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On the Controllability of Nonlinear Partial Differential Equations

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Abstract

A control system is a dynamical system on which one can act by using controls. A classical issue is the controllability problem: Is it possible to reach a desired target from a given starting point by using appropriate controls? We survey some methods to handle this problem when the control system is modeled by means of a nonlinear partial differential equation and when the nonlinearity plays a crucial role.

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1. Introduction

A control system is a dynamical system on which one can act by using suitable *controls*. Very often it is modeled by a differential equation of the following type

$$\dot{y} = f(y, u). \tag{1}$$

The variable y is the state and belongs to some space $\mathcal{Y}$. The variable u is the control and belongs to some space $\mathcal{U}$. The spaces $\mathcal{Y}$ and $\mathcal{U}$ can be of infinite dimension and the differential equation (1) can be a partial differential equation (PDE). There are many problems that appear when studying a control system. One of the most common ones is the *controllability* problem, which, roughly speaking, is the following one. Given two states, is it possible to steer the control system from the first one to the second one? In the framework of (1), this means that, given the state $a \in \mathcal{Y}$ and the state $b \in \mathcal{Y}$, does there exist

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a map $u : [0,T] \to \mathcal{U}$ such that the solution of the Cauchy problem $\dot{y} = f(y, u(t))$, $y(0) = a$, satisfies $y(T) = b$? If the answer is yes, the control system is said to be *controllable*.

The purpose of this article is to survey some results on the controllability of nonlinear control systems in the case where the nonlinearity plays a crucial role. This is, for example, the case when the linearized control system around the equilibrium of interest is not controllable. This is also the case when the nonlinearity is big at infinity and one looks for global results. For convenience, we start by recalling in Section 2 some classical controllability results for control systems in finite dimension. Then, in Section 3, we turn to systems modeled by means of nonlinear partial differential equations.

2. Controllability of Finite Dimensional Control Systems

Let, for $i \in \{0, 1, \ldots, m\}$, $f_i \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$. In this section, our control system is

$$\dot{y} = f(y, u) = f_0(y) + \sum_{i=1}^{m} u_i f_i(y), \tag{2}$$

where the state is $y = (y_1, \ldots, y_n)^{\mathrm{tr}} \in \mathbb{R}^n$ and the control is $u = (u_1, \ldots, u_m)^{\mathrm{tr}} \in \mathbb{R}^m$. We assume that $(y_e, u_e) \in \mathbb{R}^n \times \mathbb{R}^m$ is an equilibrium, i.e., $f(y_e, u_e) = 0$.

Example 1 (*Inverted pendulum on a cart*). This is the traditional example that one can find in most of the textbooks on control theory. This control system consists of a cart with an inverted pendulum on it, as represented on Figure 1. The mass of the cart is M. The pendulum rod is considered massless; its length is denoted by l. The mass of the point mass at the end of the rod is denoted by m. The force applied to the cart is the control and is denoted by F. Let $x_1 := \xi$, $x_2 := \theta$, $x_3 := \dot{\xi}$, $x_4 := \dot{\theta}$ and $u := F$. The dynamical equations governing the motion of this control system can be written in the form $\dot{y} = f(y, u)$, with $y = (y_1, y_2, y_3, y_4)^{\mathrm{tr}}$ and

$$f(y,u) := \begin{pmatrix} y_3 \\ y_4 \\ \dfrac{mly_4^2 \sin y_2 - mg \sin y_2 \cos y_2}{M + m \sin^2 y_2} + \dfrac{u}{M + m \sin^2 y_2} \\ \dfrac{-mly_4^2 \sin y_2 \cos y_2 + (M+m) g \sin y_2}{(M + m \sin^2 y_2) l} - \dfrac{u \cos y_2}{(M + m \sin^2 y_2) l} \end{pmatrix}. \tag{3}$$

Example 2 (*Baby stroller*). Let us consider the following control system, which models a baby stroller,

$$\dot{y}_1 = u_1 \cos y_3,\ \dot{y}_2 = u_1 \sin y_3,\ \dot{y}_3 = u_2, \tag{4}$$

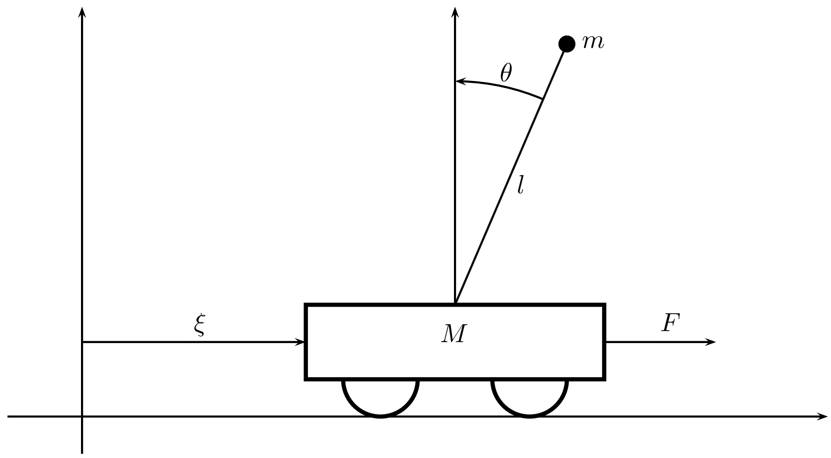

Figure 1. An inverted pendulum on a moving cart.

where the state is $(y_1, y_2, y_3)^{\mathrm{tr}} \in \mathbb{R}^3$ and the control is $(u_1, u_2)^{\mathrm{tr}} \in \mathbb{R}^2$. The variable y_3 is an angle which gives the orientation of the baby stroller and y_1, y_2 are the coordinates of the midpoint between the two back wheels; see Figure 2.

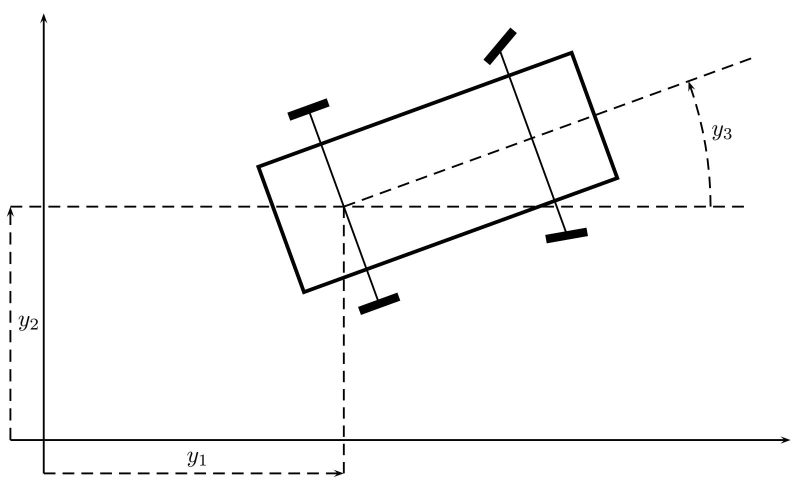

Figure 2. A baby stroller.

This control system is sometimes also called the "unicycle" or "shopping cart" control system. Note that, however, in many shops, the four wheels of a shopping cart are castor wheels. For the baby stroller control system, only the two front wheels are castor wheels: The back wheels have a fixed direction (relatively to the baby stroller). For the control system (4), $n = 3$, $m = 2$ and, for every $y = (y_1, y_2, y_3)^{\mathrm{tr}} \in \mathbb{R}^3$, $f_1(y) = (\cos y_3, \sin y_3, 0)^{\mathrm{tr}}$, $f_2(y) = (0, 0, 1)^{\mathrm{tr}}$.

There are many possible choices for natural definitions of local controllability. The most popular one is the following one.

Definition 3 (Small-Time Local Controllability (STLC)). The control system $\dot{y} = f(y, u)$ is small-time locally controllable at (y_e, u_e) if, for every real number $\varepsilon > 0$, there exists a real number $\eta > 0$ such that, for every $y^0 \in B_\eta(y_e) := \{y \in \mathbb{R}^n; |y - y_e| < \eta\}$ and for every $y^1 \in B_\eta(y_e)$, there exists $u \in L^\infty((0, \varepsilon); \mathbb{R}^m)$ satisfying $|u(t) - u_e| \leqslant \varepsilon$ for almost every $t \in (0, \varepsilon)$ and such that, if $\dot{y} = f(y, u(t))$ and $y(0) = y^0$, then $y(\varepsilon) = y^1$.

The simplest control systems are linear control systems, i.e. systems such that $f(y, u) = Ay + Bu$, for some $A \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$ and some $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$, where $\mathcal{L}(\mathbb{R}^k; \mathbb{R}^l)$ denotes the set of linear maps from $\mathbb{R}^k$ into $\mathbb{R}^l$. For linear systems, a necessary and sufficient condition for STLC is given by the Kalman rank condition that we recall in the next theorem.

Theorem 2.1 (Kalman's rank condition). *The linear control system $\dot{y} = Ay + Bu$ is small-time locally controllable at $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$ if and only if*

$$\mathit{Span}\, \{A^i Bu; u \in \mathbb{R}^m,\ i \in \{0, 1, \ldots, n-1\}\} = \mathbb{R}^n. \tag{5}$$

In "real life" there are very few linear control systems. But, by linearization, controllability of linear control systems is important to study the controllability of nonlinear systems. This is similar to the following classical result: Let $F \in C^1(\mathbb{R}^k, \mathbb{R}^l)$ and let $a \in \mathbb{R}^l$. Then, if $F'(a)$ is onto, F is locally onto at a, i.e., the image by F of every neighborhood of a is a neighborhood of $F(a)$. This just follows from the inverse mapping theorem. For the control system (2) and the equilibrium(y_e, u_e), the analog of $F'(a)$ is the linearized control system at the equilibrium (y_e, u_e), i.e. the linear control system

$$\dot{y} = \frac{\partial f}{\partial y}(y_e, u_e)y + \frac{\partial f}{\partial u}(y_e, u_e)u, \tag{6}$$

where the state is $y \in \mathbb{R}^n$ and the control is $u \in \mathbb{R}^m$. Using again the inverse mapping theorem, one has the easy but important following theorem.

Theorem 2.2 (Linear test). *If the linear control system* (6) *is small-time locally controllable at* $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$*, then* $\dot{y} = f(y, u)$ *is small-time locally controllable at* (y_e, u_e).

Example 4. We go back to the inverted pendulum on a cart, already considered in Example 1. The dynamics is $\dot{y} = f(y, u)$ where $f : \mathbb{R}^4 \times \mathbb{R}$ is defined by (3). Note that $f(0, 0) = 0$. Hence $(0, 0) \in \mathbb{R}^4 \times \mathbb{R}$ is an equilibrium of the control system $\dot{y} = f(y,u)$. The linearized control system at this equilibrium is $\dot{y} = Ay + Bu$ with

$$A := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\dfrac{mg}{M} & 0 & 0 \\ 0 & \dfrac{(M+m)g}{Ml} & 0 & 0 \end{pmatrix}, \; B := \frac{1}{Ml}\begin{pmatrix} 0 \\ 0 \\ l \\ -1 \end{pmatrix}.$$

Simple computations show that

$$\det\,(B, AB, A^2B, A^3B) = -\frac{g^2}{M^4 l^4} \neq 0.$$

Hence the linear control system $\dot{y} = Ay+Bu$ satisfies the Kalman rank condition (5) and therefore, by Theorem 2.1, is small-time locally controllable at $(0,0) \in \mathbb{R}^4 \times \mathbb{R}$. By Theorem 2.2, this implies that the cart-pendulum system is small-time locally controllable at the equilibrium $(0,0) \in \mathbb{R}^4 \times \mathbb{R}$.

Example 5. We return to the baby stroller control system (4) considered in Example 2. Note that $(0,0) \in \mathbb{R}^3 \times \mathbb{R}^2$ is an equilibrium of this control system. The linearized control system around this equilibrium is the linear control system

$$\dot{y}_1 = u_1,\ \dot{y}_2 = 0,\ \dot{y}_3 = u_2, \tag{7}$$

where the state is $(y_1, y_2, y_3)^{\mathrm{tr}} \in \mathbb{R}^3$ and the control is $(u_1, u_2)^{\mathrm{tr}} \in \mathbb{R}^2$. The linear control system (7) is clearly not controllable (one cannot control y_2).

Of course, if the linearized control system around an equilibrium is not controllable, one cannot conclude anything about the small-time local controllability of the nonlinear control system at this equilibrium. This leads naturally to the question: What to do if the linearized control system is not controllable? In finite dimension the basic tool to deal with this problem is the use of (iterated) Lie brackets. Let us recall that, if $X \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ and $Y \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ are two smooth vector fields on $\mathbb{R}^n$, the Lie bracket $[X,Y]$ of X and Y is the vector field on $\mathbb{R}^n$ defined by $[X,Y](y) := Y'(y)X(y) - X'(y)Y(y)$. Examples of iterated Lie brackets are $[X,[X,Y]]$, $[[Y,X],[X,[X,Y]]]$ etc.

Let us explain why Lie brackets are natural objects to study the local controllability problem. Let us start with the case $f_0 = 0$ (then the control system is called a driftless control system). Hence the control system is

$$\dot{y} = \sum_{i=1}^{m} u_i f_i(y), \tag{8}$$

where the state is $y = (y_1, \ldots, y_n)^{\mathrm{tr}} \in \mathbb{R}^n$ and the control is $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$. Let us fix $\eta_1 \in \mathbb{R}$, $\eta_2 \in \mathbb{R}$ and $a \in \mathbb{R}^n$. For $\varepsilon > 0$, we consider the following control $u : (0, 4\varepsilon) \to \mathbb{R}^m$

$$\begin{array}{rcll} u(t) & = & (\eta_1, 0, 0, \dots, 0)^{\mathrm{tr}}, & t \in (0, \varepsilon), \\ u(t) & = & (0, \eta_2, 0, \dots, 0)^{\mathrm{tr}}, & t \in (\varepsilon, 2\varepsilon), \\ u(t) & = & (-\eta_1, 0, 0, \dots, 0)^{\mathrm{tr}}, & t \in (2\varepsilon, 3\varepsilon), \\ u(t) & = & (0, -\eta_2, 0, \dots, 0)^{\mathrm{tr}}, & t \in (3\varepsilon, 4\varepsilon). \end{array}$$

Let $y : [0, 4\varepsilon] \to \mathbb{R}^n$ be the solution of the Cauchy problem $\dot{y} = \sum_{i=1}^m u_i(t) f_i(y)$, $y(0) = a$. Straightforward computations lead to

$$y(4\varepsilon) = a + \varepsilon^2 \eta_1 \eta_2 [f_1, f_2](a) + \mathcal{O}(\varepsilon^3) \text{ as } \varepsilon \to 0.$$

With these controls, starting from a, we have therefore succeeded to move in the directions $[f_1, f_2](a)$ and $-[f_1, f_2](a)$. This can be "iterated": suitable controls allow to move in the directions $\pm[f_1, [f_1, f_2]](a)$, $\pm[[f_2, f_1], [f_1, [f_1, f_2]]](a)$ etc. (see in particular [76]) and one has the following theorem.

Theorem 2.3 ([63, 16]). *Let $y_e \in \mathbb{R}^n$. Let us assume that*

$$\{h(y_e);\, h \in Lie(\{f_1, \dots, f_m\})\} = \mathbb{R}^n. \tag{9}$$

Then the control system (8) *is small-time locally controllable at the equilibrium* $(y_e, 0) \in \mathbb{R}^n \times \mathbb{R}^m$.

(A proof of this theorem is also given in [23, Section 3.3].) In (9) and in the following, for a nonempty subset $\mathcal{E}$ of $C^\infty(\mathbb{R}^n; \mathbb{R}^n)$, $\mathrm{Lie}(\mathcal{E})$ denotes the Lie algebra generated by $\mathcal{E}$, i.e. the smallest (for the inclusion) vector subspace $\mathcal{V}$ of $C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ containing $\mathcal{E}$ and such that $[X, Y] \in \mathcal{V}$, for every $X \in \mathcal{V}$ and for every $Y \in \mathcal{V}$.

In fact, for driftless control systems, one can also get a global controllability result relying on iterated Lie brackets. One has the following theorem.

Theorem 2.4 ([63, 16]). *Let Ω be a nonempty open connected subset of $\mathbb{R}^n$. Let us assume that*

$$\{h(y);\, h \in Lie(\{f_1, \dots, f_m\})\} = \mathbb{R}^n,\ \forall y \in \Omega. \tag{10}$$

Then the control system (8) *is globally controllable in every time in Ω in the following sense: For every $y^0 \in \Omega$, for every $y^1 \in \Omega$ and for every $T > 0$, there exists $u \in L^\infty((0, T); \mathbb{R}^m)$ such that the solution of the Cauchy problem*

$$\dot{y} = \sum_{i=1}^m u_i(t) f_i(y),\ y(0) = y^0,$$

satisfies $y(T) = y^1$ and $y([0, T]) \subset \Omega$.

(A proof of this theorem is again also given in [23, Section 3.3].) When (10) does not hold, the set of points which can be reached from a given point while remaining in Ω is an immersed submanifold of Ω whose tangent space can be precisely described: See [73].

Example 6. Let us return to the baby stroller control system (4). This control system can be written as $\dot{y} = u_1 f_1(y) + u_2 f_2(y)$, with $f_1(y) := (\cos y_3, \sin y_3, 0)^{\mathrm{tr}}$ and $f_2(y) := (0,0,1)^{\mathrm{tr}}$. One has $[f_1, f_2](y) = (\sin y_3, -\cos y_3, 0)^{\mathrm{tr}}$. Hence $f_1(y)$, $f_2(y)$ and $[f_1, f_2](y)$ span all of $\mathbb{R}^3$, for every $y \in \mathbb{R}^3$. This implies the small-time local controllability of the baby stroller at $(y, 0) \in \mathbb{R}^3 \times \mathbb{R}^2$, for every $y \in \mathbb{R}^3$ (see Theorem 2.3) and also the global controllability in every time of this control system (see Theorem 2.4).

When there is a drift term f_0, iterated Lie brackets are still useful. Let us explain, for example, how to move in the direction $\pm[f_0, f_1]$. Let $\eta \in \mathbb{R}$. Let $a \in \mathbb{R}^n$ be such that $f_0(a) = 0$. Let, for $\varepsilon > 0$, $u : (0, 2\varepsilon) \to \mathbb{R}^m$ be defined by

$$\begin{array}{lll} u(t) & := & (-\eta, 0, \ldots, 0)^{\mathrm{tr}}, \quad t \in (0, \varepsilon), \\ u(t) & := & (\eta, 0, \ldots, 0)^{\mathrm{tr}}, \quad\;\;\; t \in (\varepsilon, 2\varepsilon). \end{array}$$

Let $y : [0, 2\varepsilon] \to \mathbb{R}^n$ be the solution of the Cauchy problem

$$\dot{y} = f_0(y) + \sum_{i=1}^{m} u_i(t) f_i(y),\ y(0) = a.$$

Straightforward computations lead now to $y(2\varepsilon) = a + \varepsilon^2 \eta [f_0, f_1](a) + \mathcal{O}(\varepsilon^3)$ as $\varepsilon \to 0$. Hence, starting from a, one can move in the directions $\pm[f_0, f_1](a)$.

Let us emphasize also that the Kalman rank condition (5) is also a condition on (iterated) Lie brackets. Indeed, for $k \in \mathbb{N}$, $X \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ and $Y \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$, one defines $\mathrm{ad}_X^k Y \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ by induction on the integer k by $\mathrm{ad}_X^0 Y := Y$, $\mathrm{ad}_X^k Y := [X, \mathrm{ad}_X^{k-1} Y]$. Let us write the linear control system $\dot{y} = Ay + Bu$ as $\dot{y} = f_0(y) + \sum_{i=1}^{m} u_i f_i(y)$ with

$$f_0(y) := Ay,\ f_i(y) := B_i,\ B_i \in \mathbb{R}^n,\ (B_1, \ldots, B_m) := B.$$

Then

$$\mathrm{ad}_{f_0}^k f_i = (-1)^k A^k B_i,\ \forall k \in \mathbb{N},\ \forall i \in \{1, \ldots, m\}. \tag{11}$$

Hence the Kalman rank condition (5) can be written in the following way

$$\mathrm{Span}\ \{\mathrm{ad}_{f_0}^k f_i(0);\ k \in \{0, \ldots, n-1\},\ i \in \{1, \ldots, m\}\} = \mathbb{R}^n. \tag{12}$$

Moreover, one easily checks that Kalman rank condition (5) is also equivalent to

$$\{h(0);\ h \in \mathrm{Lie}(\{f_0, \ldots, f_m\})\} = \mathbb{R}^n. \tag{13}$$

It turns out that, for analytic systems, condition (13) is necessary for small-time local controllability at the equilibrium $(0,0)$: One has the following theorem.

Theorem 2.5 ([46, 62]). *Assume that*

$$f_0(y_e) = 0. \tag{14}$$

Assume that the control system (2) *is small-time locally controllable at the equilibrium point* $(y_e, 0)$ *and that the* f_i*'s* ($i \in \{0, \dots, m\}$) *are analytic. Then*

$$\{h(y_e);\, h \in Lie(\{f_0, \dots, f_m\})\} = \mathbb{R}^n. \tag{15}$$

Hence, condition (15) is necessary for small-time local controllability of analytic control systems (Theorem 2.5) and is also sufficient for small-time local controllability for control systems without drift (Theorem 2.3) as well as for linear control systems (Theorem 2.1 and (11)). However this condition is far from being sufficient for small-time local controllability in general. Let us give a simple example. We take $n = 2$ and $m = 1$ and consider the control system

$$\dot{y}_1 = y_2^2,\ \dot{y}_2 = u, \tag{16}$$

where the state is $y := (y_1, y_2)^{\mathrm{tr}} \in \mathbb{R}^2$ and the control is $u \in \mathbb{R}$. This control system can be written as $\dot{y} = f_0(y) + uf_1(y)$ with $f_0(y) := (y_2^2, 0)^{\mathrm{tr}}$, $f_1(y) := (0, 1)^{\mathrm{tr}}$. One has $[f_1, [f_1, f_0]] = (2, 0)^{\mathrm{tr}}$ and therefore $f_1(0)$ and $[f_1, [f_1, f_0]](0)$ span all of $\mathbb{R}^2$. However the control system (16) is clearly not small-time locally controllable at the equilibrium $(0, 0) \in \mathbb{R}^2 \times \mathbb{R}$ since $\dot{y}_1 \geqslant 0$.

One knows powerful sufficient conditions for small-time local controllability. Let us mention, in particular, [1, 2, 3, 9, 10, 54, 34, 74, 75, 77] and references therein. One knows also powerful necessary conditions which are stronger than the one given in Theorem 2.5. See, in particular, [74, Proposition 6.3] and [72]. However, one does not know an (interesting) necessary and sufficient condition for small-time local controllability. One has the following challenging open problem.

Open problem 2.6. *Let* k *be a positive integer. Let* $\mathcal{X}_k$ *be the set of vector fields in* $\mathbb{R}^n$ *whose components are polynomials of degree* k. *Let*

$$S := \{(f_0, f_1) \in \mathcal{X}_k \times \mathcal{X}_k;\ f_0(0) = 0,\ \dot{y} = f_0(y) + uf_1(y) \text{ is STLC }\}.$$

Is S *a semi-algebraic set?*

Let us recall that a semi-algebraic set is a subset of a real finite dimensional space (here $\mathcal{X}_k^2$) defined by a finite sequence of polynomial equations and polynomial inequalities on the coordinates or any finite union of such sets. Let us point out that the set of $(f_0, f_1) \in \mathcal{X}_k^2$ satisfying the Lie algebra rank condition (15) at $y_e = 0$ is a semi-algebraic set: See [64, 41, 40].

3. Controllability of PDE Control Systems

We now turn to the cases of control systems modeled by partial differential equations. Again the simplest cases concern the case of linear partial differential equations. There are many powerful tools to study the controllability of linear control systems in infinite dimension. The most popular ones are based on the duality between observability and controllability (related to the J.-L. Lions Hilbert uniqueness method [58, 59]). This leads to try to prove observability inequalities. There are many methods to prove these observability inequalities. For example, let us mention the following methods (together with the pioneering works where they have been introduced in control theory)

- Ingham's inequalities [67],
- Multipliers method [47, 58, 59],
- Microlocal analysis [5],
- Carleman's inequalities [49, 56, 50, 36]. (See also [78] in these proceedings.)

However there are still plenty of open problems on the controllability of linear partial differential equations.

Of course, when one wants to study the local controllability around an equilibrium of a control system in infinite dimension, the first step is again to study the controllability of the linearized control system. If this linearized control system is controllable, one can usually deduce the local controllability of the nonlinear control system. However this might be sometimes difficult due to some loss of derivatives issues. One may need to use suitable complicated iterative schemes. If the nonlinearity is not too big at infinity, one can get a global controllability result (see in particular [79, 55, 80] for semilinear wave equations and [31, 36, 33] for semilinear parabolic equations).

Let us now focus on cases where either the linearized control system around the equilibrium is not controllable or when the nonlinearity is too big at infinity to use this method for global controllability. Let us start with an example for the first case, namely the Euler control system. (For the second case, see the Navier-Stokes control system below.) Let Ω be a smooth connected nonempty bounded open subset of $\mathbb{R}^n$. Let Γ_0 be a nonempty open subset of the boundary $\partial\Omega$ of Ω. We denote by $\nu : \partial\Omega \to \mathbb{R}^n$ the outward unit normal vector field to Ω. The controllability problem is the following one. Let $T > 0$. Let $y^0, y^1 : \overline{\Omega} \to \mathbb{R}^n$ be such that

$$\operatorname{div} y^0 = \operatorname{div} y^1 = 0 \text{ in } \Omega \text{ and } y^0 \cdot \nu = y^1 \cdot \nu = 0 \text{ on } \partial\Omega \setminus \Gamma_0. \tag{17}$$

Does there exist $y : [0,T] \times \overline{\Omega} \to \mathbb{R}^n$ and $p : [0,T] \times \overline{\Omega} \to \mathbb{R}$ such that

$$y_t + (y \cdot \nabla)y + \nabla p = 0, \ \operatorname{div} y = 0, \ \text{ in } (0,T) \times \Omega, \tag{18}$$

$$y \cdot \nu = 0 \text{ on } [0,T] \times (\partial\Omega \setminus \Gamma_0), \tag{19}$$

$$y(0,\cdot) = y^0, \ y(T,\cdot) = y^1? \tag{20}$$

(For simplicity we do not specify the regularity of y^0, y^1, y, p etc: For these regularities, see the given references.) This system models the flow in Ω of an inviscid, incompressible fluid with constant density, which is equal to one without loss of generality. The vector $y(t,x) \in \mathbb{R}^n$ is the velocity of the fluid and $p(t,x) \in \mathbb{R}$ is the pressure, both at time t and position $x \in \Omega$. Condition (19) states that the fluid does not flow through the boundary $\partial\Omega \backslash \Gamma_0$: It slips on this boundary without friction. The first equation of (18) is Newton's second law: It states that the acceleration of a fluid particle is proportional to the pressure-force acting on it. The second equation of (18) is the incompressibility condition: It states that the volume of any part of the fluid does not change under the flow.

Note that, in the above formulation, the control does not appear explicitly: We consider a control system as an underdetermined equation. However one can specify the control if one wants to do so. Many choices are in fact possible. For example, one can take as the control $y \cdot \nu$ on Γ_0 with $\int_{\Gamma_0} y \cdot \nu = 0$ together with curl y if $n = 2$ and the tangent vector (curl y) $\times \nu$ if $n = 3$ at the points of $[0,T] \times \Gamma_0$ where $y \cdot \nu < 0$, where curl y is the vorticity of the velocity field y.

The problem of the controllability of this control system (and of the Navier-Stokes control system considered below) has been raised in [60, 61].

We start by giving an obstruction to the controllability for $n = 2$, when there is a connected component Γ_1 of $\partial\Omega$ which does not meet Γ_0. Let γ_0 be a given curve in $\overline{\Omega}$. Let, for $t \in [0,T]$, $\gamma(t)$ be the Jordan curve obtained, at time $t \in [0,T]$, from the points of the fluids which, at time 0, were on γ_0. The Kelvin law tells us that, if $\gamma(t)$ does not intersect Γ_0, $\int_{\gamma(t)} y(t,\cdot) \cdot \overrightarrow{ds} = \int_{\gamma_0} y(0,\cdot) \cdot \overrightarrow{ds}$, $\forall t \in [0,T]$. We take $\gamma_0 := \Gamma_1$. Then $\gamma(t) = \Gamma_1$ for every $t \in [0,T]$. Hence, if $\int_{\Gamma_1} y^1 \cdot \overrightarrow{ds} \neq \int_{\Gamma_1} y^0 \cdot \overrightarrow{ds}$, one cannot steer the control system from y^0 to y^1.

More generally, for every $n \in \{2,3\}$, if Γ_0 does not intersect every connected component of the boundary $\partial\Omega$ of Ω, the Euler control system is not controllable. This is the only obstruction to the controllability of the Euler control system. Indeed, one has the following theorem.

Theorem 3.1 ([18, 21] for $n = 2$ and [42, 43] for $n = 3$). *Assume that* Γ_0 *intersects every connected component of* $\partial\Omega$. *Then the Euler control system is globally controllable in every time: For every* $T > 0$, *for every* y^0, $y^1 : \Omega \to \mathbb{R}^n$ *such that* (17) *holds, there exist* $y : [0,T] \times \overline{\Omega} \to \mathbb{R}^n$ *and* $p : [0,T] \times \overline{\Omega} \to \mathbb{R}$ *such that* (18), (19) *and* (20) *hold.*

Let us sketch the main ideas of the proof of this controllability result. For simplicity we assume that $n = 2$. One first studies (as usual) the controllability of the linearized control system around 0. This linearized control system is the underdetermined system

$$y_t + \nabla p = 0, \text{div } y = 0, \text{ in } (0,T) \times \Omega, \text{ and } y \cdot \nu = 0 \text{ on } [0,T] \times (\partial\Omega \setminus \Gamma_0). \tag{21}$$

Taking the curl of the first equation of (21), on gets $(\operatorname{curl} y)_t = 0$. Thus curl y is constant along the trajectories for the linearized control system, which shows that (21) is not controllable.

In Section 2, for finite dimensional control systems, we saw that when the linearized control system is not controllable, the usual tool to use is (iterated) Lie brackets. This can also be used for some infinite dimensional control systems. See in particular [4, 68] for Euler and Navier-Stokes equations and [13] for a Schrödinger equation. However one does not know how to use this tool for numerous examples of infinite dimensional control systems (including our Euler control system). Let us explain the difficulty on the simplest PDE control system, namely

$$y_t + y_x = 0,\ t \in (0,T),\ x \in (0,L),\ \text{ and } y(t,0) = u(t),\ t \in (0,T). \tag{22}$$

This is a control system where, at time t, the state is $y(t,\cdot) : (0,L) \to \mathbb{R}$ and the control is $u(t) \in \mathbb{R}$. A natural state space is $\mathcal{Y} := L^2(0,L)$ and for the control u it then suffices to take $u \in L^2(0,T)$ in order to have a well-posed Cauchy problem. Of course, using the explicit expression of the solutions of (22), it is easy to see that this control system is controllable on $[0,T]$ if and only if $T \geqslant L$. Let us try to understand what is $[f_0, f_1]$ for the control system (22). We proceed as in Section 2. Let $\eta \in \mathbb{R}$. Let us consider, for $\varepsilon > 0$, the control defined on $[0, 2\varepsilon]$ by

$$u(t) := -\eta \text{ for } t \in (0,\varepsilon),\ u(t) := \eta \text{ for } t \in (\varepsilon, 2\varepsilon).$$

Let $y : (0,2\varepsilon) \times (0,L) \to \mathbb{R}$ be the (weak) solution of the Cauchy problem

$$\begin{gathered} y_t + y_x = 0,\ t \in (0,2\varepsilon),\ x \in (0,L), \\ y(t,0) = u(t),\ t \in (0,2\varepsilon),\ y(0,x) = 0,\ x \in (0,L). \end{gathered}$$

Then one readily gets, if $2\varepsilon \leqslant L$,

$$y(2\varepsilon,x) = \eta,\ x \in (0,\varepsilon),\ y(2\varepsilon,x) = -\eta,\ x \in (\varepsilon,2\varepsilon),\ y(2\varepsilon,x) = 0,\ x \in (2\varepsilon,L).$$

Unfortunately

$$\left| \frac{y(2\varepsilon,\cdot) - y(0,\cdot)}{\varepsilon^2} \right|_{L^2(0,L)} \to +\infty \text{ as } \varepsilon \to 0^+.$$

In fact, for every $\phi \in H^2(0,L)$, one gets, after suitable computations,

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^2} \int_0^L \phi(x)(y(2\varepsilon,x) - y(0,x))dx = -\eta\phi'(0). \tag{23}$$

So, in some sense, (23) says that, for the control system (22), $[f_0, f_1](0) = \delta_0'$. Unfortunately it is not clear how to use this derivative of a Dirac mass at 0 in the framework of our controllability problem.

Remark 3.2. *In fact the above problem already appears for* $f_1 = ad^0_{f_0} f_1$*. Proceeding as for* $[f_0, f_1]$*, one finds that, in some sense, we could say that, for the control system* (22)*,* $ad^0_{f_0} f_1(0) = \delta_0$*.*

In order to overcome this difficulty, we use the return method, a method introduced in [17] for a stabilization problem. Let us explain this method first in the framework of the local controllability of a control system in finite dimension. Thus we consider the control system $\dot{y} = f(y, u)$, where $y \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the control. We assume that f is of class C^∞ and satisfies $f(0, 0) = 0$. The return method consists in reducing the local controllability of a nonlinear control system to the existence of suitable trajectories and to the controllability of *linear* systems. The idea is the following one: Assume that, for every positive real number T and every positive real number ε, there exists a measurable bounded function $\bar{u} : [0, T] \to \mathbb{R}^m$ with $\|\bar{u}\|_{L^\infty(0,T)} \leqslant \varepsilon$ such that, if we denote by $\bar{y}$ the (maximal) solution of $\dot{\bar{y}} = f(\bar{y}, \bar{u}(t))$, $\bar{y}(0) = 0$, then

$$\bar{y}(T) = 0, \tag{24}$$

the linearized control system around $(\bar{y}, \bar{u})$ is controllable on $[0, T]$. (25)

Then, from the inverse mapping theorem, one gets the existence of $\eta > 0$ such that, for every $y^0 \in \mathbb{R}^n$ and for every $y^1 \in \mathbb{R}^n$ such that $|y^0| < \eta$ and $|y^1| < \eta$, there exists $u \in L^\infty((0, T); \mathbb{R}^m)$ such that

$$|u(t) - \bar{u}(t)| \leqslant \varepsilon,\ t \in (0, T),$$

and such that, if $y : [0, T] \to \mathbb{R}^n$ is the solution of the Cauchy problem $\dot{y} = f(y, u(t))$, $y(0) = y^0$, then $y(T) = y^1$. Since $T > 0$ and $\varepsilon > 0$ are arbitrary, one gets that $\dot{x} = f(x, u)$ is small-time locally controllable at the equilibrium $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$.

Let us show how this method works on the baby stroller control system (4). For every $\bar{u} \in C^\infty([0, T]; \to \mathbb{R}^2)$ such that, for every t in $[0, T]$, $\bar{u}(T-t) = -\bar{u}(t)$, every solution $\bar{y} : [0, T] \to \mathbb{R}^3$ of $\dot{\bar{y}}_1 = \bar{u}_1 \cos \bar{y}_3$, $\dot{\bar{y}}_2 = \bar{u}_1 \sin \bar{y}_3$, $\dot{\bar{y}}_3 = \bar{u}_2$, satisfies $\bar{y}(0) = \bar{y}(T)$. We impose $\bar{y}(0) = 0$. We then have $\bar{y}(T) = 0$. The linearized control system around $(\bar{y}, \bar{u})$ is

$$\dot{y}_1 = -\bar{u}_1 y_3 \sin \bar{y}_3 + u_1 \cos \bar{y}_3,\ \dot{y}_2 = \bar{u}_1 y_3 \cos \bar{y}_3 + u_1 \sin \bar{y}_3,\ \dot{y}_3 = u_2. \tag{26}$$

Using a Kalman rank condition for time varying linear systems (see [69] or [23, Theorem 1.18, page 11]), one can easily check that the linear control system (26) is controllable if (and only if) $\bar{u} \not\equiv 0$. Hence we have given a new proof of the small-time local controllability of the baby stroller control system (4) at $(0, 0) \in \mathbb{R}^3 \times \mathbb{R}^2$ which does not use Lie brackets: this proof uses only controllability results for linear (time-varying) control systems.

The next proposition shows some kind of converse: The return method essentially always works if the control system is small-time locally controllable. More precisely, let us go back to the control system (2) and assume that (14) holds.

We also assume that (15) holds. (Let us recall that, if the f_i's are analytic, (15) is a necessary condition for small-time local controllability at $(y_e, 0) \in \mathbb{R}^n \times \mathbb{R}^m$: See Theorem 2.5). Then one has the following proposition.

Proposition 3.3 ([70, 19]). *Let us assume that the control system* (2) *is small-time locally controllable at* $(y_e, 0) \in \mathbb{R}^n \times \mathbb{R}^m$. *Then, for every* $\varepsilon > 0$, *there exists* $\bar{u} \in L^\infty((0,\varepsilon); \mathbb{R}^m)$ *satisfying* $|u(t)| \leqslant \varepsilon$ *for almost every* $t \in (0, T)$ *such that, if* $\bar{y} : [0, \varepsilon] \to \mathbb{R}^n$ *is the solution of* $\dot{\bar{y}} = f(\bar{y}, \bar{u}(t))$, $\bar{y}(0) = y_e$, *then*

$$\bar{y}(T) = y_e,$$

the linearized control system around $(\bar{y}, \bar{u})$ *is controllable.*

However there is a fundamental drawback for the return method: it does not provide any insight on strategies to construct $(\bar{y}, \bar{u})$.

Let us show how to construct $(\bar{y}, \bar{u})$ for our Euler control system. One looks for $(\bar{y}, \bar{p}) : [0, T] \times \overline{\Omega} \to \mathbb{R}^n \times \mathbb{R}$ such that

$$\bar{y}_t + (\bar{y} \cdot \nabla \bar{y}) + \nabla \bar{p} = 0, \text{ div } \bar{y} = 0, \text{ in } (0, T) \times \Omega, \tag{27}$$

$$\bar{y} \cdot \nu = 0 \text{ on } [0, T] \times (\partial\Omega \setminus \Gamma_0), \tag{28}$$

$$\bar{y}(T, \cdot) = \bar{y}(0, \cdot) = 0, \tag{29}$$

$$\text{the linearized control system around } (\bar{y}, \bar{p}) \text{ is controllable.} \tag{30}$$

We construct $(\bar{y}, \bar{p})$ if $n = 2$ and Ω is simply connected. Let us take $\theta : \overline{\Omega} \to \mathbb{R}$ such that

$$\Delta\theta = 0 \text{ in } \Omega, \ \frac{\partial\theta}{\partial\nu} = 0 \text{ on } \partial\Omega \setminus \Gamma_0. \tag{31}$$

Then, let $\alpha : [0, T] \to [0, +\infty)$ be such that $\alpha(0) = \alpha(T) = 0$. Finally, we define $(\bar{y}, \bar{p}) : [0, T] \times \overline{\Omega} \to \mathbb{R}^2 \times \mathbb{R}$ by

$$\bar{y}(t, x) := \alpha(t)\nabla\theta(x), \ \bar{p}(t, x) := -\dot{\alpha}(t)\theta(x) - \frac{\alpha(t)^2}{2}|\nabla\theta(x)|^2.$$

Then $(\bar{y}, \bar{p})$ is a trajectory of the Euler control system which goes from 0 to 0, i.e. it satisfies (27)–(28)–(29). Let us now study (30). The linearized control system around $(\bar{y}, \bar{p})$ is

$$\begin{cases} y_t + (\bar{y} \cdot \nabla)y + (y \cdot \nabla)\bar{y} + \nabla p = 0, \ \text{div } y = 0, \ \text{in } [0, T] \times \Omega, \\ y \cdot \nu = 0 \text{ on } [0, T] \times (\partial\Omega \setminus \Gamma_0). \end{cases} \tag{32}$$

Taking once more the curl of the first equation, one gets

$$(\text{curl } y)_t + (\bar{y} \cdot \nabla)(\text{curl } y) = 0 \text{ in } [0, T] \times \Omega. \tag{33}$$

This is a simple transport equation on curl y. If there exists $a \in \overline{\Omega}$ such that $\nabla\theta(a) = 0$, then $\bar{y}(t, a) = 0$ and $(\text{curl } y)_t(t, a) = 0$, which shows that (33) is not controllable. This is the only obstruction: If

$$\nabla\theta \text{ does not vanish in } \overline{\Omega}, \tag{34}$$

one can (easily) prove that (33), and then (32), are controllable if $\int_0^T \alpha(t)dt$ is large enough. For the construction of $\theta : \overline{\Omega} \to \mathbb{R}$ satisfying (31) and (34), let Γ_+ and Γ_- be two nonempty open connected subsets of Γ_0 such that

$$\overline{\Gamma_+} \cap \overline{\Gamma_-} = \emptyset,\ \overline{\Gamma_+} \cup \overline{\Gamma_-} \subset \Gamma_0.$$

Let $g : \partial\Omega \to \mathbb{R}$ be such that

$$\{x \in \partial\Omega;\ g(x) > 0\} = \Gamma_+,\ \{x \in \partial\Omega;\ g(x) < 0\} = \Gamma_-,\ \int_{\partial\Omega} g(s)ds = 0.$$

Let $\theta : \overline{\Omega} \to \mathbb{R}$ be the solution of the following Neumann problem

$$\Delta\theta = 0 \text{ in } \Omega,\ \frac{\partial\theta}{\partial\nu} = g \text{ on } \partial\Omega,\ \int_\Omega \theta = 0.$$

Then one can check that (34) holds (apply the strong maximum principle to the harmonic conjugate of θ together with Morse or degree theory).

From the above argument, one expects only a local controllability result around 0. However this local controllability result leads to a global controllability result by using the following simple scaling argument, which works in every dimension n. If $(y, p) : [0, 1] \times \overline{\Omega} :\to \mathbb{R}^n \times \mathbb{R}$ is a solution of our Euler control system (18)-(19), then, for every $\varepsilon > 0$, $(y^\varepsilon, p^\varepsilon) : [0, \varepsilon] \times \overline{\Omega} \to \mathbb{R}^n \times \mathbb{R}$ defined by

$$y^\varepsilon(t, x) := \frac{1}{\varepsilon} y\left(\frac{t}{\varepsilon}, x\right),\ p^\varepsilon(t, x) := \frac{1}{\varepsilon^2} p\left(\frac{t}{\varepsilon}, x\right)$$

is also a solution of our Euler control system.

Let us now turn to the controllability of a Navier-Stokes control system. The Navier-Stokes control system is deduced from the Euler control system by adding the linear term $-\mu\Delta y$: The equation is now

$$y_t - \mu\Delta y + (y \cdot \nabla)y + \nabla p = 0,\ \text{div } y = 0,\ \text{ in } (0, T) \times \Omega, \tag{35}$$

where $\mu > 0$ is the viscosity of the fluid (a positive constant). For the boundary condition, one requires now that

$$y = 0 \text{ on } [0, T] \times (\partial\Omega \setminus \Gamma_0), \tag{36}$$

meaning that the viscous fluid sticks to the boundary $\partial\Omega \setminus \Gamma_0$. Here, for the control u, one can take, for example, $y = u$ on $[0, T] \times \Gamma_0$.

Due to the regularizing effect of the Navier-Stokes equations, the "right" controllability problem is not to go from a given state to another given state. The right problem is to go from a state to a given trajectory. For simplicity we assume that this given trajectory is 0. The controllability problem is then the following one. Let $T > 0$. Let $y^0 : \overline{\Omega} \to \mathbb{R}^n$ be such that

$$\text{div } y^0 = 0 \text{ in } \Omega,\ y^0 = 0 \text{ on } \partial\Omega \setminus \Gamma_0,$$

Does there exist $y : [0, T] \times \overline{\Omega} \to \mathbb{R}^n$ and $p : [0, T] \times \overline{\Omega} \to \mathbb{R}$ such that (35) (36) hold, $y(0, \cdot) = y^0$ and $y(T, \cdot) = 0$? One has the following theorem.

Theorem 3.4 ([51, 52]; see also [35, 37, 38, 32]). *Such a* (y, p) *exists if* y^0 *is small enough (in* $L^{2n-2}(\Omega)^n$*).*

A challenging open problem is the following one.

Open problem 3.5. *Does* (y, p) *exist even if* y^0 *is not small?*

One has a positive answer to this problem if $\Gamma_0 = \partial\Omega$:

Theorem 3.6 ([20, 25, 39]). *Such a* (y, p) *always exists if* $\Gamma_0 = \partial\Omega$.

Note that the linearized control system around $(0, 0)$ is controllable (this is a key point for the proof of Theorem 3.4 and this property is known to be true even if $\Gamma_0 \neq \partial\Omega$). However this result seems to give only a local controllability result (i.e. Theorem 3.4). The main idea is to consider other trajectories going from 0 to 0 which have a better controllability around them. Let us explain this in the context of a linear perturbation of a quadratic control system in finite dimension. We consider the following control system

$$\dot{y} = F(y) + Bu(t), \tag{37}$$

where the state is $y \in \mathbb{R}^n$, the control is $u \in \mathbb{R}^m$, B is a $n \times m$ matrix and $F \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ is quadratic: $F(\lambda y) = \lambda^2 F(y)$, $\forall \lambda \in [0, +\infty)$, $\forall y \in \mathbb{R}^n$. We assume that there exists a trajectory $(\bar{y}, \bar{u}) \in C^0([0, T_0]; \mathbb{R}^n) \times L^\infty((0, T_0); \mathbb{R}^m)$ of the control system (37) such that the linearized control system around $(\bar{y}, \bar{u})$ is controllable and such that $\bar{y}(0) = \bar{y}(T_0) = 0$.

Remark 3.7. *One has* $F(0) = 0$. *Hence* $(0, 0)$ *is an equilibrium of the control system* (37). *The linearized control system around this equilibrium is* $\dot{y} = Bu$, *which is not controllable if (and only if)* B *is not onto.*

Let A be a $n \times n$ matrix and let us consider the following control system

$$\dot{y} = Ay + F(y) + Bu(t), \tag{38}$$

where the state is $y \in \mathbb{R}^n$, the control is $u \in \mathbb{R}^m$. For the application to incompressible fluids, (37) is the Euler control system and (38) is the Navier-Stokes control system.

One has the following (easy) theorem.

Theorem 3.8. *Under the above assumptions, the control system* (38) *is globally controllable in arbitrary time: For every* $T > 0$, *for every* $y^0 \in \mathbb{R}^n$ *and for every* $y^1 \in \mathbb{R}^n$, *there exists* $u \in L^\infty((0, T); \mathbb{R}^m)$ *such that*

$$\left(\dot{y} = f(y, u(t)),\, y(0) = y^0\right) \Rightarrow \left(y(T) = y^1\right).$$

Proof of Theorem 3.8. Let $y^0 \in \mathbb{R}^n$ and $y^1 \in \mathbb{R}^n$. Let

$$\begin{array}{cccc} G: & \mathbb{R} \times L^\infty((0, T_0); \mathbb{R}^m) & \to & \mathbb{R}^n \\ & (\varepsilon, \tilde{u}) & \mapsto & \tilde{y}(T_0) - \varepsilon y^1 \end{array}$$

where $\tilde{y} : [0, T_0] \to \mathbb{R}^n$ is the solution of $\dot{\tilde{y}} = F(\tilde{y}) + \varepsilon A\tilde{y} + B\tilde{u}(t)$, $\tilde{y}(0) = \varepsilon y^0$. The map G is of class C^1 in a neighborhood of $(0, \bar{u})$. One has $G(0, \bar{u}) = 0$. Moreover $G'_{\tilde{u}}(0, \bar{u})v = y(T_0)$ where $y : [0, T_0] \to \mathbb{R}^n$ is the solution of $\dot{y} = F'(\bar{y})y + Bv$, $y(0) = 0$. Hence $G'_{\tilde{u}}(0, \bar{u})$ is onto. Therefore there exist $\varepsilon_0 > 0$ and a C^1-map $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \mapsto \tilde{u}^\varepsilon \in L^\infty((0, T_0); \mathbb{R}^m)$ such that

$$G(\varepsilon, \tilde{u}^\varepsilon) = 0, \ \forall \varepsilon \in (-\varepsilon_0, \varepsilon_0),$$
$$\tilde{u}^0 = \bar{u}.$$

Let $\tilde{y}^\varepsilon : [0, T_0] \to \mathbb{R}^n$ be the solution of the Cauchy problem $\dot{\tilde{y}}^\varepsilon = F(\tilde{y}^\varepsilon) + \varepsilon A\tilde{y}^\varepsilon + B\tilde{u}^\varepsilon(t)$, $\tilde{y}^\varepsilon(0) = \varepsilon y^0$. Then $\tilde{y}^\varepsilon(T_0) = \varepsilon y^1$. Let $y : [0, \varepsilon T_0] \to \mathbb{R}^n$ and $u : [0, \varepsilon T_0] \to \mathbb{R}^m$ be defined by

$$y(t) := \frac{1}{\varepsilon}\tilde{y}^\varepsilon\left(\frac{t}{\varepsilon}\right), \ u(t) := \frac{1}{\varepsilon^2}\tilde{u}^\varepsilon\left(\frac{t}{\varepsilon}\right).$$

Then $\dot{y} = F(y) + Ay + Bu$, $y(0) = y^0$ and $y(\varepsilon T_0) = y^1$. This concludes the proof of Theorem 3.8 if T is small enough. If T is not small, it suffices, with $\varepsilon > 0$ small enough, to go from y^0 to 0 during the interval of time $[0, \varepsilon]$, stay at 0 during the interval of time $[\varepsilon, T - \varepsilon]$ and finally go from 0 to y^1 during the interval of time $[T - \varepsilon, T]$.

The "morality" behind Theorem 3.8 is that the quadratic term $F(y)$ wins against the linear term Ay. Note, however, that for Euler/Navier-Stokes equations the linear term $\mu\Delta y$ contains more derivatives than the quadratic term $(y \cdot \nabla)y$. This creates many new difficulties and the proof requires important modifications. In particular, one first gets a global approximate controllability result and then concludes with a local controllability result (see Theorem 3.4).

Of course, as one can see by looking at the proof of Theorem 3.8, this method works only if we have a (good) convergence of the solution of the Navier-Stokes equations to the solution of the Euler equations when the viscosity tends to 0. Let us recall that this is not known even in dimension $n = 2$ if there is no control. More precisely, let us assume that Ω is of class C^∞, that $n = 2$ and that $\varphi \in C_0^\infty(\Omega; \mathbb{R}^2)$ is such that div $\varphi = 0$. Let $T > 0$. Let $y \in C^\infty([0, T] \times \overline{\Omega}; \mathbb{R}^2)$ and $p \in C^\infty([0, T] \times \overline{\Omega})$ be the solution to the Euler equations

$$(E) \begin{cases} y_t + (y \cdot \nabla)y + \nabla p = 0, \text{ div } y = 0, \text{ in } (0, T) \times \Omega, \\ y \cdot \nu = 0 \text{ on } [0, T] \times \partial\Omega, \\ y(0, \cdot) = \varphi \text{ on } \overline{\Omega}. \end{cases}$$

Let $\varepsilon \in (0, 1]$. Let $y^\varepsilon \in C^\infty([0, T] \times \overline{\Omega}; \mathbb{R}^2)$ and $p^\varepsilon \in C^\infty([0, T] \times \overline{\Omega})$ be the solution to the Navier-Stokes equations

$$(NS) \begin{cases} y_t^\varepsilon - \varepsilon\Delta y^\varepsilon + (y^\varepsilon \cdot \nabla)y^\varepsilon + \nabla p^\varepsilon = 0, \text{ div } y^\varepsilon = 0, \text{ in } (0, T) \times \Omega, \\ y^\varepsilon = 0 \text{ on } [0, T] \times \partial\Omega, \\ y(0, \cdot) = \varphi \text{ on } \overline{\Omega}. \end{cases}$$

One knows that there exists $C > 0$ such that $|y^\varepsilon|_{C^0([0,T]; L^2(\Omega; \mathbb{R}^2))} \leqslant C$, for every $\varepsilon \in (0, 1]$. One has the following challenging open problem.

Open problem 3.9. *(i) Does y^ε converge weakly to y in $L^2((0,T)\times\Omega;\mathbb{R}^2)$ as $\varepsilon\to 0^+$?*

(ii) Let K be a compact subset of Ω and m be a positive integer. Does $y^\varepsilon_{|[0,T]\times K}$ converge to $y_{|[0,T]\times K}$ in $C^m([0,T]\times K;\mathbb{R}^2)$ as $\varepsilon\to 0^+$? (Of course, due to the difference of boundary conditions between the Euler equations and the Navier-Stokes equations, one does not have a positive answer to this last question if $K=\overline{\Omega}$.)

The return method turns out to give controllability results on many other partial differential equations, for example, Burgers equations [48, 15, 53], Saint-Venant equations [22] (see also below), Vlasov Poisson equations [44], Isentropic Euler equations [45], Schrödinger equations [6, 7], Korteweg-de Vries equations [14], Hyperbolic equation [26], Navier-Stokes equations with a control force having a vanishing component [27], Ensemble controllability of Bloch equation [8]. For finite dimensional control systems, this method is much less interesting since one then has at one's disposal the powerful tool of iterated Lie brackets; see however [71].

As mentioned above, there is an important difficulty in the application of the return method, namely it is often difficult to construct the reference trajectory $(\bar{y},\bar{u})$ satisfying $\bar{y}(0)=0$, (24) and (25). Let us present a method to take care of this problem in some cases and that has been applied to get controllability results for the Saint-Venant equation (shallow water equation) in [22] (which is motivated by [30]) and a Schrödinger equation [6, 7] (which is motivated by [66]). Let us deal with the control system modeled by the Saint-Venant equation. It concerns the motion of a 1-D tank containing an inviscid incompressible irrotational fluid. The tank is subject to one-dimensional horizontal moves. We assume that the horizontal acceleration of the tank is small compared to the gravity constant and that the height of the fluid is small compared to the length of the tank. This motivates the use of the Saint-Venant equations (also called shallow water equations) to describe the motion of the fluid; see e.g. [29, Section 4.2]. After suitable scaling arguments, the length of the tank, the gravity constant and the height of the fluid at rest can be taken to be equal to 1; see [22]. Then the dynamics equations are (see [30])

$$\begin{cases} H_t\left(t,x\right)+\left(Hv\right)_x\left(t,x\right)=0, \\ v_t\left(t,x\right)+\left(H+\dfrac{v^2}{2}\right)_x\left(t,x\right)=-u\left(t\right), \\ v(t,0)=v(t,1)=0, \\ \dfrac{\mathrm{d}s}{\mathrm{d}t}\left(t\right)=u\left(t\right),\ \dfrac{\mathrm{d}D}{\mathrm{d}t}\left(t\right)=s\left(t\right), \end{cases} \tag{39}$$

where, see Figure 3, at time t and position $x\in[0,1]$,

- $H\left(t,x\right)$ is the height of the fluid,

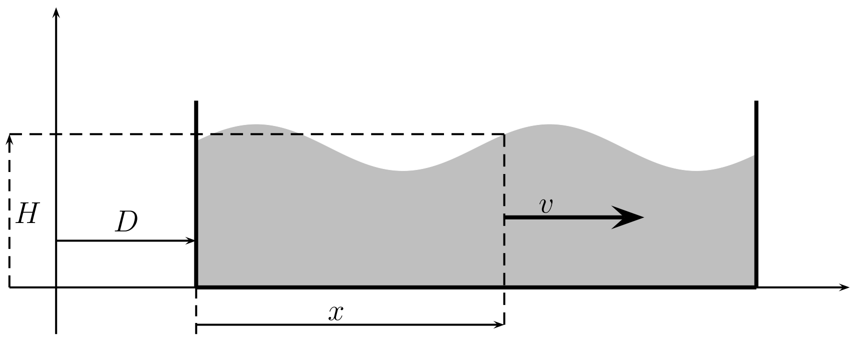

Figure 3. Fluid in the 1-D tank

- $v\,(t,x)$ is the horizontal water velocity of the fluid *in a referential attached to the tank* (in the Saint-Venant model, the points on the same vertical line have the same horizontal velocity),
- $u(t)$ is the horizontal acceleration of the tank in the absolute referential,
- $s(t)$ is the horizontal velocity of the tank,
- $D(t)$ is the horizontal displacement of the tank.

This is a control system, where at time t, the control is $u(t) \in \mathbb{R}$ and the state is $Y(t) = (H(t,\cdot), v(t,\cdot), s(t), D(t))$.

One is interested in the local controllability of the control system (39) around the equilibrium point $(Y_e, u_e) := ((1,0,0,0),0)$. Of course, the total mass of the fluid is conserved so that, for every solution of (39),

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 H\,(t,x)\,\mathrm{d}x = 0. \tag{40}$$

(One gets (40) by integrating the first equation of (39) on $[0,1]$ and by using the third equation of (39).) Moreover, if H and v are of class C^1, it follows from the second and third equation of (39) that

$$H_x(t,0) = H_x(t,1) \quad (= -u\,(t)). \tag{41}$$

Therefore, we introduce the vector space $\mathcal{E}$ of functions

$$Y = (H, v, s, D) \in C^1([0,1]) \times C^1([0,1]) \times \mathbb{R} \times \mathbb{R}$$

such that $H_x(0) = H_x(1)$ and $v(0) = v(1) = 0$. We consider the affine subspace $\mathcal{Y} \subset \mathcal{E}$ of $Y = (H, v, s, D) \in \mathcal{E}$ satisfying

$$\int_0^1 H(x)dx = 1. \tag{42}$$

For general controllability results for 1-D quasilinear hyperbolic systems, let us refer to [57]. However, the results of [57] cannot be applied here since they all deal with cases where the linearized control system around the equilibrium of interest is controllable and, as pointed in [30], the linearized control system of (39) around the equilibrium point (Y_e, u_e) is not controllable. However, as for the Euler control system (18)-(19), the nonlinearity allows to get the controllability: One has the following theorem, where $|\varphi|_1$ denotes the usual C^1-norm of $\varphi \in C^1([0,1])$.

Theorem 3.10 ([22]). *There exists $T > 0$ satisfying the following property: For every ε, there exists $\eta > 0$ such that, for every $Y_0 = (H_0, v_0, s_0, D_0) \in \mathcal{Y}$ and for every $Y_1 = (H_1, v_1, s_1, D_1) \in \mathcal{Y}$ such that*

$$|H_0 - 1|_1 + |v_0|_1 + |s_0| + |D_0| < \eta,\ |H_1 - 1|_1 + |v_1|_1 + |s_1| + |D_1| < \eta,$$

there exists (H, v, s, D, u) satisfying

$$H \text{ and } v \text{ are in } C^1([0,T] \times [0,1]),\ s \text{ and } D \text{ are in } C^1([0,T]),\ u \text{ in } C^0([0,T]),$$
$$(39) \text{ holds for every } (t,x) \in [0,T] \times [0,1],$$
$$(H(0,\cdot), v(0,\cdot), s(0), D(0)) = Y_0 \text{ and } (H(T,\cdot), v(T,\cdot), s(T), D(T)) = Y_1,$$
$$|H(t,\cdot) - 1|_1 + |v(t,\cdot)|_1 + |s(t)| + |D(t)| + |u(t)| < \varepsilon,\ \forall t \in [0,T].$$

Note that, as a consequence of this theorem, it is possible to move the tank from a given position to a desired position with the fluid at rest at the beginning and at the end (see also [30] for such motion for the linearized control system).

For simplicity, we explain some of the main ideas of the proof of Theorem 3.10 on the following toy control problem in finite dimension

$$\dot{y}_1 = y_2,\ \dot{y}_2 = -y_1 + y_2 y_3 + u,\ \dot{y}_3 = y_4,\ \dot{y}_4 = -y_3 + 2y_1 y_2, \tag{43}$$

where the state is $y = (y_1, y_2, y_3, y_4)^{\text{tr}} \in \mathbb{R}^4$ and the control is $u \in \mathbb{R}$. The linearized control system of (43) around $(0,0) \in \mathbb{R}^4 \times \mathbb{R}$ is

$$\dot{y}_1 = y_2,\ \dot{y}_2 = -y_1 + u,\ \dot{y}_3 = y_4,\ \dot{y}_4 = -y_3. \tag{44}$$

This linear control system is again not controllable (look at $(y_3, y_4)^{\text{tr}}$). The analog of Theorem 3.10 for the control system (43) is the following proposition.

Proposition 3.11. *There exists $T > 0$ such that, for every $\varepsilon > 0$, there exists $\eta > 0$ such that, for every $y^0 \in \mathbb{R}^4$ and every $y^1 \in \mathbb{R}^4$ with $|y^0| < \eta$ and $|y^1| < \eta$, there exists $u \in L^\infty((0,T);\mathbb{R})$ satisfying $|u(t)| < \varepsilon$ for almost every $t \in (0,T)$ and the following property: If $y = (y_1, y_2, y_3, y_4)^{\text{tr}} : [0,T] \to \mathbb{R}^4$ is the solution of the Cauchy problem*

$$\dot{y}_1 = y_2,\ \dot{y}_2 = -y_1 + y_2 y_3 + u,\ \dot{y}_3 = y_4,\ \dot{y}_4 = -y_3 + 2y_1 y_2,\ y(0) = y^0,$$

then $y(T) = y^1$.

Let us prove this proposition by using the return method and quasi-static deformations. (Of course, for the finite dimensional control system (43), a simpler method relying on iterated Lie brackets can be used; but one does not know how to adapt this method to the Saint-Venant control system (39).) In order to use the return method, one needs, at least, to know trajectories of the control system (43) such that the linearized control systems around these trajectories are controllable. The simplest trajectories one can consider are the trajectories

$$((y_1^\gamma, y_2^\gamma, y_3^\gamma, y_4^\gamma)^{\mathrm{tr}}, u^\gamma) := ((\gamma, 0, 0, 0)^{\mathrm{tr}}, \gamma), \tag{45}$$

where γ is any real number different from 0. These trajectories are here just equilibrium points (they could be more complicated: for the Saint-Venant and Schrödinger control systems these special trajectories do depend on time). The linearized control system around the trajectory $(y^\gamma, u^\gamma) := ((y_1^\gamma, y_2^\gamma, y_3^\gamma, y_4^\gamma)^{\mathrm{tr}}, u^\gamma)$ is the linear control system

$$\dot{y}_1 = y_2,\ \dot{y}_2 = -y_1 + u,\ \dot{y}_3 = y_4,\ \dot{y}_4 = -y_3 + 2\gamma y_2, \tag{46}$$

Using the usual Kalman rank condition for controllability (Theorem 2.1), one easily checks that this linear control system is small-time locally controllable at $(0,0) \in \mathbb{R}^4 \times \mathbb{R}$ if (and only if) $\gamma \neq 0$. Let us now choose $\gamma \neq 0$ and $\tau_1 > 0$. Then, by this controllability of (46) and Theorem 2.2, there exists $\delta_1 > 0$ such that for every $y^0 \in B(y^\gamma, \delta_1) := \{y \in \mathbb{R}^4; |y - y^\gamma| < \delta_1\}$ and for every y^1 in $B(y^\gamma, \delta_1)$ there exists $u \in L^\infty((0,\tau_1);\mathbb{R})$ such that $|u(t) - \gamma| < \gamma$ for almost every $t \in (0, \tau_1)$ and

$$\big(\dot{y}_1 = y_2,\ \dot{y}_2 = -y_1 + y_2 y_3 + u,\ \dot{y}_3 = y_4,\ \dot{y}_4 = -y_3 + 2y_1 y_2,\ y(0) = y^0\big)$$
$$\Rightarrow (y(\tau_1) = y^1).$$

Let us first deal with the weaker statement where one replaces, in Proposition 3.11, "There exists $T > 0$ such that, for every $\varepsilon > 0$, there exists $\eta > 0$ such that..." by "For every $\varepsilon > 0$, there exist $T > 0$ and $\eta > 0$ such that...". Then, by the continuity of the solutions of the Cauchy problem with respect to the initial condition, it suffices to check that

(i) there exist $\tau_2 > 0$ and a trajectory $(\tilde{y}, \tilde{u}) : [0, \tau_2] \to \mathbb{R}^4 \times \mathbb{R}$ of the control system (43) such that $\tilde{y}(0) = 0$ and $|\tilde{y}(\tau_2) - y^\gamma| < \delta_1$.

(ii) there exist $\tau_3 > \tau_2 + \tau_1$ and a trajectory $(\hat{y}, \hat{u}) : [\tau_2 + \tau_1, \tau_3] \to \mathbb{R}^4 \times \mathbb{R}$ of the control system (43) such that $\hat{y}(\tau_3) = 0$ and $|\hat{y}(\tau_2 + \tau_1) - y^\gamma| < \delta_1$.

In order to prove (i), we consider quasi-static deformations. Let $g \in C^2([0,1];\mathbb{R})$ be such that $g(0) = 0$ and $g(1) = 1$. Let $\tilde{u} : [0, 1/\varepsilon] \to \mathbb{R}$ be defined by $\tilde{u}(t) = \gamma g(\varepsilon t)$. Let $\tilde{y} := (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)^{\mathrm{tr}} : [0, 1/\varepsilon] \to \mathbb{R}^4$ be defined by

$$\dot{\tilde{y}}_1 = \tilde{y}_2,\ \dot{\tilde{y}}_2 = -\tilde{y}_1 + \tilde{y}_2\tilde{y}_3 + \tilde{u},\ \dot{\tilde{y}}_3 = \tilde{y}_4,\ \dot{\tilde{y}}_4 = -\tilde{y}_3 + 2\tilde{y}_1\tilde{y}_2,\ \tilde{y}(0) = 0.$$

One easily checks that

$$\tilde{y}(1/\varepsilon) \to (\gamma, 0, 0, 0)^{\mathrm{tr}} \text{ as } \varepsilon \to 0,$$

which ends the proof of (i).

In order to get (ii) one just needs to modify a little bit the above construction. In order to have the required statement "There exists $T > 0$ such that, for every $\varepsilon > 0$, there exists $\eta > 0$ such that...", one needs some further estimates which are omitted.

Remark 3.12. *If $g(y) := (y_2, -y_1 + \gamma + y_2 y_3, y_3, -y_3 + 2y_1 y_2)^{\mathrm{tr}}$, then the eigenvalues of $g'(y^\gamma)$ are i and $-i$. This is why the quasi-static deformations are so easy to perform. If this linear map had eigenvalues with strictly positive real part, it is still possible to perform in some cases quasi-static deformations by stabilizing the equilibriums by suitable feedbacks, as it has been pointed out in [28] for a parabolic equation.*

The method which we have used in order to prove Proposition 3.11 has still an important drawback: Due to the quasi-static deformation parts, it leads to too conservative estimates on the time T for controllability. Let us now propose another method which gives the optimal estimate on the time T for local controllability. This method, called "power series expansion" has been introduced for the first time in infinite dimension for a KdV control system in [24], a paper motivated by[65]. (For other applications of this method, see [7, 11, 12].) This method consists in looking for "higher order variations" which allows to move in the directions which are missed by the controllability of the linearized control system. These directions are $\pm(0,0,1,0)$ and $\pm(0,0,0,1)$ for the control system (43). Let us describe this method in order to show that Proposition 3.11 holds more precisely for every $T > \pi$. (Again, for the finite dimensional control system (43), a simpler method relying on iterated Lie brackets can be used; but, again, one does not know how to adapt these methods to the PDE control systems considered in [24, 7, 11, 12].)

One first looks to the linearized control system around 0, i.e. the linear control system (44). Let $T > 0$ and let $(e_i)_{i \in \{1,\dots,4\}}$ be the usual basis of $\mathbb{R}^4$. One easily sees that, for every $i \in \{1, 2\}$, there exists $u_i \in L^\infty(0,T)$ such that

$$(\dot{y}_1 = y_2,\ \dot{y}_2 = -y_1 + u_i,\ \dot{y}_3 = y_4,\ \dot{y}_4 = -y_3,\ y(0) = 0) \Rightarrow (y(T) = e_i).$$

Let us assume for the time being that, for every $i \in \{3, 4\}$, there exist $u_i^\pm \in L^\infty(0,T)$ such that

$$(\dot{y}_1 = y_2,\ \dot{y}_2 = -y_1 + u_i^\pm,\ \dot{y}_3 = y_4,\ \dot{y}_4 = -y_3 + 2y_1 y_2,\ y(0) = 0) \Rightarrow (y(T) = \pm e_i). \quad (47)$$

Note that in the left hand side of (47), we have put $\dot{y}_2 = -y_1 + u_i^\pm$ and not $\dot{y}_2 = -y_1 + y_2 y_3 + u_i^\pm$. The reason is that the y_i with $i \in \{1, 2\}$ and u

are considered to be of order 1, and the y_i with $i \in \{3,4\}$ are considered to be of order 2. With this choice of scaling, the left hand side of (47) is the approximation of order 2 of the control system (43). Then, let $b := \sum_{i=1}^{4} b_i e_i$. Let, for $i \in \{3,4\}$,

$$u_i := u_i^+ \text{ if } b_i \geq 0 \text{ and } u_i := u_i^- \text{ if } b_i < 0.$$

Let $u \in L^\infty(0,T)$ be defined by $u := \sum_{i\in\{1,2\}} b_i u_i + \sum_{i\in\{3,4\}} |b_i|^{1/2} u_i$. Let $y : [0,T] \to \mathbb{R}^4$ be the solution of the Cauchy problem

$$\dot{y}_1 = y_2,\ \dot{y}_2 = -y_1 + y_2 y_3 + u,\ \dot{y}_3 = y_4,\ \dot{y}_4 = -y_3 + 2y_1 y_2, y(0) = 0.$$

Then straightforward estimates lead to $y(T) = b + o(b)$ as $b \to 0$. Hence, using the Brouwer fixed point theorem (and standard estimates on ordinary differential equations), one gets the local controllability of (43) (around $(0,0) \in \mathbb{R}^4 \times \mathbb{R}$) in the considered time T (and therefore Proposition 3.11 for that T). It then remains to prove the existence of $u_i^\pm \in L^\infty(0,T)$ for every $i \in \{3,4\}$ and for $T > \pi$. Easy computations show that

$$(\dot{y}_1 = y_2,\ \dot{y}_3 = y_4,\ \dot{y}_4 = -y_3 + 2y_1 y_2,\ y(0) = 0)$$
$$\Rightarrow \left(y_3(T) = \int_0^T y_1^2(t)\cos(T-t)dt,\ y_4(T) = y_1^2(T) - \int_0^T y_1^2(t)\sin(T-t)dt \right).$$

Then, taking $u_i^\pm := y_1 + \dot{y}_2$, it is not hard to get that the existence of $u_i^\pm \in L^\infty(0,T)$ for every $i \in \{3,4\}$ holds if (and only if) $T > \pi$. Unfortunately, one does not know how to use this power series expansion method for the Saint-Venant control system (39) and the optimal value of T in Theorem 3.10 is not known.

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