DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **14**, Number **4**, November **2010**

doi:10.3934/dcdsb.2010.14.1337

pp. **1337–1359**

ANALYSIS OF A CONSERVATION LAW MODELING A HIGHLY RE-ENTRANT MANUFACTURING SYSTEM

Jean-Michel Coron

Institut universitaire de France and Université Pierre et Marie Curie-Paris VI
UMR 7598, Laboratoire Jacques-Louis Lions, 4, place Jussieu
Paris, F-75005, France

Matthias Kawski

Arizona State University
Tempe, Arizona 85287-1804, USA

Zhiqiang Wang

Fudan University and Université Pierre et Marie Curie-Paris VI
UMR 7598, Laboratoire Jacques-Louis Lions, 4, place Jussieu
Paris, F-75005, France

Abstract. This article studies a hyperbolic conservation law that models a highly re-entrant manufacturing system as encountered in semi-conductor production. Characteristic features are the nonlocal character of the velocity and that the influx and outflux constitute the control and output signal, respectively. We prove the existence and uniqueness of solutions for L^1-data, and study their regularity properties. We also prove the existence of optimal controls that minimizes in the L^2-sense the mismatch between the actual and a desired output signal. Finally, the time-optimal control for a step between equilibrium states is identified and proven to be optimal.

1. **Introduction and prior work.** This article studies optimal control problems governed by the scalar hyperbolic conservation law

$$\partial_t \rho(t,x) + \partial_x \left(\lambda(W(t))\, \rho(t,x)\right) = 0 \quad \text{where} \quad W(t) = \int_0^1 \rho(t,x)\, dx, \tag{1}$$

on a rectangular domain $[0,T]\times[0,1]$ or the semi-infinite strip $[0,\infty)\times[0,1]$. We assume that $\lambda(\cdot) \in C^1([0,+\infty);(0,+\infty))$ in the whole paper.

This work is motivated by problems arising in the control of semiconductor manufacturing systems which are characterized by their highly re-entrant character, see below for more details. In the manufacturing system the natural control input is the influx, which suggests the boundary conditions

$$\rho(0,x) = \rho_0(x), \text{ for } 0 \le x \le 1, \text{ and } \rho(t,0)\lambda(W(t)) = u(t), \text{ for } t \ge 0. \tag{2}$$

2000 *Mathematics Subject Classification.* Primary: 35L65, 49J20, 93C20.

Key words and phrases. Optimal control, conservation law, re-entrant manufacturing system.

The first author was partially supported by the “Agence Nationale de la Recherche” (ANR), Project C-QUID, number BLAN-3-139579. The second author was partially supported by the National Science Foundation through DMS 05-09030, the University Pierre and Marie Curie-Paris VI and the Foundation Sciences Mathématiques de Paris. The third author was partially supported by the Natural Science Foundation of China grant 10701028 and the Foundation Sciences Mathématiques de Paris.

Various different choices of the space of admissible controls are of both practical and mathematical interest, each leading to distinct mathematical problems. Motivated by this application from manufacturing systems, natural control objectives are to minimize the *error signal* that is the difference between a given demand forecast y_d and the actual out-flux $y(t) = \lambda(W(t))\rho(t, 1)$. An alternative to this problem modeling a *perishable demand*, is the similar problem that permits *backlogs*. In that case, the objective is to minimize in a suitable sense the size of the different error signal

$$\beta(t) = \int_0^t y_d(s)\, ds - \int_0^t \lambda(W(s))\rho(s, 1)\, ds, \tag{3}$$

while keeping the state $\rho(\cdot, x)$ bounded. This article only considers the problem of perishable demand and the minimization in the L^2-sense.

Partial differential equations models for such manufacturing systems are motivated by the very high volume (number of parts manufactured per unit time) and the very large number of consecutive production steps which typically number in the many hundreds. They are popular due to their superior analytic properties and the availability of efficient numerical tools for simulation. For more detailed discussions see e.g. [3, 4, 5, 6, 16, 17, 19]. In many aspects these models are very similar to those of traffic flows, compare e.g. [12].

The study of hyperbolic conservation laws, and especially of control systems governed by such laws, have a rich history. A modern introduction to the subject is the text [8]. From a mathematical perspective, the choice of spaces in which to consider the conservations laws (and their data) provides for distinct levels of challenges. Fundamental are question of wellposedness, regularity properties of solutions, controllability, existence, uniqueness and regularity of optimal controls. Existence of solutions, regularity and well-posedness of nonlinear conservation laws have been widely studied under diverse sets of hypotheses, commonly in the context of vector values systems of conservation laws, see e.g. [2, 7, 9]. Further results on uniqueness may be found in [11], while [10] introduced an a distinct notion of differentiability of the solution of hyperbolic systems. For the controllability of linear hyperbolic systems, see, in particular, the important survey [22]. The attainable sets of nonlinear conservation laws are studied in [1, 15, 18, 20, 21], while [14] provides a comprehensive survey of controllability that also includes nonlinear conservation laws.

This article is, in particular, motivated by the recent work [19] which, among others, considered the optimal control problem of minimizing $\|y - y_d\|_{L^2(0,T)}$ (the L^2 norm of the difference between a demand forecast and the actual outflux). That work derived necessary conditions and used these to numerically compute optimal controls corresponding to piecewise constant desired outputs y_d.

The organization of the following sections is as follows: First we rigorously prove the existence of weak solutions of the Cauchy problem for the conservation law (1) for the case when the initial data and boundary condition (2) lie in $L^1(0, 1)$ and $L^1(0, T)$, respectively. Next we establish the existence and uniqueness of solutions for the optimal control problem of minimizing the L^2-norm of the difference between any desired L^2-demand forecast y_d and actual outflux $y(t) = \lambda(W(t)) \cdot \rho(t, 1)$. Finally, in the classical special case where

$$\lambda(W) = \frac{1}{1+W}, \tag{4}$$

we prove that the natural candidate control for transferring the system from one equilibrium state to another one is indeed time-optimal.

While preparing the final version of this article, the authors received a copy of the related manuscript [13] which is also motivated in part by [3, 4, 19] and which addresses wellposedness for systems of hyperbolic conservation laws with a nonlocal speed on all of $\mathbb{R}^n$. It also includes a study of the solutions with respect to the initial datum and a necessary condition for the optimality of integral functionals. There are substantial differences between [13] and our paper, especially the treatment of the boundary conditions and the method of proof.

2. Existence, uniqueness, and regularity of solutions in L^1.

2.1. Technical preliminaries and notation. For any $\lambda \in C^1([0,+\infty);(0,+\infty))$ define the functions $\widetilde{\lambda}, \overline{\lambda} \in C^0([0,\infty);(0,\infty))$ and $d \in C^0([0,\infty);[0,\infty))$ with respect to λ as

$$\widetilde{\lambda}(M) := \inf_{0\le W\le M} \lambda(W),\ \overline{\lambda}(M) := \sup_{0\le W\le M} \lambda(W),\ d(M) := \sup_{0\le W\le M} |\lambda'(W)|. \tag{5}$$

For convenience we extend λ to all of $\mathbb{R}$ in such a way that this extension, still denoted λ, is in $C^1(\mathbb{R};(0,+\infty))$.

2.2. Weak solutions of the Cauchy problem. First we recall, from [14, Section 2.1], the usual definition of a weak solution to the Cauchy problem (1) and (2).

Definition 2.1. Let $T > 0$, $\rho_0 \in L^1(0,1)$ and $u \in L^1(0,T)$ be given. A weak solution of the Cauchy problem (1) and (2) is a function $\rho \in C^0([0,T];L^1(0,1))$ such that, for every $\tau \in [0,T]$ and every $\varphi \in C^1([0,\tau]\times[0,1])$ such that

$$\varphi(\tau,x) = 0, \forall x \in [0,1] \quad \text{and} \quad \varphi(t,1) = 0, \forall t \in [0,\tau], \tag{6}$$

one has

$$\begin{aligned}\int_0^\tau \int_0^1 \rho(t,x)(\varphi_t(t,x) + \lambda(W(t))\varphi_x(t,x))dxdt \\ + \int_0^\tau u(t)\varphi(t,0)dt + \int_0^1 \rho_0(x)\varphi(0,x)dx = 0.\end{aligned} \tag{7}$$

One has the following lemma, which will be useful to prove a uniqueness result for the Cauchy problem (1) and (2).

Lemma 2.2. *If $\rho \in C^0([0,T];L^1(0,1))$ is a weak solution to the Cauchy problem (1) and (2), then for every $\tau \in [0,T]$ and every $\varphi \in C^1([0,\tau]\times[0,1])$ such that*

$$\varphi(t,1) = 0, \forall t \in [0,\tau], \tag{8}$$

one has

$$\begin{aligned}\int_0^\tau \int_0^1 \rho(t,x)(\varphi_t(t,x) + \lambda(W(t))\varphi_x(t,x))dxdt + \int_0^\tau u(t)\varphi(t,0)dt \\ - \int_0^1 \rho(\tau,x)\varphi(\tau,x)dx + \int_0^1 \rho_0(x)\varphi(0,x)dx = 0.\end{aligned} \tag{9}$$

Proof. The case $\tau = 0$ is trivial. For every $\tau \in (0, T]$ and $\varepsilon \in (0, \tau)$, let $\eta_\varepsilon \in C^1([0,\tau])$ be such that

$$\eta_\varepsilon(\tau) = 0 \quad \text{and} \quad \eta_\varepsilon(t) = 1,\ \forall t \in [0, \tau - \varepsilon] \quad \text{and} \quad \eta_\varepsilon'(t) \leq 0,\ \forall t \in [0, \tau]. \tag{10}$$

It is easy to prove that, for every $h \in C^0([0,\tau])$,

$$\lim_{\varepsilon \to 0} \int_{\tau-\varepsilon}^{\tau} \eta_\varepsilon'(t) h(t) dt = -h(\tau). \tag{11}$$

Then, for every $\varphi \in C^1([0,\tau] \times [0,1])$ satisfying (8), let $\varphi_\varepsilon(t,x) := \eta_\varepsilon(t)\varphi(t,x)$. This obviously verifies

$$\varphi_\varepsilon(\tau, x) = 0, \forall x \in [0,1] \quad \text{and} \quad \varphi_\varepsilon(t,1) = 0, \forall t \in [0,\tau]. \tag{12}$$

Since $\rho \in C^0([0,T]; L^1(0,1))$ is a weak solution to the Cauchy problem (1) and (2), we have

$$\begin{aligned} \int_0^\tau \int_0^1 \rho(t,x)((\varphi_\varepsilon)_t(t,x) + \lambda(W(t))(\varphi_\varepsilon)_x(t,x))dxdt \\ + \int_0^\tau u(t)(\varphi_\varepsilon)(t,0)dt + \int_0^1 \rho_0(x)(\varphi_\varepsilon)(0,x)dx = 0. \end{aligned} \tag{13}$$

Using the definition of φ_ε, (10) and (13), one has

$$\begin{aligned} &\int_0^\tau \int_0^1 \rho(t,x)(\varphi_t(t,x) + \lambda(W(t))\varphi_x(t,x))dxdt \\ &\qquad + \int_0^\tau u(t)\varphi(t,0)dt + \int_0^1 \rho_0(x)\varphi(0,x)dx \\ &= \int_{\tau-\varepsilon}^\tau \int_0^1 (1-\eta_\varepsilon(t))\rho(t,x)(\varphi_t(t,x) + \lambda(W(t))\varphi_x(t,x))dxdt \\ &\qquad + \int_{\tau-\varepsilon}^\tau (1-\eta_\varepsilon(t))u(t)\varphi(t,0)dt - \int_{\tau-\varepsilon}^\tau \int_0^1 \eta_\varepsilon'(t)\rho(t,x)\varphi(t,x)dxdt. \end{aligned} \tag{14}$$

Observing that $\rho \in C^0([0,T]; L^1(0,1))$, $\lambda \in C^1(\mathbb{R}; (0,\infty))$ and $\varphi \in C^1([0,\tau] \times [0,1])$, we point out that the functions $W(\cdot) = \int_0^1 \rho(\cdot, x)dx$, $\int_0^1 \rho(\cdot,x)\varphi(\cdot,x)dx$ and $\lambda(W(\cdot))$ are all in $C^0([0,T])$.

We can estimate the first two terms on the right hand side of (14) as

$$\Big| \int_{\tau-\varepsilon}^\tau \int_0^1 (1-\eta_\varepsilon(t))\rho(t,x)(\varphi_t(t,x) + \lambda(W(t))\varphi_x(t,x))dxdt \Big| \leq K\varepsilon, \tag{15}$$

and

$$\Big| \int_{\tau-\varepsilon}^\tau (1-\eta_\varepsilon(t))u(t)\varphi(t,0)dt \Big| \leq K \int_{\tau-\varepsilon}^\tau u(t)dt, \tag{16}$$

where K is a constant independent of ε. While for the last term on the right hand side of (14), we get from (11) that

$$\begin{aligned} \int_{\tau-\varepsilon}^\tau \int_0^1 \eta_\varepsilon'(t)\rho(t,x)\varphi(t,x)dxdt &= \int_{\tau-\varepsilon}^\tau \eta_\varepsilon'(t)\Big(\int_0^1 \rho(t,x)\varphi(t,x)dx\Big)dt \\ &\longrightarrow -\int_0^1 \rho(\tau,x)\varphi(\tau,x)dx \quad \text{as} \quad \varepsilon \to 0. \end{aligned} \tag{17}$$

In view of (15)-(17), letting $\varepsilon \to 0$ in (14) one gets (9). ∎

Theorem 2.3. *If $\rho_0 \in L^1(0,1)$ and $u \in L^1(0,T)$ are nonnegative almost everywhere, then the Cauchy problem (1) and (2) admits a unique weak solution $\rho \in C^0([0,T];L^1(0,1))$, which is also nonnegative almost everywhere in $Q = [0,T] \times [0,1]$.*

Proof. We first prove the existence of weak solution for small time: there exists a small $\delta \in (0,T]$ such that the Cauchy problem (1) and (2) has a weak solution $\rho \in C^0([0,\delta];L^1(0,1))$. The idea is to find first the characteristic curve $\xi = \xi(t)$ passing through $(0,0)$, then construct a solution to the Cauchy problem.

Let

$$\begin{aligned} \Omega_{\delta,M} := \Big\{ &\xi \in C^0([0,\delta]) \colon \xi(0) = 0, \\ &\widetilde{\lambda}(M) \le \frac{\xi(s)-\xi(t)}{s-t} \le \overline{\lambda}(M), \forall s,t \in [0,\delta],\, s > t \Big\}, \end{aligned} \tag{18}$$

where $\widetilde{\lambda}, \overline{\lambda}$ are defined by (5) and

$$M := \|u\|_{L^1(0,T)} + \|\rho_0\|_{L^1(0,1)}. \tag{19}$$

We point out here that the case $d(M) = 0$ (by (5), λ is a constant in $[0,M]$) is trivial. We only prove Theorem 2.3 for the case $d(M) > 0$.

We define a map $F : \Omega_{\delta,M} \to C^0([0,\delta])$, $\xi \mapsto F(\xi)$, as

$$F(\xi)(t) := \int_0^t \lambda\Big(\int_0^s u(\sigma)d\sigma + \int_0^{1-\xi(s)} \rho_0(x)dx\Big)ds,\ \forall \xi \in \Omega_{\delta,M},\ \forall t \in [0,\delta]. \tag{20}$$

It is obvious that F maps into $\Omega_{\delta,M}$ itself if

$$0 < \delta < T \text{ and } \delta < \frac{1}{\overline{\lambda}(M)}. \tag{21}$$

Now we prove that, if δ is small enough, F is a contraction mapping on $\Omega_{\delta,M}$ with respect to the C^0 norm defined by

$$\|\xi\|_{C^0([0,\delta])} := \sup_{0 \le t \le \delta} |\xi(t)|.$$

Let $\xi_1, \xi_2 \in \Omega_{\delta,M}$. We define $\overline{\xi}_1 \in C^0([0,\delta])$ and $\overline{\xi}_2 \in C^0([0,\delta])$ by $\overline{\xi}_1(t) := \max\{\xi_1(t), \xi_2(t)\}$ and $\overline{\xi}_2(t) := \min\{\xi_1(t), \xi_2(t)\}$. By (5) and changing the order of the integrations (see Figure 1), we have

$$\begin{aligned} |F(\xi_2)(t) - F(\xi_1)(t)| &\le d(M) \int_0^t \Big| \int_{1-\xi_1(s)}^{1-\xi_2(s)} \rho_0(x)dx \Big| ds \\ &= d(M) \int_{1-\overline{\xi}_1(t)}^{1-\overline{\xi}_2(t)} \rho_0(x)(t - \overline{\xi}_1^{-1}(1-x))dx \\ &\quad + d(M) \int_{1-\overline{\xi}_2(t)}^{1} \rho_0(x)(\overline{\xi}_2^{-1}(1-x) - \overline{\xi}_1^{-1}(1-x))dx \\ &\le d(M) \int_{1-\overline{\xi}_1(t)}^{1-\overline{\xi}_2(t)} \rho_0(x)dx \cdot (\overline{\xi}_2^{-1}(\overline{\xi}_2(t)) - \overline{\xi}_1^{-1}(\overline{\xi}_2(t))) \\ &\quad + d(M) \int_{1-\overline{\xi}_2(t)}^{1} \rho_0(x)(\overline{\xi}_2^{-1}(1-x) - \overline{\xi}_1^{-1}(1-x))dx \\ &\le d(M) \int_{1-\overline{\xi}_1(t)}^{1} \rho_0(x)dx \cdot \sup_{0 \le y \le \overline{\xi}_2(t)} (\overline{\xi}_2^{-1}(y) - \overline{\xi}_1^{-1}(y)). \end{aligned} \tag{22}$$

Using the definitions of $\overline{\xi}_1, \overline{\xi}_2$ and of $\Omega_{\delta,M}$, we obtain that, for every $y \in [0, \overline{\xi}_2(t)]$ (see Figure 2),

$$
\begin{aligned}
0 \leq\, & \overline{\xi}_2^{-1}(y) - \overline{\xi}_1^{-1}(y) \\
=& \Big(\overline{\xi}_2^{-1}(y) - \frac{\overline{\xi}_1^{-1}(y) + \overline{\xi}_2^{-1}(y)}{2}\Big) + \Big(\frac{\overline{\xi}_1^{-1}(y) + \overline{\xi}_2^{-1}(y)}{2} - \overline{\xi}_1^{-1}(y)\Big) \\
\leq& \frac{1}{\widetilde{\lambda}(M)}\Big(y - \overline{\xi}_2\big(\frac{\overline{\xi}_1^{-1}(y) + \overline{\xi}_2^{-1}(y)}{2}\big)\Big) + \frac{1}{\widetilde{\lambda}(M)}\Big(\overline{\xi}_1\big(\frac{\overline{\xi}_1^{-1}(y) + \overline{\xi}_2^{-1}(y)}{2}\big) - y\Big) \\
=& \frac{1}{\widetilde{\lambda}(M)}\Big(\overline{\xi}_1\big(\frac{\overline{\xi}_1^{-1}(y) + \overline{\xi}_2^{-1}(y)}{2}\big) - \overline{\xi}_2\big(\frac{\overline{\xi}_1^{-1}(y) + \overline{\xi}_2^{-1}(y)}{2}\big)\Big) \\
\leq& \frac{1}{\widetilde{\lambda}(M)}\|\xi_1 - \xi_2\|_{C^0([0,\delta])}.
\end{aligned}
\tag{23}
$$

Therefore,

$$
\begin{aligned}
|F(\xi_2)(t) - F(\xi_1)(t)| &\leq \frac{d(M)}{\widetilde{\lambda}(M)} \int_{1-\overline{\xi}_1(t)}^{1} \rho_0(x)dx \cdot \|\xi_1 - \xi_2\|_{C^0([0,\delta])} \\
&\leq \frac{d(M)}{\widetilde{\lambda}(M)} \int_{1-\overline{\lambda}(M)\delta}^{1} \rho_0(x)dx \cdot \|\xi_1 - \xi_2\|_{C^0([0,\delta])}.
\end{aligned}
\tag{24}
$$

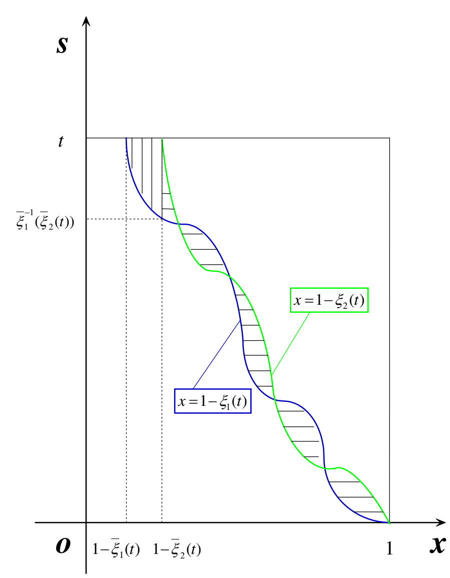

FIGURE 1. Change order of integrations for x and s in (22)

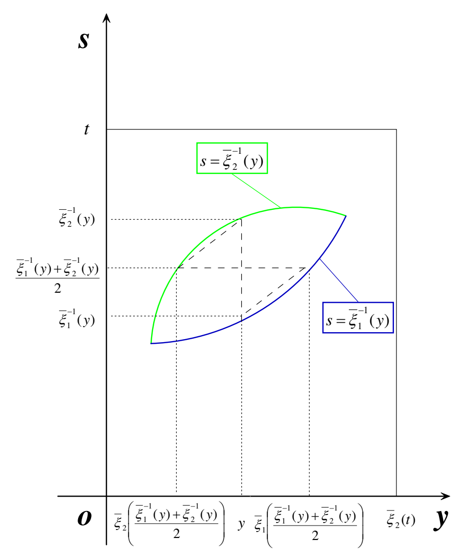

FIGURE 2. Using property of $\Omega_{\delta,M}$ in estimate (23)

Since $\rho_0 \in L^1(0,1)$, we can choose $\delta \in (0,1)$ small enough such that

$$
\int_{1-\overline{\lambda}(M)\delta}^{1} \rho_0(x)\,dx < \frac{\widetilde{\lambda}(M)}{2d(M)}. \tag{25}
$$

Then

$$\|F(\xi_1)-F(\xi_2)\|_{C^0([0,\delta])}\le \frac{1}{2}\|\xi_1-\xi_2\|_{C^0([0,\delta])}. \tag{26}$$

By means of the contraction mapping principle, there exists a unique fixed point $\xi = F(\xi)$ in $\Omega_{\delta,M}$. By (20), the fix point ξ is an increasing function in $C^1([0,\delta])$, and one has

$$\xi'(t)=\lambda(\int_0^t u(\sigma)d\sigma+\int_0^{1-\xi(t)}\rho_0(x)dx),\quad \forall t\in[0,\delta]. \tag{27}$$

Then we define a function ρ by

$$\rho(t,x)=\begin{cases}\rho_0(x-\xi(t)), & 0\le\xi(t)\le x\le 1, 0\le t\le\delta,\\ \dfrac{u(\xi^{-1}(\xi(t)-x))}{\xi'(\xi^{-1}(\xi(t)-x))}, & 0\le x\le\xi(t)\le 1, 0\le t\le\delta,\end{cases} \tag{28}$$

which is obviously nonnegative almost everywhere. Direct computations give that, for every $t\in[0,\delta]$,

$$\begin{aligned}0\le W(t)&=\int_0^1\rho(t,x)dx\\&=\int_0^{\xi(t)}\frac{u(\xi^{-1}(\xi(t)-x))}{\xi'(\xi^{-1}(\xi(t)-x))}dx+\int_{\xi(t)}^1\rho_0(x-\xi(t))dx\\&=\int_0^t u(\sigma)d\sigma+\int_0^{1-\xi(t)}\rho_0(y)dy\\&\le\|u\|_{L^1(0,T)}+\|\rho_0\|_{L^1(0,1)}=M.\end{aligned} \tag{29}$$

Using (5), (27) and (29), we obtain the following estimates of ξ' from above and below:

$$0<\widetilde{\lambda}(M)\le\xi'(t)=\lambda(W(t))\le\overline{\lambda}(M),\quad\forall t\in[0,\delta]. \tag{30}$$

We now prove that $\rho\in C^0([0,\delta];L^1(0,1))$. For every $s,t\in[0,\delta]$ with $s\ge t$,

$$\begin{aligned}&\int_0^1|\rho(s,x)-\rho(t,x)|dx\\&\le\int_0^{\xi(t)}\Big|\frac{u(\xi^{-1}(\xi(s)-x))}{\xi'(\xi^{-1}(\xi(s)-x))}-\frac{u(\xi^{-1}(\xi(t)-x))}{\xi'(\xi^{-1}(\xi(t)-x))}\Big|dx\\&\quad+\int_{\xi(t)}^{\xi(s)}|\rho(s,x)-\rho(t,x)|dx+\int_{\xi(s)}^1|\rho_0(x-\xi(s))-\rho_0(x-\xi(t))|dx.\end{aligned} \tag{31}$$

As for the first term on the right hand side of (31), we choose $\{u^n\}_{n=1}^{\infty}\subset C^1([0,T])$ which converges to u in $L^1(0,T)$, then we have

$$\begin{aligned}
&\int_0^{\xi(t)} \Big|\frac{u(\xi^{-1}(\xi(s)-x))}{\xi'(\xi^{-1}(\xi(s)-x))}-\frac{u(\xi^{-1}(\xi(t)-x))}{\xi'(\xi^{-1}(\xi(t)-x))}\Big|dx\\
\leq &\int_0^{\xi(t)} \Big|\frac{u(\xi^{-1}(\xi(s)-x))}{\xi'(\xi^{-1}(\xi(s)-x))}-\frac{u^n(\xi^{-1}(\xi(s)-x))}{\xi'(\xi^{-1}(\xi(s)-x))}\Big|dx\\
&+\int_0^{\xi(t)} \Big|\frac{u^n(\xi^{-1}(\xi(s)-x))}{\xi'(\xi^{-1}(\xi(s)-x))}-\frac{u^n(\xi^{-1}(\xi(t)-x))}{\xi'(\xi^{-1}(\xi(t)-x))}\Big|dx\\
&+\int_0^{\xi(t)} \Big|\frac{u^n(\xi^{-1}(\xi(t)-x))}{\xi'(\xi^{-1}(\xi(t)-x))}-\frac{u(\xi^{-1}(\xi(t)-x))}{\xi'(\xi^{-1}(\xi(t)-x))}\Big|dx\\
\leq &\Big(\int_{\xi^{-1}(\xi(s)-\xi(t))}^s+\int_0^t\Big)|u(\sigma)-u^n(\sigma)|d\sigma\\
&+\int_0^{\xi(t)} \Big|\frac{u^n(\xi^{-1}(\xi(s)-x))}{\xi'(\xi^{-1}(\xi(s)-x))}-\frac{u^n(\xi^{-1}(\xi(t)-x))}{\xi'(\xi^{-1}(\xi(t)-x))}\Big|dx\\
\leq &2\int_0^T |u(\sigma)-u^n(\sigma)|d\sigma\\
&+\int_0^{\xi(t)} \Big|\frac{u^n(\xi^{-1}(\xi(s)-x))}{\xi'(\xi^{-1}(\xi(s)-x))}-\frac{u^n(\xi^{-1}(\xi(t)-x))}{\xi'(\xi^{-1}(\xi(t)-x))}\Big|dx.
\end{aligned} \tag{32}$$

By (30),

$$\begin{aligned}
&\int_0^{\xi(t)} \Big|\frac{u^n(\xi^{-1}(\xi(s)-x))}{\xi'(\xi^{-1}(\xi(s)-x))}-\frac{u^n(\xi^{-1}(\xi(t)-x))}{\xi'(\xi^{-1}(\xi(t)-x))}\Big|dx\\
\leq &\int_0^{\xi(t)} \Big|\frac{u^n(\xi^{-1}(\xi(s)-x))-u^n(\xi^{-1}(\xi(t)-x))}{\xi'(\xi^{-1}(\xi(s)-x))}\Big|dx\\
&+\int_0^{\xi(t)} \Big|u^n(\xi^{-1}(\xi(t)-x))\Big(\frac{1}{\xi'(\xi^{-1}(\xi(s)-x))}-\frac{1}{\xi'(\xi^{-1}(\xi(t)-x))}\Big)\Big|dx\\
\leq &C_n|\xi(s)-\xi(t)|+C_n\int_0^{\xi(t)}\int_{\xi^{-1}(\xi(t)-x)}^{\xi^{-1}(\xi(s)-x)} u(\sigma)d\sigma dx\\
&+C_n\int_0^{\xi(t)}\int_{1-\xi(s)+x}^{1-\xi(t)+x}\rho_0(y)dydx,
\end{aligned} \tag{33}$$

where C_n is a constant independent of s and t but depending on u^n. By changing the order of integrations, we obtain furthermore (see Figure 3)

$$\begin{aligned}
&\int_0^{\xi(t)}\int_{\xi^{-1}(\xi(t)-x)}^{\xi^{-1}(\xi(s)-x)} u(\sigma)d\sigma dx\\
=&\Big(\int_0^{\xi^{-1}(\xi(s)-\xi(t))}\int_{\xi(t)-\xi(\sigma)}^{\xi(t)}+\int_{\xi^{-1}(\xi(s)-\xi(t))}^t\int_{\xi(t)-\xi(\sigma)}^{\xi(s)-\xi(\sigma)}\\
&+\int_t^s\int_0^{\xi(s)-\xi(\sigma)}\Big)u(\sigma)dxd\sigma\\
\leq&\Big(\int_0^{\xi^{-1}(\xi(s)-\xi(t))}+\int_{\xi^{-1}(\xi(s)-\xi(t))}^t+\int_t^s\Big)u(\sigma)d\sigma\cdot|\xi(s)-\xi(t)|\\
\leq&\|u\|_{L^1(0,T)}\cdot|\xi(s)-\xi(t)|
\end{aligned} \tag{34}$$

and (see Figure 4)

$$
\begin{aligned}
&\int_0^{\xi(t)}\int_{1-\xi(s)+x}^{1-\xi(t)+x}\rho_0(y)dydx\\
&=\Big(\int_{1-\xi(s)}^{1-\xi(t)}\int_0^{\xi(s)-1+y}+\int_{1-\xi(t)}^{1-\xi(s)+\xi(t)}\int_{\xi(t)-1+y}^{\xi(s)-1+y}\\
&\quad+\int_{1-\xi(s)+\xi(t)}^{1}\int_{\xi(t)-1+y}^{\xi(t)}\Big)\rho_0(y)dxdy \qquad (35)\\
&\le\Big(\int_{1-\xi(s)}^{1-\xi(t)}+\int_{1-\xi(t)}^{1-\xi(s)+\xi(t)}+\int_{1-\xi(s)+\xi(t)}^{1}\Big)\rho_0(y)dy\cdot|\xi(s)-\xi(t)|\\
&\le\|\rho_0\|_{L^1(0,1)}\cdot|\xi(s)-\xi(t)|.
\end{aligned}
$$

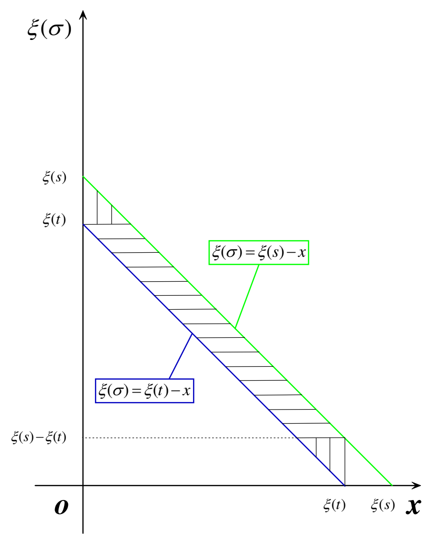

FIGURE 3. Change order of integration on σ and x in (34)

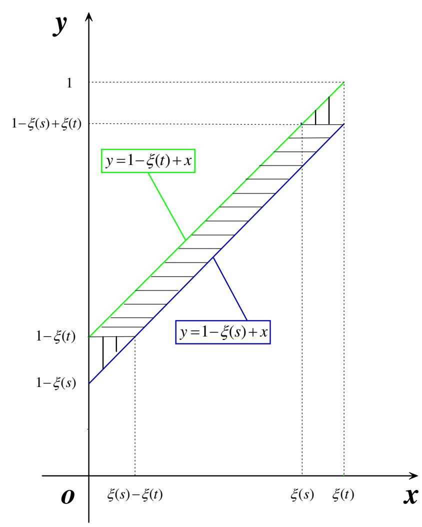

FIGURE 4. Change order of integration on y and x in (35)

As for the second term on the right hand side of (31), it is easy to get that

$$
\begin{aligned}
\int_{\xi(t)}^{\xi(s)}|\rho(s,x)-\rho(t,x)|dx &\le \int_{\xi(t)}^{\xi(s)}\rho(s,x)dx+\int_{\xi(t)}^{\xi(s)}\rho(t,x)dx\\
&=\int_0^{\xi^{-1}(\xi(s)-\xi(t))}u(\sigma)d\sigma+\int_0^{\xi(s)-\xi(t)}\rho_0(y)dy.
\end{aligned} \qquad (36)
$$

As for the last term on the right hand side of (31), we choose $\{\rho_0^n\}_{n=1}^\infty\subset C^1([0,1])$ which converges to ρ_0 in $L^1(0,1)$, then we have

$$
\begin{aligned}
&\int_{\xi(s)}^{1} |\rho_0(x-\xi(s)) - \rho_0(x-\xi(t))|dx \\
\leq &\int_{\xi(s)}^{1} |\rho_0(x-\xi(s)) - \rho_0^n(x-\xi(s))|dx \\
&+ \int_{\xi(s)}^{1} |\rho_0^n(x-\xi(s)) - \rho_0^n(x-\xi(t))|dx \\
&+ \int_{\xi(s)}^{1} |\rho_0^n(x-\xi(t)) - \rho_0(x-\xi(t))|dx \\
\leq &\Big(\int_0^{1-\xi(s)} + \int_{\xi(s)-\xi(t)}^{1-\xi(t)} \Big) |\rho_0(y) - \rho_0^n(y)|dy + D_n|\xi(s)-\xi(t)| \\
\leq &\, 2\int_0^1 |\rho_0(y) - \rho_0^n(y)|dy + D_n|\xi(s)-\xi(t)|,
\end{aligned} \tag{37}
$$

where D_n is a constant independent of s and t but depending on ρ_0^n.

Using (19) together with the estimates (31) to (37), we obtain for any $s, t \in [0, \delta]$ with $s \geq t$,

$$
\begin{aligned}
&\int_0^1 |\rho(s,x) - \rho(t,x)|dx \\
\leq &\, 2\int_0^T |u(\sigma) - u^n(\sigma)|d\sigma + C_n|\xi(s)-\xi(t)| + M|\xi(s)-\xi(t)| \\
&+ \int_0^{\xi^{-1}(\xi(s)-\xi(t))} u(\sigma)d\sigma + \int_0^{\xi(s)-\xi(t)} \rho_0(y)dy \\
&+ 2\int_0^1 |\rho_0(y) - \rho_0^n(y)|dy + D_n|\xi(s)-\xi(t)|.
\end{aligned} \tag{38}
$$

We can choose u^n and ρ_0^n such that $\int_0^T |u(\sigma) - u^n(\sigma)|d\sigma$ and $\int_0^1 |\rho_0(y) - \rho_0^n(y)|dy$ are small as we want. Then according to (30) and the fact that $u \in L^1(0,T)$ and $\rho_0 \in L^1(0,1)$, the right hand side of (38) is sufficiently small if s and t are close enough to each other. This proves that the function ρ defined by (28) belongs to $C^0([0,\delta]; L^1(0,1))$.

Next, we prove that ρ defined by (28) is a weak solution to the Cauchy problem (1) and (2). Let $\delta' \in [0,\delta]$. For any $\varphi \in C^1([0,\delta'] \times [0,1])$ with $\varphi(\delta', x) \equiv 0$ and $\varphi(t,1) \equiv 0$, let

$$
A := \int_0^{\delta'} \int_0^1 \rho(t,x)(\varphi_t(t,x) + \lambda(W(t))\varphi_x(t,x))dxdt. \tag{39}
$$

Then we have

$$
\begin{aligned}
A = &\int_0^{\delta'} \int_0^{\xi(t)} \frac{u(\xi^{-1}(\xi(t)-x))}{\xi'(\xi^{-1}(\xi(t)-x))} (\varphi_t(t,x) + \lambda(W(t))\varphi_x(t,x))dxdt \\
&+ \int_0^{\delta'} \int_{\xi(t)}^1 \rho_0(x-\xi(t))(\varphi_t(t,x) + \lambda(W(t))\varphi_x(t,x))dxdt,
\end{aligned} \tag{40}
$$

and thus

$$\begin{aligned}
A &= \int_0^{\delta'} \int_0^t u(\sigma)(\varphi_t(t,\xi(t)-\xi(\sigma)) + \lambda(W(t))\varphi_x(t,\xi(t)-\xi(\sigma))d\sigma dt \\
&\quad + \int_0^{\delta'} \int_0^{1-\xi(t)} \rho_0(y)(\varphi_t(t,\xi(t)+y) + \lambda(W(t))\varphi_x(t,\xi(t)+y))dydt \\
&= \int_0^{\delta'} \int_\sigma^{\delta'} u(\sigma)\frac{d\varphi(t,\xi(t)-\xi(\sigma))}{dt}dtd\sigma \\
&\quad + \Big(\int_0^{1-\xi(\delta')} \int_0^{\delta'} + \int_{1-\xi(\delta')}^1 \int_0^{\xi^{-1}(1-y)}\Big)\rho_0(y)\frac{d\varphi(t,\xi(t)+y)}{dt}dtdy \\
&= -\int_0^{\delta'} u(\sigma)\varphi(\sigma,0)d\sigma - \int_0^1 \rho_0(y)\varphi(0,y)dy.
\end{aligned} \tag{41}$$

This proves the existence of weak solutions to the Cauchy problem (1) and (2) for small time.

Now we turn to prove the uniqueness of the weak solution. Let us assume that $\overline{\rho} \in C^0([0,\delta]; L^1(0,1))$ is a weak solution to the Cauchy problem (1) and (2). Then by Lemma 2.2, for any $\tau \in [0,\delta]$ and $\psi \in C^1([0,\tau]\times[0,1])$ with $\psi(t,1) \equiv 0$,

$$\begin{aligned}
\int_0^\tau \int_0^1 \overline{\rho}(t,x)(\psi_t(t,x) + \lambda(\overline{W}(t))\psi_x(t,x))dxdt + \int_0^\tau u(t)\psi(t,0)dt \\
- \int_0^1 \overline{\rho}(\tau,x)\psi(\tau,x)dx + \int_0^1 \rho_0(x)\psi(0,x)dx = 0,
\end{aligned} \tag{42}$$

where $\overline{W}(t) := \int_0^1 \overline{\rho}(t,x)dx$.

Let $\overline{\xi}(t) := \int_0^t \lambda(\overline{W}(s))ds$ and $\psi_0 \in C_0^1(0,1)$ (i.e. a C^1 function with compact support in $(0,1)$). Then we choose the test function

$$\psi(t,x) = \begin{cases} \psi_0(\overline{\xi}(\tau) - \overline{\xi}(t) + x), & 0 \le x \le \overline{\xi}(t) - \overline{\xi}(\tau) + 1, 0 \le t \le \tau, \\ 0, & 0 \le \overline{\xi}(t) - \overline{\xi}(\tau) + 1 \le x \le 1, 0 \le t \le \tau, \end{cases} \tag{43}$$

which obviously belongs to $C^1([0,\tau]\times[0,1])$ and satisfies the following backward Cauchy problem:

$$\begin{cases} \psi_t + \lambda(\overline{W}(t))\psi_x = 0, & 0 \le t \le \tau, 0 \le x \le 1, \\ \psi(\tau,x) = \psi_0(x), & 0 \le x \le 1, \\ \psi(t,1) = 0, & 0 \le t \le \tau. \end{cases} \tag{44}$$

In view of (42), we compute

$$\begin{aligned}
&\int_0^1 \overline{\rho}(\tau,x)\psi_0(x)dx \\
&= \int_0^\tau u(t)\psi_0(\overline{\xi}(\tau) - \overline{\xi}(t))dt + \int_0^{1-\overline{\xi}(\tau)} \rho_0(x)\psi_0(\overline{\xi}(\tau)+x)dx \\
&= \int_0^{\overline{\xi}(\tau)} \frac{u(\overline{\xi}^{-1}(\overline{\xi}(\tau)-y))}{\overline{\xi}'(\overline{\xi}^{-1}(\overline{\xi}(\tau)-y))}\psi_0(y)dy + \int_{\overline{\xi}(\tau)}^1 \rho_0(y-\overline{\xi}(\tau))\psi_0(y)dy.
\end{aligned} \tag{45}$$

Since $\psi_0 \in C_0^1(0,1)$ and $\tau \in [0,\delta]$ were arbitrary, we obtain in $C^0([0,\delta]; L^1(0,1))$ that

$$\overline{\rho}(t,x)=\begin{cases}\rho_0(x-\overline{\xi}(t)), & 0\le\overline{\xi}(t)\le x\le 1, 0\le t\le\delta,\\ \dfrac{u(\overline{\xi}^{-1}(\overline{\xi}(t)-x))}{\overline{\xi}'(\overline{\xi}^{-1}(\overline{\xi}(t)-x))}, & 0\le x\le\overline{\xi}(t)\le 1, 0\le t\le\delta,\end{cases}\tag{46}$$

which hence gives

$$\begin{aligned}\overline{\xi}(t)&=\int_0^t\lambda(\int_0^1\overline{\rho}(s,x)dx)ds\\ &=\int_0^t\lambda(\int_0^{\overline{\xi}(s)}\frac{u(\overline{\xi}^{-1}(\overline{\xi}(t)-x))}{\overline{\xi}'(\overline{\xi}^{-1}(\overline{\xi}(t)-x))}dx+\int_{\overline{\xi}(t)}^1\rho_0(x-\overline{\xi}(s))dx)ds\\ &=\int_0^t\lambda(\int_0^s u(\sigma)d\sigma+\int_0^{1-\overline{\xi}(s)}\rho_0(y)dy)ds\\ &=F(\overline{\xi})(t).\end{aligned}\tag{47}$$

It is easy to check that $\overline{\xi}\in\Omega_{\delta,M}$ when δ is small enough, which implies that $\overline{\xi}=\xi$ since ξ is the unique fixed point of F in $\Omega_{\delta,M}$ for δ small enough, and then $\overline{\rho}=\rho$ by comparing (28) and (46). This gives us the uniqueness of the weak solution for small time.

Now we suppose that we have solved the Cauchy problem (1) and (2) to the moment $\tau\in(0,T)$ with the weak solution $\rho\in C^0([0,\tau];L^1(0,1))$. Similar to the obtention of local solution, we know that this weak solution is given by

$$\rho(t,x)=\begin{cases}\rho_0(x-\xi(t)), & \text{if } 0\le\xi(t)\le x\le 1, 0\le t\le\tau,\\ \dfrac{u(\xi^{-1}(\xi(t)-x))}{\xi'(\xi^{-1}(\xi(t)-x))}, & \text{else.}\end{cases}\tag{48}$$

Moreover, the following uniform a priori estimate holds for every $t\in[0,\tau]$:

$$0\le W(t)=\int_0^1\rho(t,x)dx\le M.\tag{49}$$

Hence by (49) and the fact that $\rho_0\in L^1(0,1)$, $u\in L^1(0,T)$, one can find a suitably small $\delta_0\in(0,T)$ independent of τ such that (21) holds and for every $t\in[0,\tau]$ that

$$\begin{aligned}&\int_{1-\overline{\lambda}(M)\delta_0}^1\rho(t,x)dx\\ \le&\sup_{t\in[0,T-\frac{\overline{\lambda}(M)\delta_0}{\lambda(M)}]}\int_t^{t+\frac{\overline{\lambda}(M)\delta_0}{\lambda(M)}}u(\sigma)d\sigma+\sup_{x\in[0,1-\overline{\lambda}(M)\delta_0]}\int_x^{x+\overline{\lambda}(M)\delta_0}\rho_0(y)dy\\ \le&\frac{\widetilde{\lambda}(M)}{2d(M)}.\end{aligned}\tag{50}$$

Applying the previous results on the weak solution for small time, the weak solution $\rho\in C^0([0,\tau];L^1(0,1))$ is extended to the time interval $[\tau,\tau+\delta_0]\cap[\tau,T]$. Step by step, we finally have a unique global weak solution $\rho\in C^0([0,T];L^1(0,1))$. This concludes the proof of Theorem 2.3. □

2.3. **Remarks of Theorem 2.3.**

Remark 2.4. Let ρ be the weak solution in Theorem 2.3. Let $W\in C^0([0,T])$ be defined by $W(t):=\int_0^1\rho(t,x)dx$ and let $\xi\in C^1([0,T])$ be defined by requiring

$$\xi(0)=0,\ \dot{\xi}(t)=\lambda(W(t)),\ \forall t\in[0,T].$$

Then, it follows from our proof of Theorem 2.3 that

$$\rho(t,x) = \begin{cases} \rho_0(x-\xi(t)), & 0 \le \xi(t) \le x \le 1, 0 \le t \le \xi^{-1}(1), \\ \dfrac{u(\xi^{-1}(\xi(t)-x))}{\xi'(\xi^{-1}(\xi(t)-x))}, & 0 \le x \le \xi(t) \le 1, 0 \le t \le \xi^{-1}(1), \\ \dfrac{u(\xi^{-1}(\xi(t)-x))}{\xi'(\xi^{-1}(\xi(t)-x))}, & 0 \le x \le 1, t \ge \xi^{-1}(1). \end{cases} \tag{51}$$

Moreover, $W(t)$ can be expressed as

$$W(t) = \int_0^1 \rho(t,x)dx = \begin{cases} \displaystyle\int_0^t u(\sigma)d\sigma + \int_0^{1-\xi(t)} \rho_0(y)dy, & 0 \le t \le \xi^{-1}(1), \\ \displaystyle\int_{\xi^{-1}(\xi(t)-1)}^t u(\sigma)d\sigma & t \ge \xi^{-1}(1), \end{cases} \tag{52}$$

which implies that

$$0 \le W(t) = \int_0^1 \rho(t,x)dx \le M, \quad \forall t \in [0,T] \tag{53}$$

and

$$0 < \widetilde{\lambda}(M) \le \xi'(t) = \lambda(W(t)) \le \overline{\lambda}(M), \quad \forall t \in [0,T]. \tag{54}$$

Finally, W is absolutely continuous:

$$W(t) = W(0) + \int_0^t W'(s)ds \tag{55}$$

with

$$W'(t) = \begin{cases} u(t) - \xi'(t)\rho_0(1-\xi(t)), & 0 \le t \le \xi^{-1}(1), \\ u(t) - \dfrac{\xi'(t)u(\xi^{-1}(\xi(t)-1))}{\xi'(\xi^{-1}(\xi(t)-1))}, & t \ge \xi^{-1}(1) \end{cases} \tag{56}$$

and

$$0 \le \int_0^T |W'(t)|dt \le 2M. \tag{57}$$

Remark 2.5. (Hidden regularity.) From the definition of the weak solution, we can expect $\rho \in L^1(0,1;L^1(0,T))$. However, the weak solution is more regular than expected. In fact, under the assumptions of Theorem 2.3, we have the hidden regularity that $\rho \in C^0([0,1];L^1(0,T))$ so that the function $t \mapsto \rho(t,x) \in L^1(0,T)$ is well defined for any fixed $x \in [0,1]$. The proof of the hidden regularity is quite similar to our proof of $\rho \in C^0([0,T];L^1(0,1))$ by means of the explicit expression of ρ (see also (55)-(57) that we use when T is large).

Remark 2.6. If $\rho_0 \in L^p(0,1)$ and $u \in L^p(0,T)$ $(p > 1)$ are nonnegative almost everywhere, then the Cauchy problem (1) and (2) admits a unique weak solution $\rho \in C^0([0,T];L^p(0,1)) \cap C^0([0,1];L^p(0,T))$, which is also nonnegative almost everywhere in $Q = [0,T] \times [0,1]$. In fact, the uniqueness of the weak solution comes directly from Theorem 2.3. And the expression of the solution $\rho \in C^0([0,T];L^1(0,1)) \cap C^0([0,1];L^1(0,T))$ given by (51) shows that ρ belongs to $C^0([0,T];L^p(0,1)) \cap C^0([0,1];L^p(0,T))$.

Remark 2.7. If $\rho_0 \in C^1([0,1])$ and $u \in C^1([0,T])$ are nonnegative and the C^1 compatibility conditions are satisfied at the origin:

$$\begin{cases} \dfrac{u(0)}{\lambda(W(0))} - \rho_0(0) = 0, \\ \dfrac{u'(0)\lambda(W(0)) - u(0)\lambda'(W(0))W'(0)}{|\lambda(W(0))|^2} + \lambda(W(0))\rho_0'(0) = 0, \end{cases} \tag{58}$$

where $W(0) = \int_0^1 \rho_0(x)dx$ and $W'(0) = u(0) - \rho_0(1)\lambda(W(0))$, then the Cauchy problem (1) and (2) admits a unique classical solution $\rho \in C^1([0,T] \times [0,1])$, which is also nonnegative.

3. L^2-optimal control for demand tracking problem.

Let $\rho_0 \in L^2(0,1)$ be nonnegative almost everywhere and let $T > 0$ be given. Let us define

$$L^2_+(0,T) := \{u \in L^2(0,T); u \text{ is nonnegative almost everywhere}\}.$$

According to Remark 2.6, for every $u \in L^2_+(0,T)$, the Cauchy problem (1) and (2) admits a unique solution $\rho \in C^0([0,T], L^2(0,1)) \cap C^0([0,1], L^2(0,T))$.

For any fixed *demand signal* $y_d \in L^2(0,T)$ and initial data ρ_0, define a functional on $L^2_+(0,T)$ by

$$J(u) := \int_0^T |u(t)|^2 dt + \int_0^T |y(t) - y_d(t)|^2 dt,\ u \in L^2_+(0,T), \tag{59}$$

where

$$y(t) := \rho(t,1)\lambda(W(t)) \tag{60}$$

is the out-flux corresponding to the in-flux $u \in L^2_+(0,T)$ and initial data ρ_0.

Theorem 3.1. *The infimum of the functional J in $L^2_+(0,T)$ is achieved, i.e., there exists $u_\infty \in L^2_+(0,T)$ such that*

$$J(u_\infty) = \inf_{u \in L^2_+(0,T)} J(u). \tag{61}$$

Proof. Let $\{u_n\}_{n=1}^\infty \subset L^2_+(0,T)$ be a minimizing sequence of the functional J, i.e.

$$\lim_{n\to\infty} J(u_n) = \inf_{u \in L^2_+(0,T)} J(u). \tag{62}$$

Then we have

$$\|u_n\|_{L^2(0,T)} + \|y_n\|_{L^2(0,T)} \le C, \quad \forall n \in \mathbb{Z}^+. \tag{63}$$

In (63) and hereafter, we denote by C various constants which do not depend on n.

The uniform boundedness of u_n in $L^2(0,T)$ shows that there exists $u_\infty \in L^2_+(0,T)$ and a subsequence of $\{u_{n_k}\}_{k=1}^\infty$ such that $u_{n_k} \rightharpoonup u_\infty$ in $L^2_+(0,T)$. For simplicity, we still denote the subsequence as $\{u_n\}_{n=1}^\infty$.

Let ρ_n be the weak solution to the Cauchy problem of equation (1) with the initial and boundary conditions

$$\begin{cases} \rho(t,0)\lambda(W(t)) = u_n(t), & 0 \le t \le T, \\ \rho(0,x) = \rho_0(x), & 0 \le x \le 1. \end{cases} \tag{64}$$

Let $W_n : [0,T] \mapsto \mathbb{R}$ and $\xi_n : [0,T] \mapsto \mathbb{R}$ be defined by

$$W_n(t) := \int_0^1 \rho_n(t,x)dx, \quad \xi_n(t) := \int_0^t \lambda(W_n(s))ds. \tag{65}$$

Thus by (52), we have

$$\xi_n(t) = \int_0^t \lambda(\int_0^s u_n(\sigma)d\sigma + \int_0^{1-\xi_n(s)} \rho_0(x)dx)ds, \quad 0 \le t \le \min\{\xi_n^{-1}(1), T\}. \tag{66}$$

In view of (63) and (65), we can derive from (52) that

$$\|W_n\|_{C^0([0,T])} \leq C, \quad \forall n \in \mathbb{Z}^+, \tag{67}$$

which in turn gives with (65) that

$$\|\xi_n\|_{C^1([0,T])} \leq C, \quad \forall n \in \mathbb{Z}^+. \tag{68}$$

Moreover, let us point out that ξ_n' is uniformly bounded from above and below:

$$0 < \widetilde{\lambda}(\overline{C}) \leq \xi_n'(t) = \lambda(W_n(t)) \leq \overline{\lambda}(\overline{C}), \quad \forall t \in [0,T], \ \forall n \in \mathbb{Z}^+, \tag{69}$$

where $\widetilde{\lambda}, \overline{\lambda}$ are defined by (5) with

$$\overline{C} := \sup_{n \in \mathbb{Z}+} \|u_n\|_{L^1(0,T)} + \|\rho_0\|_{L^1(0,1)} < \infty. \tag{70}$$

Then it follows from Arzelà-Ascoli Theorem that there exists $\overline{\xi}_\infty \in C^0([0,T])$ and a subsequence $\{\xi_{n_l}\}_{l=1}^\infty$ such that $\xi_{n_l} \to \overline{\xi}_\infty$ in $C^0([0,T])$. Now we choose the corresponding subsequence $\{u_{n_l}\}_{l=1}^\infty$ and again, denote it as $\{u_n\}_{n=1}^\infty$. Thus we have

$$u_n \rightharpoonup u_\infty \quad \text{in} \quad L^2(0,T), \quad \text{as } n \to \infty \tag{71}$$

and

$$\xi_n \to \overline{\xi}_\infty \quad \text{in} \quad C^0([0,T]), \quad \text{as } n \to \infty. \tag{72}$$

Then one has

$$\begin{aligned} &\widetilde{\lambda}(\overline{C})|\xi_n^{-1}(x) - \overline{\xi}_\infty^{-1}(x)| \leq |\xi_n(\xi_n^{-1}(x)) - \xi_n(\overline{\xi}_\infty^{-1}(x))| \\ &\quad = |x - \xi_n(\overline{\xi}_\infty^{-1}(x))| = |\overline{\xi}_\infty(\overline{\xi}_\infty^{-1}(x)) - \xi_n(\overline{\xi}_\infty^{-1}(x))| \to 0, \quad \text{as } n \to \infty \end{aligned} \tag{73}$$

uniformly for $x \in [0, \overline{\xi}_\infty(T))$. Thus we get for any $x_0 \in [0, \overline{\xi}_\infty(T))$,

$$\xi_n^{-1} \to \overline{\xi}_\infty^{-1} \quad \text{in } C^0([0,x_0]), \quad \text{as } n \to \infty, \tag{74}$$

and therefore, by passing the limit $n \to \infty$ in (66),

$$\overline{\xi}_\infty(t) = \int_0^t \lambda(\int_0^s u_\infty(\sigma)d\sigma + \int_0^{1-\overline{\xi}_\infty(s)} \rho_0(x)dx)ds, \ 0 \leq t \leq \min\{\overline{\xi}_\infty^{-1}(1), T\}. \tag{75}$$

Let ρ_∞ be the weak solution to the Cauchy problem of equation (1) with the initial and boundary conditions

$$\begin{cases} \rho(t,0)\lambda(W(t)) = u_\infty(t), & 0 \leq t \leq T, \\ \rho(0,x) = \rho_0(x), & 0 \leq x \leq 1, \end{cases} \tag{76}$$

and denote

$$W_\infty(t) := \int_0^1 \rho_\infty(t,x)dx, \quad \xi_\infty(t) := \int_0^t \lambda(W_\infty(s))ds. \tag{77}$$

We claim that $\xi_\infty = \overline{\xi}_\infty$. In fact,

$$\xi_\infty(t) = \int_0^t \lambda(\int_0^s u_\infty(\sigma)d\sigma + \int_0^{1-\xi_\infty(s)} \rho_0(x)dx)ds, \ 0 \leq t \leq \min\{\xi_\infty^{-1}(1), T\}. \tag{78}$$

As in the proof of Theorem 2.3, there exists $\delta > 0$ small enough which is depending only on u_∞ and ρ_0 such that

$$\xi(t) = F_\infty(\xi)(t) := \int_0^t \lambda(\int_0^s u_\infty(\sigma)d\sigma + \int_0^{1-\xi(s)} \rho_0(x)dx)ds \tag{79}$$

has a unique fixed point in $\Omega_{\delta,\overline{C}}$ (replacing M by $\overline{C}$ in (18)). This implies from (75) and (78) that $\xi_\infty(t) \equiv \overline{\xi}_\infty(t)$ on $[0,\delta]$. Moreover, with the help of (50), there exists $\delta_0 > 0$ independent of $\tau \in (0,T)$ such that if $\xi_\infty(\tau) = \overline{\xi}_\infty(\tau)$ then $\xi_\infty(t) \equiv \overline{\xi}_\infty(t)$ on $[\tau, \tau+\delta_0] \cap [\tau, T]$.

Therefore

$$\xi_\infty \equiv \overline{\xi}_\infty \quad \text{and} \quad \xi_n \to \xi_\infty \quad \text{in } C^0([0,T]), \quad \text{as } n \to \infty, \tag{80}$$

and it follows that

$$W_n \to W_\infty, \quad \xi_n' \to \xi_\infty' \quad \text{in } C^0([0,T]), \quad \text{as } n \to \infty \tag{81}$$

and, for any $x_0 \in [0, \xi_\infty(T))$,

$$\xi_n^{-1} \to \xi_\infty^{-1} \quad \text{in } C^0([0,x_0]), \quad \text{as } n \to \infty. \tag{82}$$

Next prove that $y_n(t) = \lambda(W_n(t))\rho_n(t,1)$ converges to $y_\infty(t) = \lambda(W_\infty(t))\rho_\infty(t,1)$ weakly in $L^2(0,T)$. By (63), $\{y_n\}_{n=1}^\infty$ is bounded in $L^2(0,T)$. Hence, it is suffices to prove that for any $g \in C^1([0,T])$,

$$\lim_{n\to\infty} \int_0^T (y_n(t) - y_\infty(t))g(t)dt = 0. \tag{83}$$

If $\xi_\infty(T) < 1$, then $\xi_n(T) < 1$ for n large enough. By (51), (65) and (77), for every $x_0 \in [0, \xi_\infty(T))$, we have

$$\begin{aligned}
&\Big| \int_0^T (y_n(t) - y_\infty(t))g(t)dt \Big| \\
=&\Big| \int_0^T (\rho_0(1-\xi_n(t))\lambda(W_n(t)) - \rho_0(1-\xi_\infty(t))\lambda(W_\infty(t)))g(t)dt \Big| \\
=&\Big| \int_{1-\xi_n(T)}^1 \rho_0(y)g(\xi_n^{-1}(1-y))dy - \int_{1-\xi_\infty(T)}^1 \rho_0(y)g(\xi_\infty^{-1}(1-y))dy \Big| \\
\leq&\Big| \int_{1-\xi_\infty(T)}^1 \rho_0(y)(g(\xi_n^{-1}(1-y)) - g(\xi_\infty^{-1}(1-y)))dy \Big| \\
&+ \Big| \int_{1-\xi_n(T)}^{1-\xi_\infty(T)} \rho_0(y)g(\xi_n^{-1}(1-y))dy \Big| \\
\leq& C \sup_{0\leq x\leq x_0} |\xi_n^{-1}(x) - \xi_\infty^{-1}(x)| + C|\xi_\infty(T) - x_0|^{\frac{1}{2}} + C|\xi_n(T) - \xi_\infty(T)|^{\frac{1}{2}}.
\end{aligned} \tag{84}$$

By (80) and (82), it is easy to get (83) from (84).

If $\xi_\infty(T) = 1$ (i.e., $T = \xi_\infty^{-1}(1)$), for every $\tau \in [0, \xi_\infty^{-1}(1))$, we have

$$\begin{aligned}
&\Big| \int_0^{\xi_\infty^{-1}(1)} (y_n(t) - y_\infty(t))g(t)dt \Big| \\
=&\Big| \int_0^\tau (y_n(t) - y_\infty(t))g(t)dt + \int_\tau^{\xi_\infty^{-1}(1)} (y_n(t) - y_\infty(t))g(t)dt \Big| \\
\leq&\Big| \int_0^\tau (y_n(t) - y_\infty(t))g(t)dt \Big| + C(\xi_\infty^{-1}(1) - \tau)^{\frac{1}{2}}.
\end{aligned} \tag{85}$$

Since it is known that for every $\tau \in [0,T)$

$$\Big| \int_0^\tau (y_n(t) - y_\infty(t))g(t)dt \Big| \to 0 \quad \text{as } n \to \infty,$$

one has (83) for $T = \xi_\infty^{-1}(1)$ from (85).

If $\xi_\infty(T) > 1$, then $\xi_n(T) > 1$ for n large enough and we have

$$\int_0^T (y_n(t) - y_\infty(t))g(t)dt = \Big(\int_0^{\xi_\infty^{-1}(1)} + \int_{\xi_\infty^{-1}(1)}^T \Big)(y_n(t) - y_\infty(t))g(t)dt. \tag{86}$$

From the above study, we need only to estimate the last term in (86). Assuming $\xi_n^{-1}(1) \leq \xi_\infty^{-1}(1)$ (the case $\xi_n^{-1}(1) \geq \xi_\infty^{-1}(1)$ can be treated similarly), we get from (51) that

$$\begin{aligned}
&\Big| \int_{\xi_\infty^{-1}(1)}^T (y_n(t) - y_\infty(t))g(t)dt\Big| \\
=&\Big| \int_{\xi_\infty^{-1}(1)}^T (u_n(\xi_n^{-1}(\xi_n(t) - 1)) - u_\infty(\xi_\infty^{-1}(\xi_\infty(t) - 1)))g(t)dt\Big| \\
=&\Big| \int_{\xi_n^{-1}(\xi_n(\xi_\infty^{-1}(1))-1)}^{\xi_n^{-1}(\xi_n(T)-1)} \frac{u_n(\sigma)g(\xi_n^{-1}(\xi_n(\sigma)+1))\xi_n'(\sigma)}{\xi_n'(\xi_n^{-1}(\xi_n(\sigma)+1))}d\sigma \\
&- \int_0^{\xi_\infty^{-1}(\xi_\infty(T)-1)} \frac{u_\infty(\sigma)g(\xi_\infty^{-1}(\xi_\infty(\sigma)+1))\xi_\infty'(\sigma)}{\xi_\infty'(\xi_\infty^{-1}(\xi_\infty(\sigma)+1))}d\sigma\Big| \\
=&\Big| \int_{\tau_n(\xi_\infty^{-1}(1))}^{\tau_n(T)} \frac{u_n(\sigma)g(\eta_n(\sigma))\xi_n'(\sigma)}{\xi_n'(\eta_n(\sigma))}d\sigma - \int_0^{\tau_\infty(T)} \frac{u_\infty(\sigma)g(\eta_\infty(\sigma))\xi_\infty'(\sigma)}{\xi_\infty'(\eta_\infty(\sigma))}d\sigma\Big|,
\end{aligned} \tag{87}$$

where we denote

$$\tau_n(t) := \xi_n^{-1}(\xi_n(t) - 1), \quad \eta_n(t) := \xi_n^{-1}(\xi_n(t) + 1), \tag{88}$$

$$\tau_\infty(t) := \xi_\infty^{-1}(\xi_\infty(t) - 1), \quad \eta_\infty(t) := \xi_\infty^{-1}(\xi_\infty(t) + 1). \tag{89}$$

From (80) and (82), we get

$$\tau_n \to \tau_\infty \quad \text{in } C^0([0, T]), \quad \text{as } n \to \infty \tag{90}$$

and, for every $t_0 \in [0, \tau_\infty(T))$,

$$\eta_n \to \eta_\infty \quad \text{in } C^0([0, t_0]), \quad \text{as } n \to \infty. \tag{91}$$

Therefore, by (87), one has for every $t_0 \in [0, \tau_\infty(T))$ and for n large enough,

$$\begin{aligned}
&\Big| \int_{\xi_\infty^{-1}(1)}^T (y_n(t) - y_\infty(t))g(t)dt\Big| \\
\leq&\Big|\Big(\int_{\tau_\infty(T)}^{\tau_n(T)} - \int_0^{\tau_n(\xi_\infty^{-1}(1))} \Big)\frac{u_n(\sigma)g(\eta_n(\sigma))\xi_n'(\sigma)}{\xi_n'(\eta_n(\sigma))}d\sigma\Big| \\
&+ \Big| \int_0^{\tau_\infty(T)} \Big(\frac{u_n(\sigma)g(\eta_n(\sigma))\xi_n'(\sigma)}{\xi_n'(\eta_n(\sigma))} - \frac{u_\infty(\sigma)g(\eta_\infty(\sigma))\xi_\infty'(\sigma)}{\xi_\infty'(\eta_\infty(\sigma))}\Big)d\sigma\Big|
\end{aligned} \tag{92}$$

and thus

$$
\begin{aligned}
&\Big|\int_{\xi_\infty^{-1}(1)}^{T}(y_n(t)-y_\infty(t))g(t)dt\Big|\\
\le& C|\tau_n(T)-\tau_\infty(T)|^{\frac12}+C|\tau_n(\xi_\infty^{-1}(1))|^{\frac12}\\
&+\Big|\int_0^{\tau_\infty(T)}\frac{u_n(\sigma)\xi_n'(\sigma)}{\xi_n'(\eta_n(\sigma))}(g(\eta_n(\sigma))-g(\eta_\infty(\sigma))d\sigma\Big|\\
&+\Big|\int_0^{\tau_\infty(T)}u_n(\sigma)g(\eta_\infty(\sigma))\Big(\frac{\xi_n'(\sigma)}{\xi_n'(\eta_n(\sigma))}-\frac{\xi_\infty'(\sigma)}{\xi_\infty'(\eta_\infty(\sigma))}\Big)d\sigma\Big|\\
&+\Big|\int_0^{\tau_\infty(T)}(u_n(\sigma)-u_\infty(\sigma))\frac{g(\eta_\infty(\sigma))\xi_\infty'(\sigma)}{\xi_\infty'(\eta_\infty(\sigma))}d\sigma\Big|\\
\le& C|\tau_n(T)-\tau_\infty(T)|^{\frac12}+C|\tau_n(\xi_\infty^{-1}(1))|^{\frac12}+C|\tau_\infty(T)-t_0|^{\frac12}\\
&+C\sup_{0\le\sigma\le t_0}|\eta_n(\sigma)-\eta_\infty(\sigma)|+C\sup_{0\le\sigma\le t_0}\Big|\frac{\xi_n'(\sigma)}{\xi_n'(\eta_n(\sigma))}-\frac{\xi_\infty'(\sigma)}{\xi_\infty'(\eta_\infty(\sigma))}\Big|\\
&+\Big|\int_0^{\tau_\infty(T)}(u_n(\sigma)-u_\infty(\sigma))\frac{g(\eta_\infty(\sigma))\xi_\infty'(\sigma)}{\xi_\infty'(\eta_\infty(\sigma))}d\sigma\Big|.
\end{aligned}
\tag{93}
$$

By (71),(81), (90)-(91) and the arbitrariness of $t_0 \in [0, \tau_\infty(T))$, we have (83) for the case $\xi_\infty(T) > 1$. This concludes the proof of (83).

As a result,

$$
\begin{aligned}
J(u_\infty) &= \int_0^T |u_\infty(t)|^2dt + \int_0^T |y_\infty(t)-y_d(t)|^2dt\\
&\le \liminf_{n\to\infty}\int_0^T |u_n(t)|^2dt + \liminf_{n\to\infty}\int_0^T |y_n(t)-y_d(t)|^2dt\\
&\le \liminf_{n\to\infty} J(u_n) = \lim_{n\to\infty} J(u_n) = \inf_{u\in L^2_+(0,T)} J(u).
\end{aligned}
\tag{94}
$$

This shows u_∞ is a minimizer of $J(u)$ in $L^2_+(0,T)$, and it proves also that u_n tends to u_∞ strongly in $L^2(0,T)$. □

4. **Time-optimal transition between equilibria.** In this section, we focus on the specific model that relates the nonlocal speed to the total mass according to the assumption (4).

It is immediate that constant boundary data $\rho(\cdot,0) = \rho_{\text{in}} \ge 0$ eventually drive the state to the equilibrium $\rho \equiv \rho_{\text{in}}$. Together with the symmetry $(t,x,\rho(t,x)) \longrightarrow (T-t, 1-x, \rho(T-t,1-x))$ of the conservation law (1) this establishes (long-time state) controllability. Of particular interest is the question of how long it takes to drive the system from one equilibrium state ρ_0 to another equilibrium state ρ_1, compare also the numerical studies of transfers between equilibria in [19].

We first explicitly calculate all quantities for the corresponding piecewise constant boundary data $\rho(\cdot,0)$, and subsequently prove that this boundary control is indeed time-optimal.

Suppose $\rho_1 \ge \rho_0 \ge 0$ are constant, the initial density is the equilibrium $\rho(0,x) = \rho_0$ for $x \in (0,1]$, and the desired terminal density is $\rho(T,x) = \rho_1$ for $x \in [0,1]$ and some minimal $T > 0$. The case $\rho_0 \ge \rho_1 \ge 0$ is similar.

A natural choice for the boundary values is $\rho(t,0) = \rho_1$ for $t \ge 0$. This determines for $0 \le t \le T$ the control influx and the outflux via $u(t) = \rho_1\lambda(W(t))$ and $y(t) = \rho_0\lambda(W(t))$, where W is a solution of the initial value problem

$$W'(t) = \frac{\rho_1 - \rho_0}{1 + W(t)}, \quad W(0) = \int_0^1 \rho(0,x)\,dx = \rho_0. \tag{95}$$

This can be integrated in closed form, yielding

$$W(t) = -1 + \sqrt{(1+\rho_0)^2 + 2t(\rho_1 - \rho_0)} \tag{96}$$

and similar expressions for the fluxes and the speed. All characteristic curves are translations of the solution of the initial value problem

$$\xi'(t) = \lambda(W(t)) = \frac{1}{\sqrt{(1+\rho_0)^2 + 2t(\rho_1 - \rho_0)}}, \quad \xi(0) = 0. \tag{97}$$

which has the explicit solution

$$\xi(t) = \frac{\sqrt{(1+\rho_0)^2 + 2t(\rho_1 - \rho_0)} - (1+\rho_0)}{\rho_1 - \rho_0}. \tag{98}$$

The time T to achieve this transition between equilibria is uniquely determined by $\xi(T) = 1$ and evaluates to

$$T = 1 + \frac{\rho_0 + \rho_1}{2}. \tag{99}$$

In the sequel we prove that this time is indeed minimal.

Note that W is a continuous function, and, in particular $W(T) = \rho_1$. It is convenient to extend ρ, u, v, and W to negative times by setting $\rho(t,x) = W(t) = \rho_0$ and $u(t) = y(t) = \frac{\rho_0}{1+\rho_0} = u_0 = y_0$ for all $t < 0$. Then u is continuous except for a jump at $t = 0$, and y is continuous except for a jump at T. Note that the height of the jump of u at $t = 0$ is larger than the corresponding jump of y at T.

$$u(0^+) - u(0^-) = \frac{\rho_1 - \rho_0}{1+\rho_0}, \quad \text{whereas} \quad y(T^+) - y(T^-) = \frac{\rho_1 - \rho_0}{1+\rho_1}. \tag{100}$$

Supposing a jump of the reference demand from $y_d(t) = \rho_0\lambda(\rho_0)$ for $t < T$ to $y_d(t) = \rho_1\lambda(\rho_1)$ for $t \geq T$ at this earliest feasible time, the total backlog at any $t \geq T$ is, due to the *inverse response*,

$$\beta(t) = \int_0^T (y_d(s) - y(s))\,ds = \frac{\rho_0 T}{1+\rho_0} - \rho_0 \underbrace{\int_0^T \lambda(W(s))\,ds}_{=1} = \frac{\rho_0 T - \rho_0 - \rho_0^2}{1+\rho_0}. \tag{101}$$

Using the expression (99) for T, this simplifies for $t \geq T$ to

$$\beta(t) = \frac{(\rho_1 - \rho_0)\rho_0}{1+\rho_0} = y_0(\rho_1 - \rho_0). \tag{102}$$

Correspondingly, for $0 < t < T$ the total mass $W(t) < \rho_1$ continues to grow, and hence the speed is further decreasing. Therefore, the influx $u(t) = \rho_1\lambda(W(t)))$ is larger than the eventual new equilibrium influx $u_1 = \rho_1\lambda(\rho_1)$. The total excess in influx evaluates to

$$\alpha(T) = \int_0^T (\rho_1\lambda(W(s)) - \rho_1\lambda(\rho_1))ds = \frac{(\rho_1 - \rho_0)\rho_1}{1+\rho_1} = u_1(\rho_1 - \rho_0) \tag{103}$$

Together with the nominal difference $(y_1 - y_0)T$ between the accumulated equilibrium fluxes over the time interval $[0,T]$, these add up the difference in total mass, compare the three shaded regions in Figure 5,

$$\begin{aligned} W(T) - W(0) = &\int_0^T (u(s) - y(s))\, ds \\ =&\alpha(T) + \beta(T) + (\rho_1\lambda(\rho_1) - \rho_0\lambda(\rho_0))T = \rho_1 - \rho_0. \end{aligned} \tag{104}$$

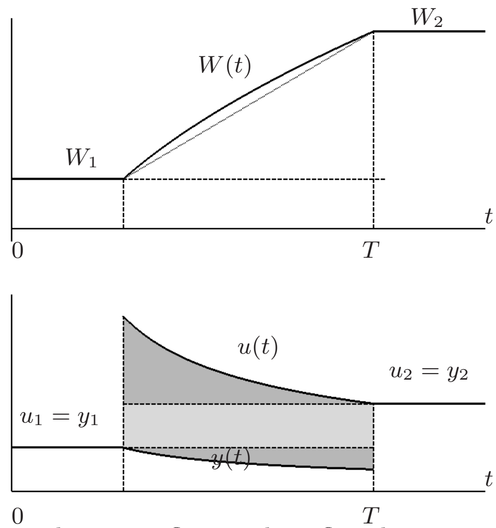

FIGURE 5. Total mass, influx, and outflux during optimal transition between equilibria

While it may seem intuitive that this control is time-optimal, we need to rigorously prove that it is indeed not possible to improve on this time by e.g. temporarily increasing the speed via smaller influxes.

Proposition 4.1. *The minimum time to transfer the state from one equilibrium $\rho(0,x) = \rho_0$, $x \in (0,1]$ to the equilibrium $\rho(x,T) = \rho_1 > \rho_0$, $x \in [0,1]$ using influx $u \in L^1([0,\infty),[0,\infty))$ is $T = 1 + \frac{\rho_0+\rho_1}{2}$.*

Proof. Suppose $T > 0$ and $\rho(t,0)$ is an integrable function on $[0,T]$ such that the solution of (1) satisfies $\rho(T,\cdot) = \rho_1$.

Since ρ is constant along the characteristic curves, there exists $t_0 \in [0,T]$ such that for all $t \in [t_0,T]$, $\rho(t,0) = \rho_1$. Let $\xi\colon [0,T] \mapsto [0,\infty)$ be the unique function satisfying $\xi(0) = 0$ and $\xi'(t) = \lambda(\int_0^1 \rho(t,x)\, dx)$. Then there exists a unique $t_1 \in (0,T]$ such that $\xi(t_1) = 1$.

For $0 < t < t_0$, $\xi'(t)$ is bounded above by

$$\xi'(t) = \frac{1}{1 + \int_0^1 \rho(t,x)dx} \leq \frac{1}{1 + \int_{\xi(t)}^1 \rho(t,x)dx} = \frac{1}{1 + \rho_0(1 - \xi(t))}. \tag{105}$$

Rewrite as $((1+\rho_0) - \rho_0\xi(t))\xi'(t) \leq 1$ and integrate from $t = 0$ to $t = t_0$ to obtain a lower bound for t_0.

$$(1+\rho_0)\xi(t_0) - \frac{1}{2}\rho_0\xi(t_0)^2 \leq t_0. \tag{106}$$

The primary interest is the case of $t_0 < t_1$ (see the left subfigure of Figure 6). For $t_0 \leq t \leq t_1$ estimate

$$\begin{aligned}\xi'(t) &\le \frac{1}{1+\int_0^{\xi(t)-\xi(t_0)}\rho(t,x)dx+\int_{\xi(t)}^1\rho(t,x)dx} \\ &= \frac{1}{1+\rho_1(\xi(t)-\xi(t_0))+\rho_0(1-\xi(t))}.\end{aligned} \tag{107}$$

and integrate from t_0 to t_1 to obtain

$$(1+\rho_0-\rho_0\xi(t_0))(\xi(t_1)-\xi(t_0))+\frac{1}{2}(\rho_1-\rho_0)(\xi(t_1)-\xi(t_0))^2 \le t_1-t_0. \tag{108}$$

Analogously, for $t_1 \le t \le T$, the bound $\xi'(t) \le 1/(1+\rho_1(\xi(t)-\xi(t_0)))$ yields

$$\xi(T)-\xi(t_1)+\frac{1}{2}\rho_1\left((\xi(T)-\xi(t_0))^2-(\xi(t_1)-\xi(t_0))^2\right) \le T-t_1. \tag{109}$$

After combining the estimates (106), (108), and (109), elementary simplifications yield

$$\begin{aligned}T \ge\ &(1+\rho_0)\xi(t_0)-\frac{1}{2}\rho_0\xi(t_0)^2 \\ &+(1+\rho_0-\rho_0\xi(t_0))(\xi(t_1)-\xi(t_0))+\frac{1}{2}(\rho_1-\rho_0)(\xi(t_1)-\xi(t_0))^2 \\ &+\xi(T)-\xi(t_1)+\frac{1}{2}\rho_1\left((\xi(T)-\xi(t_0))^2-(\xi(t_1)-\xi(t_0))^2\right).\end{aligned} \tag{110}$$

Noting that $\xi(t_1)=\xi(T)-\xi(t_0)=1$, (110) simplifies to

$$T \ge 1+\frac{\rho_0+\rho_1}{2}+\xi(t_0). \tag{111}$$

This shows that the optimal choice is $t_0=0$, i.e. $\rho(t,0)=\rho_1$ for all $t \ge 0$.

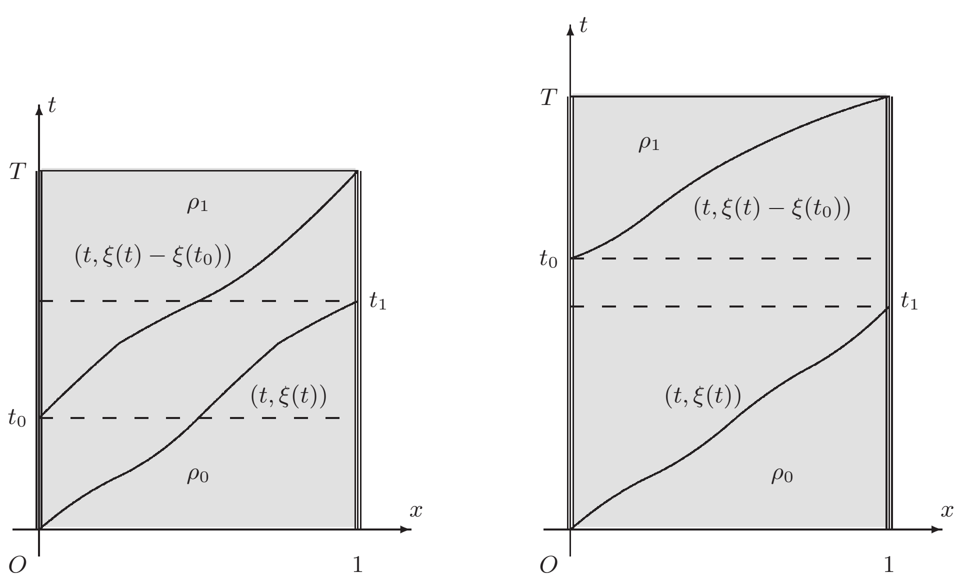

FIGURE 6. Time optimal transfer between equilibrium states

It remains to dispose of the case when $t_1 < t_0$ (see the right subfigure of Figure 6). For $0 \leq t \leq t_1$, use $\xi'(t) \leq 1/(1+\rho_0(1-\xi(t)))$ and $\xi(t_1) = 1$ to obtain

$$t_1 \geq \xi(t_1)(1+\rho_0) - \frac{1}{2}\rho_0\xi(t_1)^2 = 1 + \frac{1}{2}\rho_0. \tag{112}$$

Similarly, for $t_0 \leq t \leq T$, use $\xi'(t) \leq 1/(1+\rho_1(\xi(t)-\xi(t_0)))$ and $\xi(T) - \xi(t_0) = 1$ to obtain

$$T - t_0 \geq (\xi(T) - \xi(t_0)) + \frac{1}{2}\rho_1(\xi(T)-\xi(t_0))^2 = 1 + \frac{1}{2}\rho_1. \tag{113}$$

Combining (112) and (113) together with $t_0 > t_1$ yields

$$T = (T-t_0) + (t_0 - t_1) + t_1 \geq (1 + \frac{1}{2}\rho_1) + 0 + (1+\frac{1}{2}\rho_0) \geq 2 + \frac{\rho_0+\rho_1}{2}. \tag{114}$$

This shows that any controls for which $t_1 < t_0$ will perform even worse than the ones in the first case. $\square$

Acknowledgments. The authors thank François Murat for interesting discussions. The second and third authors thank the Laboratoire Jacques-Louis Lions at the University Pierre and Marie Curie-Paris VI for its hospitality and the Foundation Sciences Mathématiques de Paris for its support.

REFERENCES

[1] F. Ancona and A. Marson, *On the attainable set for scalar nonlinear conservation laws with boundary control*, SIAM J. Control Optim., **36** (1998), 290–312 (electronic).

[2] ———, *Existence theory by front tracking for general nonlinear hyperbolic systems*, Arch. Ration. Mech. Anal., **185** (2007), 287–340.

[3] D. Armbruster, P. Degond and C. Ringhofer, *A model for the dynamics of large queuing networks and supply chains*, SIAM J. Appl. Math., **66** (2006), 896–920 (electronic).

[4] D. Armbruster, D. Marthaler and C. Ringhofer, *Kinetic and fluid model hierarchies for supply chains*, Multiscale Model. Simul., **2** (2003), 43–61 (electronic).

[5] D. Armbruster, D. Marthaler, C. Ringhofer, K. Kempf and T.-C. Jo, *A continuum model for a re-entrant factory*, Oper. Res., **54** (2006), 933–950.

[6] D. Armbruster and C. Ringhofer, *Thermalized kinetic and fluid models for reentrant supply chains*, Multiscale Model. Simul., **3** (2005), 782–800 (electronic).

[7] P. Baiti, P. LeFloch and B. Piccoli, *Uniqueness of classical and nonclassical solutions for nonlinear hyperbolic systems*, J. Differential Equations, **172** (2001), 59–82.

[8] A. Bressan, "Hyperbolic Systems of Conservation Laws. The one-dimensional Cauchy problem," vol. **20** of Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, Oxford, 2000.

[9] A. Bressan, G. Crasta and B. Piccoli, *Well-posedness of the Cauchy problem for $n \times n$ systems of conservation laws*, Mem. Amer. Math. Soc., **146** (2000), viii+134.

[10] A. Bressan and G. Guerra, *Shift-differentiability of the flow generated by a conservation law*, Discrete Contin. Dynam. Systems, **3** (1997), 35–58.

[11] A. Bressan and P. LeFloch, *Uniqueness of weak solutions to systems of conservation laws*, Arch. Rational Mech. Anal., **140** (1997), 301–317.

[12] G. M. Coclite, M. Garavello and B. Piccoli, *Traffic flow on a road network*, SIAM J. Math. Anal., **36** (2005), 1862–1886 (electronic).

[13] R. Colombo, M. Herty and M. Mercier, *Control of the continuity equation with a non local flow*, to appear in ESAIM Control Optim. Calc. Var., (2010).

[14] J.-M. Coron, "Control and Nonlinearity," vol. **136** of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2007.

[15] J.-M. Coron, O. Glass and Z. Wang, *Exact boundary controllability for 1-d quasilinear hyperbolic systems with a vanishing characteristic speed*, SIAM J. Control Optim., **48** (2009/10), 3105–3122.

[16] P. Degond, S. Göttlich, M. Herty and A. Klar, *A network model for supply chains with multiple policies*, Multiscale Model. Simul., **6** (2007), 820–837.

[17] M. Herty, A. Klar and B. Piccoli, *Existence of solutions for supply chain models based on partial differential equations*, SIAM J. Math. Anal., **39** (2007), 160–173.

[18] T. Horsin, *On the controllability of the Burgers equation*, ESAIM Control Optim. Calc. Var., **3** (1998), 83–95 (electronic).

[19] M. La Marca, D. Armbruster, M. Herty and C. Ringhofer, *Control of continuum models of production systems*, to appear in IEEE Trans. Automat. Control, (2010).

[20] T. Li, "Controllability and Observability for Quasilinear Hyperbolic Systems," vol. **3** of AIMS Series on Applied mathematics, American Institute of Mathematical Sciences (AIMS), Springfield, MO; Higher Education Press, Beijing, 2010.

[21] T. Li and B. Rao, *Exact boundary controllability for quasi-linear hyperbolic systems*, SIAM J. Control Optim., **41** (2003), 1748–1755 (electronic).

[22] D. L. Russell, *Controllability and stabilizability theory for linear partial differential equations: Recent progress and open questions*, SIAM Rev., **20** (1978), 639–739.

Received July 2009; revised March 2010

E-mail address: `coron@ann.jussieu.fr`
E-mail address: `kawski@asu.edu`
E-mail address: `wzq@fudan.edu.cn`