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Local null controllability of the two-dimensional Navier–Stokes system in the torus with a control force having a vanishing component

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Abstract

In this paper we deal with the two-dimensional Navier–Stokes system in a torus. The main result establishes the local null controllability with internal controls having one vanishing component. The linearized control system around 0 is not null controllable: the nonlinear term is essential to get this null controllability. Our proof uses the return method together with previous results by Fursikov and Imanuvilov.

Résumé

Dans cet article on considère le système de Navier–Stokes dans un tore bidimensionnel. Notre résultat principal établit la contrôlabilité locale à zéro avec des contrôles internes ayant une composante nulle. Le système de contrôle linéarisé autour de 0 n'est pas contrôlable : le terme non linéaire est donc essentiel pour obtenir ce résultat. Notre démonstration utilise la méthode du retour combinée avec des résultats précédents de Fursikov et Imanuvilov.

Keywords: Navier–Stokes system; Controllability; Return method

1. Introduction

Let $T > 0$, $L_1 > 0$ and $L_2 > 0$ and let $\mathbb{T}_2$ be the flat torus $(\mathbb{R}/L_1\mathbb{Z}) \times (\mathbb{R}/L_2\mathbb{Z})$. We will use the notation $Q := (0, T) \times \mathbb{T}_2$. Let ω be a nonempty open subset of $\mathbb{T}_2$, which is the control domain. For $y = (y_1, y_2) : \mathbb{T}_2 \to \mathbb{R}^2$ (resp. $y = (y_1, y_2) : (0, T) \times \mathbb{T}_2 \to \mathbb{R}^2$), let $\nabla \cdot y : \mathbb{T}_2 \to \mathbb{R}$ (resp. $\nabla \cdot y : (0, T) \times \mathbb{T}_2 \to \mathbb{R}$) be defined by:

$$\nabla \cdot y := \operatorname{div} y := \partial_{x_1} y_1 + \partial_{x_2} y_2. \tag{1.1}$$

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In (1.1) and in the following, ∂_{x_i} denotes the partial derivatives with respect to x_i, $i \in \{1,2\}$, x_1 being the first component of the current point $x \in \mathbb{T}_2$, x_2 being the second component of x (in other words $x = (x_1, x_2)$). For $y = (y_1, y_2) : \mathbb{T}_2 \to \mathbb{R}^2$ and $z = (z_1, z_2) : \mathbb{T}_2 \to \mathbb{R}^2$ (resp. $y = (y_1, y_2) : (0,T) \times \mathbb{T}_2 \to \mathbb{R}^2$ and $z = (z_1, z_2) : (0,T) \times \mathbb{T}_2 \to \mathbb{R}^2$), let $(y \cdot \nabla)z : \mathbb{T}_2 \to \mathbb{R}^2$ (resp. $(y \cdot \nabla)z : (0,T) \times \mathbb{T}_2 \to \mathbb{R}^2$) be defined by:

$$(y \cdot \nabla)z = (y_1 \partial_{x_1} z_1 + y_2 \partial_{x_2} z_1, y_1 \partial_{x_1} z_2 + y_2 \partial_{x_2} z_2).$$

In this paper, we deal with the following Navier–Stokes control system:

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla)y + \nabla p = 1_\omega (v_1, 0) & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q. \end{cases} \tag{1.2}$$

Here, and in the following, $1_\omega : \mathbb{T}_2 \to \mathbb{R}$ denotes the characteristic function of ω, i.e. $1_\omega(x) := 0$ if $x \in \mathbb{T}_2 \setminus \omega$, $1_\omega(x) := 1$ if $x \in \omega$. System (1.2) is a control system where, at time $t \in [0,T]$, the control is the scalar function,

$$v_1(t, \cdot)1_\omega : \mathbb{T}_2 \to \mathbb{R},\ (x_1, x_2) \mapsto 1_\omega(x_1, x_2) v_1\big(t, (x_1, x_2)\big),$$

and the state is:

$$y(t) := y(t, \cdot) : \mathbb{T}_2 \to \mathbb{R}^2, \quad (x_1, x_2) \mapsto y\big(t, (x_1, x_2)\big).$$

Let us recall that the Cauchy problem associated to (1.2) is well posed for $y^0 \in L^2(\mathbb{T}_2)^2$ satisfying,

$$\nabla \cdot y^0 = 0 \quad \text{in } \mathbb{T}_2, \tag{1.3}$$

$v_1 \in L^2(Q)$ and $y \in L^2((0,T); H^1(\mathbb{T}_2))^2 \cap H^1((0,T); H^{-1}(\mathbb{T}_2))^2$. That is, for every $T > 0$, every $y^0 \in L^2(\mathbb{T}_2)^2$ satisfying (1.3) and every $v_1 \in L^2(Q)$, there exists a unique $y \in L^2((0,T); H^1(\mathbb{T}_2))^2 \cap H^1((0,T); H^{-1}(\mathbb{T}_2))^2$ such that

$$y_t - \Delta y + (y \cdot \nabla)y + \nabla p = 1_\omega (v_1, 0) \quad \text{in } Q, \tag{1.4}$$

$$\nabla \cdot y = 0 \quad \text{in } Q, \tag{1.5}$$

$$y(0, x) = y^0(x), \quad x \in \mathbb{T}_2, \tag{1.6}$$

hold for some $p \in L^2((0,T); L^2(\mathbb{T}_2)) = L^2(Q)$ (p is unique up to a function depending only on $t \in (0,T)$). See, e.g., [24, Theorem 3.1] or [26, Theorem 3.1, p. 282, Theorem 3.2, p. 294]. Moreover, $y \in C^0([0,T]; L^2(\mathbb{T}_2))^2$ (see, e.g., [24, Theorem 3.1]) and, for every $\eta > 0$, $y \in L^2((\eta,T); H^2(\mathbb{T}_2))^2 \cap H^1((\eta,T); L^2(\mathbb{T}_2))^2$. Finally, if $y^0 \in H^1(\mathbb{T}_2)^2$, then $y \in L^2((0,T); H^2(\mathbb{T}_2))^2 \cap H^1((0,T); L^2(\mathbb{T}_2))^2$ (see, e.g., [26, Theorem 3.10, Section 3.7.2, p. 314]).

Let

$$H := \big\{ y = (y_1, y_2) \in L^2(\mathbb{T}_2)^2 \colon \nabla \cdot y = 0 \big\}, \tag{1.7}$$

$$H_0 := \left\{ y = (y_1, y_2) \in H \colon \int_{\mathbb{T}_2} y_2 \, dx = 0 \right\}. \tag{1.8}$$

The linear space H equipped with the $L^2(\mathbb{T}_2)^2$ scalar product is a Hilbert space and H_0 is a closed linear subspace of H. Integrating the second component of equality (1.4), using (1.5) and simple integrations by parts, one sees that H_0 is invariant for the control system (1.2), i.e., for every $y^0 \in H_0$ and for every $v_1 \in L^2(Q)$, the solution y of (1.4)–(1.5)–(1.6) is such that $y(t, \cdot) \in H_0$ for every $t \in [0,T]$.

Our main result is the small-time null local controllability of the control system (1.2) in the invariant subspace H_0. More precisely, our main result is

Theorem 1. *For every $T > 0$ and for every $\varepsilon > 0$, there exists $\eta > 0$ such that, for every $y^0 \in H_0$ satisfying $|y^0|_{L^2(\mathbb{T}_2)^2} < \eta$, there exists a control $v_1 \in L^2(Q)$ satisfying $|v_1|_{L^2(Q)} \leqslant \varepsilon$ such that the solution $y \in L^2((0,T); H^1(\mathbb{T}_2))^2 \cap H^1((0,T); H^{-1}(\mathbb{T}_2))^2$ to the Cauchy problem* (1.4)–(1.5)–(1.6) *satisfies*

$$y(T, \cdot) = 0 \quad \text{in } \mathbb{T}_2. \tag{1.9}$$

In order to get a local controllability result around an equilibrium as in Theorem 1, the usual first thing to look at is the controllability of the linearized control system around this equilibrium. Here, this linearized control system is the following one:

$$\begin{cases} y_t - \Delta y + \nabla p = 1_\omega (v_1, 0) & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \end{cases} \tag{1.10}$$

where, as for (1.2), at time $t \in [0, T]$, the control is the scalar function,

$$1_\omega v_1(t, \cdot) : \mathbb{T}_2 \to \mathbb{R}, \qquad (x_1, x_2) \mapsto 1_\omega(x_1, x_2) v_1\big(t, (x_1, x_2)\big),$$

and the state is

$$y(t, \cdot) : \mathbb{T}_2 \to \mathbb{R}^2, \qquad (x_1, x_2) \mapsto y\big(t, (x_1, x_2)\big).$$

As for (1.2) again, the Cauchy problem (1.10) is well posed for an initial data $y^0 \in L^2(\mathbb{T}_2)^2$, $v_1 \in L^2(Q)$ and $y \in L^2((0, T); H^1(\mathbb{T}_2))^2 \cap H^1((0, T); H^{-1}(\mathbb{T}_2))^2$. Again, H_0 is invariant for (1.10). However the linear control system (1.10) is far from being null controllable in H_0. Indeed, let $n \in \mathbb{Z}$, let $(\lambda_1, \lambda_2) \in \mathbb{R}^2$, and let $\zeta \in C^\infty(\mathbb{T}_2)$ be defined by:

$$\zeta(x_1, x_2) := \lambda_1 \sin\left(\frac{2n\pi x_1}{L_1}\right) + \lambda_2 \cos\left(\frac{2n\pi x_1}{L_1}\right), \quad \forall (x_1, x_2) \in \mathbb{T}_2. \tag{1.11}$$

Let us multiply the second component of the first equality of (1.10) by ζ. Integrating this new equality on $\mathbb{T}_2$, simple integrations by parts show that, whatever is $v_1 \in L^2(Q)$, any solution y to (1.10) satisfies:

$$\frac{d}{dt} \int_{\mathbb{T}_2} \zeta y_2 \, dx = -\frac{4n^2\pi^2}{L_1^2} \int_{\mathbb{T}_2} \zeta y_2 \, dx. \tag{1.12}$$

In particular

$$\left(\int_{\mathbb{T}_2} \zeta y_2^0 \, dx \neq 0 \right) \quad \Rightarrow \quad \big(y(T, \cdot) \neq 0\big).$$

Hence one needs to use the nonlinear term $(y \cdot \nabla) y$ in (1.2) in order to get Theorem 1 (in particular the method of [12] cannot be applied here).

The strategy is to use the return method, i.e., in our context, find a trajectory $\bar{y}$ of the control system (1.2) (i.e. a solution $y = \bar{y}$ of (1.2) for some $v_1 : Q \to \mathbb{R}$ and some $p : Q \to \mathbb{R}$) such that

1. the linearized control system around $\bar{y}$ is null controllable (in H_0),
2. the trajectory $\bar{y}$ starts from 0 and arrives at 0 in time T: $\bar{y}(0, \cdot) = \bar{y}(T, \cdot) = 0$.

With such a trajectory $\bar{y}$, using some inverse mapping theorem, one can expect to steer the control system (1.2) in time T from $y^0 \in H_0$ to $0 = \bar{y}(T, \cdot)$ by following a trajectory close to $\bar{y}$ at least if y^0 is small enough. If moreover $\bar{y}$ can itself be chosen close to 0 then the trajectory going from y^0 to 0 will be itself close to 0.

Remark 1. The return method has been introduced in [6] for a stabilization problem. It has been used for the first time in order to get the controllability of nonlinear partial differential equations in [7,9]. For other applications of the return method to Navier–Stokes equations (or its one-dimensional analog, namely the viscous Burgers equations), we refer to [3–5,11,19,20]. For applications to other partial differential equations and more references, see [10, Chapters 6, 7, 9].

Remark 2. The null controllability of the Navier–Stokes equations with a control force of the form $1_\omega(v_1, v_2)$ has been obtained for the first time in [16] when the initial data y^0 is small and in [8,11] without any restriction on the initial data. See also [14,21,22] when the initial data y^0 is small but for manifolds with a boundary. For prior null controllability results when one of the components of the force vanishes, see [15]. Let us emphasize that in the above papers, the linearized control system is controllable, which is not the case here. In contrast with these papers, where the nonlinearity is considered as a 'disturbing' term, it is the nonlinear term which helps us to obtain here the controllability. For global approximate controllability results with a finite number of control forces (but with a support equal to all of $\mathbb{T}_2$), we refer to [1,25]. In [13] and [23], there are examples of three-dimensional Stokes systems (i.e. the linearized control systems of the Navier–Stokes control system at 0) which are not null controllable with a force term having two vanishing components (even if ω is the full domain). It would be interesting to know if our method can be adapted to show that in these cases the nonlinear term $(y \cdot \nabla)y$ helps again to recover the local null controllability. We conjecture that this is indeed the case. Let us point out that in [23] it is also proved that, for generic bounded vertical cylinders in $\mathbb{R}^3$, the approximate controllability holds for the Stokes control system with one scalar control vertical force distributed on the full cylinder.

This paper is organized as follows:

1. In Section 2, we construct the trajectory $\bar{y}$ mentioned above.
2. In Section 3, we study the controllability of the linearized control system around $\bar{y}$.
3. Finally, in Section 4, we show how to deduce Theorem 1 from the controllability of the linearized control system around $\bar{y}$ and a suitable inverse mapping theorem.

2. Construction of the trajectory $\bar{y}$

In this section, we construct a specific trajectory of the control system (1.2) going from 0 (at $t = 0$) to 0 (at $t = T$). Let $\mu \in (0, 1]$, $\delta \in \mathbb{R}$. We define $d \in C^\infty([0, T])$ by

$$d(t) := \delta e^{-\mu/t(T-t)}, \quad \forall t \in (0, T), \qquad d(0) := 0, \qquad d(T) := 0. \tag{2.1}$$

Let γ_j, $j \in \{1, 2, 3, 4\}$, and $\delta_1, \delta_{1,1}, \delta_2, \delta_{2,1}$ be eight real numbers such that

$$0 < \gamma_1 < \gamma_2 < \gamma_3 < \gamma_4 < L_1, \tag{2.2}$$

$$0 < \delta_1 < \delta_{1,1} < \delta_2 < \delta_{2,1} < L_2, \tag{2.3}$$

$$[\gamma_1, \gamma_4] \times [\delta_1, \delta_{2,1}] \subset \omega. \tag{2.4}$$

See Fig. 1. Let, for $i \in \{1, 2\}$,

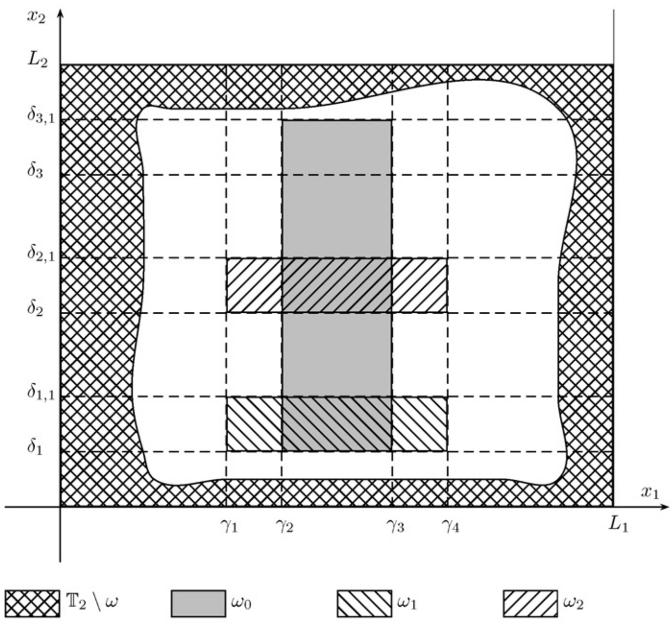

Fig. 1. $\omega, \omega_0, \omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \delta_1, \delta_2, \delta_3, \delta_{1,1}, \delta_{2,1}, \delta_{3,1}$.

$$\omega_i := \{x = (x_1, x_2) \in \mathbb{T}_2\colon\ \gamma_1 \leqslant x_1 \leqslant \gamma_4,\ \delta_i \leqslant x_2 \leqslant \delta_{i,1}\} \subset \omega, \quad \forall i \in \{1,2\}. \tag{2.5}$$

(The inclusions in (2.5) follow from (2.3) and (2.4).) See Fig. 1. Let $a_1 \in C^\infty([0, L_1])$, $a_2 \in C^\infty([0, L_1])$ be such that

$$\text{the support of } a_i \text{ is included in } (\gamma_1, \gamma_4), \quad \forall i \in \{1,2\}, \tag{2.6}$$

$$a_1(x_1) = x_1, \quad \forall x_1 \in [\gamma_2, \gamma_3], \tag{2.7}$$

$$a_2(x_1) = 1, \quad \forall x_1 \in [\gamma_2, \gamma_3]. \tag{2.8}$$

Let $b_1 \in C^\infty([0, L_2])$ and $b_2 \in C^\infty([0, L_2])$ be such that

$$\text{the support of } b_i \text{ is included in } (\delta_i, \delta_{i,1}), \quad \forall i \in \{1,2\}, \tag{2.9}$$

$$b_i \text{ does not vanish identically}, \quad \forall i \in \{1,2\}, \tag{2.10}$$

$$\int_{\delta_i}^{\delta_{i,1}} b_i(x_2)\,dx_2 = 0, \quad \forall i \in \{1,2\}. \tag{2.11}$$

Let $\varphi \in C^\infty([0,T] \times \mathbb{T}_2)$ and $\bar{y} \in C^\infty([0,T] \times \mathbb{T}_2)^2$ be defined by:

$$\varphi(t,x) := d(t) \sum_{i=1}^{2} a_i(x_1) b_i(x_2), \quad \forall t \in [0,T],\ \forall x = (x_1, x_2) \in \mathbb{T}_2, \tag{2.12}$$

$$\bar{y} := (\partial_{x_2}\varphi, -\partial_{x_1}\varphi). \tag{2.13}$$

From (2.13), one gets:

$$\nabla \cdot \bar{y} = 0 \quad \text{in } Q. \tag{2.14}$$

From (2.1), (2.12) and (2.13) one gets:

$$\bar{y}(0,\cdot) = \bar{y}(T,\cdot) = 0. \tag{2.15}$$

Let $\overline{R} \in C^\infty([0,T] \times \mathbb{T}_2)^2$ be defined by:

$$\overline{R} := (\overline{R}_1, \overline{R}_2) := \bar{y}_t - \Delta\bar{y} + (\bar{y}\cdot\nabla)\bar{y}. \tag{2.16}$$

From (2.3), (2.9), (2.12), (2.13), (2.16), one gets, for every $t \in [0,T]$ and for every $(x_1, x_2) \in \mathbb{T}_2$,

$$\begin{aligned}\overline{R}_2(t,x_1,x_2) = {} & -\dot{d}(t)\sum_{i=1}^{2} a_i'(x_1)b_i(x_2) + d(t)\left(\sum_{i=1}^{2}\big(a_i'''(x_1)b_i(x_2) + a_i'(x_1)b_i''(x_2)\big)\right. \\ & \left. - d(t)^2\left(\sum_{i=1}^{2}\big(a_i(x_1)a_i''(x_1) - \big(a_i'(x_1)\big)^2\big)\right)b_i(x_2)b_i'(x_2)\right).\end{aligned} \tag{2.17}$$

By (2.5), (2.6), (2.9), (2.11) and (2.17),

$$\int_0^{x_2} \overline{R}_2(t,x_1,s)\,ds = 0, \quad \forall\big(t,(x_1,x_2)\big) \in [0,T] \times \big(\mathbb{T}_2 \setminus (\omega_1 \cup \omega_2)\big). \tag{2.18}$$

From (2.18), one gets that $\bar{p} : [0,T] \times \mathbb{T}_2 \to \mathbb{R}$ defined by,

$$\bar{p}(t,x_1,x_2) := -\int_0^{x_2} \overline{R}_2(t,x_1,s)\,ds, \quad \forall t \in [0,T],\ \forall x_1 \in [0,L_1],\ \forall x_2 \in [0,L_2), \tag{2.19}$$

is of class C^∞ on $[0,T] \times \mathbb{T}_2$ and that

$$\text{the support of } \bar{p} \text{ is included in } [0,T] \times (\overline{\omega_1 \cup \omega_2}). \tag{2.20}$$

Let $f_1 \in C^\infty([0,T] \times \mathbb{T}_2)$ be defined by:

$$f_1 := \overline{R}_1 + \partial_{x_1} \overline{p}. \tag{2.21}$$

From (2.16), (2.19) and (2.21), one has:

$$\overline{y}_t - \Delta \overline{y} + (\overline{y} \cdot \nabla)\overline{y} + \nabla \overline{p} = (f_1, 0). \tag{2.22}$$

Moreover, from (2.5), (2.6), (2.9), (2.12), (2.13), (2.16), (2.20) and (2.21), one sees that

$$\text{the support of } f_1 \text{ is included in } [0,T] \times \omega, \tag{2.23}$$

which, together with (2.22), shows that

$$\overline{y}_t - \Delta \overline{y} + (\overline{y} \cdot \nabla)\overline{y} + \nabla \overline{p} = 1_\omega (f_1, 0).$$

In conclusion, $\overline{y}$ is indeed a trajectory of the control system (1.2) going from 0 at time 0 to 0 at time T (the associated control being $1_\omega f_1$).

From now on and until the end of the paper, a_1, a_2, b_1, b_2, γ_1, γ_2, γ_3, γ_4, δ_1, $\delta_{1,1}$, δ_2 and $\delta_{2,1}$ are fixed as above. The only parameters which are not fixed for the moment are $\mu \in (0,1]$ and $\delta \in \mathbb{R}$.

3. Controllability of the linearized control system around $\overline{y}$

In this section, we prove a null controllability result for the linearized control system of (1.2) around the trajectory $\overline{y}$. More precisely, our goal in this section is to prove the following controllability result.

Proposition 1. *There exist $C_0 > 0$ and $C_1 > 0$ such that, if*

$$\mu := \frac{C_0}{6}, \tag{3.1}$$

then, for every $y^0 \in H^1(\mathbb{T}_2)^2$ satisfying (1.3)*, for every $\delta \in \mathbb{R}$, and for every $h \in L^2(Q)^2$ satisfying,*

$$\delta \in (0, 1/C_1], \tag{3.2}$$

$$e^{C_0/(T-t)} h \in L^2(Q)^2, \tag{3.3}$$

$$\int_{\mathbb{T}_2} h_2(t,x)\,dx = 0, \quad t \in (0,T), \tag{3.4}$$

$$\int_{\mathbb{T}_2} y_2^0(x)\,dx = 0, \tag{3.5}$$

there exist

$$u \in L^2\big((0,T); H^2(\mathbb{T}_2)\big)^2 \cap H^1\big((0,T); L^2(\mathbb{T}_2)\big)^2,$$
$$q \in L^2\big((0,T); H^1(\mathbb{T}_2)\big),$$
$$v_1 \in L^2(Q),$$

such that

$$v_1(t,x) = 0, \quad (t,x) \in Q \setminus \big([T/2, T] \times \omega\big), \tag{3.6}$$

$$u_t - \Delta u + (\overline{y} \cdot \nabla) u + (u \cdot \nabla)\overline{y} + \nabla q = (v_1, 0) + h \quad \textit{in } Q, \tag{3.7}$$

$$\nabla \cdot u = 0 \quad \textit{in } Q, \tag{3.8}$$

$$u(0,\cdot) = y^0 \quad \textit{in } \mathbb{T}_2, \tag{3.9}$$

$$e^{2C_0/(3(T-t))} u \in L^2\big((0,T); H^2(\mathbb{T}_2)\big)^2, \tag{3.10}$$

$$e^{2C_0/(3(T-t))} q \in L^2\big((0,T); H^1(\mathbb{T}_2)\big), \tag{3.11}$$

$$e^{2C_0/(3(T-t))} v_1 \in L^2(Q). \tag{3.12}$$

This controllability result relies on the following two steps:

- First, in Section 3.1, we prove a controllability result for the linearized control system around $\bar{y}$ with a control with two components but with an integral constraint on the second component.
- Then, in Section 3.2, we show how one can explicitly eliminate the second component of the control.

3.1. Null controllability of the linear system with a control with two components

Let ω_0 be a non-empty open subset of ω. Our goal in this section is to prove the following proposition:

Proposition 2. *For every $r \in (0,1)$, there exist $C_2 > 0$ and $C_3 > 0$, such that, if*

$$\mu = C_2(1-r)T \tag{3.13}$$

and if $\delta \in [0, 1/C_3]$, then, for every $h \in L^2(Q)^2$ and for every $y^0 \in H^1(\mathbb{T}_2)^2$ satisfying (1.3), (3.4), (3.5) *and*

$$e^{C_2/(T-t)} h \in L^2(Q)^2, \tag{3.14}$$

there exist

$$\widetilde{u} \in L^2\big((0,T); H^2(\mathbb{T}_2)\big)^2 \cap H^1\big((0,T); L^2(0,T)\big)^2,$$
$$\widetilde{q} \in L^2\big((0,T); H^1(\mathbb{T}_2)\big),$$
$$\widetilde{v} = (\widetilde{v}_1, \widetilde{v}_2) \in H^1\big((0,T); L^2(\mathbb{T}_2)\big)^2,$$

such that

$$\int_{\mathbb{T}_2} \widetilde{v}_2(t,x)\,dx = 0, \quad t \in (0,T), \tag{3.15}$$

$$\widetilde{v}(t,x) = 0, \quad (t,x) \in Q \setminus \big([T/2,T] \times \omega_0\big), \tag{3.16}$$

$$\widetilde{u}_t - \Delta \widetilde{u} + (\bar{y} \cdot \nabla)\widetilde{u} + (\widetilde{u} \cdot \nabla)\bar{y} + \nabla \widetilde{q} = \widetilde{v} + h \quad \text{in } Q, \tag{3.17}$$

$$\nabla \cdot \widetilde{u} = 0 \quad \text{in } Q, \tag{3.18}$$

$$\widetilde{u}(0,\cdot) = y^0 \quad \text{in } \mathbb{T}_2, \tag{3.19}$$

$$e^{C_2 r/(T-t)} \widetilde{u} \in L^2\big((0,T); H^2(\mathbb{T}_2)\big)^2, \tag{3.20}$$

$$e^{C_2 r/(T-t)} \widetilde{q} \in L^2\big((0,T); H^1(\mathbb{T}_2)\big), \tag{3.21}$$

$$e^{C_2 r/(T-t)} \widetilde{v} \in L^2\big((0,T); H^2(\mathbb{T}_2)\big)^2, \tag{3.22}$$

$$e^{C_2 r/(T-t)} \widetilde{v}_t \in L^2(Q)^2. \tag{3.23}$$

Proof. Proposition 2 has already been proved in [18] except for the two following properties:

- In [18], the control $\widetilde{v}$ is “less regular”. Instead of (3.22) and (3.23), it only satisfies

$$e^{C_2 r/(T-t)} \widetilde{v} \in L^2(Q)^2. \tag{3.24}$$

- In [18], condition (3.15) is not required.

In Appendix A, we show how to modify [18] in order to take care of (3.22) and (3.23). More precisely, in this appendix, we prove the following proposition:

Proposition 3. *Let*

$$r \in (0,1). \tag{3.25}$$

There exists $C_4 > 0$ such that for every $h \in L^2(Q)^2$ satisfying

$$e^{C_4/(T-t)} h \in L^2(Q)^2, \tag{3.26}$$

and for every $y^0 \in H^1(\mathbb{T}_2)^2$ satisfying (1.3)*, there exist*

$$u^* \in L^2\big((0,T); H^2(\mathbb{T}_2)\big)^2 \cap H^1\big((0,T); L^2(\mathbb{T}_2)\big)^2,$$
$$q^* \in L^2\big((0,T); H^1(\mathbb{T}_2)\big),$$
$$v^* = \big(v_1^*, v_2^*\big) \in L^2\big((0,T); H^2(\mathbb{T}_2)\big)^2 \cap H^1\big((0,T); L^2(\mathbb{T}_2)\big)^2,$$

such that

$$v^*(t,x) = 0, \quad (t,x) \in Q \setminus \big([T/2,T] \times \omega_0\big), \tag{3.27}$$
$$u_t^* - \Delta u^* + \nabla q^* = v^* + h \quad \text{in } Q, \tag{3.28}$$
$$\nabla \cdot u^* = 0 \quad \text{in } Q, \tag{3.29}$$
$$u^*(0,\cdot) = y^0 \quad \text{in } \mathbb{T}_2, \tag{3.30}$$
$$e^{C_4 r/(T-t)} u^* \in L^2\big((0,T); H^2(\mathbb{T}_2)\big)^2, \tag{3.31}$$
$$e^{C_4 r/(T-t)} q^* \in L^2\big((0,T); H^1(\mathbb{T}_2)\big), \tag{3.32}$$
$$e^{C_4 r/(T-t)} v^* \in L^2\big((0,T); H^2(\mathbb{T}_2)\big)^2, \tag{3.33}$$
$$e^{C_4 r/(T-t)} v_t^* \in L^2\big((0,T); L^2(\mathbb{T}_2)\big)^2. \tag{3.34}$$

Let us explain how to construct $\widetilde{v}$ from v^* when $\bar{y} = 0$. Let $Z := (Z_1, Z_2) \in C^\infty(\mathbb{T}_2)^2$ be such that

$$\text{the support of } Z \text{ is included in } \omega_0, \tag{3.35}$$
$$\int_{\mathbb{T}_2} Z_2 \, dx = 1. \tag{3.36}$$

Since $\int_{\mathbb{T}_2} \nabla \cdot Z \, dx = 0$, there exists $\theta \in C^\infty(\mathbb{T}_2)$ such that

$$-\Delta\theta = \nabla \cdot Z. \tag{3.37}$$

Let

$$Y := \nabla\theta + Z \in C^\infty(\mathbb{T}_2)^2. \tag{3.38}$$

From (3.37) and (3.38), one gets:

$$\nabla \cdot Y = 0. \tag{3.39}$$

Let $f : [0,T] \to \mathbb{R}$ be defined by:

$$f(t) = -\int_0^t \int_{\mathbb{T}_2} v_2^*(s,x) \, dx. \tag{3.40}$$

Note that, by (3.27) and (3.40),

$$f(t) = 0, \quad \forall t \in [0, T/2]. \tag{3.41}$$

From (3.28), one has:

$$\int_0^t \int_{\mathbb{T}_2} u_{2,t}^* \, dx \, dt - \int_0^t \int_{\mathbb{T}_2} v_2^* \, dx \, dt = \int_0^t \int_{\mathbb{T}_2} h_2 \, dx \, dt. \tag{3.42}$$

From (3.4), (3.5), (3.30), (3.31), (3.40) and (3.42), one gets:

$$e^{C_4 r/(T-t)} f \in L^2(0,T), \tag{3.43}$$

so, in particular, $f(T)=0$. Let

$$\widetilde{u} := u^* + f(t)Y. \tag{3.44}$$

From (3.29), (3.39) and (3.44), one has (3.18). From (3.30), (3.40) and (3.44), one gets (3.19). Let

$$R := \widetilde{u}_t - \Delta\widetilde{u} + \nabla q^* - v^* - h. \tag{3.45}$$

From (3.28), (3.38), (3.44) and (3.45), one gets:

$$R = \nabla(\dot{f}\theta - f\Delta\theta) + \dot{f}Z - f\Delta Z. \tag{3.46}$$

Let us define $\widetilde{q}$ and $\widetilde{v}$ by:

$$\widetilde{q} := q^* - \dot{f}\theta + f\Delta\theta, \tag{3.47}$$

$$\widetilde{v} := v^* + \dot{f}Z - f\Delta Z. \tag{3.48}$$

From (3.27), (3.35), (3.41) and (3.48), one gets (3.16). From (3.45)–(3.48), one gets (3.17) for $\bar{y}=0$. From (3.36), (3.40) and (3.48), one gets (3.15). Finally, choosing

$$C_2 := C_4, \tag{3.49}$$

straightforward estimates, together with (3.31)–(3.34), (3.40), (3.43), (3.44), (3.47) and (3.48), show that (3.20)–(3.23) hold. This proves that Proposition 2 holds for $\delta=0$.

It remains to deal with the case where δ is small enough but not 0. We make it by a perturbation argument with the case $\delta=0$. Let $\mathcal{E}$ be the set of (u,q,v) such that

$$e^{C_4 r/(T-t)} u \in L^2\big((0,T); H^2(\mathbb{T}_2)\big)^2, \tag{3.50}$$

$$\int_{\mathbb{T}_2} u_2(t,x)\,dt = 0, \quad t \in (0,T), \tag{3.51}$$

$$\nabla\cdot u = 0 \quad \text{in } Q, \tag{3.52}$$

$$e^{C_4 r/(T-t)} q \in L^2\big((0,T); H^1(\mathbb{T}_2)\big), \tag{3.53}$$

$$v(t,x) = 0, \quad (t,x) \in Q \setminus \big([T/2,T]\times\omega_0\big), \tag{3.54}$$

$$e^{C_4 r/(T-t)} v \in L^2\big((0,T); H^2(\mathbb{T}_2)\big)^2, \tag{3.55}$$

$$e^{C_4 r/(T-t)} v_t \in L^2(Q)^2, \tag{3.56}$$

$$\int_{\mathbb{T}_2} v_2(t,x)\,dx = 0, \quad t \in (0,T), \tag{3.57}$$

$$e^{C_4/(T-t)}(u_t - \Delta u + \nabla q - v) \in L^2(Q)^2, \tag{3.58}$$

$$u(0,\cdot) \in H^1(\mathbb{T}_2)^2. \tag{3.59}$$

The linear space $\mathcal{E}$ is equipped with the norm $|\cdot|_{\mathcal{E}}$ defined by:

$$\begin{aligned}\big|(u,q,v)\big|_{\mathcal{E}}^2 :={}& \big|e^{C_4 r/(T-t)}u\big|^2_{L^2((0,T);H^2(\mathbb{T}_2))^2} + \big|e^{C_4 r/(T-t)}q\big|^2_{L^2((0,T);H^1(\mathbb{T}_2))} \\ &+ \big|e^{C_4 r/(T-t)}v\big|^2_{L^2((0,T);H^2(\mathbb{T}_2))^2} + \big|e^{C_4 r/(T-t)}v_t\big|^2_{L^2(Q)^2} \\ &+ \big|e^{C_4/(T-t)}(u_t - \Delta u + \nabla q - v)\big|^2_{L^2(Q)^2} + \big|u(0,\cdot)\big|^2_{H^1(\mathbb{T}_2)^2}.\end{aligned}$$

This norm is associated to a scalar product and $\mathcal{E}$ with this scalar product is a Hilbert space. Let $\mathcal{F}$ be the set of (h, y^0) such that

$$e^{C_4/(T-t)}h \in L^2(Q)^2 \quad \text{and} \quad y^0 \in H^1(\mathbb{T}_2)^2,$$
$$\int_{\mathbb{T}_2} h_2(t,x)\,dx = 0, \quad t \in (0,T),$$
$$\int_{\mathbb{T}_2} y_2^0\,dx = 0,$$
$$\nabla \cdot y^0 = 0 \quad \text{in } \mathbb{T}_2.$$

The linear space $\mathcal{F}$ is equipped with the norm $|\cdot|_{\mathcal{F}}$ defined by:

$$\left|\left(h, y^0\right)\right|_{\mathcal{F}}^2 := \left|e^{C_4/(T-t)}h\right|_{L^2(Q)^2}^2 + \left|y^0\right|_{H^1(\mathbb{T}_2)^2}^2.$$

This norm is associated to a scalar product and $\mathcal{F}$ with this scalar product is a Hilbert space. Let $\mathcal{L} : \mathcal{E} \to \mathcal{F}$ be defined by:

$$\mathcal{L}(u,q,v) := \left(u_t - \Delta u + \nabla q - v, u(0,\cdot)\right).$$

One easily sees that $\mathcal{L}$ is well defined and continuous. We have proved above that $\mathcal{L}$ is onto. Let $\mathcal{G} : \mathcal{E} \to \mathcal{F}$ be the linear map defined by:

$$\mathcal{G}(u,q,v) := \left((\bar{y}\cdot\nabla)u + (u\cdot\nabla)\bar{y}, 0\right).$$

One easily checks, using (2.1), (2.12)–(2.14), (3.13), (3.49) and (3.52) that $\mathcal{G}$ is well defined, continuous and that there exists $K > 0$ such that its norm is less than $K\delta$. Hence, if δ is small enough, $\mathcal{L} + \mathcal{G}$ is onto. This concludes the proof of Proposition 2. □

3.2. Elimination of the second component of the control

In this section we conclude the proof of Proposition 1. Let δ_3 and $\delta_{3,1}$ be two real numbers such that

$$0 < \delta_{2,1} < \delta_3 < \delta_{3,1} < L_2, \tag{3.60}$$
$$[\gamma_2, \gamma_3] \times [\delta_1, \delta_{3,1}] \subset \omega. \tag{3.61}$$

See Fig. 1. For r and ω_0, we choose,

$$r := \frac{6}{7}, \tag{3.62}$$
$$\omega_0 := (\gamma_2, \gamma_3) \times (\delta_1, \delta_{3,1}). \tag{3.63}$$

See Fig. 1. Note that (3.61) and (3.63) lead to

$$\omega_0 \subset \omega. \tag{3.64}$$

We apply Proposition 2 with these data and take $\delta \in (0, 1/C_3]$. Let $h \in L^2(Q)^2$ and $y^0 \in H^1(\mathbb{T}_2)^2$ be such that (1.3), (3.4), (3.5), and (3.14) hold. Then, by Proposition 2, there exist:

$$\widetilde{u} \in L^2\left((0,T); H^2(\mathbb{T}_2)\right)^2 \cap H^1\left((0,T); L^2(\mathbb{T}_2)\right)^2,$$
$$\widetilde{q} \in L^2\left((0,T); H^1(\mathbb{T}_2)\right),$$
$$\widetilde{v} = (\widetilde{v}_1, \widetilde{v}_2) \in L^2(Q)^2,$$

such that (3.15) to (3.23) hold.

We define C_0 by:

$$C_0 := C_2 T. \tag{3.65}$$

Let us assume, for the moment being, that there exist $\xi = (\xi_1, \xi_2) : Q \to \mathbb{R}^2$ and $\pi : Q \to \mathbb{R}$ such that

$$e^{2C_2/(3(T-t))}\xi \in L^2\big((0,T);H^2(\mathbb{T}_2)\big)^2, \tag{3.66}$$

$$e^{2C_2/(3(T-t))}\xi_t \in L^2(Q)^2, \tag{3.67}$$

$$e^{2C_2/(3(T-t))}\pi \in L^2\big((0,T);H^1(\mathbb{T}_2)\big), \tag{3.68}$$

$$\xi(t,x)=0, \quad (t,x)\in Q\setminus\big((T/2,T)\times\omega_0\big), \tag{3.69}$$

$$\pi(t,x)=0, \quad (t,x)\in Q\setminus\big((T/2,T)\times\omega_0\big), \tag{3.70}$$

$$\nabla\cdot\xi=0, \tag{3.71}$$

$$\xi_{2,t}-\Delta\xi_2+(\bar{y}\cdot\nabla)\xi_2+(\xi\cdot\nabla)\bar{y}_2=-\widetilde{v}_2-\partial_{x_2}\pi. \tag{3.72}$$

Then, if wet let

$$u:=\widetilde{u}+\xi, \tag{3.73}$$

$$q:=\widetilde{q}+\pi, \tag{3.74}$$

$$v_1:=\widetilde{v}_1+\xi_{1,t}-\Delta\xi_1+(\bar{y}\cdot\nabla)\xi_1+(\xi\cdot\nabla)\bar{y}_1+\partial_{x_1}\pi, \tag{3.75}$$

one readily sees that (3.6)–(3.12) hold. Indeed (3.6) follows from (3.16), (3.64), (3.69), (3.70) and (3.75). Equality (3.7) follows from (3.17), (3.72)–(3.75). Equality (3.8) follows from (3.18), (3.71) and (3.73). Equality (3.9) follows from (3.19), (3.69) and (3.73). Property (3.10) follows from (3.20), (3.62), (3.65), (3.66) and (3.73). Property (3.11) follows from (3.21), (3.62), (3.65), (3.68) and (3.74). Finally, (3.12) follows from (2.1), (2.12), (2.13), (3.13), (3.22), (3.62), (3.65)–(3.68) and (3.75).

It remains to construct ξ. We look for a function ξ of the form,

$$\xi:=(\xi_1,\xi_2):=(\partial_{x_2}\psi,-\partial_{x_1}\psi), \tag{3.76}$$

where $\psi: Q\to\mathbb{R}$ will be defined later on (see (3.84) below). Note that (3.76) gives (3.71). Let $D: Q\to\mathbb{R}$ be defined by:

$$D:=\xi_{2,t}-\Delta\xi_2+\bar{y}_1\partial_{x_1}\xi_2+\xi_1\partial_{x_1}\bar{y}_2+\widetilde{v}_2. \tag{3.77}$$

Note that, from (3.76) and (3.77), one gets:

$$\xi_{2,t}-\Delta\xi_2+(\bar{y}\cdot\nabla)\xi_2+(\xi\cdot\nabla)\bar{y}_2=-\widetilde{v}_2+D-\partial_{x_2}(\bar{y}_2\partial_{x_1}\psi), \tag{3.78}$$

and

$$D=-\partial_{x_1}\psi_t+\partial_{x_1}\Delta\psi-\bar{y}_1\partial^2_{x_1}\psi+\partial_{x_1}\bar{y}_2\partial_{x_2}\psi+\widetilde{v}_2. \tag{3.79}$$

Using (2.13) and (3.79), one gets:

$$D=-\partial_{x_1}\psi_t+\partial^3_{x_1}\psi+\partial_{x_1}\partial^2_{x_2}\psi-\partial_{x_2}\varphi\partial^2_{x_1}\psi-\partial^2_{x_1}\varphi\partial_{x_2}\psi+\widetilde{v}_2. \tag{3.80}$$

Let β_1,β_2,β_3 be three functions of class C^∞ on $[0,L_2]$ such that (see (2.9) and (2.10) for (3.83))

$$\text{the support of } \beta_i \text{ is included in } (\delta_i,\delta_{i,1}), \quad \forall i\in\{1,2,3\}, \tag{3.81}$$

$$\int_0^{L_2}\beta_i(x_2)\,dx_2=1, \quad \forall i\in\{1,2,3\}, \tag{3.82}$$

$$\int_0^{L_2}b_i'(x_2)\beta_i(x_2)\,dx_2=1, \quad \forall i\in\{1,2\}. \tag{3.83}$$

We take the function ψ in the following form:

$$\psi(t,x)=\sum_{i=1}^{3}\alpha_i(t,x_1)\beta_i(x_2), \quad \big(t,(x_1,x_2)\big)\in[0,T]\times\mathbb{T}_2, \tag{3.84}$$

where

$$\alpha_3(t,x_1) := -\alpha_1(t,x_1) - \alpha_2(t,x_1), \quad (t,x_1) \in (0,T) \times (0,L_1). \tag{3.85}$$

From (2.3), (2.9), (2.12) and (3.80)–(3.85), one gets:

$$\frac{1}{d}\int_0^{L_2} D(t,x_1,s)\,ds = a_1''\alpha_1 + a_2''\alpha_2 - a_1\partial_{x_1}^2\alpha_1 - a_2\partial_{x_1}^2\alpha_2 + V(t,x_1), \tag{3.86}$$

where $V : (0,T) \times [0,L_1] \to \mathbb{R}$ is defined by

$$V(t,x_1) := \frac{1}{d(t)}\int_0^{L_2} \widetilde{v}_2(t,x_1,x_2)\,dx_2, \quad t \in (0,T),\ x_1 \in [0,L_1]. \tag{3.87}$$

Let us show how to choose α_1 and α_2 so that the left-hand side of (3.86) vanishes. Note that, by (3.15) and (3.87),

$$\int_0^{L_1} V(t,x_1)\,dx_1 = 0, \quad t \in (0,T). \tag{3.88}$$

By (3.16), (3.63) and (3.87),

$$V(t,x_1) = 0, \quad \forall (t,x_1) \in \big((0,T) \times (0,L_1)\big) \setminus \big([T/2,T] \times [\gamma_2,\gamma_3]\big). \tag{3.89}$$

Let $\bar{\alpha}_1 \in C^\infty([0,L_1])$ be such that

$$\text{the support of } \bar{\alpha}_1 \text{ is included in } (\gamma_2,\gamma_3), \tag{3.90}$$

$$\int_0^{L_1} \bar{\alpha}_1(x_1)\,dx_1 = -\frac{1}{2}. \tag{3.91}$$

We define $W : (0,T) \times [0,L_1] \to \mathbb{R}$, α_1 and α_2 by

$$W(t,x_1) := \int_0^{x_1} V(t,s)\,ds, \quad (t,x_1) \in (0,T) \times [0,L_1], \tag{3.92}$$

$$\alpha_1(t,x_1) := \bar{\alpha}_1(x_1)\int_0^{L_1} W(t,s)\,ds, \quad (t,x_1) \in (0,T) \times [0,L_1], \tag{3.93}$$

$$\alpha_2(t,x_1) := \int_0^{x_1} \big(W(t,s) - s\partial_{x_1}\alpha_1(t,s) + \alpha_1(t,s)\big)\,ds, \quad (t,x_1) \in (0,T) \times [0,L_1]. \tag{3.94}$$

From (3.85), (3.88)–(3.94), one gets:

$$\alpha_i(t,x_1) = 0, \qquad (t,x_1) \in \big((0,T) \times [0,L_1]\big) \setminus \big([T/2,T] \times (\gamma_2,\gamma_3)\big), \quad \forall i \in \{1,2,3\}. \tag{3.95}$$

From (3.63), (3.76), (3.81), (3.84), (3.85) and (3.95) one gets (3.69). From (2.7), (2.8), (3.86), (3.89) and (3.92)–(3.95), one has:

$$\int_0^{L_2} D(t,x_1,s)\,ds = 0, \quad (t,x_1) \in (0,T) \times [0,L_1]$$

(observe that the support of D is included in $[0,T] \times [\gamma_2,\gamma_3]$). This allows us to define $\pi : (0,T) \times \mathbb{T}_2 \to \mathbb{R}$ by

$$\pi(t,x_1,x_2) := \bar{y}_2\partial_{x_1}\psi - \int_0^{x_2} D(t,x_1,s)\,ds, \quad (t,x_1,x_2)\in(0,T)\times[0,L_1]\times[0,L_2]. \tag{3.96}$$

From (3.78) and (3.96), one gets (3.72). From (2.1), (2.12), (2.13), (3.1), (3.22), (3.23), (3.62), (3.76), (3.84), (3.85), (3.87) and (3.92)–(3.95), one gets (3.66) and (3.67) (observe that $6/7 - 1/6 > 2/3$). From (2.1), (2.12), (2.13), (3.1), (3.21), (3.62), (3.84), (3.87), (3.92)–(3.96), one gets (3.68).

Remark 3. The construction of α_1, α_2 and W having an appropriate support is inspired by a finite-dimension technique: when a system,

$$\dot{x} = A(t)x + B(t)u, \quad x\in\mathbb{R}^n,\ t\in\mathbb{R},\ u\in\mathbb{R}^m,$$

is controllable in the interval $[0,T]$, then, for given $f:\mathbb{R}\to\mathbb{R}^n$ supported in $(0,T)$, one can construct $x:\mathbb{R}\to\mathbb{R}^n$ and $u:\mathbb{R}\to\mathbb{R}^m$ both having support in $[0,T]$ a solution of:

$$\dot{x} - A(t)x - B(t)u = f.$$

See, for example, [10, Section 1.3].

4. Local controllability around $\bar{y}$

In this section, inspired by [17, Chapter 3], we deduce from an inverse mapping theorem and the controllability of the linearized control system around the trajectory $\bar{y}$ (see Proposition 1) the local exact controllability around $\bar{y}$ (which together with (2.15) implies Theorem 1).

Let us first point out that it is sufficient to prove Theorem 1 with $y^0\in H^1(\mathbb{T}_2)^2$ and $|y^0|_{L^2(\mathbb{T}_2)^2} < \eta$ replaced by $|y^0|_{H^1(\mathbb{T}_2)^2} < \eta$ (with the classical method consisting of taking the control to be zero during some time). Indeed, this is a straightforward consequence of the following classical results:

- For every $T>0$, for every $y^0\in L^2(\mathbb{T}_2)^2$ satisfying (1.3) and for every $\varepsilon\in(0,T]$, the solution $y\in L^2((0,T);H^1(\mathbb{T}_2))^2\cap H^1((0,T);H^{-1}(\mathbb{T}_2))^2$ to the Cauchy problem (1.4)–(1.5)–(1.6) is such that $y\in C^0([\varepsilon,T];H^1(\mathbb{T}_2))^2$.
- For every $T>0$, for every $\varepsilon\in(0,T]$, there exist $\nu>0$ and $C>0$ such that, for every $y^0\in L^2(\mathbb{T}_2)^2$ satisfying (1.3) and $|y^0|_{L^2(\mathbb{T}_2)^2} < \nu$, the solution y to the Cauchy problem (1.4)–(1.5)–(1.6) satisfies $|y(t,\cdot)|_{H^1(\mathbb{T}_2)^2} \leqslant C|y^0|_{L^2(\mathbb{T}_2)^2}$, $\forall t\in[\varepsilon,T]$.

See, e.g., (the proof of) [26, Theorem 3.10, Section 3.7.2, p. 314].

The inverse mapping theorem that we are going to use is the following one (see, for instance, [2, Section 2.3]).

Theorem 2. *Let E and F be two Banach spaces. Let $e_0\in E$ and let $\mathcal{A}:E\to F$ be of class C^1 on an open neighborhood of e_0. We assume that $\mathcal{A}'(e_0):E\to G$ is surjective. Then, there exists $C>0$ and $\eta>0$ such that, for every $g\in G$ satisfying $|g-\mathcal{A}(e_0)|_G<\eta$, there exists $e\in E$ such that*

$$\mathcal{A}(e) = g \quad \text{and} \quad |e-e_0|_E \leqslant K\big|g-\mathcal{A}(e_0)\big|_G.$$

Remark 4. If E is a Hilbert space (which will be the case for our application of Theorem 2), then, under the assumptions of Theorem 2, there exist η_0 and an application $\Psi:\{g\in|g-\mathcal{A}(e_0)|_G<\eta_0\}\to E$ of class C^1 such that

$$\Psi\big(\mathcal{A}(e_0)\big) = e_0, \qquad \mathcal{A}\big(\Psi(g)\big) = g, \quad \forall g\in G \text{ such that } \big|g-\mathcal{A}(e_0)\big|_G < \eta_0.$$

However, if E is a general Banach space, such a (η_0,Ψ) may not exist.

We apply Theorem 2 with the following E, G and $\mathcal{A}$. Let $\delta\in\mathbb{R}$ be such that (3.2) holds. The space E is the set of (u,q,v_1) such that (see Proposition 1)

$$e^{2C_0/(3(T-t))}u \in L^2\big((0,T);H^2(\mathbb{T}_2)\big)^2,$$
$$\int\limits_{\mathbb{T}_2} u_2(t,x)\,dt = 0, \quad t \in (0,T),$$
$$\nabla\cdot u = 0 \quad \text{in } Q,$$
$$e^{2C_0/(3(T-t))}q \in L^2\big((0,T);H^1(\mathbb{T}_2)\big),$$
$$v_1(t,x) = 0, \quad (t,x) \in Q\setminus\big([T/2,T]\times\omega_0\big),$$
$$e^{2C_0/(3(T-t))}v_1 \in L^2(Q),$$
$$e^{C_0/(T-t)}\big(u_t - \Delta u + (\bar{y}\cdot\nabla)u + (u\cdot\nabla)\bar{y} + \nabla q - (v_1,0)\big) \in L^2(Q)^2,$$
$$u(0,\cdot) \in H^1(\mathbb{T}_2)^2.$$

The linear space E is equipped with the norm $|\cdot|_E$ defined by:

$$\begin{aligned}\big|(u,q,v)\big|_E^2 := {} & \big|e^{2C_0/(3(T-t))}u\big|^2_{L^2((0,T);H^2(\mathbb{T}_2))^2} + \big|e^{2C_0/(3(T-t))}q\big|^2_{L^2((0,T);H^1(\mathbb{T}_2))} \\ & + \big|e^{2C_0/(3(T-t))}v_1\big|^2_{L^2(Q)} \\ & + \big|e^{C_0/(T-t)}\big(u_t - \Delta u + (\bar{y}\cdot\nabla)u + (u\cdot\nabla)\bar{y} + \nabla q - (v_1,0)\big)\big|^2_{L^2(Q)^2} \\ & + \big|u(0,\cdot)\big|^2_{H^1(\mathbb{T}_2)^2}.\end{aligned}$$

This norm is associated to a scalar product and E with this scalar product is a Hilbert space. Let F be the set of (h, y^0) such that

$$e^{C_0/(T-t)}h \in L^2(Q)^2 \quad \text{and} \quad y^0 \in H^1(\mathbb{T}_2)^2,$$
$$\int\limits_{T_2} h_2(t,x)\,dx = 0, \quad t \in (0,T),$$
$$\int\limits_{\mathbb{T}_2} y_2^0\,dx = 0,$$
$$\nabla\cdot y^0 = 0 \quad \text{in } \mathbb{T}_2.$$

The linear space F is equipped with the norm $|\cdot|_F$ defined by:

$$\big|(h,y^0)\big|_F^2 := \big|e^{C_0/(T-t)}h\big|^2_{L^2(Q)^2} + \big|y^0\big|^2_{H^1(\mathbb{T}_2)^2}.$$

This norm is associated to a scalar product and F with this scalar product is a Hilbert space. Let $\mathcal{A}: E \to F$ be defined by:

$$\mathcal{A}(u,q,v) := \big(u_t - \Delta u + (\bar{y}\cdot\nabla)u + (u\cdot\nabla)\bar{y} + (u\cdot\nabla)u + \nabla q - (v_1,0), u(0,\cdot)\big).$$

One has the following lemma.

Lemma 1. *The map $\mathcal{A}: E \to F$ is of class C^1.*

Proof. Clearly the map,

$$(u,q,v) \in E \mapsto \big(u_t - \Delta u + (\bar{y}\cdot\nabla)u + (u\cdot\nabla)\bar{y} + \nabla q - (v_1,0), u(0)\big) \in F,$$

is well defined, linear and continuous. Since $(u\cdot\nabla)u$ is quadratic with respect to u, it then suffices to check that there exists $K > 0$ such that, for every $(u,v,q) \in E$,

$$\big|e^{C_0/(T-t)}(u\cdot\nabla)u\big|_{L^2(Q)^2} \leqslant K\big|(u,v,q)\big|_E^2. \tag{4.1}$$

Let $u^\flat: Q \to \mathbb{R}^2$ be defined by:

$$u^\flat(t,x) := e^{C_0/2(T-t)}u(t,x), \quad t \in (0,T),\ x \in \mathbb{T}_2. \tag{4.2}$$

One easily sees that there exists K_0 independent of $(u,q,v) \in E$ such that

$$|u^\flat|_{H^1((0,T);L^2(\mathbb{T}_2))^2} + |u^\flat|_{L^2((0,T);H^2(\mathbb{T}_2))^2} \leqslant K_0 |(u,v,q)|_E.$$

Hence, by classical interpolation inequalities, there exists $K_1 > 0$ independent of $(u,q,v) \in E$ such that

$$|u^\flat|_{L^\infty((0,T);L^4(\mathbb{T}_2))^2} \leqslant K_1 |(u,v,q)|_E. \tag{4.3}$$

Moreover, by the continuous Sobolev embedding $H^1(\mathbb{T}_2) \hookrightarrow L^4(\mathbb{T}_2)$, there exists $K_2 > 0$ independent of $(u,q,v) \in E$ such that

$$|\nabla u^\flat|_{L^2((0,T);L^4(\mathbb{T}_2))^4} \leqslant K_2 |(u,v,q)|_E. \tag{4.4}$$

From (4.3) and (4.4), one gets that

$$|(u^\flat \cdot \nabla)u^\flat|_{L^2(Q)^2} \leqslant K_1 K_2 |(u,v,q)|_E,$$

which, together with (4.2), shows that (4.1) holds with $K := K_1 K_2$. This concludes the proof of Lemma 1. □

We choose $e_0 := (0,0,0)$. Note that the map $\mathcal{A}'(e_0)$ is the following one:

$$\mathcal{A}'(e_0)(u,q,v) = \big(u_t - \Delta u + (\bar{y} \cdot \nabla)u + (u \cdot \nabla)\bar{y} + \nabla q - (v_1, 0), u(0)\big).$$

Hence, by Proposition 1, $\mathcal{A}'(e_0)$ is surjective. Note that, if $\mathcal{A}(u,q,v) = (0, y^0)$, if $y := \bar{y} + u$ and $p := \bar{p} + q$, then (1.4), (1.5), (1.6), (1.9), and

$$y_t - \Delta y + (y \cdot \nabla)y + \nabla p = 1_\omega(v_1 + f_1, 0) \quad \text{in } Q$$

hold. Therefore Theorem 1 readily follows by taking $g := (0, y^0)$ in Theorem 2. □

Remark 5. With some extra straightforward estimates, our proof of Theorem 1 shows that one can estimate η in terms of ε: on can replace in this theorem,

$$\begin{cases} \text{for every } \varepsilon > 0, \text{ there exists } \eta > 0 \text{ such that for every } y^0 \in H_0 \text{ satisfying} \\ |y^0|_{L^2(\mathbb{T}_2)^2} < \eta, \text{ there exists } \ldots \end{cases} \tag{4.5}$$

by

$$\begin{cases} \text{there exists } c > 0 \text{ such that, for every } \varepsilon \in (0,1) \text{ and for every } y^0 \in H_0 \text{ satisfying} \\ |y^0|_{L^2(\mathbb{T}_2)^2} < c\varepsilon^2, \text{ there exists } \ldots \end{cases} \tag{4.6}$$

Note that a linear growth of η in terms of ε is not possible. More precisely one cannot replace in Theorem 1 (4.5) by,

$$\begin{cases} \text{there exists } c > 0 \text{ such that, for every } \varepsilon \in (0,1) \text{ and for every } y^0 \in H_0 \text{ satisfying} \\ |y^0|_{L^2(\mathbb{T}_2)^2} < c\varepsilon, \text{ there exists} \ldots \end{cases}$$

This follows from the fact that, with ζ defined in (1.11), whatever is $v_1 \in L^2(Q)$, the solution y to the Cauchy problem (1.4)–(1.5)–(1.6) satisfies (compare with (1.12) for the linearized system (1.10))

$$\frac{d}{dt}\int_{\mathbb{T}_2} \zeta y_2\,dx = -\frac{4n^2\pi^2}{L_1^2}\int_{\mathbb{T}_2} \zeta y_2\,dx + \int_{\mathbb{T}_2} (\partial_{x_1}\zeta + \partial_{x_2}\zeta) y_1 y_2\,dx, \tag{4.7}$$

together with straightforward estimates.

Remark 6. Taking $\delta = C|y_0|^{1/2}_{L^2(\mathbb{T}_2)^2}$ with $C > 0$ large enough and using some extra estimates, one can adapt our proof of Theorem 1 to prove the existence of $\eta_1 > 0$ and of a map,

$$\mathcal{V} : B := \{y^0 \in H_0;\ |y_0|_{L^2(\mathbb{T}_2)^2} < \eta_1\} \subset H_0 \subset L^2(\mathbb{T}_2)^2 \to L^2(Q),$$

of class C^1 in $B \setminus \{0\}$ such that

- For every $y^0 \in B$, the solution y to the Cauchy problem (1.4)–(1.5)–(1.6) with $v_1 := \mathcal{V}(y^0)$ satisfies (1.9),
- There exists $C_5 > 0$ such that

$$\big|\mathcal{V}(y^0)\big|_{L^2(Q)} \leqslant C_5\big(|y_0|_{L^2(\mathbb{T}_2)^2}\big)^{1/2}, \quad \forall y^0 \in B. \tag{4.8}$$

Again, as in Remark 5, one cannot replace the exponent $1/2$ by 1 in (4.8).

Appendix A. Proof of Proposition 3

Let ω_0' be a non-empty open subset of $\mathbb{T}_2$ such that its closure is included in ω_0 and let $\chi \in C^\infty([0,T] \times \mathbb{T}_2)$ be such that

$$0 \leqslant \chi(t,x) \leqslant 1, \quad \forall (t,x) \in [0,T] \times \mathbb{T}_2, \tag{A.1}$$

$$\text{the support of } \chi \text{ is included in } (T/2, T] \times \omega_0, \tag{A.2}$$

$$\chi = 1 \quad \text{in } [2T/3, T] \times \omega_0'. \tag{A.3}$$

Before starting with the proof, let us recall a Carleman inequality which holds for the solutions of,

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi = f & \text{in } Q, \\ \nabla \cdot \varphi = 0 & \text{in } Q, \\ \varphi(T,\cdot) = \varphi_T & \text{in } \mathbb{T}_2. \end{cases} \tag{A.4}$$

This inequality, which readily follows from [17,18], is the following one. There exists $C_4 > 0$ and there exists $C > 0$ independent of (φ, π, f) satisfying (A.4) such that

$$\begin{aligned} &\int_{\mathbb{T}_2} \big|\varphi(0,x)\big|^2\,dx + \iint_Q e^{-2C_4/(T-t)}|\varphi|^2\,dx\,dt \\ &\quad \leqslant C \iint_Q e^{-(3+r)C_4/(2(T-t))}|f|^2\,dx\,dt + C \iint_{(2T/3,T)\times\omega_0'} e^{-(3+r)C_4/(2(T-t))}|\varphi|^2\,dx\,dt. \end{aligned} \tag{A.5}$$

(Observe that, by (3.25), we have $(3+r)/2 < 2$.)

The proof of Proposition 3 is inspired by [17]. Let us introduce the heat operator:

$$Lw := w_t - \Delta w$$

and its dual operator

$$L^*\phi := -\phi_t - \Delta\phi.$$

Let us set, with $\overline{Q} := [0,T] \times \mathbb{T}_2$,

$$X := \left\{(\varphi,\pi) \in C^\infty(\overline{Q})^2 \times C^\infty(\overline{Q}):\ \nabla \cdot \varphi = 0 \text{ in } Q,\ \int_{\mathbb{T}_2} \pi(t,x)\,dx = 0,\ \forall t \in [0,T]\right\}.$$

Consider the bilinear form,

$$a\big((\widehat{\varphi},\widehat{\pi}),(\varphi,\pi)\big) := \iint_Q e^{-(3+r)C_4/(2(T-t))}\big(\big(L^*\widehat{\varphi} + \nabla\widehat{\pi}\big)\cdot\big(L^*\varphi + \nabla\pi\big) + \chi(x)^2\widehat{\varphi}\cdot\varphi\big)\,dx\,dt,$$

for every $(\widehat{\varphi},\widehat{\pi}), (\varphi,\pi) \in X$. We also introduce the linear form,

$$\ell\big((\varphi,\pi)\big) := \iint_Q h\cdot\varphi\,dx\,dt + \int_{\mathbb{T}_2} y^0(x)\cdot\varphi(0,x)\,dx, \quad \forall(\varphi,\pi)\in X.$$

Let $\overline{X}$ be the completion of X for the norm $(\varphi,\pi)\mapsto a((\varphi,\pi),(\varphi,\pi))^{1/2}$ (it is a norm thanks to (A.5)).

Now, we are going to prove that the following variational problem,

$$a\big((\widehat{\varphi},\widehat{\pi}),(\varphi,\pi)\big) = \ell(\varphi) \quad \forall(\varphi,\pi)\in\overline{X}, \tag{A.6}$$

has a unique solution $(\widehat{\varphi},\widehat{\pi})\in\overline{X}$. Since $\overline{X}$ is a Hilbert space for the scalar product $a(\cdot,\cdot)$, the proof of the existence and uniqueness of solution of (A.6) is reduced to prove that ℓ is a continuous linear form on $\overline{X}$. From (A.5), one gets the continuity of the linear form $(\varphi,\pi)\in\overline{X}\mapsto\varphi(0,\cdot)\in L^2(\mathbb{T}_2)^2$. Moreover, from the Cauchy–Schwarz inequality, we have:

$$\left|\iint_Q h\cdot\varphi\,dx\,dt\right| \leqslant \left|e^{C_4/(T-t)}h\right|_{L^2(Q)^2}\left|e^{-C_4/(T-t)}\varphi\right|_{L^2(Q)^2},$$

so ℓ is continuous. Hence, there exists a unique $(\widehat{\varphi},\widehat{\pi})\in\overline{X}$ satisfying (A.6).

Let us set:

$$u^* := e^{-(3+r)C_4/(2(T-t))}\big(L^*\widehat{\varphi}+\nabla\widehat{\pi}\big) \quad \text{in } Q, \tag{A.7}$$

$$v^* := -\chi e^{-(3+r)C_4/(2(T-t))}\widehat{\varphi} \quad \text{in } Q. \tag{A.8}$$

Note that (A.2) and (A.8) imply (3.27). In view of the (variational) identity satisfied by $(\widehat{\varphi},\widehat{\pi})$, we readily have (3.29) and the existence of $q^*\in L^2((0,T);H^{-1}(\mathbb{T}_2))$ such that

$$Lu^*+\nabla q^* = v^*+h \quad \text{in } Q \quad \text{and} \quad u^*(0,\cdot)=y^0,$$

which gives (3.28) and (3.30). Using (A.7)–(A.8), we have:

$$\left|e^{(3+r)C_4/(4(T-t))}u^*\right|^2_{L^2(Q)^2} + \left|e^{(3+r)C_4/(4(T-t))}v^*\right|^2_{L^2(Q)^2} = a\big((\widehat{\varphi},\widehat{\pi}),(\widehat{\varphi},\widehat{\pi})\big) < +\infty. \tag{A.9}$$

Let $(u^\natural,q^\natural) := e^{rC_4/(T-t)}(u^*,q^*)$. It satisfies:

$$\begin{cases} Lu^\natural+\nabla q^\natural = e^{rC_4/(T-t)}(h+v^*)+(e^{rC_4/(T-t)})_t u^* & \text{in } Q,\\ \nabla\cdot u^\natural = 0 & \text{in } Q,\\ u^\natural(0,\cdot) = e^{rC_4/T}y^0 & \text{in } \mathbb{T}_2. \end{cases} \tag{A.10}$$

Thanks to (3.25)–(3.26) and (A.9), the right-hand side of the first equation of (A.10) belongs to $L^2(Q)^2$. Hence

$$u^\natural \in L^2\big((0,T);H^2(\mathbb{T}_2)\big)^2 \cap H^1\big((0,T);L^2(\mathbb{T}_2)\big)^2,$$
$$q^\natural \in L^2\big((0,T);H^1(\mathbb{T}_2)\big),$$

which gives (3.31), (3.32) and (3.34) (observe that $(e^{rC_4/(T-t)})_t u^*\in L^2(Q)^2$ thanks to (3.25)) and (A.9).

Let us finally prove (3.33). For this, we introduce:

$$\big(\varphi^\natural,\pi^\natural\big) := e^{-(3-r)C_4/(2(T-t))}(\widehat{\varphi},\widehat{\pi}). \tag{A.11}$$

Denoting $f := L^*\widehat{\varphi}+\nabla\widehat{\pi}$, we have:

$$\begin{cases} L^*\varphi^\natural+\nabla\pi^\natural = e^{-(3-r)C_4/(2(T-t))}f - (e^{-(3-r)C_4/(2(T-t))})_t\widehat{\varphi} & \text{in } Q,\\ \nabla\cdot\varphi^\natural = 0 & \text{in } Q,\\ \varphi^\natural(T,\cdot) = 0 & \text{in } \mathbb{T}_2. \end{cases} \tag{A.12}$$

From (3.25), (A.5) and (A.9), one gets that the right hand side of the first equation in (A.12) belongs to $L^2(Q)^2$, so that

$$\varphi^\natural \in L^2\big((0,T);H^2(\mathbb{T}_2)\big)^2 \cap H^1\big((0,T);L^2(\mathbb{T}_2)\big)^2, \tag{A.13}$$

which, together with (3.25), (A.8) and (A.11), gives (3.33). This concludes the proof of Proposition 3. □

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