0 \text{ in } \mathbb{R}. \tag{2.33}$$

From (2.25), the second equality of (2.30) and (2.33), one has

$$x_1(1 + 2x_4) = V \text{ in } \mathbb{R}. \tag{2.34}$$

From the first equality of (2.30) and (2.33), one gets that

$$\dot{x}_1 = 0 \text{ in } \mathbb{R}. \tag{2.35}$$

Differentiating (2.34) with respect to time, and using (2.31) and (2.35), one gets that

$$x_1 \dot{x}_4 = 0 \text{ in } \mathbb{R}. \tag{2.36}$$

Note that, if there exists $t_0 \in \mathbb{R}$ is such that $x_1(t_0) = 0$, then, by (2.34), $V(x(t_0)) = 0$ which, together with (2.23), implies that $x(t_0) = 0$ and therefore (2.32). Hence we may assume that

$$x_1(t) \neq 0, \forall t \in \mathbb{R}. \tag{2.37}$$

From (2.36) and (2.37), one gets that

$$\dot{x}_4 = 0 \text{ in } \mathbb{R}. \tag{2.38}$$

From the last equality of (2.30), (2.33) and (2.38), one has

$$x_3 = 0 \text{ in } \mathbb{R}. \tag{2.39}$$

From the third equality of (2.30) and (2.39), one has

$$x_4 = 0 \text{ in } \mathbb{R}. \tag{2.40}$$

From (2.34) and (2.40), one has

$$x_1 = V \text{ in } \mathbb{R}. \tag{2.41}$$

Finally, from (2.20), (2.33), (2.39), (2.40) and (2.41), one has $V = 0$ in $\mathbb{R}$ and therefore, by (2.23), $x = 0$ in $\mathbb{R}$.

Here the above method lead to an explicit feedback law since we can find explicitly V satisfying (2.19)–(2.20) (see (2.22)). In many situations V (and therefore u also) will not be known explicitly. However one can circumvent this problem by the following two methods

(i) Use a suitable approximation of V,
(ii) Use a dynamic extension of the control system (2.3) (see [9, Section 11.3]).

(i) Approximation method. Clearly

$$V(x) = |x|^2 + O(|x|^3) \text{ as } |x| \to 0. \tag{2.42}$$

Using (2.42) in the right hand side of (2.20), one gets

$$V(x) = |x|^2 - 2x_1|x|^2 + O(|x|^4) \text{ as } |x| \to 0.$$

Hence it is natural to redefine V by simply taking

$$V(x) := |x|^2(1 - 2x_1),\ \forall x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4. \tag{2.43}$$

Note that

$$V(x) > V(0) = 0,\ \forall x \in B_{1/2}, \tag{2.44}$$

where $B_{1/2}$ denotes the open ball in $\mathbb{R}^4$ of center $0 \in \mathbb{R}^4$ and radius $1/2$. The time derivative of V (defined now by (2.43)) along the trajectories of (2.3) is

$$\dot{V} = 2x_2((u + 2x_1x_4)(1 - 2x_1) - |x|^2). \tag{2.45}$$

This leads to choose the following feedback law $u : B_{1/2} \to \mathbb{R}$ by

$$u(x) := -2x_1x_4 + \frac{|x|^2 - x_2}{1 - 2x_1},\ \forall x = (x_1, x_2, x_3, x_4) \in B_{1/2}. \tag{2.46}$$

From (2.45) and (2.46), one has

$$\dot{V} = -2x_2^2,\ \forall x = (x_1, x_2, x_3, x_4) \in B_{1/2}. \tag{2.47}$$

Using the LaSalle invariance principle as above, in order to prove that $0 \in \mathbb{R}^4$ is locally asymptotically stable for the closed loop system (2.3) and (2.46), it suffices to check that, if $r > 0$ is small enough, then every $x = (x_1, 0, x_3, x_4) : \mathbb{R} \to B_{1/2}$ of class C^1 such that

$$\dot{x}_1 = 0,\ \dot{x}_3 = x_4,\ \dot{x}_4 = -x_3 \text{ in } \mathbb{R}, \tag{2.48}$$

$$x_1 = -2x_1x_4 + \frac{|x|^2}{1 - 2x_1} \text{ in } \mathbb{R}, \tag{2.49}$$

$$|x(t)| < r,\ \forall t \in \mathbb{R}, \tag{2.50}$$

satisfies (2.32). Differentiating (2.49) with respect to time and using (2.48), one gets

$$x_1x_3 = 0 \text{ in } \mathbb{R}. \tag{2.51}$$

Note that, if $t_0 \in \mathbb{R}$ is such that $x_1(t_0) = 0$, then, by (2.49), $|x(t_0)|^2 = 0$ which implies (2.32). Hence we may assume that, for every $t \in \mathbb{R}$, $x_1(t) \neq 0$, which together with (2.51), implies that

$$x_3 = 0 \text{ in } \mathbb{R}. \tag{2.52}$$

From the second equation of (2.48) and (2.52), one has

$$x_4 = 0 \text{ in } \mathbb{R}. \tag{2.53}$$

From (2.49), (2.52) and (2.53), one gets

$$x_1 = \frac{x_1^2}{1 - 2x_1} \text{ in } \mathbb{R}. \tag{2.54}$$

From (2.50) and (2.54) one gets, if $r \in (0, 1/3)$,

$$x_1(t) = 0 \text{ in } \mathbb{R}. \tag{2.55}$$

Property (2.32) follows from (2.52), (2.53) and (2.55). This concludes the proof that $0 \in \mathbb{R}^4$ is locally asymptotically stable for the control system (2.3) with the feedback law u defined by (2.46).

(ii) Dynamic extension. One considers the control system

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + u,\ \dot{x}_3 = x_4,\ \dot{x}_4 = -x_3 + 2x_1x_2,\ \dot{\gamma} = v, \tag{2.56}$$

where the state is $(x_1, x_2, x_3, x_4, \gamma) \in \mathbb{R}^5$ and the control is $(u, v) \in \mathbb{R}^2$. The control system (2.56) is an extension of the original control system (2.3) and a feedback law stabilizing (2.56) allows to stabilize the original control system (2.3). For $z = (x_1, x_2, x_3, x_4, \gamma) \in \mathbb{R}^5$, we define

$$\varphi(z) := (x_1 - \gamma)^2 + x_2^2 + x_3^2 + x_4^2, \tag{2.57}$$

$$W(z) := \varphi(z) + (\gamma - \varphi(z))^2. \tag{2.58}$$

The term $(\gamma - \varphi(z))^2$ is used to penalize $\gamma \neq \varphi(z)$, which is related to (2.20). Indeed, if $\gamma = \varphi(z)$, then, with (2.57), one has $\gamma := (x_1 - \gamma)^2 + x_2^2 + x_3^2 + x_4^2$. This explains the choice of W as a control Lyapunov function for the control system (2.56). The time derivative of φ and W along the trajectories of (2.56) is

$$\dot{\varphi} = 2(x_2(-\gamma + u + 2x_1x_4) + v(-x_1 + \gamma)), \tag{2.59}$$

$$\dot{W} = \dot{\varphi} + 2(\gamma - \varphi)(v - \dot{\varphi}). \tag{2.60}$$

It is natural to choose for u the following fonction:

$$u := \gamma - 2x_1x_4 - x_2. \tag{2.61}$$

From now on we assume that u is defined by (2.61). From (2.59), (2.60) and (2.61), one has

$$\dot{W} = -2x_2^2(1 + 2(\varphi - \gamma)) + 2v(-x_1 + \gamma + (\gamma - \varphi)(1 + 2x_1 - 2\gamma)). \tag{2.62}$$

We define v in a neighborhood of $0 \in \mathbb{R}^5$ by

$$v := -(-x_1 + \gamma + (\gamma - \varphi)(1 + 2x_1 - 2\gamma)). \tag{2.63}$$

From (2.62) and (2.63) one has

$$\dot{W} = -2x_2^2(1 + 2(\varphi - \gamma)) - 2v^2. \tag{2.64}$$

In particular, in a neighborhood of $0 \in \mathbb{R}^5$,

$$\dot{W} \leqslant 0, \tag{2.65}$$

$$(\dot{W} = 0) \Rightarrow (x_2 = 0 \text{ and } x_1 = \gamma + (\gamma - \varphi)(1 + 2x_1 - 2\gamma)). \tag{2.66}$$

The closed loop control system is

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + \gamma - 2x_1x_4 - x_2,\ \dot{x}_3 = x_4,\ \dot{x}_4 = -x_3 + 2x_1x_2, \tag{2.67}$$

$$\dot{\gamma} = -(-x_1 + \gamma + (\gamma - \varphi)(1 + 2x_1 - 2\gamma)). \tag{2.68}$$

Let us check that

$$\left\{ \begin{array}{l} 0 \in \mathbb{R}^5 \text{ is locally asymptotically stable} \\ \text{for the dynamical system (2.67)–(2.68).} \end{array} \right. \tag{2.69}$$

By the LaSalle invariance principle, it suffices to check that if $(x_1, x_3, x_4, \gamma) \in C^1(\mathbb{R}; \mathbb{R}^4)$ satisfies

$$\dot{x}_1 = 0,\ \dot{x}_3 = x_4,\ \dot{x}_4 = -x_3,\ \dot{\gamma} = 0 \text{ in } \mathbb{R}, \tag{2.70}$$

$$x_1 + 2x_1x_4 = \gamma \text{ in } \mathbb{R}, \tag{2.71}$$

$$-x_1 + \gamma + (\gamma - \varphi)(1 + 2x_1 - 2\gamma) = 0 \text{ in } \mathbb{R}, \tag{2.72}$$

then there exists $t_0 \in \mathbb{R}$ such that

$$x_1(t_0) = x_3(t_0) = x_4(t_0) = \gamma(t_0) = 0. \tag{2.73}$$

Differentiating (2.71) with respect to time and using (2.70) one has

$$x_1x_3 = 0 \text{ in } \mathbb{R}. \tag{2.74}$$

If $t_0 \in \mathbb{R}$ is such that $x_1(t_0) = 0$, then, using (2.57), (2.71) and (2.72), one gets (2.73). Hence we may assume that, for every $t \in \mathbb{R}$, $x_1(t) \neq 0$. Then, using (2.74), one has

$$x_3 = 0 \text{ in } \mathbb{R}. \tag{2.75}$$

From the second equality of (2.70) and (2.75), one has

$$x_4 = 0 \text{ in } \mathbb{R}. \tag{2.76}$$

From (2.71) and (2.76), one has

$$x_1 = \gamma \text{ in } \mathbb{R}. \tag{2.77}$$

Using (2.57), (2.72), (2.75), (2.76) and (2.77), one gets

$$x_1 = \gamma = 0 \text{ in } \mathbb{R}, \tag{2.78}$$

which, with (2.75) and (2.76), concludes the proof of (2.69).

Remark 1. In the examples given in this section (as in the ones given in sections 3 and 4, the control Lyapunov functions are not strict and we conclude by means of the LaSalle invariance principle. For control systems modeled by means of partial differential equations, the LaSalle invariance principle is usually difficult to apply since it requires the precompactness of the trajectory (in positive time), a property which is usually difficult to check. However, if we start with strict control Lyapunov functions V^γ, then the constructed V's are also strict Lyapunov functions and the LaSalle invariance principle is no more needed.

3. **Phantom tracking and homogeneity.** In many situations it is important to find stabilizing feedback laws having a prescribed homogeneity. This is for example useful to have a fast stabilization or if the control system $\dot{x} = f(x, u)$ is only an homogeneous approximation of the real control system. For homogeneous feedback laws, we refer to [2, 11, 12, 16, 17, 18, 21, 24] and [9, Section 12.3]. In this section we show how the phantom tracking method can be used to construct stabilizing feedback laws having a prescribed homogeneity.

3.1. **A $3-D$ example.** The control system is

$$\dot{x}_1 = u,\ \dot{x}_3 = x_4,\ \dot{x}_4 = -x_3 + 2x_1 u, \tag{3.1}$$

where the control is $u \in \mathbb{R}$ and the state is $x = (x_1, x_3, x_4) \in \mathbb{R}^3$. We want to stabilize (asymptotically) this control system (at $x = 0$) by means of a continuous feedback $u : \mathbb{R}^3 \to \mathbb{R}$ such that

$$u(\lambda x_1, \lambda^2 x_3, \lambda^2 x_4) = \lambda u(x_1, x_3, x_4),\ \forall \lambda \in [0, +\infty),\ \forall (x_1, x_3, x_4) \in \mathbb{R}^3. \tag{3.2}$$

The equilibria of this control system are $x^\gamma = (\gamma, 0, 0)$, $u^\gamma = 0$, with $\gamma \in \mathbb{R}$. If one wants to stabilize x^γ with $\gamma \neq 0$ thanks to a feedback law having the good homogeneity (which still has to be defined...), one easily sees that one can use the homogeneous control Lyapunov function (and the LaSalle invariance principle)

$$V^\gamma = \frac{1}{4}(x_1 - \gamma)^4 + \frac{1}{2}(x_3^2 + x_4^2). \tag{3.3}$$

Hence, with the implicit formulation of the phantom tracking method, one might try, for the control system (3.1), the control Lyapunov function V_{i} defined implicitly by

$$V_{\mathrm{i}} = \frac{1}{4}\left(x_1 - cV_{\mathrm{i}}^{1/4}\right)^4 + \frac{1}{2}(x_3^2 + x_4^2), \tag{3.4}$$

where $c > 0$ is a constant to be chosen. In order to have an explicit control, using the approximation method described above, we replace (3.4) by

$$V := \frac{1}{4}\left(x_1 - c\theta\right)^4 + \frac{1}{2}(x_3^2 + x_4^2), \tag{3.5}$$

with

$$\theta := \left(\frac{1}{4}x_1^4 + \frac{1}{2}(x_3^2 + x_4^2)\right)^{1/4}. \tag{3.6}$$

One has

$$V(x) \geqslant V(0) = 0,\ \forall x \in \mathbb{R}^3, \tag{3.7}$$

$$V(\lambda x_1, \lambda^2 x_3, \lambda^2 x_4) = \lambda^4 V(x_1, x_3, x_4),\ \forall \lambda \in [0, +\infty),\ \forall (x_1, x_3, x_4) \in \mathbb{R}^3. \tag{3.8}$$

Moreover

$$V(x) > V(0) = 0,\ \forall x \in \mathbb{R}^3 \setminus \{0\} \text{ and } \lim_{|x| \to +\infty} V(x) = +\infty \tag{3.9}$$

if (and only if)

$$c \neq \sqrt{2}. \tag{3.10}$$

One has, along the trajectories of (3.1),

$$\begin{aligned} \dot{V} &= (x_1 - c\theta)^3(u - c\dot{\theta}) + 2x_1 x_4 u \\ &= u\left((x_1 - c\theta)^3 + 2x_1 x_4\right) - c(x_1 - c\theta)^3\dot{\theta}. \end{aligned} \tag{3.11}$$

We have, still along the trajectories of (3.1),

$$\dot{\theta} = \frac{1}{4\theta^3} u\left(x_1^3 + 2x_1 x_4\right). \tag{3.12}$$

From (3.11) and (3.12), one gets

$$\dot{V} = u\left((x_1 - c\theta)^3 + 2x_1 x_4 - c\frac{(x_1 - c\theta)^3}{4\theta^3}\left(x_1^3 + 2x_1 x_4\right)\right). \tag{3.13}$$

Hence we choose

$$u := -\left((x_1 - c\theta)^3 + 2x_1x_4 - c\frac{(x_1 - c\theta)^3}{4\theta^3}\left(x_1^3 + 2x_1x_4\right)\right)^{1/3} \text{ if } x \neq 0, \tag{3.14}$$

$$u(0) = 0. \tag{3.15}$$

Note that this u is continuous and satisfies the homogeneity property (3.2). (If one wants a smoother feedback law outside the origin, it is also possible to use a strict Lyapunov function by changing V^γ and then V accordingly). Let us check that 0 is globally asymptotically stable for the control system (3.1) with this feedback law u. One uses the LaSalle invariance principle (which also works with our continuous u). Since $\dot{V} = -u^4$, we assume that we have $x : \mathbb{R} \to \mathbb{R}^3 \setminus \{0\}$ such that

$$\dot{x}_1 = 0,\ \dot{x}_3 = x_4,\ \dot{x}_4 = -x_3, \tag{3.16}$$

$$(x_1 - c\theta)^3 + 2x_1x_4 - c\frac{(x_1 - c\theta)^3}{4\theta^3}\left(x_1^3 + 2x_1x_4\right) = 0. \tag{3.17}$$

Differentiating (3.17) twice successively with respect to time, and using (3.12) together with (3.16), we get

$$x_1x_3(4\theta^3 - c(x_1 - c\theta)^3) = 0, \tag{3.18}$$

$$x_1x_4(4\theta^3 - c(x_1 - c\theta)^3) = 0. \tag{3.19}$$

We fix $c > 0$ small enough so that (3.10) holds and

$$4\theta^3 > c(x_1 - c\theta)^3,\ \forall x \in \mathbb{R}^3 \setminus \{0\}. \tag{3.20}$$

Note that, by (3.16), x_1 and $x_3^2 + x_4^2$ are constant functions. There are two cases:
Case 1. The constant function x_1 is vanishing identically on $\mathbb{R}$. Going back to (3.17), we get

$$0 = \theta = \left(\frac{1}{2}(x_3^2 + x_4^2)\right)^{1/4}. \tag{3.21}$$

Hence $x_3 = x_4 = 0$, leading to a contradiction with our assumption $x \neq 0$.
Case 2. The constant function $x_1 \neq 0$ does not vanish. Then, using (3.18), (3.19) and (3.20), one has

$$x_3 = x_4 = 0. \tag{3.22}$$

From (3.6), (3.17) and (3.22), one has

$$\left(\sqrt{2}x_1 - c|x_1|\right)^3 (\sqrt{2}|x_1|^3 - cx_1^3) = 0.$$

and therefore, with (3.10) and (3.22),

$$x = 0,$$

which, again, leads to a contradiction.

3.2. **A $4-D$ control system.** We go back to the control system studied in section 2. Hence, the control system is

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + u,\ \dot{x}_3 = x_4,\ \dot{x}_4 = -x_3 + 2x_1x_2, \tag{3.23}$$

where the control is $u \in \mathbb{R}$ and the state is $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. We want to stabilize (asymptotically) this control system (at $x = 0$) by means of a continuous feedback $u : \mathbb{R}^4 \to \mathbb{R}$ such that

$$u(\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda^2 x_4) = \lambda u(x_1, x_2, x_3, x_4),\ \forall \lambda \in [0, +\infty),\ \forall (x_1, x_2, x_3, x_4) \in \mathbb{R}^4. \quad (3.24)$$

In order to construct a continuous stabilizing feedback law $u : \mathbb{R}^4 \to \mathbb{R}$ satisfying (3.24) one could use section 3.1, a backstepping approach together with a desingularization technique as we will do in section 5 for a different control system. We use a more direct approach. Let us first try the control Lyapunov function V defined implicitly by

$$V = \frac{1}{4}\left((x_1 - cV^{1/4})^2 + x_2^2\right)^2 + \frac{1}{2}(x_3^2 + x_4^2), \quad (3.25)$$

where $c > 0$ is a small constant to be chosen. One has, if $c > 0$ is small enough, V is well defined and

$$V(x) > V(0) = 0,\ \forall x \in \mathbb{R}^4 \setminus \{0\} \text{ and } \lim_{|x| \to +\infty} V(x) = +\infty. \quad (3.26)$$

Along the trajectories of (3.23), one has

$$\dot{V} = \left((x_1 - cV^{1/4})^2 + x_2^2\right)\left(\left(x_1 - cV^{1/4}\right)\left(x_2 - \frac{c}{4V^{3/4}}\dot{V}\right) + x_2\left(-x_1 + u\right)\right) + 2x_1x_2x_4, \quad (3.27)$$

which gives

$$\left(1 + \frac{c}{4V^{3/4}}\left((x_1 - cV^{1/4})^2 + x_2^2\right)\left(x_1 - cV^{1/4}\right)\right)\dot{V} = - x_2\left(\left((x_1 - cV^{1/4})^2 + x_2^2\right)\left(cV^{1/4} - u\right) - 2x_1x_4\right). \quad (3.28)$$

This leads to define u by

$$u := cV^{1/4} - \frac{2x_1x_4 + \mu x_2^2}{(x_1 - cV^{1/4})^2 + x_2^3}, \quad (3.29)$$

where $\mu > 0$ is a constant. Indeed, with this u,

$$\left(1 + \frac{c}{4V^{3/4}}\left((x_1 - cV^{1/4})^2 + x_2^2\right)^2\left(x_1 - cV^{1/4}\right)\right)\dot{V} = -\mu x_2^4. \quad (3.30)$$

Unfortunately the u defined by (3.29) has singularities outside 0 (it is locally unbounded). So this u does not work.

We try another natural control Lyapunov function. It is defined implicitly by

$$V := \frac{1}{2}\left((x_1 - cV^{1/2})^2 + x_2^2\right) + m\left((x_1^2 + x_2^2)^2 + x_3^2 + x_4^2\right)^{1/2}. \quad (3.31)$$

with $m \in (0, +\infty)$. (One could also use the approximation technique explained in section 1 in order to have an explicit V.) Again, one can check that, if $c > 0$ is small enough, V is well defined and satisfies (3.26). With this V, along the trajectories of (3.23), one has, for $x \neq 0$,

$$\dot{V} = (x_1 - cV^{1/2})(x_2 - c\frac{\dot{V}}{2V^{1/2}}) - x_1x_2 + x_2u + \frac{2m}{((x_1^2+x_2^2)^2+x_3^2+x_4^2)^{1/2}}\left((x_1^2+x_2^2)x_2u + x_1x_2x_4\right). \quad (3.32)$$

Hence, for $x \neq 0$,

$$\left(1 + (x_1 - cV^{1/2})\frac{c}{2V^{1/2}})\right)\dot{V} = x_2(Au+B), \quad (3.33)$$

with

$$A := 1 + \frac{2m(x_1^2+x_2^2)}{((x_1^2+x_2^2)^2+x_3^2+x_4^2)^{1/2}}, \quad (3.34)$$

$$B := -cV^{1/2} + \frac{2mx_1x_4}{((x_1^2+x_2^2)^2+x_3^2+x_4^2)^{1/2}}. \quad (3.35)$$

Note that now A does not vanish. From (3.33), it is natural to define $u : \mathbb{R}^4 \to \mathbb{R}$ by

$$u(x) = -\frac{B+x_2}{A}, \; \forall x \in \mathbb{R}^4 \setminus \{0\} \text{ and } u(0) = 0. \quad (3.36)$$

One readily sees that u is continuous and satisfies the homogeneity condition (3.24). Moreover, from (3.33) and (3.36), one has

$$\dot{V} = -x_2^2. \quad (3.37)$$

As above, using the LaSalle invariance principle, one then checks that $0 \in \mathbb{R}^4$ is globally asymptotically stable for the closed loop system (3.23) and (3.36).

Remark 2. The homogeneity gives not only global stabilization but also an estimate on the decay rate. Indeed, it follows from the homogeneity and from a result due to Lionel Rosier [24] that there exists $C > 0$ and $\nu > 0$ such that, for every solution of the closed loop system (3.23) and (3.36), one has

$$(x_1^2(t)+x_2^2(t))^2 + x_3^2(t) + x_4^2(t) \leqslant Ce^{-\nu t}\left((x_1^2(0)+x_2^2(0))^2 + x_3^2(0) + x_4^2(0)\right), \; \forall t \in [0,+\infty). \quad (3.38)$$

4. **Phantom tracking and global stabilization.** In section 2, the phantom tracking method presented there leads only to local stabilization. In order to get global stabilization homogeneity is an interesting solution in some cases. In this section we show how the phantom tracking method can also be used in order get global stabilization without any homogeneity requirement. This was already shown in [1] using an implicit control Lyapunov. Here we show how one can also use an explicit Lyapunov approach, leading to explicit feedback laws.

We go back to the control system (2.3). Let $\mu \in C^1([0,+\infty);[0,+\infty))$ be such that

$$\mu(0) = 0, \quad (4.1)$$

$$|\mu'(s)| < 1/(4\sqrt{s}), \; \forall s \in (0,+\infty), \quad (4.2)$$

$$\mu(s) > 0, \forall s \in (0,+\infty). \quad (4.3)$$

We define the following control Lyapunov function

$$V(x) := |x|^2 - \mu(|x|^2)x_1, \ \forall x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4. \tag{4.4}$$

(Roughly speaking, the phantom is at $(\mu(|x|^2)x_1/2, 0, 0, 0)$ and we omit $\mu(|x|^2)^2 x_1^2/4$ μ being small.) Note that (4.1) and (4.2) imply that

$$\mu(s) \in [0, \sqrt{s}/2], \forall s \in [0, +\infty). \tag{4.5}$$

From (4.1), (4.4) and (4.5), one has

$$V(x) > V(0) = 0, \ \forall x \in \mathbb{R}^4 \setminus \{0\} \text{ and } \lim_{|x| \to +\infty} V(x) = +\infty. \tag{4.6}$$

One computes the time derivative of V along the trajectories of (2.3). One gets

$$\begin{aligned} \dot{V} &= 2ux_2 + 4x_1x_2x_4 - 2\mu'(|x|^2)(ux_2 + 2x_1x_2x_4)x_1 - \mu(|x|^2)x_2 \\ &= 2x_2\,(uA + B)\,, \end{aligned} \tag{4.7}$$

with

$$A := 1 - \mu'(|x|^2)x_1, \tag{4.8}$$

$$B := 2x_1x_4(1 - \mu'(|x|^2)x_1) - \frac{1}{2}\mu(|x|^2). \tag{4.9}$$

Note that, by (4.2) and (4.8),

$$A = 1 - \mu'(|x|^2)x_1 > 0, \forall x \in \mathbb{R}^4. \tag{4.10}$$

This allows to define $u \in C^0(\mathbb{R}^4)$ by

$$u(x) := -\frac{x_2 + B}{A}. \tag{4.11}$$

With this feedback law, one gets, with (4.7),

$$\dot{V} = -2x_2^2. \tag{4.12}$$

One uses again the LaSalle invariance principle. In order to prove that $0 \in \mathbb{R}^4$ is globally asymptotically stable for the closed loop system (2.3) and (4.11) it suffices to check that, if $x = (x_1, x_2, x_3, x_4) : \mathbb{R} \to \mathbb{R}^4$ is such that

$$\dot{x}_1 = 0, \ x_2 = 0, \ \dot{x}_3 = x_4, \ \dot{x}_4 = -x_3, \tag{4.13}$$

$$(1 - \mu'(|x|^2)x_1)x_1 + 2x_1x_4(1 - \mu'(|x|^2)x_1) - \frac{1}{2}\mu(|x|^2) = 0, \tag{4.14}$$

then

$$\exists t_0 \in \mathbb{R} \text{ such that } x(t_0) = 0. \tag{4.15}$$

Differentiating (4.14) twice successively with respect to time and using (4.10) and (4.13), one gets

$$x_1x_3 = 0, \ x_1x_4 = 0. \tag{4.16}$$

Again, we distinguish between two cases.

Case 1. The constant function x_1 is vanishing identically on $\mathbb{R}$. Going back to (4.14) and using (4.3) we get that

$$x(t) = 0, \ \forall t \in \mathbb{R}. \tag{4.17}$$

Hence (4.15) holds.

Case 2. The constant function $x_1 \neq 0$ does not vanish. Then, using (4.2) and (4.16), one has

$$x_3 = x_4 = 0. \tag{4.18}$$

From (4.14) and (4.18), one has

$$(1 - \mu'(x_1^2)x_1)x_1 - \frac{1}{2}\mu(x_1^2) = 0,$$

which, together with (4.2) and (4.5), implies that

$$x_1 = 0.$$

This concludes the proof that $0 \in \mathbb{R}^4$ is globally asymptotically stable for the closed loop system (2.3) and (4.11).

5. **A quadratic system.** Our control system is now

$$\dot{x} = u,\ \dot{y} = x(Ay + Bv), \tag{5.1}$$

where the state is $(x, y) \in \mathbb{R} \times \mathbb{R}^n$ and the control is $(u, v) \in \mathbb{R} \times \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. This control system can serve as a (very) simplified toy model of the Euler equation of incompressible fluids as studied in [7]. Roughly speaking the x part represents fluids with no vorticity (potential flows) and the y part represents the vorticity. The vorticity satisfies a transport equation, which is simply replaced by the second equation of (5.1). We assume that (A, B) satisfies the Kalman rank condition:

$$\text{Span}\ \{A^i Bv;\ i \in \{0, 1, \dots, n-1\},\ v \in \mathbb{R}^m\} = \mathbb{R}^n. \tag{5.2}$$

In other words, we assume that the control system

$$\dot{y} = Ay + Bv, \tag{5.3}$$

where the state is $y \in \mathbb{R}^n$ and the control is $v \in \mathbb{R}$, is controllable. Our next theorem is the following one.

Theorem 5.1. *The control system (5.1) is globally controllable in small time, i.e., for every $T > 0$, for every $(x^0, y^0) \in \mathbb{R} \times \mathbb{R}^n$ and for every $(x^1, y^1) \in \mathbb{R} \times \mathbb{R}^n$, there exist $u \in L^\infty(0, T)$ and $v \in L^\infty(0, T)^m$ such that the solution $(x, y) : [0, T] \to \mathbb{R} \times \mathbb{R}^n$ of the Cauchy problem*

$$\dot{x} = u,\ \dot{y} = x(Ay + Bv),\ x(0) = x^0,\ y(0) = y^0, \tag{5.4}$$

satisfies

$$x(T) = x^1,\ y(T) = y^1. \tag{5.5}$$

Proof of Theorem 5.1. On this toy model of the Euler control system of incompressible fluids, we apply the strategy used in [6, 13] to prove the controllability of this Euler control system. We first prove the local controllability in time $T/2$. For this purpose, as for the controllability of the Euler control system of incompressible fluids [6, 13], we use the return method (see, e.g., [9, Chapter 6] or [10]). Let $\bar{u} \in C^\infty([0, T/2])$ be such that

$$\int_0^{T/2} \bar{u}(t)dt = 0, \tag{5.6}$$

$$\bar{u} \text{ do not vanish identically.} \tag{5.7}$$

Let $\bar{v} \in L^\infty(0, T/2)$ be defined by

$$\bar{v}(t) = 0, \forall t \in (0, T/2) \tag{5.8}$$

Let $(\bar{x}, \bar{y}) : [0, T/2] \to \mathbb{R} \times \mathbb{R}^n$ be the solution to the Cauchy problem

$$\dot{\bar{x}} = \bar{u},\ \dot{\bar{y}} = \bar{x}(A\bar{y} + B\bar{v}),\ \bar{x}(0) = 0,\ \bar{y}(0) = 0. \tag{5.9}$$

From (5.8) and (5.9), one has

$$\bar{y}(t) = 0,\ \forall t \in [0, T/2]. \tag{5.10}$$

From (5.6) and (5.9),

$$\bar{x}(T/2) = 0. \tag{5.11}$$

From (5.7) and (5.9),

$$\bar{x} \text{ do not vanish identically.} \tag{5.12}$$

The linearized control system around the trajectory $(\bar{x}, \bar{y}, \bar{u}, \bar{v})$ of the control system (5.1) is

$$\dot{x} = u,\ \dot{y} = \bar{x}(Ay + Bv), \tag{5.13}$$

where the state is $(x, y) \in \mathbb{R} \times \mathbb{R}^n$ and the control is $(u, v) \in \mathbb{R} \times \mathbb{R}^m$. From (5.2) and (5.12) and a classical result on the controllability of time-varying linear systems (see, e.g., [9, Theorem 1.18, page 11]), the linear control (5.13) is controllable on $[0, T/2]$. Hence (see e.g. [9, Theorem 3.6, page 127]) the nonlinear control system (5.1) is locally controllable along the trajectory $(\bar{x}, \bar{y}, \bar{u}, \bar{v})$. Then, using (5.10) and (5.11), we get the existence of $\varepsilon > 0$ such that, for every $(x^0, y^0) \in \mathbb{R} \times \mathbb{R}^n$ such that $|x^0| + |y^0| < \varepsilon$, there exist $u \in L^\infty(0, T/2)$ and $v \in L^\infty(0, T/2)^m$ such that the solution to the Cauchy problem (5.4) satisfies

$$x(T/2) = 0,\ y(T/2) = 0. \tag{5.14}$$

We now notice that if (x, y, u, v) is a trajectory of the control system (5.1) on $[0, T/2]$, then, for every $\delta > 0$, $(x_\delta, y_\delta, u_\delta, v_\delta) : [0, \delta T/2] \to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m$ defined by

$$x_\delta(t) = \frac{1}{\delta} x\left(\frac{t}{\delta}\right),\ y_\delta(t) = \frac{1}{\delta}\left(\frac{t}{\delta}\right),\ \forall t \in [0, \delta T/2], \tag{5.15}$$

$$u_\delta(t) = \frac{1}{\delta^2} u\left(\frac{t}{\delta}\right),\ v_\delta(t) = \frac{1}{\delta} v\left(\frac{t}{\delta}\right),\ \forall t \in [0, \delta T/2], \tag{5.16}$$

is also a trajectory of the control system (5.1). Note also that, if (5.14) holds, then

$$(x_\delta(\delta T/2), y_\delta(\delta T/2)) = (0, 0). \tag{5.17}$$

Moreover, if we impose $x_\delta(0) = a$ and $y_\delta(0) = b$, where (a, b) are given in $\mathbb{R} \times \mathbb{R}^n$, then

$$x(0) = \delta x_\delta(0) \to 0 \text{ and } y(0) = \delta y_\delta(0) \text{ as } \delta \to 0^+. \tag{5.18}$$

If $\delta \in (0, 1)$ and (5.17) holds, we extend $(x_\delta, y_\delta, u_\delta, v_\delta)$ by

$$(x_\delta, y_\delta, u_\delta, v_\delta) = (0, 0, 0, 0) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m \text{ in } [\delta T/2, T/2]. \tag{5.19}$$

Then $(x_\delta, y_\delta, u_\delta, v_\delta)$ is a trajectory of (5.1) on [0,T/2] such that

$$(x_\delta(T/2), y_\delta(T/2)) = (0, 0). \tag{5.20}$$

Hence, for every $x^0 \in \mathbb{R}^n$ and for every $y^0 \in \mathbb{R}^n$, there exits $(u, v) \in L^\infty(0, T/2) \times L^\infty(0, T/2)^m$ such that the solution to the Cauchy problem (5.4) satisfies (5.14).

Reversing time and doing a translation, one also get that for every $x^1 \in \mathbb{R}^n$ and for every $y^1 \in \mathbb{R}^n$, there exits $(u,v) \in L^\infty(T/2,T) \times L^\infty(T/2,T)^m$ such that the solution to the Cauchy problem

$$\dot{x} = u,\ \dot{y} = x(Ay + Bv),\ x(T) = x^1,\ y(T) = y^1, \tag{5.21}$$

satisfies (5.14). Gluing together the trajectories defined on $[0, T/2]$ and on $[T/2, T]$, one concludes the proof of Theorem 5.1.

Remark 3. As pointed to us by Emmanuel Trélat, the above proof can be simplified by using that x is of dimension 1. It allows to perform the change of time $\mathrm{d}s/\mathrm{d}t = x(t)$ when $x(t) > 0$ and rewrite $\dot{y} = x(Ay + Bv)$ as $\mathrm{d}y/\mathrm{d}s = Ay + Bv$ in the region where $x > 0$. Moreover one can always move from x_0 to x_1 and simultaneously impose to go through the region $x > 0$ during part of this trajectory. However this change of time does not seem to be adaptable to the Euler control system.

We now consider the problem of fast stabilization of the control system (5.1) for large states. We have the following theorem.

Theorem 5.2. *Let $\varepsilon > 0$. Then there exist a continuous map $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and a continuous map $v : \mathbb{R}^n \to \mathbb{R}^m$ such that*

(i) the origin $(0,0) \in \mathbb{R} \times \mathbb{R}^n$ is globally asymptotically stable for the closed loop system

$$\dot{x} = u(x,y),\ \dot{y} = x(Ay + Bv(y)), \tag{5.22}$$

(ii) the continuous maps u and v satisfy the following homogeneity properties:

$$u(sx, sy) = s^2 u(x,y),\ v(sy) = sv(y),\ \forall s \in [0, +\infty),\ \forall x \in \mathbb{R},\ \forall y \in \mathbb{R}^n, \tag{5.23}$$

(iii) For every solution $(x,y) : [0, +\infty) \to \mathbb{R} \times \mathbb{R}^n$ of the closed loop system (5.22),

$$|x(t)| + |y(t)| \leqslant \frac{\varepsilon}{t},\ \forall t \in [0, +\infty). \tag{5.24}$$

Proof of Theorem 5.2. For our control system (5.1), the phantom can be taken as

$$(x^\gamma, y^\gamma, u^\gamma, v^\gamma) := (\gamma, 0, 0, 0). \tag{5.25}$$

The linearized control system around the equilibrium $(x^\gamma, y^\gamma, u^\gamma, v^\gamma)$ is

$$\dot{x} = u,\ \dot{y} = \gamma(Ay + Bv), \tag{5.26}$$

where the state is $(x,y) \in \mathbb{R} \times \mathbb{R}^n$ and the control is $(u,v) \in \mathbb{R} \times \mathbb{R}^m$. From (5.2) it follows that the linear control system (5.26) is controllable (and therefore stabilizable) if (and only if) $\gamma \neq 0$. Since the dynamic on the set $\{(x^\gamma, y^\gamma); \gamma \in \mathbb{R}\}$ can be arbitrary thanks to the control $u \in \mathbb{R}$ and $v = 0$, we may first consider the control system

$$\dot{y} = \gamma(Ay + Bv), \tag{5.27}$$

where the state is $y \in \mathbb{R}^n$ and the control is $(\gamma, v) \in \mathbb{R} \times \mathbb{R}^m$ and try to stabilize $0 \in \mathbb{R}^n$ for this control system, with feedback laws $\gamma : \mathbb{R}^n \to \mathbb{R}$ and $v : \mathbb{R}^n \to \mathbb{R}^m$ having the following homogeneity

$$\gamma(sy) = s\gamma(y),\ v(sy) = sv(y),\ \forall s \in [0, +\infty),\ \forall y \in \mathbb{R}^n. \tag{5.28}$$

Let $\lambda > 0$. By (5.2) and the pole shifting theorem (see, e.g., [9, Theorem 10.1, page 275]), there exist $K \in \mathbb{R}^{n\times m}$ and $C > 0$ (both depending on λ) such that

$$|e^{t(A+BK)}y| \leqslant Ce^{-(1+\lambda)t}|y|,\ \forall y \in \mathbb{R}^n,\ \forall t \in [0, +\infty). \tag{5.29}$$

Let $\rho > 0$. Using (5.29), we can define $Q \in \mathbb{R}^{n\times n}$ by

$$Q = \rho \int_0^{+\infty} e^{2\lambda s} e^{s(A^{\mathrm{tr}}+K^{\mathrm{tr}}B^{\mathrm{tr}})} e^{s(A+BK)} ds. \tag{5.30}$$

This matrix Q is symmetric positive definite. Taking $\rho > 0$ large enough, we may assume that

$$|y|^2 \leqslant y^{\mathrm{tr}}Qy,\ \forall y \in \mathbb{R}^n. \tag{5.31}$$

From (5.30) and integration by parts, one readily gets that

$$(A^{\mathrm{tr}} + K^{\mathrm{tr}}B^{\mathrm{tr}})Q + Q(A + BK) = -2\lambda Q - \rho \mathrm{Id}, \tag{5.32}$$

where Id denotes the identity matrix of $\mathbb{R}^{n\times n}$. We now define our feedback laws for the control system (5.27):

$$\gamma(y) := \left(y^{\mathrm{tr}}Qy\right)^{1/2},\ v(y) = Ky,\ \forall y \in \mathbb{R}^n. \tag{5.33}$$

Note that they satisfy (5.28). Let $W : \mathbb{R}^n \to [0, +\infty)$ be defined by

$$W(y) := y^{\mathrm{tr}}Qy,\ \forall y \in \mathbb{R}^n. \tag{5.34}$$

For the control system (5.27) with the feedback laws (5.33), one has

$$\dot{W} = \left(y^{\mathrm{tr}}Qy\right)^{1/2} y^{\mathrm{tr}} \left((A^{\mathrm{tr}} + K^{\mathrm{tr}}B^{\mathrm{tr}})Q + Q(A + BK)\right) y, \tag{5.35}$$

which, together with (5.30) and (5.32), gives

$$\dot{W} \leqslant -2\lambda W^{3/2}. \tag{5.36}$$

Integrating (5.36) along a solution $y : [0, +\infty) \to \mathbb{R}^n$ of the closed loop system (5.27) and (5.33), one gets

$$\left(y(t)^{\mathrm{tr}}Qy(t)\right)^{1/2} \leqslant \frac{\left(y(0)^{\mathrm{tr}}Qy(0)\right)^{1/2}}{1 + \lambda t\left(y(0)^{\mathrm{tr}}Qy(0)\right)^{1/2}},\ \forall t \in [0, +\infty). \tag{5.37}$$

Note that, from (5.33) and (5.37), one has

$$|\gamma(y(t))| + |y(t)| \leqslant \frac{2}{\lambda t}, \forall t \in [0, +\infty), \tag{5.38}$$

which, if we take $\lambda = 2/\varepsilon$, is strongly related to (5.24).

Let us now turn to the original control system, namely (5.1). We apply the backstepping method, a method introduced independently by Christopher Byrnes and Alberto Isidori in [5], Daniel Koditschek in [20] and John Tsinias in [26]; see, e.g. [9, Section 12.5]. To overcome the fact that γ defined in (5.33) is not of class C^1, we use the desingularization technique introduced in [23]; see also [9, Section 12.5.1] (one could alternatively use [22] by Pascal Morin and Claude Samson). We note that

$$\left(x = W(y)^{1/2}\right) \Leftrightarrow \left(x^3 - W(y)^{3/2} = 0\right), \tag{5.39}$$

$$\int_{W(y)^{1/2}}^{x} s^3 - W(y)^{3/2} ds = \frac{x^4}{4} - W(y)^{3/2}x + \frac{3}{4}W(y)^2. \tag{5.40}$$

Let $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be defined

$$V(x,y) := \frac{x^4}{4} - W(y)^{3/2}x + W(y)^2, \ \forall x \in \mathbb{R}, \ \forall y \in \mathbb{R}^n. \tag{5.41}$$

Noticing that

$$\frac{x^4}{4} - W(y)^{3/2}x + \frac{3}{4}W(y)^2 = \frac{1}{4}\left(x - W(y)^{1/2}\right)\left(x^2 + 2W(y)^{1/2}x + 3W(y)\right), \tag{5.42}$$

one sees that

$$V(x,y) > V(0,0) = 0, \ \forall (x,y) \in (\mathbb{R} \times \mathbb{R}^n) \setminus \{(0,0)\}, \tag{5.43}$$

$$\lim_{|x|+|y| \to +\infty} V(x,y) = +\infty. \tag{5.44}$$

Let us now define $\Phi : \mathbb{R}^n \to \mathbb{R}$ by

$$\Phi(y) := y^{\mathrm{tr}}(A^{\mathrm{tr}} + K^{\mathrm{tr}}B^{\mathrm{tr}})Qy + y^{\mathrm{tr}}Q(A+BK)y, \forall y \in \mathbb{R}^n. \tag{5.45}$$

For the feedback law v, we define it by

$$v(y) := Ky, \ \forall y \in \mathbb{R}^n. \tag{5.46}$$

Let us compute the time derivative $\dot{V}$ of V along the trajectories of (5.1). One gets

$$\begin{aligned} \dot{V} &= x^3u - uW^{3/2} + \frac{3}{2}x^2W^{1/2}\Phi - 2xW\Phi \\ &= u\left(x^3 - W^{3/2}\right) + \frac{3}{2}W^{3/2}\Phi + \frac{3}{2}\left(x^2 - W\right)W^{1/2}\Phi \\ &\quad -2W^{3/2}\Phi - 2\left(x - W^{1/2}\right)W\Phi \\ &= -\frac{1}{2}W^{3/2}\Phi + \left(x - W^{1/2}\right)(uA + B) \end{aligned} \tag{5.47}$$

with

$$A := x^2 + xW^{1/2} + W, \ B := \frac{3}{2}\left(x + W^{1/2}\right)W^{1/2}\Phi - 2W\Phi. \tag{5.48}$$

We define the feedback law u in the following way

$$u := -\frac{B + \lambda\left|x - W^{1/2}\right|^3\left(x - W^{1/2}\right)}{A} \text{ in } \mathbb{R} \times \mathbb{R}^n \setminus \{0,0\}, \tag{5.49}$$

$$u(0,0) = 0. \tag{5.50}$$

Then u is continuous and satisfies the first equality of (5.23). Moreover, from (5.47) and (5.49), one has

$$\dot{V} = -\lambda\left|x - W(y)^{1/2}\right|^5 - \frac{1}{2}W(y)^{3/2}\Phi(y). \tag{5.51}$$

From (5.32), (5.45) and (5.51), one gets

$$\dot{V} \leqslant -\lambda\left|x - W(y)^{1/2}\right|^5 - \lambda W(y)^{5/2}. \tag{5.52}$$

Let $r_1 > 0$ (independent of λ) such that

$$|a-b|^5 + b^5 \geqslant r_1\left(\frac{a^4}{4} - ab^3 + b^4\right)^{5/4}, \ \forall (a,b) \in \mathbb{R} \times [0, +\infty). \tag{5.53}$$

From (5.41), (5.52) and (5.53), one gets that

$$\dot{V} \leqslant -\lambda r_1 V^{5/4}. \tag{5.54}$$

From (5.43), (5.44) and (5.54), one gets (i) of Theorem 5.2. From (5.43) and (5.54), one gets that, if $(x, y) : [0, +\infty) \to \mathbb{R} \times \mathbb{R}^n$ is a solution of (5.1) with u and v given by (5.49)-(5.50) and (5.46) respectively,

$$V^{1/4}(x(t), y(t)) \leqslant \frac{4V^{1/4}(x(0), y(0))^{1/4}}{4 + \lambda rt V^{1/4}(x(0), y(0))^{1/4}}, \ \forall t \in [0, +\infty), \tag{5.55}$$

which implies that

$$V^{1/4}(x(t), y(t)) \leqslant \frac{4}{\lambda r_1 t}, \ \forall t \in [0, +\infty), \tag{5.56}$$

Let $r_2 > 0$ (independent of λ) such that (see (5.42))

$$\frac{a^4}{4} - b^3 a + a^4 \geqslant r_2 \left(|a| + b\right)^4, \ \forall (a, b) \in \mathbb{R} \times [0, +\infty). \tag{5.57}$$

From (5.31), (5.34), (5.41), (5.56) and (5.57), one has

$$|x(t)| + |y(t)| \leqslant \frac{4}{\lambda r_1 r_2^{1/4} t}, \ \forall t \in [0, +\infty). \tag{5.58}$$

Since $\lambda > 0$ can be taken arbitrary large and $r_1 > 0$ as well as $r_2 > 0$ do not depend on $\lambda > 0$, this concludes the proof of Theorem 5.2.

Acknowledgments. We thank Malik Drici, Tommaso Mingazzini and Emmanuel Trélat for their useful comments.

REFERENCES

[1] Karine Beauchard, Jean Michel Coron, Mazyar Mirrahimi and Pierre Rouchon, *Implicit lyapunov control of finite dimensional schrödinger equations,* Systems Control Lett., **56** (2007), 388–395.

[2] Sanjay P. Bhat and Dennis S. Bernstein, *Geometric homogeneity with applications to finite-time stability,* Math. Control Signals Systems, **17** (2005), 101–127.

[3] Bernard Bonnard, *Contrôlabilité de systemes mécaniques sur les groupes de lie*, (French) [Controllability of mechanical systems on Lie groups] SIAM J. Control Optim., **22** (1984), 711–722.

[4] Bernard Bonnard, Ludovic Faubourg and Emmanuel Trélat, "Mécanique CÉleste Et ContrÔle Des VÉhicules Spatiaux," (French) [Celestial mechanics and the control of space vehicles] Mathmatiques & Applications (Berlin) [Mathematics & Applications], Springer-Verlag, Berlin, 2006.

[5] Christopher I. Byrnes and Alberto Isidori, *New results and examples in nonlinear feedback stabilization*, Systems Control Lett., **12** (1989), 437–442.

[6] Jean-Michel Coron, *On the controllability of* 2*-D incompressible perfect fluids,* J. Math. Pures Appl., **75** (1996), 155–188.

[7] Jean-Michel Coron, *On the null asymptotic stabilization of the two-dimensional incompressible euler equations in a simply connected domain*, SIAM J. Control Optim., **37** (1999), 1874–1896.

[8] Jean-Michel Coron, *Local controllability of a 1-D tank containing a fluid modeled by the shallow water equations*, A tribute to J. L. Lions. ESAIM Control Optim. Calc. Var., **8** (2002), 513–554.

[9] Jean-Michel Coron, "Control and Nonlinearity," Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2007.

[10] Jean-Michel Coron, *On the controllability of nonlinear partial differential equations,* In "Proceedings of the International Congress of Mathematicians. Volume I," pages 238–264, Hindustan Book Agency, New Delhi, 2010.

[11] Wijesuriya P. Dayawansa, Clyde F. Martin and Sandra L. Samelson Samelson, *Asymptotic stabilization of a generic class of three-dimensional homogeneous quadratic systems*, Systems Control Lett., **24** (1995), 115–123.

[12] (1765428) Ludovic Faubourg and Jean-Baptiste Pomet, *Control lyapunov functions for homogeneous "Jurdjevic-Quinn" systems*, ESAIM Control Optim. Calc. Var., **5** (2000), 293–311(electronic).

[13] Olivier Glass, *Exact boundary controllability of 3-D euler equation,* ESAIM Control Optim. Calc. Var., **5** (2000), 1–44 (electronic).

[14] Olivier Glass, *Asymptotic stabilizability by stationary feedback of the two-dimensional euler equation: The multiconnected case*, SIAM J. Control Optim., **44** (2005), 1105–1147 (electronic).

[15] Andreea Grigoriu, *Implicit lyapunov control for schrödinger equations with dipole and polarizability term*, Proc. of the 50th IEEE Conf. on Decision and Control, USA, December 2011.

[16] Henry Hermes, *Homogeneous coordinates and continuous asymptotically stabilizing feedback controls*, In "Differential Equations" (Colorado Springs, CO, 1989), Lecture Notes in Pure and Appl. Math., pages 249–260. Dekker, New York, 1991.

[17] Henry Hermes, *Smooth homogeneous asymptotically stabilizing feedback controls*, ESAIM Control Optim. Calc. Var., **2** (1997), 13–32 (electronic).

[18] Hamadi Jerbi, *On the stabilizability of homogeneous systems of odd degree*, ESAIM Control Optim. Calc. Var., **9** (2003), 343–352 (electronic).

[19] Hassan K. Khalil, "Nonlinear Systems," Prentice Hall, New Jersey, 2002.

[20] Daniel E. Koditschek, *Adaptive techniques for mechanical systems*, In "Proc. 5th. Yale University Conference, New Haven" (1987), pages 259–265. Yale University, New Haven, 1987.

[21] Pascal Morin, Jean-Baptiste Pomet and Claude Samson, *Design of homogeneous time-varying stabilizing control laws for driftless controllable systems via oscillatory approximation of lie brackets in closed loop*, SIAM J. Control Optim., **38** (1999), 22–49 (electronic).

[22] Pascal Morin and Claude Samson, *Time-varying exponential stabilization of a rigid spacecraft with two control torques*, IEEE Trans. Automat. Control, **42** (1997), 528–534.

[23] Laurent Praly, Brigitte d'Andréa-Novel and Jean-Michel Coron, *Lyapunov design of stabilizing controllers for cascaded systems*, IEEE Trans. Automat. Control, **36** (1991), 1177–1181.

[24] Lionel Rosier, *Homogeneous lyapunov function for homogeneous continuous vector fields*, Systems Control Lett., **19** (1992), 467–473.

[25] Eduardo D. Sontag, "Mathematical Control Theory," Deterministic finite-dimensional systems, Second edition, Texts in Applied Mathematics, Springer-Verlag, New York, 1998.

[26] John Tsinias, *Sufficient Lyapunov-like conditions for stabilization*, Math. Control Signals Systems, **2** (1989), 343–357.

Received January 2013; revised April 2013.

E-mail address: coron@ann.jussieu.fr