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CONTROL FOR FAST AND STABLE LAMINAR-TO-HIGH-REYNOLDS-NUMBERS TRANSFER IN A 2D NAVIER-STOKES CHANNEL FLOW

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Abstract. We consider the problem of generating and tracking a trajectory between two arbitrary parabolic profiles of a periodic 2D channel flow, which is linearly unstable for high Reynolds numbers. Also known as the Poiseuille flow, this problem is frequently cited as a paradigm for transition to turbulence. Our procedure consists in generating an exact trajectory of the nonlinear system that approaches exponentially the objective profile. Using a backstepping method, we then design boundary control laws guaranteeing that the error between the state and the trajectory decays exponentially in L^2, H^1, and H^2 norms. The result is first proved for the linearized Stokes equations, then shown to hold locally for the nonlinear Navier-Stokes system.

1. Introduction.

One of the few situations in which analytic expressions for solutions of the stationary flow field are available is the channel flow problem. Also known as the Poiseuille flow, this problem is frequently cited as a paradigm for transition to turbulence. Poiseuille flow requires an *imposed* external pressure gradient for being created and sustained (see [5]). The magnitude of the pressure gradient determines the value of the centerline velocity, which parameterizes the whole flow.

It is very well known that this solution goes *linearly* unstable for Reynolds numbers greater than the so-called critical Reynolds number, $Re_{CR} \approx 5772$ (see [34]), even though the non-normality of the problem [32] may lead to large transient growth and enable a transition to turbulence at substantially smaller Reynolds number. Stabilization of Navier-Stokes equation for general geometries has been widely studied (see, e.g., [17,18,4,30,31] and the references therein, see also [12,21] for the Euler equations of incompressible fluids). For channel flow geometry, there are some particular results. The problem of locally stabilizing the equilibrium has been considered by means of *discretized* optimal control (see [25]), Lyapunov analysis (see [1,2]), spectral decomposition and pole placement (see [3,41]), and using the backstepping technique (see [42,44]). Observers have been developed as well using dual methods (see [24, 43]).

However, all prior works in channel flow consider a constant pressure gradient and a developed flow which is already close to the desired solution. The problem of tracking time varying profiles generated by unsteady pressure gradients has, so far, not been considered from a control point of view. Stability for channel flow driven by unsteady pressure gradient has been previously studied (see [27]). Velocity tracking problems have been considered in an optimal control framework (see [22]).

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In this paper, we consider the problem of *moving* the state from one Poiseuille equilibrium to another, following a pre-determined flow trajectory that should be "nice" in some sense. For example, we may wish to smoothly accelerate fluid at rest up to a given Reynolds number, probably over the critical value, avoiding transition to turbulence. The means at our disposal are the imposed pressure gradient and boundary control of the velocity field. We consider velocity actuation at one of the walls.

This is a problem of practical interest which, to the best of our knowledge, has not been solved or even been considered so far, since all control laws in the literature are designed for one given Poiseuille flow (fixed Reynolds number).

A possible solution for the problem would be to apply quasi-static deformation theory; this would require to modify the pressure gradient very slowly, and simultaneously gain-schedule a fixed Reynolds number boundary controller (see [42]) for tracking a (slowly) time varying trajectory, which in general would not be an exact solution of the system. This idea has been already used for moving between equilibria of a semilinear heat equation (see [13]), a semilinear wave equation (see [14]), or a Schrödinger's equation (see [6, 7]). Other applications include the shallow water problem (see [11]) and the Couette-Taylor flow controllability problem (see [35]). In this paper, however, we follow an alternative approach, finding analytically an *exact*, *fast* trajectory of the system which is then stabilized by means of boundary control. The advantage of this approach is that it reaches the objective profile requiring substantially less time and, apart from the imposed pressure gradient which steers the system, the boundary velocity control effort is only necessary for stabilization and will be zero in the absence of perturbations.

The procedure used for stabilization is similar to the method used in [44] for local stabilization of a steady Poiseuille profile, and it is based on the backstepping method (see [37,39]), which has been also employed in other flow control problems (see [45]). The method requires to solve a nonstandard partial integro-differential boundary value problem, and we provide a proof of its solvability. A simpler and similar looking equation was used, for other purposes, in [10] and in [36], where time analyticity of coefficients is assumed for obtaining solvability. Later, in [26] it is shown that for general $\mathcal{C}^\infty$ coefficients the equation has no solutions. We settle the issue by showing that the most natural class of functions for which the equation is solvable is the Gevrey class (see [20]).

The organization of the paper is as follows. Section 2 contains a detailed presentation of the results: the model is described in Section 2.1; an expression for the control laws designed using a backstepping method, the main result and the strategy of the proof are provided in Section 2.4. Section 3 is devoted to the proof of the main result: mathematical preliminaries are given in Section 3.1; then, a stabilization result is proved for the linearized system in Section 3.2; the main result (Theorem 2.1) is finally derived in Section 3.3. Section 4 is an appendix containing technical results needed in the proof.

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2. Main result.

2.1. Channel flow model.

We consider a 2D incompressible fluid filling a region Ω between two infinite planes separated from each other by a distance L. The exact setting is depicted on Figure 1, on which an example of an equilibrium profile is shown. Define U_c as a velocity scale, where U_c is the maximum centerline velocity, ρ and ν as the density and the kinematic viscosity of the fluid, respectively, and the Reynolds number, Re, as $Re = U_c h/\nu$. Then, using L, L/U_c and $\rho\nu U_c/L$ as length, time and pressure scales respectively, the nondimensional 2D Navier-Stokes equations are

$$u_t = \frac{\triangle u}{Re} - p_x - uu_x - vu_y, \tag{1}$$

$$v_t = \frac{\triangle v}{Re} - p_y - uv_x - vv_y, \tag{2}$$

where u is the streamwise velocity, v the wall-normal velocity, and p the pressure, with boundary conditions

$$u(t, x, 0) = v(t, x, 0) = 0, \qquad u(t, x, 1) = U(t, x), \quad v(t, x, 1) = V(t, x). \tag{3}$$

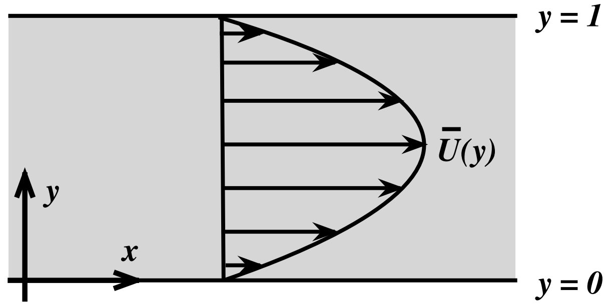

FIGURE 1. 2D Channel Flow and equilibrium profile (actuation is on the top wall).

In (3), U and V are the actuators located at the upper wall, which can be actuated independently for each x. The fluid is considered incompressible, so that the velocity field must verify the divergence-free condition

$$u_x + v_y = 0. \tag{4}$$

In these nondimensional coordinates, Ω is defined by

$$\Omega = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq 1\}, \tag{5}$$

with boundary $\partial\Omega = \partial\Omega_0 \cup \partial\Omega_1$, where $\partial\Omega_i = \{(x, y) \in \mathbb{R}^2 : y = i\}$, $i = 0, 1$. We refer to $\partial\Omega_0$ as the uncontrolled boundary, and to $\partial\Omega_1$ as the controlled boundary.

2.2. **Poiseuille flows.** A family of stationary solutions of (1)–(3) is the one-parameter Poiseuille family of parabolic profiles ($\mathcal{P}^\delta$) defined by

$$\mathcal{P}^\delta = (u^\delta, v^\delta, p^\delta) = \left(4\delta y(1-y), 0, -\frac{8\delta}{Re}x\right),$$

where the parameter δ stands for the maximum centerline velocity. Note that the velocity actuation at the wall is zero for $\mathcal{P}^\delta$, since both u^δ and v^δ are zero at the boundaries. The pressure gradient $p^\delta_x = -\frac{8\delta}{Re}$ must be externally sustained for $\mathcal{P}^\delta$ to be a stationary solution (see, e.g. [5, p. 182-183]).

We next describe in Section 2.3 a trajectory steering the system from a given (arbitrary) Poiseuille flow $\mathcal{P}^{\delta_0}$ to another one $\mathcal{P}^{\delta_1}$. In general this trajectory is unstable, and must be stabilized by means of boundary controls. In Section 2.4, we provide explicit feedback laws which stabilize this trajectory exponentially. We also make precise the analytic functional framework in which one has existence and uniqueness of a solution for the closed loop system together with exponential stability (Theorem 2.1).

2.3. **Generation of the trajectory to be tracked.** Given δ_0 and δ_1, we first generate a trajectory $\Theta(t) = (u(t), v(t), p(t))$ (where space dependence is omitted for clarity) connecting $\mathcal{P}^{\delta_0}$ to $\mathcal{P}^{\delta_1}$. We assume $\delta_0 = 0$ and $\delta_1 = 1$ for simplicity. Consider the trajectory $\Theta^q(t)$ defined by

$$\Theta^q(t) = (u^q(t), v^q(t), p^q(t)) = (g(t, y), 0, -xq(t)), \tag{6}$$

where q is the *chosen* external pressure gradient. Then, $\Theta^q(t)$ verifies (1)–(4) with $U = V = 0$ if

$$g_t = \frac{g_{yy}}{Re} + q. \tag{7}$$

Since $\mathcal{P}^0 \equiv 0$, we set $\Theta^q(0) = 0$, which implies $g(0, y) = q(0) = 0$. We impose $g(t, 0) = g(t, 1) = 0$ so no velocity control effort is needed to steer the control system, only to stabilize it. Given these initial-boundary data, choosing q completely determines g from (7) and consequently $\Theta^q(t)$, so $q(t)$ parameterizes $\Theta^q(t)$. Setting

$$q(t) = \frac{8}{Re}\left(1 - \mathrm{e}^{-ct}\right), \tag{8}$$

with $c > 0$ a design parameter, one has $q(0) = 0$ and $\lim_{t\to\infty} q(t) = 8/Re$. This selection of q determines a value g in (7) that verifies

$$\lim_{t\to\infty} g(t, y) = 4y(1-y)$$

(see Lemma 4.1 in Section 4.1 in the Appendix, where other properties of g are derived). It follows that $\Theta^q(t)$ is a solution of the trajectory generation problem, since its components are smooth and solve (1)–(4), and one has $\Theta^q(0) = \mathcal{P}^0$ and $\lim_{t\to\infty} \Theta^q(t) = \mathcal{P}^1$, so $\Theta^q(t)$ connects the chosen Poiseuille profiles (reaching $\mathcal{P}^1$ in infinite time, however by construction through rapidly decaying exponentials, Θ^q closely approaches $\mathcal{P}^1$ after a short time; in this sense, we consider Θ^q a *fast* trajectory).

Remark 2.1. The fact that an exact trajectory is obtained from a linear parabolic equation (Equation (7)) can be exploited to move between equilibria in *arbitrary finite time*, since it is known (see [16]) that this kind of equations have finite-time zero controllability for even initial data (i.e., $g(0, 1-y) = g(0, y)$, for every $y \in [0, 1]$). Motion planning theory for the heat equation (see [29]) allows to define an *explicit* finite-time trajectory, in the framework of Gevrey functions. We do not pursue a finite-time result (exponential stability is enough for practical purposes), however we present in Section 4.2 in the appendix a proof guaranteeing that our method allows tracking of trajectories defined in Gevrey spaces.

2.4. **Construction of stabilizing feedback laws and main result.** In general, the trajectory Θ^q, $t \in [0, +\infty)$, is unstable. Our goal is to design control feedback laws permitting to track the trajectory $\Theta^q(t)$. Define first the error variables

$$(\tilde{u}, \tilde{v}, \tilde{p}) = (u, v, p) - \Theta^q(t) = (u - g(t, y), v, p + xq(t)).$$

In these new variables, one has, dropping tildes for ease of notation,

$$u_t = \frac{\triangle u}{Re} - p_x - uu_x - vu_y - g(t, y)u_x - g_y(t, y)v, \tag{9}$$

$$v_t = \frac{\triangle v}{Re} - p_y - uv_x - vv_y - g(t, y)v_x, \tag{10}$$

and the same divergence-free condition and boundary conditions as before, i.e.,

$$u_x + v_y = 0, \tag{11}$$

$$u(t, x, 0) = v(t, x, 0) = 0, \qquad u(t, x, 1) = U(t, x), \quad v(t, x, 1) = V(t, x). \tag{12}$$

Our new control objective is to stabilize the equilibrium at the origin in (9)–(12) by means of suitable feedback laws U and V. Linearizing (9)–(12) around 0, we obtain the unsteady Stokes equations

$$u_t = \frac{\triangle u}{Re} - p_x - g(t, y)u_x - g_y(t, y)v, \tag{13}$$

$$v_t = \frac{\triangle v}{Re} - p_y - g(t, y)v_x, \tag{14}$$

with the divergence-free condition

$$u_x + v_y = 0, \tag{15}$$

and the boundary conditions

$$u(t, x, 0) = v(t, x, 0) = 0, \qquad u(t, x, 1) = U(t, x), \quad v(t, x, 1) = V(t, x). \tag{16}$$

Our strategy consists in using a backstepping method in order to design control laws stabilizing the origin of (13)–(16). Then, we prove that these control laws stabilize *locally* the origin of (9)–(12).

2.4.1. *Functional framework.* In our main result (Theorem 2.1), an assumption of periodicity in x is required for the initial condition. Combined with a uniqueness argument, this permits to show that the velocity field (u, v) and the pressure p are periodic in x with some period, say, $2h > 0$. This fact is essential in our analysis, and standard in the study of Stokes or Navier-Stokes equations (see, e.g., [40, 35]).

In order to take into account this periodicity, we set

$$\Omega_h = \{(x, y) \in \Omega : -h < x < h\}.$$

Let $L^2(\Omega_h)$ be the usual space of square-integrable functions on Ω_h, endowed with the scalar product

$$(\phi, \psi)_{L^2(\Omega_h)} = \int_{-h}^{h} \int_0^1 \phi(x, y)\psi(x, y)dydx.$$

Similarly, consider the spaces $H^1(\Omega_h)$ and $H^2(\Omega_h)$, defined as usual. Define $L^2_h(\Omega)$ as the closure of the set of continuous, $2h$-periodic in x, functions on Ω with respect to the norm induced by the scalar product

$$(\phi,\psi)_{L^2_h(\Omega)} = (\phi|_{\Omega_h},\psi|_{\Omega_h})_{L^2(\Omega_h)}.$$

In other words,

$$L^2_h(\Omega) = \{f \in L^2_{loc}(\Omega):\ f|_{\Omega_h} \in L^2(\Omega_h),\ f(x+2h,y) = f(x,y) \text{ for a.e. } (x,y)\in\Omega\}.$$

Furthermore, define the spaces

$$\begin{aligned} H^1_h(\Omega) &= \{f \in L^2_h(\Omega):\ f|_{\Omega_h} \in H^1(\Omega_h),\ f|_{x=-h} = f|_{x=h} \text{ in the trace sense}\},\\ H^2_h(\Omega) &= \{f \in H^1_h(\Omega):\ f|_{\Omega_h} \in H^2(\Omega_h),\ \nabla f|_{x=-h} = \nabla f|_{x=h} \text{ in the trace sense}\}, \end{aligned}$$

endowed with the corresponding norms. Similarly, denoting $\mathbf{w} = (u,v)$, define

$$\begin{aligned} \mathbf{H}^0_{0h}(\Omega) &= \{\mathbf{w} \in [L^2_h(\Omega)]^2 : u_x + v_y = 0, v|_{\partial\Omega_0} = 0\},\\ \mathbf{H}^1_{0h}(\Omega) &= \{\mathbf{w} \in [H^1_h(\Omega)]^2 : u_x + v_y = 0, \mathbf{w}|_{\partial\Omega_0} = 0\},\\ \mathbf{H}^2_{0h}(\Omega) &= \mathbf{H}^1_{0h}(\Omega) \cap [H^2_h(\Omega)]^2, \end{aligned}$$

endowed with the scalar product of, respectively, $[L^2_h(\Omega)]^2$, $[H^1_h(\Omega)]^2$ and $[H^2_h(\Omega)]^2$ (see, e.g., [40, page 9] for the precise meaning of $v|_{\partial\Omega_0} = 0$ for $\mathbf{w} \in [L^2_h(\Omega)]^2$ satisfying $u_x + v_y = 0$; note that $u|_{\partial\Omega_0} = 0$ has, in general, no meaning for $\mathbf{w} \in [L^2_h(\Omega)]^2$ even satisfying $u_x + v_y = 0$). These are the spaces for the velocity field where the main result is proved.

2.4.2. *Design of stabilizing controls.* We now define the stabilizing control laws for the controllers V and U. The way they are designed relies on a backstepping method, as explained in details in Section 3. The controller $V(t,x)$ is a dynamic controller, found as the unique solution of the forced parabolic equation

$$\begin{aligned} V_t &= \frac{V_{xx}}{Re} - \sum_{0<|n|<M} \int_{-h}^{h} \mathrm{e}^{i\gamma_n(\xi-x)} \left(2i\int_0^1 g_y(t,\eta)\cosh\left(\gamma_n(1-\eta)\right) v(t,\xi,\eta)d\eta \right.\\ &\qquad\qquad \left. -i\frac{u_y(t,\xi,0)-u_y(t,\xi,1)}{Re}\right) d\xi, \end{aligned} \tag{17}$$

initialized at zero[1], with the periodicity conditions $V(t,x+h) = V(t,x)$. The control law U is defined by

$$U(t,x) = \sum_{0<|n|<M} \int_{-h}^{h}\int_0^1 \mathrm{e}^{i\gamma_n(\xi-x)} K_n(t,1,\eta) u(t,\xi,\eta) d\eta d\xi, \tag{18}$$

where $M = \frac{2h\sqrt{Re}}{\pi}$, and $\gamma_n = \pi n/h$. For every integer n such that $0<|n|<M$, K_n in (18) is the solution of the kernel equation

$$K_{nt} = \frac{1}{Re}\left(K_{nyy} - K_{n\eta\eta}\right) - \lambda_n(t,\eta)K_n + f_n(y,\eta) - \int_\eta^y f_n(\xi,\eta)K_n(t,y,\xi)d\xi, \tag{19}$$

which is a linear partial integro-differential equation in the region $\Gamma = \{(t,y,\eta) \in (0,\infty)\times\mathcal{T}\}$, where $\mathcal{T} = \{(y,\eta)\in\mathbb{R}^2 : 0 \le \eta \le y \le 1\}$, with boundary conditions

$$K_n(t,y,y) = -Re\left(\int_0^y \frac{\lambda_n(\sigma)}{2}d\sigma + \mu_n(0)\right), \tag{20}$$

$$K_n(t,y,0) = Re\left(\int_0^y \mu_n(\sigma)K_n(t,y,\sigma)d\sigma - \mu_n(y)\right), \tag{21}$$

[1] If the velocity field initial conditions at the boundary were not zero, then it is required that $V(0,x) = v(0,x,1)$. We assume for simplicity $v(0,x,1) = 0$.

and where the coefficients in (19)–(21) are

$$\lambda_n(t,y) \;=\; -i\gamma_n g(t,y), \tag{22}$$

$$f_n(t,y,\eta) \;=-\; i\gamma_n\left(g_y(t,y)+2\gamma_n\int_\eta^y g_y(t,\sigma)\sinh\left(\gamma_n(y-\sigma)\right)d\sigma\right), \tag{23}$$

$$\mu_n(y) \;=\; \frac{\gamma_n}{Re}\frac{\cosh\left(\gamma_n(1-y)\right)-\cosh\left(\gamma_n y\right)}{\sinh\gamma_n}. \tag{24}$$

The solvability of (19)–(21) is stated in Proposition 3.3, and investigated in details in Section 4.2 (Appendix).

Remark 2.2. Averaging (in x) Equation (17), it can be seen that the mean component of V is zero (provided it is initialized at zero), thus the physical constraint of zero net flux is enforced. This can be written as $\int_{-h}^{h}V(t,\xi)d\xi=0$. Verifying this condition is crucial, since its violation would imply not satisfying mass conservation in the channel.

2.4.3. *Statement of the stability result.*

THEOREM 2.1. *There exist $\varepsilon>0$, $C_1>0$ and $C_2>0$, both depending only on c, δ_0, δ_1, h and R_e, such that, for every $\mathbf{w}_0=(u_0,v_0)\in\mathbf{H}^2_{0h}(\Omega)$ satisfying $\|\mathbf{w}_0\|_{\mathbf{H}^0_{0h}(\Omega)}<\varepsilon$ and the compatibility conditions*

$$u_0(x,1)=\sum_{0<|n|<M}\int_{-h}^{h}\int_0^1 \mathrm{e}^{i\gamma_n(\xi-x)}K_n(0,1,\eta)u_0(\xi,\eta)d\eta d\xi, \qquad v_0(x,1)=0,$$

there exists a unique

$$\mathbf{w}=(u,v)\in L^2(0,\infty;\mathbf{H}^2_{0h}(\Omega)),\ \textit{with}\ \mathbf{w}_t\in L^2(0,\infty;H^1_h(\Omega)^2),$$

such that

$$u(0,x,y)=u_0(x,y),\quad v(0,x,y)=v_0(x,y),$$

and, for some $p\in L^2(0,\infty;H^1_h(\Omega))$, Equations (9)-(12) hold with U and V defined by (17)-(24). Moreover,

$$\|\mathbf{w}(t)\|_{\mathbf{H}^i_{0h}(\Omega)}\le C_1\mathrm{e}^{-C_2t}\|\mathbf{w}_0\|_{\mathbf{H}^i_{0h}(\Omega)},\ \forall t\ge0,\ \forall i\in\{0,1,2\}.$$

Remark 2.3. If, in the previous results, the initial data $\mathbf{w}_0$ belongs to the space $\mathbf{H}^1_{0h}(\Omega)$, then we still have a unique solution $\mathbf{w}\in L^2(0,\infty;\mathbf{H}^1_{0h}(\Omega))$, with now $p\in L^2(0,\infty;L^2_h(\Omega))$, and the exponential decay property in $\mathbf{H}^i_{0h}(\Omega)$-norm, for $i\in\{0,1\}$.

Remark 2.4. The exponential decay rate C_2 in the theorem can be made as large as desired, just increasing as much as necessary M and λ_n in (22), so that

$$M=\frac{2h\sqrt{Re}}{\pi}+\bar{M},\quad \lambda_n=-i\gamma_n g(t,y)+\bar{\lambda},$$

for $\bar{M},\bar{\lambda}>0$ large enough. Increasing M means that more modes are controlled, whereas the uncontrolled modes (see Section 3.2.1) are more damped, except the mode $n=0$ (see the beginning of Section 3.2.1). To control and damp more the mode $n=0$, one can use, for example, the pole-shifting method described in [13]. Increasing λ means that more damping is added in the target system (49), so that controlled modes (see Section 3.2.2) decay faster.

Remark 2.5. Even though the controller (17)–(24) looks rather involved, it is not hard to compute and implement. One has to solve a finite set of linear PIDE equations (19)–(21) for computing the K_n's; we provide an symbolically computable solution via a convergent infinite series, whose partial sums provides an approximation to the controller. The kernel equations can be solved numerically as well, which can be done fast and efficiently compared, for example, with LQR—where nonlinear time dependent Riccati equations appear (see [37] for a numerical comparison between LQR and backstepping).

Remark 2.6. The result can be extended in a number of ways. Control laws (17)–(18) are defined by state feedback laws, so Theorem 2.1 requires knowledge of the full state. An output feedback design is possible applying a backstepping observer methodology dual to the techniques we follow here (see [38,43]). Then, only measurements of pressure and skin friction are required. A 3D channel flow, periodic in two directions, is also tractable, adding some refinements which include actuation of the spanwise velocity at the wall (see [9] for the new techniques and difficulties involved). All cited references consider the steady problem of stabilizing a given Poiseuille profile, therefore some modifications to account for the unsteady coefficients have to be done, in the same way the present paper extends the results of [42].

3. **Proof of the main result.**

The structure of the proof is the following. We first start with mathematical preliminaries (Section 3.1), then design control laws using backstepping theory and prove the stability result for Stokes equations (Section 3.2) which represent the linearized system of the Navier-Stokes equations around the Poiseuille profile. The main result (Theorem 2.1) is finally proved in Section 3.3.

3.1. **Mathematical preliminaries.**

3.1.1. *Fourier series expansion.* The complex Fourier coefficients $(\phi_n(y))_{n\in\mathbb{Z}}$ of a given integrable $2h$-periodic in x function ϕ defined on Ω are given by

$$\phi_n(y) = \frac{1}{2h}\int_{-h}^{h} \phi(x,y)\mathrm{e}^{\frac{in\pi}{h}x}dx, \quad n \in \mathbb{Z}.$$

We will write simply ϕ_n in the sequel. It is well known that if $\phi \in L^2_h(\Omega)$, then $(\phi_n(\cdot))_{n\in\mathbb{Z}} \in \ell^2\left(\mathbb{Z}, L^2(0,1)\right)$, i.e.,

$$\sum_{n\in\mathbb{Z}} \int_0^1 |\phi_n(y)|^2 dy < \infty.$$

Conversely, if $(\phi_n(\cdot))_{n\in\mathbb{Z}} \in \ell^2\left(\mathbb{Z}, L^2(0,1)\right)$, then one can recover ϕ from its Fourier series by $\phi(x,y) = \sum_{n\in\mathbb{Z}} \phi_n(y)\mathrm{e}^{-\frac{in\pi}{h}x}$.

Let $\phi,\psi \in L^2_h(\Omega)$. Recall that, according to Parseval's formula,

$$(\phi,\psi)_{L^2_h(\Omega)} = 2h\left((\phi_n),(\psi_n)\right)_{\ell^2(\mathbb{Z},L^2(0,1))}$$

where the scalar product in $\ell^2\left(\mathbb{Z}, L^2(0,1)\right)$ is $\left((\phi_n),(\psi_n)\right)_{\ell^2(\mathbb{Z},L^2(0,1))} = \sum_{n\in\mathbb{Z}}\int_0^1 \phi_n(y)\overline{\psi}_n(y)dy$, and the bar denotes the complex conjugate.

Given $\psi \in L^2(\Omega_h)$, one has

$$\|\psi\|^2_{L^2(\Omega_h)} = 2h\|\psi_n\|^2_{\ell^2(\mathbb{Z},L^2(0,1))} = 2h\sum_{n\in\mathbb{Z}}\|\psi_n\|^2_{L^2(0,1)},$$

where $\|\psi_n\|^2_{L^2(0,1)} = \int_0^1 |\psi_n(y)|^2 dy$. In the sequel we omit the subindexes when clear from the context.

3.1.2. *Poincaré inequalities.*

Lemma 3.1. *(Poincaré's inequality in $H^2(0,1)$). Suppose that f is a complex valued function belonging to $H^2(0,1)$, such that $f(0) = f(1) = 0$. Then $\|f_y\|^2_{L^2(0,1)} \le \|f_{yy}\|^2_{L^2(0,1)}$.*

Proof. Set $f_1(y) = \Re(f)$ and $f_2 = \Im(f)$. Since $f_1(0) = f_1(1) = 0$, there must exist $a \in (0,1)$ such that $f_{1y}(a) = 0$. Therefore,

$$f_{1y}(y) = \begin{cases} \int_a^y f_{1yy}(\eta)d\eta & \text{for} \quad y \in (a,1), \\ -\int_y^a f_{1yy}(\eta)d\eta & \text{for} \quad y \in (1,a). \end{cases}$$

Hence, by Cauchy-Schwarz inequality, $|f_{1y}(y)|^2 \le \int_0^1 f^2_{1yy}(\eta)d\eta$, and integrating, the inequality follows for f_1 (and analogously for f_2). □

Lemma 3.2. *(Trace inequality in $H^2(0,1)$). Suppose that f is a complex valued function belonging to $H^2(0,1)$. Then for any $A > 0$, $|f_y(1)|^2 + |f_y(0)|^2 \le (2+A)\|f_y\|^2_{L^2(0,1)} + \frac{1}{A}\|f_{yy}\|^2_{L^2(0,1)}$.*

Proof. We can write

$$\begin{aligned}|f_y(1)|^2+|f_y(0)|^2 &= \int_0^1 \frac{d}{dy}\left((2y-1)|f_y|^2\right)dy \\ &= 2\|f_y\|^2_{L^2(0,1)}+\int_0^1 (2y-1)\left(f_y\overline{f_{yy}}+\overline{f_y}f_{yy}\right)dy \\ &\leq 2\|f_y\|^2_{L^2(0,1)}+2\int_0^1 |f_y||f_{yy}|dy\end{aligned}$$

and since $ab \leq \frac{A}{2}a^2+\frac{1}{2A}b^2$, we get

$$|f_y(1)|^2+|f_y(0)|^2 \leq (2+A)\|f_y\|^2_{L^2(0,1)}+\frac{1}{A}\|f_{yy}\|^2_{L^2(0,1)}$$

□

LEMMA 3.3. *(Poincaré's inequalities in $H^1_h(\Omega)$ and $H^2_h(\Omega)$). Let $\phi \in H^1_h(\Omega)$ be such that $\phi|_{\partial\Omega_0} \equiv 0$, and $\psi \in H^2_h(\Omega)$ such that $\psi|_{\partial\Omega_i} \equiv 0$ for $i=0,1$. Then*

$$\|\phi\|^2_{L^2_h(\Omega)} \leq \|\phi_y\|^2_{L^2_h(\Omega)}, \tag{25}$$

$$\|\psi_y\|^2_{L^2_h(\Omega)} \leq \|\psi_{yy}\|^2_{L^2_h(\Omega)}. \tag{26}$$

Proof. Using Parseval's formula, $\|\phi\|^2_{L^2_h(\Omega)} = h\sum\|\phi_n\|^2_{L^2(0,1)} \leq h\sum\|\phi_{ny}\|^2_{L^2(0,1)}$, where we have used the classical Poincaré's formula for functions of $H^1(0,1)$ vanishing at 0, since $\phi \in H^1_h(\Omega)$ implies $\phi_n \in H^1(0,1)$, and $\phi|_{\partial\Omega_0} \equiv 0$ implies $\phi_n(0)=0$.

By the same reasoning, $\psi \in H^2_h(\Omega)$ implies $\psi_n \in H^2(0,1)$, and $\psi|_{\partial\Omega_i} \equiv 0$ implies $\psi_n(i)=0$, for $i=0,1$. Applying Lemma 3.1 for every n leads to $\|\psi_y\|^2_{L^2_h(\Omega)} = h\sum\|\psi_{ny}\|^2_{L^2(0,1)} \leq h\sum\|\phi_{nyy}\|^2_{L^2(0,1)} = \|\psi_{yy}\|^2_{L^2_h(\Omega)}$, thus proving the lemma. □

3.1.3. *Transformations of L^2 functions.* The following definitions establish facts and notations useful for designing our control laws, based on the backstepping method (see [37]). This method consists in finding an invertible transformation of the original variables into others whose stability properties are easy to establish.

DEFINITION 3.1. Let $\mathcal{T}=\{(y,\eta)\in\mathbb{R}^2 : 0\leq\eta\leq y\leq 1\}$. Given complex valued functions $f\in L^2(0,1)$ and $K\in L^\infty(\mathcal{T})$, we define the *transformed* variable $g=(\mathrm{I}-K)f$, where the operator Kf is defined by

$$Kf(y)=\int_0^y K(y,\eta)f(\eta)d\eta,$$

i.e., a Volterra operator. We call $\mathrm{I}-K$ the *direct* transformation with *kernel* K. Now, if there exists a function $L\in L^\infty(\mathcal{T})$ such that $f=(\mathrm{I}+L)g$, then we say that the transformation is invertible, and we call $\mathrm{I}+L$ the *inverse* transformation, and L the *inverse kernel* (or the inverse of K).

PROPOSITION 3.1. *One has the following properties.*

1. *The transformation $\mathrm{I}-K$ is invertible for every $K\in L^\infty(\mathcal{T})$. Moreover,*

$$L(y,\eta)=K(y,\eta)+\int_\eta^y K(y,\sigma)L(\sigma,\eta)d\sigma = K(y,\eta)+\int_\eta^y L(y,\sigma)K(\sigma,\eta)d\sigma.$$

2. *If $f\in L^2(0,1)$ then $g=(\mathrm{I}-K)f\in L^2(0,1)$, and $\|g\|^2_{L^2(0,1)}\leq(1+\|K\|_{L^\infty})^2\|f\|^2_{L^2(0,1)}$. Similarly, if $g\in L^2(0,1)$ then $f=(\mathrm{I}+L)g\in L^2(0,1)$, and $\|f\|^2_{L^2(0,1)}\leq(1+\|L\|_{L^\infty})^2\|g\|^2_{L^2(0,1)}$.*

The first point of this proposition is immediate from the theory of Volterra integral equations (see [23]). The second point follows easily from the Cauchy-Schwarz inequality.

This result allows to define a norm equivalent to the L^2 norm,

$$\|f\|^2_{KL^2(0,1)}=\|(\mathrm{I}-K)f\|^2_{L^2(0,1)}=\|g\|^2_{L^2(0,1)}. \tag{27}$$

For $\mathcal{C}^1(\mathcal{T})$ and $\mathcal{C}^2(\mathcal{T})$ kernels K and L, one has an equivalent version of Proposition 3.1, allowing to define respectively a $KH^1(0,1)$ and $KH^2(0,1)$ norm, respectively equivalent to the $H^1(0,1)$ and $H^2(0,1)$ norm:

$$\|f\|^2_{KH^1(0,1)} = \|(I-K)f\|^2_{H^1(0,1)} = \|g\|^2_{H^1(0,1)}, \tag{28}$$

$$\|f\|^2_{KH^2(0,1)} = \|(I-K)f\|^2_{H^2(0,1)} = \|g\|^2_{H^2(0,1)}, \tag{29}$$

where higher derivatives are calculated as follows:

$$\begin{aligned} g_y &= f_y - K(y,y)f(y) - \int_0^y K_y(y,\eta)f(\eta)d\eta, \\ g_{yy} &= f_{yy} - K(y,y)f_y(y) - 2K_y(y,y)f(y) - K_\eta(y,y)f(y) - \int_0^y K_{yy}(y,\eta)f(\eta)d\eta, \end{aligned}$$

and similarly for the inverse transformation. In particular, one has

$$(1+\|L\|_{L^\infty}+\|L_y\|_{L^\infty})^{-2}\|f\|^2_{H^1(0,1)} \le \|f\|^2_{KH^1(0,1)} \le (1+\|K\|_{L^\infty}+\|K_y\|_{L^\infty})^2\|f\|^2_{H^1(0,1)},$$

and other similar estimates hold for the H^2 norm.

3.1.4. *Transformations of the velocity field.* We define transformations of functions in $\mathbf{H}^0_{0h}(\Omega)$.

Definition 3.2. Consider a finite set $A = \{a_1, \dots, a_j\} \subset \mathbb{Z}$, and $\mathcal{K} = (K_n(y,\eta))_{n\in A}$ a family of $L^\infty(\mathcal{T})$ kernels. Then, for $\mathbf{w} = (u,v) \in \mathbf{H}^0_{0h}(\Omega)$, the *transformed* variable $\omega = (\alpha,\beta) = (\mathrm{I}-\mathcal{K})\mathbf{w}$ is defined through its Fourier components by

$$\omega_n = \begin{cases} ((\mathrm{I}-K_n)u_n, 0) & \text{for } n \in A, \\ \mathbf{w}_n, & \text{otherwise.} \end{cases}$$

The inverse transformation, $\mathbf{w} = (I+\mathcal{L})\omega$, is defined by

$$\mathbf{w} = \begin{cases} ((\mathrm{I}+L_n)\alpha_n, \hat{L}_n\alpha_n) & \text{for } n \in A, \\ \omega_n, & \text{otherwise,} \end{cases}$$

where the new operator $\hat{L}_n$ is defined by $\hat{L}_n f = -\pi i \frac{n}{h}\int_0^y \left(f(\eta) + \int_0^\eta L(\eta,\sigma)f(\sigma)d\sigma\right)d\eta$.

It is straightforward to show that $\mathbf{w}$ is well defined. Indeed, for $n \in A$, the second component of $\mathbf{w}$ is

$$\hat{L}_n\alpha_n = -\pi i\frac{n}{h}\int_0^y \left(\alpha_n(\eta) + \int_0^\eta L(\eta,\sigma)\alpha_n(\sigma)d\sigma\right)d\eta,$$

and, by definition of α_n from the direct transformation, and after some manipulation,

$$\hat{L}_n\alpha_n = -\pi i\frac{n}{h}\int_0^y \left(u_n(\eta) - \int_0^\eta \left(K_n(\eta,\sigma) - L_n(\eta,\sigma) + \int_\sigma^\eta L_n(\eta,\delta)K_n(\delta,\sigma)d\delta\right)u_n(\delta)d\sigma\right)d\eta.$$

By Proposition 3.1, one gets $\hat{L}_n\alpha_n = -\pi i\frac{n}{h}\int_0^y u_n(\eta)d\eta$. Since the divergence-free condition in Fourier space is $\pi i\frac{n}{h}u_n + v_{ny} = 0$ and $v_n(0) = 0$, one gets $\hat{L}_n\alpha_n = \int_0^y v_{ny}(\eta)d\eta = v_n(y)$. This way, even though the second component of the velocity seems to be lost in the direct transformation, it can be recovered and the transformation is still invertible. Using a similar argument as in Proposition 3.1,

$$\|\omega\|^2_{\mathbf{H}^0_{0h}(\Omega)} \le (1+\|\mathcal{K}\|_{L^\infty})^2\|\mathbf{w}\|^2_{\mathbf{H}^0_{0h}(\Omega)}, \quad \|\mathbf{w}\|^2_{\mathbf{H}^0_{0h}(\Omega)} \le (1+N^2)(1+\|\mathcal{L}\|_{L^\infty})^2\|\omega\|^2_{\mathbf{H}^0_{0h}(\Omega)},$$

where $N = \max_{n\in A}\{\pi\frac{n}{h}\}$, and $\|\mathcal{K}\|_{L^\infty} = \max_{n\in A}\|K_n\|_{L^\infty}$, $\|\mathcal{L}\|_{L^\infty} = \max_{n\in A}\|L_n\|_{L^\infty}$. This allows the definition of a norm, as in (27), equivalent to the $\mathbf{H}^0_{0h}(\Omega)$, that we call $\mathcal{K}\mathbf{H}^0_{0h}(\Omega)$,

$$\|\mathbf{w}\|_{\mathcal{K}\mathbf{H}^0_{0h}(\Omega)} = \|\omega\|_{\mathbf{H}^0_{0h}(\Omega)}. \tag{30}$$

For $\mathcal{C}^1(\mathcal{T})$ and $\mathcal{C}^2(\mathcal{T})$ kernel families one can define as well $\mathcal{K}\mathbf{H}^1_{0h}(\Omega)$ and $\mathcal{K}\mathbf{H}^2_{0h}(\Omega)$ norms, respectively equivalent to the regular $\mathbf{H}^1_{0h}(\Omega)$ and $\mathbf{H}^2_{0h}(\Omega)$ norms.

Remark 3.1. All previous results hold for transformation kernels depending on time, as long as they are uniformly bounded on the time interval (finite or infinite) considered (see Proposition 3.3 for such a statement).

3.2. **Stabilization result for the linearized system.** In this section we focus on the linearized system (13)–(16) which consists of Stokes equations. We show how to design, using a backstepping method, the control laws (17)–(18), and prove the following result, which is the analogous of Theorem 2.1 but for the Stokes systems instead of the Navier-Stokes system.

PROPOSITION 3.2. *There exist $C_1 > 0$ and $C_2 > 0$, both depending only on c, δ_0, δ_1, h and R_e, such that, for every $\mathbf{w}_0 = (u_0, v_0) \in \mathbf{H}^2_{0h}$ satisfying the compatibility conditions*

$$u_0(x,1) = \sum_{0<|n|<M} \int_{-h}^{h} \int_0^1 \mathrm{e}^{i\gamma_n(\xi - x)} K_n(0,1,\eta) u_0(\xi,\eta) d\eta d\xi, \qquad v_0(x,1) = 0,$$

there exists a unique

$$\mathbf{w} = (u,v) \in L^2(0,\infty; \mathbf{H}^2_{0h}(\Omega)), \text{ with } \mathbf{w}_t \in L^2(0,\infty; H^1_h(\Omega)^2),$$

such that $u(0,x,y) = u_0(x,y)$, $v(0,x,y) = v_0(x,y)$, and, for some $p \in L^2(0,\infty; H^1_h(\Omega))$, Equations (13)–(16) of the linear Stokes system hold with U and V defined by (17)-(24). Moreover,

$$\|\mathbf{w}(t)\|_{\mathbf{H}^i_{0h}(\Omega)} \leq C_1 \mathrm{e}^{-C_2 t} \|\mathbf{w}_0\|_{\mathbf{H}^i_{0h}(\Omega)}, \forall t \geq 0, \forall i \in \{0,1,2\}.$$

The remaining part of this section is devoted to the proof of that result. The proof requires Lyapunov methods. For denoting some positive constants that arise from various inequalities and norm equivalences, we will repeatedly use C with some subscript.

Equations (13)–(14) written in Fourier space are

$$u_{nt} = \frac{\triangle_n u_n}{Re} - i\gamma_n(p_n + g(t,y)u_n) - g_y(t,y)v_n, \tag{31}$$

$$v_{nt} = \frac{\triangle_n v_n}{Re} - p_{ny} - i\gamma_n g(t,y) v_n, \tag{32}$$

where $\triangle_n = \partial_{yy} - \gamma_n^2$ has been introduced for simplifying the expressions, and where $\gamma_n = \pi n/h$. The boundary conditions are

$$u_n(t,0) = v_n(t,0) = 0, \qquad u_n(t,1) = U_n(t), \quad v_n(t,1) = V_n(t), \tag{33}$$

and the divergence-free condition is

$$i\gamma_n u_n + v_{ny} = 0. \tag{34}$$

From (31)–(32) an equation for the pressure can be derived,

$$p_{nyy} - \gamma_n^2 p_n = -2i\gamma_n g_y(t,y) v_n, \tag{35}$$

with boundary conditions obtained from evaluating (32) at the boundaries and using (33),

$$p_{ny}(t,0) = -i\gamma_n \frac{u_{ny}(t,0)}{Re}, \qquad p_{ny}(t,1) = -i\gamma_n \frac{u_{ny}(t,1)}{Re} - \dot{V}_n - \gamma_n^2 \frac{V_n}{Re}. \tag{36}$$

Equations for different n are uncoupled due to linearity and spatial invariance, allowing separate consideration for each mode n. Most modes, which we refer to as *uncontrolled*, are naturally stable and thus left without control. A finite set of modes are unstable and require to be controlled.

3.2.1. *Uncontrolled modes.* These are $n = 0$ and large modes that verify $|n| \geq M$, where $M > 0$ will be made precise.

Mode $n = 0$ (mean velocity field). From (33) and (34), $v_0 \equiv 0$. Then, u_0 verifies $u_{0t} = \frac{u_{0yy}}{Re}$, with $u_0(t,0) = u_0(t,1) = 0$. Applying Lemma 3.1, we have $\frac{d}{dt}\|u_0\|^2 \leq -\frac{2}{Re}\|u_0\|^2$, implying $\|u_0(t)\|^2 \leq \mathrm{e}^{-\frac{2}{Re}t}\|u_0(0)\|^2$, where $\| \ \|$ stands for the $L^2(0,1)$, $H^1(0,1)$, or $H^2(0,1)$ norm.

Modes for large $|n|$. If $\mathbf{w}_n = (u_n, v_n)$, then, with $V_n = U_n = 0$,

$$\begin{aligned}\frac{d}{dt}\|\mathbf{w}_n\|^2_{L^2(0,1)^2} \quad = \quad & -2\frac{\|\mathbf{w}_{ny}\|^2_{L^2(0,1)^2}}{Re} - 2\gamma_n^2\frac{\|\mathbf{w}_n\|^2_{L^2(0,1)^2}}{Re} - (g_y u_n, v_n)_{L^2(0,1)^2} - (g_y v_n, u_n)_{L^2(0,1)^2} \\ & - (u_n, i\gamma_n p_n)_{L^2(0,1)^2} - (i\gamma_n p_n, u_n)_{L^2(0,1)^2} - (v_n, p_{ny})_{L^2(0,1)^2} - (p_{ny}, v_n)_{L^2(0,1)^2}. \end{aligned} \quad (37)$$

Consider the pressure terms like those in the second line of (37). Using the divergence-free condition $i\gamma_n u_n + v_{ny} = 0$, and integrating by parts,

$$-(u_n, i\gamma_n p_n)_{L^2(0,1)^2} = -(v_{ny}, p_n)_{L^2(0,1)^2} = (v_n, p_{ny})_{L^2(0,1)^2}.$$

Therefore, the pressure terms in (37) cancel each other. Then, using the Cauchy-Schwarz inequality and the inequality $ab \le (a^2+b^2)/2$, one gets

$$\frac{d}{dt}\|\mathbf{w}_n\|^2_{L^2(0,1)^2} \quad \le \quad -2\frac{\|\mathbf{w}_{ny}\|^2_{L^2(0,1)^2}}{Re} - 2\gamma_n^2\frac{\|\mathbf{w}_n\|^2_{L^2(0,1)^2}}{Re} + \|g_y\|_{L^\infty(0,1)}\|\mathbf{w}_n\|^2_{L^2(0,1)^2}. \quad (38)$$

Since $|g_y(t,y)| \le 4$ (see Lemma 4.1 in Section 4.1), choosing $|\gamma_n| \ge 2\sqrt{Re}$, i.e., $|n| \ge M = \frac{2h\sqrt{Re}}{\pi}$, yields

$$\frac{d}{dt}\|\mathbf{w}_n\|^2_{L^2(0,1)^2} \le -2\frac{\|\mathbf{w}_{ny}\|^2_{L^2(0,1)^2}}{Re} - \gamma_n^2\frac{\|\mathbf{w}_n\|^2_{L^2(0,1)^2}}{Re}, \quad (39)$$

therefore achieving L^2 exponential stability for large modes ($|n| \ge M$).

The H^1 exponential stability is proved for the same set of modes. Indeed, compute

$$\begin{aligned}\frac{d}{dt}\|\mathbf{w}_{ny}\|^2_{L^2(0,1)^2}, \quad = \quad & (\mathbf{w}_{ny}, \mathbf{w}_{nyt})_{L^2(0,1)^2} + (\mathbf{w}_{nyt}, \mathbf{w}_{ny})_{L^2(0,1)^2} \\ = \quad & -(\mathbf{w}_{nyy}, \mathbf{w}_{nt})_{L^2(0,1)^2} - (\mathbf{w}_{nt}, \mathbf{w}_{nyy})_{L^2(0,1)^2} \\ = \quad & -2\frac{\|\mathbf{w}_{nyy}\|^2_{L^2(0,1)^2}}{Re} + 2\gamma_n^2\frac{\|\mathbf{w}_{ny}\|^2_{L^2(0,1)^2}}{Re} - i\gamma_n(\mathbf{w}_{nyy}, g\mathbf{w}_n)_{L^2(0,1)^2} \\ & + i\gamma_n(g\mathbf{w}_n, \mathbf{w}_{nyy})_{L^2(0,1)^2} - i\gamma_n(u_{nyy}, p_n)_{L^2(0,1)} + i\gamma_n(p_n, u_{nyy})_{L^2(0,1)} \\ & + (v_{nyy}, p_{ny})_{L^2(0,1)} + (p_{ny}, v_{nyy})_{L^2(0,1)} + (u_{nyy}, g_y v_n)_{L^2(0,1)} + (g_y v_n, u_{nyy})_{L^2(0,1)}.\end{aligned}$$

Let us study first the terms without pressure. Using integrations by parts, the Cauchy-Schwarz inequality and the divergence-free condition, one gets

$$\begin{aligned} & -i\gamma_n(\mathbf{w}_{nyy}, g\mathbf{w}_n)_{L^2(0,1)^2} + i\gamma_n(g\mathbf{w}_n, \mathbf{w}_{nyy})_{L^2(0,1)^2} + (u_{nyy}, g_y v_n)_{L^2(0,1)} + (g_y v_n, u_{nyy})_{L^2(0,1)} \\ = \quad & i\gamma_n(\mathbf{w}_{ny}, g\mathbf{w}_{ny} + g_y\mathbf{w}_n)_{L^2(0,1)^2} - i\gamma_n(g\mathbf{w}_{ny} + g_y\mathbf{w}_n, \mathbf{w}_{ny})_{L^2(0,1)^2} \\ & -(u_{ny}, g_{yy}v_n + g_y v_{ny})_{L^2(0,1)} - (g_{yy}v_n + g_y v_{ny}, u_{ny})_{L^2(0,1)} \\ = \quad & i\gamma_n(\mathbf{w}_{ny}, g_y\mathbf{w}_n)_{L^2(0,1)^2} - i\gamma_n(g_y\mathbf{w}_n, \mathbf{w}_{ny})_{L^2(0,1)^2} - (u_{ny}, g_{yy}v_n)_{L^2(0,1)} \\ & -(g_{yy}v_n, u_{ny})_{L^2(0,1)} - i\gamma_n(u_{ny}, g_y u_n)_{L^2(0,1)} + i\gamma_n(g_y u_n, u_{ny})_{L^2(0,1)} \\ = \quad & i\gamma_n(v_{ny}, g_y v_n)_{L^2(0,1)^2} - i\gamma_n(g_y v_n, v_{ny})_{L^2(0,1)^2} - (u_{ny}, g_{yy}v_n)_{L^2(0,1)} - (g_{yy}v_n, u_{ny})_{L^2(0,1)} \\ \le \quad & \frac{\gamma_n^2+1}{Re}\|\mathbf{w}_{ny}\|^2_{L^2(0,1)^2} + Re\left(\|g_y\|^2_{L^\infty(0,1)} + \|g_{yy}\|^2_{L^\infty(0,1)}\right)\|\mathbf{w}_n\|^2_{L^2(0,1)^2}.\end{aligned}$$

Concerning the pressure terms, we have

$$\begin{aligned} & -i\gamma_n(u_{nyy}, p_n)_{L^2(0,1)} + i\gamma_n(p_n, u_{nyy})_{L^2(0,1)} + (v_{nyy}, p_{ny})_{L^2(0,1)} + (p_{ny}, v_{nyy})_{L^2(0,1)} \\ & = \Big[v_{nyy}(t,y)\overline{p}_n(t,y) + \overline{v}_{nyy}(t,y)p_n(t,y)\Big]_{y=0}^{y=1} = Re\Big[p_{ny}(y)\overline{p}_n(y) + \overline{p}_{ny}(y)p_n(y)\Big]_{y=0}^{y=1}, \end{aligned} \quad (40)$$

where the last equality is deduced from (32) evaluated at the boundaries. Hence

$$\begin{aligned} & \left|-i\gamma_n(u_{nyy}, p_n)_{L^2(0,1)} + i\gamma_n(p_n, u_{nyy})_{L^2(0,1)} + (v_{nyy}, p_{ny})_{L^2(0,1)} + (p_{ny}, v_{nyy})_{L^2(0,1)}\right| \\ & \le 2Re\left(|p_{ny}(1)||\overline{p}_n(1)| + |p_{ny}(0)||\overline{p}_n(0)|\right)\end{aligned}$$

LEMMA 3.4. $|p_{ny}(1)||\overline{p}_n(1)| + |p_{ny}(0)||\overline{p}_n(0)| \le \dfrac{\|\mathbf{w}_{nyy}\|^2_{L^2(0,1)}}{2Re^2} + C_1\dfrac{1+\gamma_n^2}{2}\|\mathbf{w}_{ny}\|^2_{L^2(0,1)}$, *for some constant* $C_1 > 0$.

Proof. Solving the Poisson pressure equation (35) as a function of the boundary conditions $p_{ny}(1)$ and $p_{ny}(0)$ we get

$$\begin{aligned} p_n(y) &= 2i\int_0^y \frac{\cosh(\gamma_n\eta)\cosh(\gamma_n(1-y))}{\sinh\gamma_n} g_y(\eta)v_n(\eta)d\eta + 2i\int_y^1 \frac{\cosh(\gamma_n y)\cosh(\gamma_n(1-\eta))}{\sinh\gamma_n} g_y(\eta)v_n(\eta)d\eta \\ &\quad + \frac{\cosh(\gamma_n y)}{\gamma_n\sinh\gamma_n}p_{ny}(1) - \frac{\cosh(\gamma_n(1-y))}{\gamma_n\sinh\gamma_n}p_{ny}(0) \end{aligned}$$

Hence

$$|p_n(y)| \leq 2\|g_y\|_{L^\infty(0,1)} \left|\frac{\cosh(\gamma_n y)\cosh(\gamma_n(1-y))}{\sinh\gamma_n}\right| \int_0^1 |v_n(\eta)|d\eta + \left|\frac{\cosh\gamma_n}{\gamma_n\sinh\gamma_n}\right| (|p_{ny}(1)| + |p_{ny}(0)|)$$

and since $\cosh(\gamma_n y)\cosh(\gamma_n(1-y)) \leq \cosh\gamma_n$, and for $|n| \geq M$ we have that $|\tanh\gamma_n| \geq \tanh\gamma_M$, it follows that

$$|p_n(y)| \leq \frac{2}{\tanh\gamma_M}\|g_y\|_{L^\infty(0,1)}\|v_n\|_{L^2(0,1)} + \frac{1}{|\gamma_n|\tanh\gamma_M}(|p_{ny}(1)| + |p_{ny}(0)|)$$

Then,

$$|p_{ny}(1)||p_n(1)| + |p_{ny}(0)||p_n(0)| \leq \frac{|p_{ny}(1)| + |p_{ny}(0)|}{\tanh\gamma_M}\left(2\|g_y\|_{L^\infty(0,1)}\|v_n\|_{L^2(0,1)} + \frac{|p_{ny}(1)| + |p_{ny}(0)|}{|\gamma_n|}\right)$$

Hence,

$$|p_{ny}(1)||p_n(1)| + |p_{ny}(0)||p_n(0)| \leq \frac{1}{\tanh\gamma_M}\left(|\gamma_n|\|g_y\|^2_{L^\infty(0,1)}\|v_n\|^2_{L^2(0,1)} + 3\frac{|p_{ny}(1)|^2 + |p_{ny}(0)|^2}{|\gamma_n|}\right)$$

Noting that $|\gamma_n| \leq 1 + \gamma_n^2$ and the pressure boundary conditions (36)

$$|p_{ny}(1)||p_n(1)| + |p_{ny}(0)||p_n(0)| \leq \frac{(1+\gamma_n^2)}{\tanh\gamma_M}\|g_y\|^2_{L^\infty(0,1)}\|v_n\|^2_{L^2(0,1)} + 3|\gamma_n|\frac{|u_{ny}(1)|^2 + |u_{ny}(0)|^2}{Re^2\tanh\gamma_M}$$

then applying Lemma 3.2 with $A = \frac{6|\gamma_n|}{\tanh\gamma_M}$ we obtain

$$\begin{aligned} |p_{ny}(1)||p_n(1)| + |p_{ny}(0)||p_n(0)| &\leq \frac{\|u_{nyy}\|^2_{L^2(0,1)}}{2Re^2} + \frac{18\gamma_n^2}{\tanh^2\gamma_M}\frac{\|u_{ny}\|^2_{L^2(0,1)}}{Re^2} \\ &\quad + \frac{(1+\gamma_n^2)}{\tanh\gamma_M}\|g_y\|^2_{L^\infty(0,1)}\|v_n\|^2_{L^2(0,1)} \end{aligned}$$

and applying Poincare's inequality we obtain the result. Note that C_1 depends only on Re and h (since M depends only on Re and h). ☐

From the previous estimates and applying Lemma 3.4, one gets

$$\frac{d}{dt}\|\mathbf{w}_{ny}\|^2_{L^2(0,1)^2} \leq -\frac{\|\mathbf{w}_{nyy}\|^2_{L^2(0,1)^2}}{Re} + C_2(1+\gamma_n^2)\|\mathbf{w}_{ny}\|^2_{L^2(0,1)^2} + C_3\|\mathbf{w}_n\|^2_{L^2(0,1)^2},$$

for some $C_2, C_3 > 0$. Setting $L = \frac{1+ReC_3+ReC_2(1+\gamma_n^2)}{2}\|\mathbf{w}_n\|^2_{L^2(0,1)^2} + \|\mathbf{w}_{ny}\|^2_{L^2(0,1)^2}$, which is obviously equivalent to the H^1 norm, one has

$$\begin{aligned}
\frac{d}{dt}L &\leq -\frac{1+ReC_3+ReC_2(1+\gamma_n^2)}{2}\left(2\frac{\|\mathbf{w}_{ny}\|^2_{L^2(0,1)^2}}{Re}+\gamma_n^2\frac{\|\mathbf{w}_n\|^2_{L^2(0,1)^2}}{Re}\right)\\
&\quad -\frac{\|\mathbf{w}_{nyy}\|^2_{L^2(0,1)^2}}{Re}+C_2(1+\gamma_n^2)\frac{\|\mathbf{w}_{ny}\|^2_{L^2(0,1)^2}}{Re}+C_3\|\mathbf{w}_n\|^2_{L^2(0,1)^2}\\
&\leq -\frac{1}{2}\gamma_n^2\frac{\|\mathbf{w}_n\|^2_{L^2(0,1)^2}}{Re}-\frac{\|\mathbf{w}_{ny}\|^2_{L^2(0,1)^2}}{Re}-\frac{\|\mathbf{w}_{nyy}\|^2_{L^2(0,1)^2}}{Re}\\
&\quad -C_3\|\mathbf{w}_{ny}\|^2_{L^2(0,1)^2}+C_3\|\mathbf{w}_n\|^2_{L^2(0,1)^2}\\
&\leq -\left(\frac{2+\gamma_n^2}{2Re}\|\mathbf{w}_n\|^2_{L^2(0,1)^2}+\frac{\|\mathbf{w}_{ny}\|^2_{L^2(0,1)^2}}{Re}\right)\\
&\leq -C_4L,
\end{aligned}$$

where $C_4 > 0$ depends on Re and h, but not on n. This establishes a H^1 stability property for $\mathbf{w}_n$ with a decay rate independent of n.

We next prove H^2 stability. For $|n| \geq M$, one has

$$\|\mathbf{w}_n\|^2_{H^2(0,1)^2} = \|u_{nyy}\|^2_{L^2(0,1)}+\|v_{nyy}\|^2_{L^2(0,1)}+\gamma_n^2(\|u_{ny}\|^2_{L^2(0,1)}+\|v_{ny}\|^2_{L^2(0,1)})+\gamma_n^4(\|u_n\|^2_{L^2(0,1)}+\|v_n\|^2_{L^2(0,1)}).$$

Integrating by parts, one gets

$$\begin{aligned}
\|\triangle_n u_n\|^2_{L^2(0,1)} &= (u_{nyy}-\gamma_n^2u_n, u_{nyy}-\gamma_n^2u_n)_{L^2(0,1)}\\
&= \|u_{nyy}\|^2_{L^2(0,1)}+\gamma_n^4\|u_n\|^2_{L^2(0,1)}-\gamma_n^2(u_{nyy},u_n)_{L^2(0,1)}-\gamma_n^2(u_{nyy},u_n)_{L^2(0,1)},\\
&= \|u_{nyy}\|^2_{L^2(0,1)}+\gamma_n^4\|u_n\|^2_{L^2(0,1)}+2\gamma_n^2\|u_{ny}\|^2_{L^2(0,1)},
\end{aligned}$$

and hence, $\|\triangle_n\mathbf{w}_n\|_{L^2(0,1)^2}$ is equivalent to $\|\mathbf{w}_n\|_{H^2(0,1)^2}$.

LEMMA 3.5. *For $\mathbf{w}$ verifying (31)–(32), the norm $\|\triangle_n\mathbf{w}_n\|_{L^2(0,1)^2}$ (and therefore $\|\mathbf{w}_n\|_{H^2(0,1)^2}$) is equivalent to the norm $((1+\gamma_n^2)\|\mathbf{w}_n\|^2_{H^1(0,1)^2}+\|\mathbf{w}_{nt}\|^2_{L^2(0,1)^2})^{1/2}$.*

Proof. From (31)–(32), one has

$$\|\mathbf{w}_{nt}\|^2_{L^2(0,1)^2} = \frac{\|\triangle_n\mathbf{w}_n\|^2_{L^2(0,1)^2}}{Re^2}+\Lambda,$$

where

$$\begin{aligned}
\Lambda &= -i\gamma_n(p_n,u_{nt})_{L^2(0,1)}-i\gamma_n(g(t,y)u_n,u_{nt})_{L^2(0,1)}-(g_y(t,y)v_n,u_{nt})_{L^2(0,1)}-(p_{ny},v_{nt})_{L^2(0,1)}\\
&\quad -i\gamma_n(g(t,y)v_n,v_{nt})_{L^2(0,1)}+\frac{1}{Re}\left(i\gamma_n(\triangle_nu_n,p_n)_{L^2(0,1)}+i\gamma_n(\triangle_nu_n,g(t,y)u_n)_{L^2(0,1)}\right.\\
&\quad \left.-(\triangle_nu_n,g_y(t,y)v_n)_{L^2(0,1)}-(\triangle_nv_n,p_{ny})_{L^2(0,1)}+i\gamma_n(\triangle_nv_n,g(t,y)v_n)_{L^2(0,1)}\right).
\end{aligned}$$

Integrating by parts and using the divergence-free condition,

$$-i\gamma_n(p_n,u_{nt})-(p_{ny},v_{nt})_{L^2(0,1)} = (p_n, i\gamma_nu_{nt}+v_{nyt})_{L^2(0,1)} = 0. \tag{41}$$

Then,

$$i\gamma_n(\triangle_nu_n,p_n)_{L^2(0,1)}-(\triangle_nv_n,p_{ny})_{L^2(0,1)} = -\left(\triangle_nv_n(t,y)\overline{p}_n(t,y)\right)_{y=0}^{y=1},$$

and using Equation (40) and Lemma 3.4,

$$|\triangle_nv_n(t,y)\overline{p}_n(t,y)|_{y=0}^{y=1} = Re\,|p_{ny}(t,y)\overline{p}_n(t,y)|_{y=0}^{y=1} \leq \frac{\|\triangle_n\mathbf{w}_n\|^2_{L^2(0,1)^2}}{2Re}+\frac{C_1}{2}(1+\gamma_n^2)\|\mathbf{w}_n\|^2_{H^1(0,1)^2}.$$

Hence, we can obtain

$$|\Lambda| \leq C_5(1+\gamma_n^2)\|\mathbf{w}_n\|^2_{H^1(0,1)^2}+\frac{3}{4}\left(\|\mathbf{w}_{nt}\|^2_{L^2(0,1)^2}+\frac{\|\triangle_n\mathbf{w}_n\|^2_{L^2(0,1)^2}}{Re^2}\right),$$

for some $C_5 > 0$. Taking into account that $(1+\gamma_n^2)\|\mathbf{w}_n\|^2_{H^1(0,1)^2}$ is bounded by $C_6\|\triangle_n \mathbf{w}_n\|^2_{L^2(0,1)^2}$ for some $C_6 > 0$, the lemma follows. □

Now, taking a time derivative in Equations (31)–(32), and applying the same argument as in the proof of L^2 stability, one gets

$$\frac{d}{dt}\|\mathbf{w}_{nt}\|^2_{L^2(0,1)^2} \leq -2\frac{\|\mathbf{w}_{nt}\|^2_{L^2(0,1)^2}}{Re} + C_7\|\mathbf{w}_{nt}\|_{L^2(0,1)^2}\|\mathbf{w}_n\|_{L^2(0,1)^2},$$

where the last term is due to the time-varying coefficients. Combining with the previous estimates for the L^2 and H^1 norms and Lemma 3.5, the H^2 stability property follows.

3.2.2. *Controlled modes, and design of control laws.* The remaining modes, such that $0 < |n| < M$, are open-loop unstable and must be controlled. We design the control laws in several steps.

Pressure shaping. Solving (35)–(36), one gets

$$\begin{aligned} p_n &= -2i\int_0^y g_y(t,\eta)\sinh\left(\gamma_n(y-\eta)\right)v_n(t,\eta)d\eta + 2i\frac{\cosh\left(\gamma_n y\right)}{\sinh\gamma_n}\int_0^1 g_y(t,\eta)\cosh\left(\gamma_n(1-\eta)\right)v_n(t,\eta)d\eta \\ &\quad + i\frac{\cosh\left(\gamma_n(1-y)\right)}{\sinh\gamma_n}\frac{u_{ny}(t,0)}{Re} - \frac{\cosh\left(\gamma_n y\right))}{\sinh\gamma_n}\left(i\frac{u_{ny}(t,1)}{Re} + \frac{\dot{V}_n}{\gamma_n} + \gamma_n\frac{V_n}{Re}\right). \end{aligned} \tag{42}$$

Note that V_n appears in (42), allowing to "shape" it. We design V_n to enforce in (42) a strict-feedback structure in y (see [28]). This structural property is a sort of "spatial causality", requiring that in the expression for, say, $f(t,y)$, no value of $f(t,s)$ for $s > y$ appears. It is a technical requirement in the backstepping method for parabolic equations (see [37, 39]) used next. Seeking the strict-feedback structure in (42), we choose V_n such that

$$\frac{\dot{V}_n}{\gamma_n} = -\gamma_n\frac{V_n}{Re} - i\frac{u_{ny}(t,0)-u_{ny}(t,1)}{Re} - 2i\int_0^1 g_y(t,\eta)\cosh\left(\gamma_n(1-\eta)\right)v_n(t,\eta)d\eta, \tag{43}$$

i.e.,

$$V_n = -i\int_0^t \mathrm{e}^{-\gamma_n^2\tau}\left(\gamma_n\frac{u_{ny}(\tau,0)-u_{ny}(\tau,1)}{Re} + 2\int_0^1 g_y(\tau,\eta)\cosh\left(\gamma_n(1-\eta)\right)v_n(\tau,\eta)d\eta\right)d\tau. \tag{44}$$

Plugging (43) into (42), the pressure reduces to

$$p_n = -2i\int_0^y g_y(t,\eta)\sinh\left(\gamma_n(y-\eta)\right)v_n(t,\eta)d\eta + i\frac{\cosh\left(\gamma_n(1-y)\right)-\cosh\left(\gamma_n y\right)}{\sinh\gamma_n}\frac{u_{ny}(t,0)}{Re}. \tag{45}$$

Substituting (45) into (31)–(32) yields

$$\begin{aligned} u_{nt} &= \frac{u_{nyy}}{Re} - \frac{\gamma_n^2 u_n}{Re} - i\gamma_n g(t,y)u_n - g_y(t,y)v_n - 2\gamma_n\int_0^y g_y(t,\eta)\sinh\left(\gamma_n(y-\eta)\right)v_n(t,\eta)d\eta \\ &\quad + \gamma_n\frac{\cosh\left(\gamma_n(1-y)\right)-\cosh\left(\gamma_n y\right)}{Re\sinh\gamma_n}u_{ny}(t,0), \end{aligned} \tag{46}$$

$$\begin{aligned} v_{nt} &= \frac{v_{nyy}}{Re} - \frac{\gamma_n^2 v_n}{Re} - i\gamma_n g(t,y)v_n + 2i\gamma_n\int_0^y g_y(t,\eta)\cosh\left(\gamma_n(y-\eta)\right)v_n(t,\eta)d\eta \\ &\quad + i\gamma_n\frac{\sinh\left(\gamma_n(1-y)\right)+\sinh\left(\gamma_n y\right)}{Re\sinh\gamma_n}u_{ny}(t,0). \end{aligned} \tag{47}$$

Control of velocity field. Our objective is now to control (46)–(47) by means of U_n. By (34), v_n can be computed as $v_n(y,t) = -i\gamma_n\int_0^y u_n(t,\eta)d\eta$. Then, only (46) has to be considered. Using (34), we express (46) as an autonomous equation in u_n,

$$u_{nt} = \frac{\triangle_n u_n}{Re} + \lambda_n(t,y)u_n + \int_0^y f_n(t,y,\eta)u_n(t,\eta)d\eta + \mu_n(y)u_{ny}(t,0),$$

with boundary conditions

$$u_n(t,0) = 0, \quad u_n(t,1) = U_n(t), \tag{48}$$

where λ_n, f_n and μ_n were defined in (22)–(24). This is a boundary control problem for a parabolic PIDE with time-dependent coefficients, solvable by backstepping (see [39]) thanks to the strict-feedback structure. Following [39], we map u_n, for each mode $0 < |n| < M$, into the family of heat equations

$$\alpha_{nt} = \frac{1}{Re}\left(-\gamma_n^2\alpha_n + \alpha_{nyy}\right), \quad \alpha_n(k,0) = \alpha_n(k,1) = 0, \tag{49}$$

where

$$\alpha_n = (I - K_n)u_n, \qquad u_n = (I + L_n)\alpha_n, \tag{50}$$

are respectively the direct and inverse transformation. The kernel K_n is found to verify Equations (19)–(21), and L_n verifies a similar equation, or can be derived from K_n using Proposition 3.1. For (19)–(21), the following result holds.

Proposition 3.3. *For every $n \in A$, there exists a solution $K_n(t,y,\eta)$ of (19)–(21) defined in $\Gamma = \{(t,y,\eta) \in (0,\infty)\times\mathcal{T}\}$ and such that, for every $k \in \mathbb{N}$, $K_n \in L^\infty(0,\infty;\mathcal{C}^k(\mathcal{T}))$, where $\mathcal{T} = \{(y,\eta) \in \mathbb{R}^2 : 0 \leq \eta \leq y \leq 1\}$.*

A proof of this Proposition is provided in Section 4.2 (Appendix).

The control law is, from (50), (49) and (48),

$$U_n = \int_0^1 K_n(t,1,\eta)u_n(t,k,\eta)d\eta, \tag{51}$$

Stability properties of the closed loop system follow from (49) and (50). From (49), one infers

$$\begin{aligned}\|\alpha_n(t)\|^2_{L^2(0,1)} &\leq \mathrm{e}^{-\frac{2}{Re}t}\|\alpha_n(0)\|^2_{L^2(0,1)}, \quad \|\alpha_n(t)\|^2_{H^1(0,1)} \leq \mathrm{e}^{-\frac{2}{Re}t}\|\alpha_n(0)\|^2_{H^1(0,1)},\\ \|\alpha_n(t)\|^2_{H^2(0,1)} &\leq \mathrm{e}^{-\frac{1}{Re}t}\|\alpha_n(0)\|^2_{H^2(0,1)}.\end{aligned}$$

Hence, from (50) and using the norms (28)–(29), we obtain

$$\begin{aligned}\|u_n(t)\|^2_{K_nL^2(0,1)} &\leq \mathrm{e}^{-\frac{2}{Re}t}\|u_n(0)\|^2_{K_nL^2(0,1)}, \quad \|u_n(t)\|^2_{K_nH^1(0,1)} \leq \mathrm{e}^{-\frac{2}{Re}t}\|u_n(0)\|^2_{K_nH^1(0,1)},\\ \|u_n(t)\|^2_{K_nH^2(0,1)} &\leq \mathrm{e}^{-\frac{1}{Re}t}\|u_n(0)\|^2_{K_nH^2(0,1)}.\end{aligned}$$

3.2.3. *Stability for the whole system.* Set $A = \{n \in \mathbb{Z} : 0 < |n| < M\}$ and $\mathcal{K} = K_n(t,y,\eta)_{n\in A}$. Applying the control laws (51) and (44) in physical space (which yield (17)–(18)), we next prove stability in $\mathcal{K}\mathbf{H}^0_{0h}(\Omega)$ norm, defined by (30), estimating

$$\begin{aligned}\|\mathbf{w}\|^2_{\mathcal{K}\mathbf{H}^0_{0h}(\Omega)} &= \sum_{n\notin A}\|\mathbf{w}_n\|^2_{L^2(0,1)^2} + \sum_{n\in A}\|u_n\|^2_{K_nL^2(0,1)}\\ &\leq \mathrm{e}^{-\frac{2}{Re}t}\left(\sum_{n\notin A}\|\mathbf{w}_n(0)\|^2_{L^2(0,1)^2} + \sum_{n\in A}\|u_n(0)\|^2_{K_nL^2(0,1)}\right) \leq \mathrm{e}^{-\frac{2}{Re}t}\|\mathbf{w}(0)\|^2_{\mathcal{K}\mathbf{H}^0_{0h}(\Omega)}.\end{aligned} \tag{52}$$

By norm equivalence, this proves the L^2 part of Proposition 3.2. Similarly,

$$\begin{aligned}\|\mathbf{w}\|^2_{\mathcal{K}\mathbf{H}^1_{0h}(\Omega)} &= \|u_0\|^2_{H^1(0,1)} + \sum_{0<|n|<M}\|u_n\|^2_{K_nH^1(0,1)} + \sum_{|n|\geq M}\|\mathbf{w}_n\|^2_{H^1(0,1)^2}\\ &\leq \mathrm{e}^{-\frac{2}{Re}t}\|u_0(0)\|^2_{H^1(0,1)} + \sum_{0<|n|<M}\mathrm{e}^{-\frac{2}{Re}t}\|u_n(0)\|^2_{K_nH^1(0,1)} + \sum_{|n|\geq M}C_1\mathrm{e}^{-C_2t}\|\mathbf{w}_n(0)\|^2_{H^1(0,1)^2}\\ &\leq C_3\mathrm{e}^{-C_4t}\|\mathbf{w}(0)\|^2_{\mathcal{K}\mathbf{H}^1_{0h}(\Omega)}.\end{aligned}$$

A similar argument shows H^2 stability.

3.2.4. *Well-posedness.* It remains to prove the well-posedness of the Stokes equations (13)–(16) with control laws (17)–(18). Define the spaces

$$H^1_{\mathrm{per}}(-h,h) = \left\{\phi \in H^1(-h,h) : \phi(h) = \phi(-h)\right\}$$

and

$$H^2_{\mathrm{per}}(-h,h) = \left\{\phi \in H^2(-h,h) : \phi(h) = \phi(-h), \phi_x(h) = \phi_x(-h)\right\}.$$

Proposition 3.4. *Given $T > 0$, assume that the velocity field (u, v), solution of (13)–(16), verifies $(u, v) \in L^2(0, T; \mathbf{H}^2_{0h}(\Omega))$. Then, the control laws V and U respectively defined by (17) and (18) verify*

$$U, V \in L^2(0, T; H^2_{\rm per}(-h, h)) \cap H^1(0, T; H^1_{\rm per}(-h, h)). \tag{53}$$

Proof. From (17), one has $V_t = \frac{V_{xx}}{Re} - f(t, x)$, where

$$f(t, x) = \sum_{0<|n|<M} \int_{-h}^{h} \mathrm{e}^{i\gamma_n(\xi - x)} \left(2i \int_0^1 g_y(t, \eta) \cosh\left(\gamma_n(1 - \eta)\right) v(t, \xi, \eta) d\eta - i \frac{u_y(t, \xi, 0) - u_y(t, \xi, 1)}{Re} \right) d\xi,$$

with $V(t, h) = V(t, -h)$ and initial conditions $V(0, x) = 0$. Since f is defined as a finite sum of convolutions in the periodic domain of certain functions with the smooth function $\mathrm{e}^{i\gamma_n x}$, we get that $f \in L^2(0, T; \mathcal{C}^p_{\rm per}([-h, h])$ for every integer p, where $g \in \mathcal{C}^p_{per}([-h, h])$ means that g is of class $\mathcal{C}^p$ and $g^{(i)}(-h) = g^{(i)}(h)$ for every $i \in \{0, \ldots, p\}$. Therefore, by standard properties of the heat equation (see for example [15, pg. 360, Theorem 5] for the non-periodic case), we get $V \in H^1([0, T), \mathcal{C}^p_{\rm per}([-h, h])$ for every integer p, and the conclusion follows for V. Similarly, the definition of U is

$$U(t, x) = \sum_{0<|n|<M} \int_{-h}^{h} \int_0^1 \mathrm{e}^{i\gamma_n(\xi - x)} K_n(t, 1, \eta) u(t, \xi, \eta) d\eta d\xi,$$

and the same kind of argument applies. □

We use a slightly modified version of [22, Theorem 2.1] (see also [19, Theorem 4.4] for a similar argument). Note that, from Remark 2.2 and the assumptions of Proposition 3.2, the following compatibility conditions are verified:

$$u_0(x, 1) = U(0, x), \quad v_0(x, 1) = V(0, x), \quad \int_{-h}^{h} V(t, x) dx = 0.$$

Then, for U and V satisfying (53), there exists a unique solution of the Stokes equations (13)–(16) such that $(u, v) \in L^2(0, T; \mathbf{H}^2_{0h}(\Omega))$. This fact, combined with Proposition 3.4 and estimates of Section 3.2.3 guaranteeing the decay of the $\mathbf{H}^2_{0h}(\Omega)$ norm of the velocity field, yields existence and uniqueness for the Stokes equations (13)–(16) with control laws (17)–(18) in $L^2(0, \infty; \mathbf{H}^2_{0h}(\Omega))$.

3.3. **Proof of Theorem 2.1.** Proposition 3.2 proved in the previous subsection deals with the linearized system, and is actually valid for any initial condition. If we now consider the Navier-Stokes (9)–(11), then, due to the nonlinear terms, we obtain just local stability.

Denote the nonlinear term in the Navier-Stokes equations (9)–(11) by $\mathbf{N} = (N^u, N^v)$, i.e.,

$$N^u = -uu_x - vu_y, \quad N^v = -uv_x - vv_y.$$

It follows from [40, Lemma 3.4 p. 292] that, for some C depending only on h,

$$(\mathbf{w}, \mathbf{N})_{\mathbf{H}^0_{0h}(\Omega)} \ \leq \ C \|\mathbf{w}\|_{\mathcal{K}\mathbf{H}^0_{0h}(\Omega)} \|\mathbf{w}\|^2_{\mathcal{K}\mathbf{H}^1_{0h}(\Omega)}. \tag{54}$$

The bound above is valid not only for $(\mathbf{w}, \mathbf{N})_{\mathbf{H}^0_{0h}(\Omega)}$ but for any partial sum of $(\mathbf{w}_n, \mathbf{N}_n)_{L^2(0,1)^2}$, by the same argument.

The application of pressure shaping and backstepping transformation to the nonlinear system yields a new term in the target system, which appears as

$$\alpha_{nt} = \frac{1}{Re}\left(-\gamma_n^2 \alpha_n + \alpha_{nyy}\right) + N_n^\alpha,$$

where N_n^α is defined by $N_n^\alpha = (I - K_n) N_n^u + (I - K_n) N_n^p$. The term N_n^p is due to pressure shaping and is defined by

$$N_n^p = 2 \sum_{j \in \mathbb{Z}} \left(\frac{\cosh\left(\gamma_n y\right)}{\sinh \gamma_n} \int_0^1 N^q_{nj} \cosh\left(\gamma_n(1 - \eta)\right) d\eta + \int_0^y N^q_{nj} \sinh\left(\gamma_n(y - \eta)\right) d\eta \right),$$

where $N^q_{nj} = -\gamma_j\gamma_{n-j}u_ju_{n-j} - i\gamma_{n-j}u_{jy}v_{n-j}$. Then, for $n \in A$,

$$\begin{aligned}(\alpha_n, N^\alpha_n)_{L^2(0,1)} &\le C_2\left((|\alpha_n|, |N^u_n|)_{L^2(0,1)} + (|\alpha_n|, |N^p_n|)_{L^2(0,1)}\right)\\ &\le C_2\|\alpha_n\|_{L^2(0,1)}\sum_{j\in\mathbb{Z}}\left(|\gamma_j|\|u_ju_{n-j}\|_{L^2(0,1)} + \|u_{jy}v_{n-j}\|_{L^2(0,1)}\right)(1 + C_3|\gamma_{n-j}|), \end{aligned} \quad (55)$$

where

$$C_2 = 1 + \|\mathcal{K}\|_{L^\infty} \quad \text{and} \quad C_3 = 2\frac{\sinh(\gamma_1)\sinh(\gamma_M) + \cosh^2(\gamma_M)}{\sinh(\gamma_1)}.$$

Bounding (55) further, one gets

$$\begin{aligned}(\alpha_n, N^\alpha_n) \le\ & \frac{C_2}{2}\|\alpha_n\|_{L^2(0,1)}\sum_{j\in\mathbb{Z}}\Big\{2|\gamma_j|^2\|u_j\|^2_{L^2(0,1)} + 2\|u_{jy}\|^2_{L^2(0,1)}\\ &+ \left(1 + C_3^2|\gamma_{n-j}|^2\right)\left(\|u_{(n-j)}\|^2_{L^2(0,1)} + \|v_{(n-j)}\|^2_{L^2(0,1)}\right)\Big\}\\ \le\ & C_4\|\alpha_n\|_{L^2(0,1)}\,\|\mathbf{w}\|^2_{\mathcal{K}\mathbf{H}^1_{0h}(\Omega)}, \end{aligned} \quad (56)$$

for some constant $C_4 > 0$. For the $\mathcal{K}L^2$ norm of the velocity field, as in (52),

$$\|\mathbf{w}\|^2_{\mathcal{K}L^2_{0h}(\Omega)} = \sum_{n\notin A}\|\mathbf{w}_n\|_{L^2(0,1)^2} + \sum_{n\in A}\|u_n\|^2_{K_nL^2(0,1)}.$$

Let us estimate the derivatives for each term of the right-hand side of this equality. We have

$$\frac{d}{dt}\sum_{n\notin A}\|\mathbf{w}_n\|^2_{L^2(0,1)^2} \le \sum_{n\notin A}\left(\frac{-2}{Re}\|\mathbf{w}_{ny}\|^2_{L^2(0,1)^2} - \frac{\gamma_n^2}{Re}\|\mathbf{w}_n\|^2_{L^2(0,1)^2} + (\mathbf{w}_n, \mathbf{N}_n)_{L^2(0,1)^2}\right), \quad (57)$$

and for $n \in A$, since $\|u_n\|_{K_nL^2(0,1)} = \|\alpha_n\|_{L^2(0,1)}$, one has

$$\frac{d}{dt}\|u_n\|^2_{K_nL^2(0,1)} = \frac{d}{dt}\|\alpha_n\|^2_{L^2(0,1)} \le \frac{-2}{Re}\|\alpha_{ny}\|^2_{L^2(0,1)} - \frac{2\gamma_n^2}{Re}\|\alpha_n\|^2 + (\alpha_n, N^\alpha_n)_{L^2(0,1)}. \quad (58)$$

Then, summing (58) for $n \in A$, adding (57), and applying norm equivalences and the estimate (54), we get, for some $C_0 > 0$,

$$\begin{aligned}\frac{d}{dt}\|\mathbf{w}\|^2_{\mathcal{K}\mathbf{H}^0_{0h}(\Omega)} &\le -C_0\|\mathbf{w}\|^2_{\mathcal{K}\mathbf{H}^1_{0h}(\Omega)} + \sum_{n\notin A}(\mathbf{w}_n, \mathbf{N}_n)_{L^2(0,1)^2} + \sum_{n\in A}(\alpha_n, N^\alpha_n)_{L^2(0,1)}\\ &\le \|\mathbf{w}\|^2_{\mathcal{K}\mathbf{H}^1_{0h}(\Omega)}\left(C_4\|\mathbf{w}\|_{\mathcal{K}\mathbf{H}^0_{0h}(\Omega)} + \|\mathbf{w}\|_{\mathcal{K}\mathbf{H}^0_{0h}(\Omega)} - C_0\right).\end{aligned}$$

Suppose that $\|\mathbf{w}\|_{\mathcal{K}\mathbf{H}^0_{0h}(\Omega)} < \epsilon$. Then

$$\frac{d}{dt}\|\mathbf{w}\|^2_{\mathcal{K}\mathbf{H}^0_{0h}(\Omega)} \le ((C_4 + 1)\epsilon - C_0)\,\|\mathbf{w}\|^2_{\mathcal{K}\mathbf{H}^1_{0h}(\Omega)},$$

and choosing $\epsilon < \frac{C_0}{2(C_4+1)}$,

$$\frac{d}{dt}\|\mathbf{w}\|^2_{\mathcal{K}\mathbf{H}^0_{0h}(\Omega)} \le \frac{-C_0}{2}\|\mathbf{w}\|^2_{\mathcal{K}\mathbf{H}^1_{0h}(\Omega)} \le -C_5\|\mathbf{w}\|^2_{\mathcal{K}\mathbf{H}^0_{0h}(\Omega)},$$

by Poincaré's inequality, where $C_5 > 0$. This proves local exponential stability in the $\mathcal{K}\mathbf{H}^0_{0h}(\Omega)$ norm and therefore in the $\mathbf{H}^0_{0h}(\Omega)$ norm.

A similar argument applies with the $\mathbf{H}^1_{0h}(\Omega)$ and $\mathbf{H}^2_{0h}(\Omega)$ norms for proving local exponential stability; we skip the details since it is clear that the extra nonlinear convection terms that would appear in the proof (in a similar way to the proof for the $\mathbf{H}^0_{0h}(\Omega)$ norm) can be bounded by the linear terms in $\mathbf{w}$, $\mathbf{w}_x$ and $\mathbf{w}_y$ for small $\mathbf{H}^1_{0h}(\Omega)$ and $\mathbf{H}^2_{0h}(\Omega)$ norms[2]. Well-posedness follows in the same way as in Section 3.2.4, since the argument of [22] applies to nonlinear Navier-Stokes equations.

[2]The only extra detail required would be a minor modification of Lemma 3.4 to account for the extra nonlinear terms in the pressure Poisson equation.

4. **Appendix.**

4.1. **Properties of the function g.**

LEMMA 4.1. *Let $\kappa_m = \pi(2m+1)$. Consider $g(t,y)$ defined by (7) where q is given by (8), boundary conditions $g(t,0) = g(t,1) = 0$ and initial conditions $g(0,y) \equiv 0$. Assume as well that $cRe \neq \kappa_m^2$ for any $m \in \mathbb{N}$. Then g has the following properties.*

i. *The explicit expression for g in $(0,\infty) \times [0,1]$ is given by*

$$g \;=\; 16 \sum_{m=0}^{\infty} \frac{\sin(\kappa_m y)}{\kappa_m} \left(\frac{1 - \mathrm{e}^{-\frac{\kappa_m^2}{Re} t}}{\kappa_m^2} - \frac{\mathrm{e}^{-ct} - \mathrm{e}^{-\frac{\kappa_m^2}{Re} t}}{\kappa_m^2 - cRe} \right). \tag{59}$$

ii. *It holds that*

$$\lim_{t\to\infty} g(t,y) = 4y(1-y).$$

iii. *The function g belongs to the space $C^\omega(0,\infty) \times C^\infty[0,1]$ (i.e., analytic in t and smooth in x).*

iv. *The estimates*

$$0 < g(t,y) \le 1, \quad |g_y(t,y)| \le 4, \quad -8 < g_{yy}(t,y) \le 0,$$

hold for every $t \ge 0$ and every $y \in [0,1]$.

Proof. In the proof we make use of many properties of the heat equation [15].

Point i is obtained by a Fourier expansion and application of Duhamel's Principle for solving (7). That yields the solution

$$g(t,y) = 2\sum_{m=0}^{\infty} \frac{\sin(\kappa_m y)}{\kappa_m} \int_0^t \mathrm{e}^{-\frac{\kappa_m^2}{Re}(t-\tau)} q(\tau) d\tau,$$

and plugging in the expression (8) for q and solving explicitly the integral (where the assumption on c is used), (59) is found.

Point ii is obtained by passing to the limit in (59) as t goes to infinity. Then

$$\lim_{t\to\infty} g(t,y) = 16 \sum_{m=0}^{\infty} \frac{\sin(\kappa_m y)}{\kappa_m^3} = 4y(1-y), \tag{60}$$

which can be verified by computing the Fourier series of $4y(1-y)$ which coincides with the infinite sum.

Point iii is a standard property of the solutions of the heat equation, taking into account that q itself is $C^\omega(0,\infty) \times C^\infty(0,1)$.

Point iv is proved using the maximum principle for the heat equation. Having proved smoothness in Point iii, we can first consider the equation that g_{yy} verifies by differentiation of (7)

$$(g_{yy})_t = \frac{1}{Re}(g_{yy})_{yy}. \tag{61}$$

The boundary conditions for (61) can be determined plugging (8) in (7), and taking limit as y goes to 0 and 1. Then, using the fact that $g(t,0) = g(t,1) = 0$, it follows that $g_{yy}(t,0) = g_{yy}(t,0) = -8(1 - \mathrm{e}^{-ct})$. The initial condition is $g_{yy}(0,y) = 0$, and it holds that $\lim_{t\to\infty} g_{yy}(t,y) = -8$. By the maximum and minimum principle, and since $-8 < g_{yy}(t,0) < 0$, it follows that $-8 < g_{yy} < 0$.

Consider now g_y. The fact that the boundary conditions of g are $g(t,0) = g(t,1) = 0$, the initial condition is zero, and (7) has constant coefficients in y, implies that g is symmetric around $y = 1/2$, i.e., $g(y) = g(1-y)$. Hence, it follows that $g_y(y) = -g_y(1-y)$, which implies $g_y(1/2) = 0$. Then,

$$g_y(t,y) = \begin{cases} \int_{1/2}^{y} g_{yy}(t,\eta) d\eta & \text{for} \quad y \in (1/2,1), \\ -\int_{y}^{1/2} g_{yy}(t,\eta) d\eta & \text{for} \quad y \in (0,1/2), \end{cases}$$

and the bound $|g_y(t,y)| \le 4$ follows. For g, one has

$$g(t,y) = \begin{cases} -\int_y^1 g_y(\eta) d\eta = -\int_y^1 \int_{1/2}^{\eta} g_{yy}(t,\sigma) d\sigma d\eta & \text{for} \quad y \in (1/2,1), \\ \int_0^y g_y(\eta) d\eta = -\int_0^y \int_{\eta}^{1/2} g_{yy}(t,\sigma) d\sigma d\eta & \text{for} \quad y \in (0,1/2), \end{cases}$$

and the bound $|g(t,y)| \le 1$ follows, thus finishing the proof of Point iv. ∎

4.2. **Kernel equations, and proof of Proposition 3.3.** We actually derive a more general statement which includes Proposition 3.3 as a particular case. We first recall the definition of the Gevrey class of functions, which plays an important role in studying solutions of the heat equation (see [20, 8]). As shown later, solutions to kernel partial integro-differential equations that appear in unsteady backstepping theory are members of some Gevrey class.

DEFINITION 4.1. A smooth function f defined on $(0,T)$, for $T \in (0,\infty]$, is Gevrey of order α, and we denote $f \in \mathcal{G}^\alpha(0,T)$, if there exists numbers $Q, R > 0$ such that, for every positive integer k,

$$\sup_{t\in(0,T)} \left| \frac{d^k f}{dt^k} \right| \le Q \frac{(k!)^\alpha}{R^k}.$$

For a nonempty open subset $\mathcal{O}$ of $\mathbb{R}^n$, and for $m \in \mathbb{N}$, we denote by $H^{m,\infty}(\mathcal{O})$ the set of functions $\varphi \in L^\infty(\mathcal{O})$ whose derivatives $\partial^\alpha \varphi$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ with $|\alpha| = \sum_{i=1}^n \alpha_i \le m$, are in $L^\infty(\mathcal{O})$ (with the agreement that $H^{0,\infty}(\mathcal{O}) = L^\infty(\mathcal{O})$). This space, endowed with its usual norm, is a Banach space. For functions of time and space, we define the following classes.

DEFINITION 4.2. Given $m \ge 0$ integer, a function $f : (0,T) \times \mathcal{O} \to \mathbb{R}$, $(t,y) \mapsto f(t,y)$ is said to be Gevrey of order α in time $t \in (0,T)$ and $H^{m,\infty}(\mathcal{O})$ in space, and we denote $f \in \mathcal{G}^\alpha(0,T; H^{m,\infty}(\mathcal{O}))$, if $f(t,\cdot) \in H^{m,\infty}(\mathcal{O})$ for every time $t \in (0,T)$, $f(t,\cdot)$ possesses time derivatives of every order which also belong to $H^{m,\infty}(\mathcal{O})$, and there exist numbers $Q, R > 0$ such that, for every positive integer k,

$$\sup_{t\in(0,T)} \left\| \frac{d^k f}{dt^k} \right\|_{H^{m,\infty}(\mathcal{O})} \le Q \frac{(k!)^\alpha}{R^k}.$$

Consider now the kernel equation

$$\epsilon K_{yy} - \epsilon K_{\eta\eta} - K_t(t,y,\eta) = \lambda(t,\eta) K(t,y,\eta) - f(t,y,\eta) + \int_\eta^y K(t,y,\xi) f(t,\xi,\eta) d\xi, \tag{62}$$

with boundary conditions

$$K(t,y,y) = \frac{-1}{2\epsilon} \int_0^y \lambda(t,\sigma) d\sigma - \frac{g(t,0)}{\epsilon}, \qquad K(t,y,0) = \int_0^y K(t,y,\sigma) \frac{g(t,\sigma)}{\epsilon} d\sigma - \frac{g(t,y)}{\epsilon}, \tag{63}$$

in the domain $(0,T) \times \mathcal{T}$, where $\epsilon > 0$ and $T > 0$. The following result hold, where $\mathring{\mathcal{T}}$ denotes the interior of $\mathcal{T}$ in $\mathbb{R}^2$.

PROPOSITION 4.1. 1. *(Finite time) Assume $T \in (0,\infty)$. For coefficients f, g, λ satisfying*

$$f \in \mathcal{G}^\alpha(0,T; H^{m-1,\infty}(\mathring{\mathcal{T}})), \quad \lambda \in \mathcal{G}^\alpha(0,T; H^{m,\infty}(0,1)), \quad g \in \mathcal{G}^\alpha(0,T; H^{m+1,\infty}(0,1)), \tag{64}$$

the problem (62)–(63) has a unique solution $K \in \mathcal{G}^\alpha(0,T; H^{m+1,\infty}(\mathring{\mathcal{T}}))$.

2. *(Infinite time) For coefficients f, g, λ verifying*

$$\begin{aligned} f &\in \mathcal{G}^\alpha(0,\infty; H^{m-1,\infty}(\mathring{\mathcal{T}})) \cap L^\infty(0,\infty; H^{m-1,\infty}(\mathring{\mathcal{T}})), \\ \lambda &\in \mathcal{G}^\alpha(0,\infty; H^{m,\infty}(0,1)) \cap L^\infty(0,\infty; H^{m,\infty}(0,1)), \\ g &\in \mathcal{G}^\alpha(0,\infty; H^{m+1,\infty}(0,1)) \cap L^\infty(0,\infty; H^{m+1,\infty}(0,1)), \end{aligned}$$

the problem (62)–(63) has a unique solution $K \in \mathcal{G}^\alpha(0,\infty; H^{m+1,\infty}(\mathring{\mathcal{T}})) \cap L^\infty(0,\infty; H^{m+1,\infty}(\mathring{\mathcal{T}}))$.

This result includes Proposition 3.3 as a particular case, with $\alpha = 1$, and passing to the limit $m \to \infty$.

Proof. Let us prove the first item. In the following it will be assumed that $1 \le \alpha < 2$. For $\alpha < 1$ one has to substitute everywhere in the section α by 1. We follow [37] to transform the PIDE into an integral equation. Applying the change of variables $\xi = y + \eta$ and $\beta = y - \eta$, and denoting $G(t,\xi,\beta) = K(t,y,\eta) = K\left(t, \frac{\xi+\beta}{2}, \frac{\xi-\beta}{2}\right)$, the PIDE (62) is transformed into

$$\begin{aligned} 4\epsilon G_{\xi\beta} &= G_t(t,\xi,\beta) + A(t,\xi,\beta) G(t,\xi,\beta) - B(t,\xi,\beta) \\ &\quad + \int_{\frac{\xi-\beta}{2}}^{\frac{\xi+\beta}{2}} G\left(t, \frac{\xi+\beta}{2} + \sigma, \frac{\xi+\beta}{2} - \sigma\right) f\left(t, \sigma + \frac{\xi-\beta}{2}, \sigma - \frac{\xi-\beta}{2}\right) d\sigma, \end{aligned} \tag{65}$$

with boundary conditions

$$G_\xi(t,\xi,0) = \frac{-A(t,\xi,0)}{4\epsilon}, \quad G(t,\xi,\xi) = \frac{-g(t,\xi)}{\epsilon} + \int_0^\xi G\left(t,\xi+\sigma,\xi-\sigma\right)\frac{g(t,\sigma)}{\epsilon}d\sigma, \tag{66}$$

in the domain $(0,T)\times\mathcal{T}_1$, where $\mathcal{T}_1 = \{(\xi,\beta) : 0 \le \xi \le 2, 0 \le \beta \le \min\{\xi, 2-\xi\}\}$, and where now

$$\begin{aligned} A(t,\xi,\beta) &= \lambda\left(t,\frac{\xi-\beta}{2}\right) \in \mathcal{G}^\alpha(0,T); H^{m,\infty}(\mathring{\mathcal{T}}_1)), \\ B(t,\xi,\beta) &= f\left(t,\frac{\xi+\beta}{2},\frac{\xi-\beta}{2}\right) \in \mathcal{G}^\alpha(0,T;H^{m-1,\infty}(\mathring{\mathcal{T}}_1)). \end{aligned}$$

Changing the integration variable, we can rewrite Equation (65) as

$$4\epsilon G_{\xi\beta} = G_t(t,\xi,\beta) + A(t,\xi,\beta)G(t,\xi,\beta) - B(t,\xi,\beta) + \int_0^\beta G\left(t,\xi+\sigma,\beta-\sigma\right)f\left(t,\sigma+(\xi-\beta),\sigma\right)d\sigma.$$

Integrating, and using the boundary conditions (66), we reach

$$\begin{aligned} G(t,\xi,\beta) &= \frac{1}{4\epsilon}\int_\beta^\xi\int_0^\beta G_t(t,\tau,\sigma)d\sigma d\tau + \frac{1}{4\epsilon}\int_\beta^\xi\int_0^\beta A(t,\tau,\sigma)G(t,\tau,\sigma)d\sigma d\tau - \frac{1}{4\epsilon}\int_\beta^\xi\int_0^\beta B(t,\tau,\sigma)d\sigma d\tau \\ &\quad + \frac{1}{4\epsilon}\int_\beta^\xi\int_0^\beta\int_0^\sigma G\left(t,\tau+\mu,\sigma-\mu\right)f\left(t,\mu+(\tau-\sigma),\mu\right)d\mu d\sigma d\tau \\ &\quad - \frac{1}{4\epsilon}\int_\beta^\xi A(t,\tau,0)d\tau - \frac{1}{\epsilon}g(t,\beta) + \frac{1}{\epsilon}\int_0^\beta g(t,\sigma)G\left(t,\beta+\sigma,\beta-\sigma\right)d\sigma, \end{aligned} \tag{67}$$

an integro-differential equation that only contains time derivatives and spatial integrals. Following [37], we seek a successive series approximation solution

$$G = \sum_{n=0}^{\infty} G_n(t,\xi,\beta), \tag{68}$$

with

$$G_0(t,\xi,\beta) = -\frac{1}{4\epsilon}\int_\beta^\xi\int_0^\beta B(t,\tau,\sigma)d\sigma d\tau - \frac{1}{4\epsilon}\int_\beta^\xi A(t,\tau,0)d\tau - \frac{1}{\epsilon}g(t,\beta),$$

and for $n > 0$,

$$\begin{aligned} G_n(t,\xi,\beta) &= \frac{1}{4\epsilon}\int_\beta^\xi\int_0^\beta (G_{n-1})_t(t,\tau,\sigma)d\sigma d\tau + \frac{1}{4\epsilon}\int_\beta^\xi\int_0^\beta A(t,\tau,\sigma)G_{n-1}(t,\tau,\sigma)d\sigma d\tau \\ &\quad + \frac{1}{4\epsilon}\int_\beta^\xi\int_0^\beta\int_0^\sigma G_{n-1}\left(t,\tau+\mu,\sigma-\mu\right)f\left(t,\mu+(\tau-\sigma),\mu\right)d\mu d\sigma d\tau \\ &\quad + \frac{1}{\epsilon}\int_0^\beta g(t,\sigma)G_{n-1}\left(t,\beta+\sigma,\beta-\sigma\right)d\sigma. \end{aligned}$$

Since A, B, f and g are Gevrey are in $\mathcal{G}^\alpha(0,T;L^\infty(\mathcal{T}_1))$, there exists $R > 0$ and $M > 0$ such that

$$\left\|\frac{\partial^k}{\partial t^k}A(t,\cdot)\right\|_{L^\infty(\mathcal{T}_1)} + \left\|\frac{\partial^k}{\partial t^k}B(t,\cdot)\right\|_{L^\infty(\mathcal{T}_1)} + \left\|\frac{\partial^k}{\partial t^k}f(t,\cdot)\right\|_{L^\infty(\mathcal{T}_1)} \left\|\frac{\partial^k}{\partial t^k}g(t,\cdot)\right\|_{L^\infty(\mathcal{T}_1)} \le M\frac{(k!)^\alpha}{R^k}, \tag{69}$$

for every $t \in (0,T)$ and every $k \in \mathbb{N}$. Define now $h(t,t_0) = \frac{1}{1-\frac{t-t_0}{R^{1/\alpha}}}$, for $t_0 \in [0,T)$. Then, $1 \le |h(t,t_0)|$ whenever $t \in [t_0, \frac{R^{1/\alpha}}{2} + t_0)$. Since

$$\frac{\partial^k h(t,t_0)}{\partial t^k} = \frac{k!}{\left(1-\frac{t-t_0}{R^{1/\alpha}}\right)^{k+1}R^{k/\alpha}} = \frac{k! h(t,t_0)^{k+1}}{R^{k/\alpha}},$$

it is clear that, for $t \in [t_0, \frac{R^{1/\alpha}}{2} + t_0)$, one has $\frac{k!}{R^{k/\alpha}} \leq \frac{\partial^k h(t,t_0)}{\partial t^k}$, and hence, by (69),

$$\left\| \frac{\partial^k}{\partial t^k} A(t,\cdot) \right\|_{L^\infty(\mathcal{T}_1)} + \left\| \frac{\partial^k}{\partial t^k} B(t,\cdot) \right\|_{L^\infty(\mathcal{T}_1)} + \left\| \frac{\partial^k}{\partial t^k} f(t,\cdot) \right\|_{L^\infty(\mathcal{T}_1)} \left\| \frac{\partial^k}{\partial t^k} g(t,\cdot) \right\|_{L^\infty(\mathcal{T}_1)} \leq M \left(\frac{\partial^k h(t,t_0)}{\partial t^k} \right)^\alpha, \quad (70)$$

for every $k \in \mathbb{N}$. We consider a uniform subdivision of $(0,T)$ into m subintervals,

$$(0,T) = \left(0, \frac{R^{1/\alpha}}{2}\right) \cup \left(\frac{R^{1/\alpha}}{2}, R^{1/\alpha}\right) \cup \cdots \cup \left((m-1)\frac{R^{1/\alpha}}{2}, T\right), \quad (71)$$

where m is chosen so that the length of the last subinterval is less than or equal to $\frac{R^{1/\alpha}}{2}$. For each subinterval, set t_0 as the infimum of the subinterval.

We show the proof for the subinterval $t \in (0, \frac{R^{1/\alpha}}{2})$; it proceeds equally for the rest of the subintervals because t_0 does not appear explicitly in the computations. This means that the bounds obtained below for the interval $(0, \frac{R^{1/\alpha}}{2})$ uniformly holds in the whole interval $(0,T)$. Hence, it suffices to prove the result for $t \in (0, \frac{R^{1/\alpha}}{2})$.

Denote $h(t,0) = h(t)$ for simplicity. We prove the existence of the solution defined by the successive approximation series using a variant of the classical method of majorants (see [29] for a similar proof).

We claim that for all $n \geq 0$, $k \geq 0$, and $(t,\xi,\beta) \in (0, \frac{R^{1/\alpha}}{2}) \times \mathring{\mathcal{T}}_1$,

$$\left| \frac{\partial^k}{\partial t^k} G_n(t,\xi,\beta) \right| \leq \left(\frac{\partial^k}{\partial t^k} h(t)^{n+1} \right)^\alpha \frac{C^{n+1}\sqrt{\beta^n(\xi+\beta)^n}}{(n!)^\gamma}, \quad (72)$$

where $\gamma = 2 - \alpha > 0$ and $C = \frac{2}{\epsilon R} + \frac{5M}{\epsilon}$.

Assume the above formula is true (it is proved next). Then, substituting in the successive approximation series (68), one has, for $k \geq 0$,

$$\left| \frac{\partial^k}{\partial t^k} G(t,\xi,\beta) \right| \leq \sum_{n=0}^{\infty} \left(\frac{\partial^k}{\partial t^k} h(t)^{n+1} \right)^\alpha \frac{C^{n+1}\sqrt{\beta^n(\xi+\beta)^n}}{(n!)^\gamma} \leq \left(\frac{\partial^k}{\partial t^k} H(t,\xi,\beta) \right)^\alpha, \quad (73)$$

where

$$H(t,\xi,\beta) = \sum_{n=0}^{\infty} h(t)^{n+1} \frac{C^{(n+1)/\alpha}\sqrt[2\alpha]{(1+\beta)^n(1+\xi+\beta)^n}}{(n!)^{\gamma/\alpha}}$$

is an analytic function of all its variables in $(t,\xi,\beta) \in [0, \frac{R^{1/\alpha}}{2}] \times \mathring{\mathcal{T}}_1$, whenever $\alpha < 2$. This is easily seen for ξ and β. To see it for t, substitute ξ and β by their maximum (2 and 1 respectively). Then,

$$H(t,\xi,\beta) = \sum_{n=0}^{\infty} h(t)^{n+1} \frac{C^{(n+1)/\alpha}\sqrt[2\alpha]{12^n}}{(n!)^{\gamma/\alpha}} = \sum_{n=0}^{\infty} h(t)^{n+1} \frac{D^{n+1}}{(n!)^\delta}.$$

To check analyticity on $[0, \frac{R^{1/\alpha}}{2}]$, since all terms in the sum are already analytic, we extend H to a disk of radius $R^{1/\alpha}$ in the complex plane, i.e. $t \in \mathbb{C}, |t| \in [0, \frac{R^{1/\alpha}}{2}]$ and check convergence for t on compact subsets [33] of the disk. Set then $t = \frac{R^{1/\alpha}}{2}(1-\sigma)$, where $\sigma \in [0,1]$. Then,

$$H(t,\xi,\beta) \leq \sum_{n=0}^{\infty} \left(\frac{2D}{\sigma+1} \right)^{n+1} \frac{1}{(n!)^\delta},$$

which converges for all values of D, σ, δ. Therefore H is also analytic in t. Then, by using (73), it follows that G is in $\mathcal{G}^\alpha(0,T;L^\infty(\mathcal{T}_1))$. Note that, as was stated before, the proof holds as well when using $h(t,t_0)$ instead of $h(t,0)$, for $t \in [t_0, t_0 + \frac{R^{1/\alpha}}{2})$.

It remains to prove the estimate (72), by induction on n. For $n = 0$ and $k = 0$, one has, using (70),

$$|G_0(t,\xi,\beta)| \leq \frac{1}{4\epsilon} \int_\beta^\xi \int_0^\beta M h(t)^\alpha d\sigma d\tau + \frac{1}{4\epsilon} \int_\beta^\xi M h(t)^\alpha d\tau + \frac{1}{\epsilon} M h(t)^\alpha \frac{2M}{\epsilon} \leq C h(t)^\alpha,$$

and for $n=0$, $k>0$,

$$\begin{aligned}\left|\frac{\partial^k G_0(t,\xi,\beta)}{\partial t^k}\right| &\le \frac{1}{4\epsilon}\int_\beta^\xi\int_0^\beta M\left(\frac{\partial^k}{\partial t^k}h(t)\right)^\alpha d\sigma d\tau + \frac{1}{4\epsilon}\int_\beta^\xi M\left(\frac{\partial^k}{\partial t^k}h(t)\right)^\alpha d\sigma + \frac{1}{\epsilon}M\left(\frac{\partial^k}{\partial t^k}h(t)\right)^\alpha d\sigma d\tau \\ &\le \frac{2M}{\epsilon}\left(\frac{\partial^k}{\partial t^k}h(t)\right)^\alpha \le C\left(\frac{\partial^k}{\partial t^k}h(t)\right)^\alpha,\end{aligned}$$

so (72) is true for $n=0$.

Suppose now it is true for $n-1$. Then, for $k=0$,

$$\begin{aligned}|G_n(t,\xi,\beta)| \le \frac{1}{4\epsilon}\int_\beta^\xi\int_0^\beta\left|\frac{\partial G_{n-1}}{\partial t}\right|(t,\tau,\sigma)d\sigma d\tau + \frac{1}{4\epsilon}\int_\beta^\xi\int_0^\beta Mh(t)^\alpha\left|G_{n-1}\right|(t,\tau,\sigma)d\sigma d\tau \\ +\frac{1}{4\epsilon}\int_\beta^\xi\int_0^\beta\int_0^\sigma\left|G_{n-1}\right|(t,\tau+\mu,\sigma-\mu)\,Mh(t)^\alpha d\mu d\sigma d\tau + \frac{1}{\epsilon}\int_0^\beta Mh(t)^\alpha\left|G_{n-1}\right|(t,\beta+\sigma,\beta-\sigma)d\sigma,\end{aligned}$$

and, using the induction hypothesis (72),

$$\begin{aligned}|G_n(t,\xi,\beta)| &\le \frac{1}{4\epsilon}\int_\beta^\xi\int_0^\beta\left(\frac{\partial}{\partial t}h(t)^n\right)^\alpha\frac{C^n\sqrt{\sigma^{n-1}(\tau+\sigma)^{n-1}}}{((n-1)!)^\gamma}d\sigma d\tau \\ &+\frac{1}{4\epsilon}\int_\beta^\xi\int_0^\beta M\left(h(t)^{n+1}\right)^\alpha\frac{C^n\sqrt{\sigma^{n-1}(\tau+\sigma)^{n-1}}}{((n-1)!)^\gamma}d\sigma d\tau \\ &+\frac{1}{4\epsilon}\int_\beta^\xi\int_0^\beta\int_0^\sigma M\left(h(t)^{n+1}\right)^\alpha\frac{C^n\sqrt{(\sigma-\mu)^{n-1}(\tau+\sigma)^{n-1}}}{((n-1)!)^\gamma}d\mu d\sigma d\tau \\ &+\frac{1}{\epsilon}\int_0^\beta M\left(h(t)^{n+1}\right)^\alpha\frac{C^n\sqrt{(\beta-\sigma)^{n-1}(2\beta)^{n-1}}}{((n-1)!)^\gamma}d\sigma.\end{aligned}$$

We have the following estimates:

$$\int_\beta^\xi\int_0^\beta\sqrt{\sigma^{n-1}(\tau+\sigma)^{n-1}}d\sigma d\tau \le 2\int_0^\beta\frac{\sqrt{\sigma^{n-1}(\xi+\sigma)^{n+1}}}{n+1}d\sigma \le 4\frac{\sqrt{\beta^{n+1}(\xi+\beta)^{n+1}}}{(n+1)^2} \le 8\frac{\sqrt{\beta^n(\xi+\beta)^n}}{(n+1)^2},$$

$$\int_\beta^\xi\int_0^\beta\int_0^\sigma\sqrt{(\sigma-\mu)^{n-1}(\tau+\sigma)^{n-1}}d\mu d\sigma d\tau = 8\frac{\sqrt{(\beta)^{n+3}(\xi+\beta)^{n+1}}}{(n+1)^2(n+3)} \le 4\frac{\sqrt{\beta^n(\xi+\beta)^n}}{(n+1)^2},$$

$$\int_0^\beta\sqrt{(\beta-\sigma)^{n-1}(2\beta)^{n-1}}d\sigma = 2\frac{\sqrt{(2\beta)^{n-1}(\beta)^{n+1}}}{n+1} \le 2\frac{\sqrt{(\beta+\xi)^n(\beta)^n}}{n+1},$$

from which it follows that

$$|G_n(t,\xi,\beta)| \le \left(\frac{2}{R}+5M\right)\left(h(t)^{n+1}\right)^\alpha\frac{C^n\sqrt{\beta^n(\xi+\beta)^n}}{\epsilon(n!)^\gamma} \le \left(h(t)^{n+1}\right)^\alpha\frac{C^{n+1}\sqrt{\beta^n(\xi+\beta)^n}}{(n!)^\gamma}.$$

Similarly, for $k>0$, using Leibnitz's formula together with (70) and the assumption hypothesis (72),

$$\begin{aligned}\left|\frac{\partial^k G_n}{\partial t^k}\right| &\le \frac{1}{4\epsilon}\int_\beta^\xi\int_0^\beta\left(\frac{\partial^{k+1}}{\partial t^{k+1}}h(t)^n\right)^\alpha\frac{C^n\sqrt{\sigma^{n-1}(\tau+\sigma)^{n-1}}}{((n-1)!)^\gamma}d\sigma d\tau \\ &+\frac{M}{4\epsilon}\int_\beta^\xi\int_0^\beta\left(\sum_{i=0}^k\binom{k}{i}\left(\frac{\partial^i}{\partial t^i}h(t)\right)^\alpha\left|\frac{\partial^{k-i}G_{n-1}}{\partial t^{k-i}}\right|(t,\tau,\sigma)\right)d\sigma d\tau \\ &+\frac{M}{4\epsilon}\int_\beta^\xi\int_0^\beta\int_0^\sigma\left(\sum_{i=0}^k\binom{k}{i}\left(\frac{\partial^i}{\partial t^i}h(t)\right)^\alpha\left|\frac{\partial^{k-i}G_{n-1}}{\partial t^{k-i}}\right|(t,\tau+\mu,\sigma-\mu)\right)d\mu d\sigma d\tau \\ &+\frac{M}{\epsilon}\int_0^\beta\left(\sum_{i=0}^k\binom{k}{i}\left(\frac{\partial^i}{\partial t^i}h(t)\right)^\alpha\left|\frac{\partial^{k-i}G_{n-1}}{\partial t^{k-i}}\right|(t,\beta+\sigma,\beta-\sigma)\right)d\sigma. \qquad (74)\end{aligned}$$

Using the previous estimates, we get

$$\begin{aligned}\left|\frac{\partial^k G_n}{\partial t^k}\right| &\leq \frac{2}{\epsilon}\left(\frac{\partial^{k+1}}{\partial t^{k+1}}h(t)^n\right)^\alpha \frac{C^n\sqrt{\beta^n(\xi+\beta)^n}}{(n+1)^2((n-1)!)^\gamma} \\ &+\left(\frac{3M}{\epsilon(n+1)}+\frac{2M}{\epsilon}\right)\left(\sum_{i=0}^k\binom{k}{i}\left(\frac{\partial^i}{\partial t^i}h(t)\right)^\alpha\left(\frac{\partial^{k-i}}{\partial t^{k-i}}h(t)^n\right)^\alpha\right)\frac{C^n\sqrt{\beta^n(\xi+\beta)^n}}{(n+1)(n-1)!)^\gamma}. \quad (75)\end{aligned}$$

For the first line, we will use

$$\frac{\partial^i}{\partial t^i}h(t)^n=\frac{n}{R^{1/\alpha}}\frac{\partial^{i-1}}{\partial t^{i-1}}h(t)^{n+1}. \quad (76)$$

For the second line, using

$$\left(\frac{\partial^i}{\partial t^i}h(t)\right)\left(\frac{\partial^{k-i}}{\partial t^{k-i}}h(t)^n\right)=n\frac{i!}{(n+k)(n+k-1)\dots(n+k-i)}\left(\frac{\partial^k}{\partial t^k}h(t)^{n+1}\right),$$

we get

$$\sum_{i=0}^k\binom{k}{i}\left(\frac{\partial^i}{\partial t^i}h(t)\right)^\alpha\left(\frac{\partial^{k-i}}{\partial t^{k-i}}h(t)^n\right)^\alpha=\left(\frac{\partial^k}{\partial t^k}h(t)^{n+1}\right)^\alpha n^\alpha\sum_{i=0}^k\frac{(i!)^{\alpha-1}k(k-1)\dots(k-i+1)}{(n+k)^\alpha\dots(n+k-i)^\alpha}. \quad (77)$$

Lemma 4.2. *For $n,k\geq 1$, we have*

$$\sum_{i=0}^k\frac{(i!)^{\alpha-1}k(k-1)\dots(k-i+1)}{(n+k)^\alpha\dots(n+k-i)^\alpha}\leq\frac{1}{n}.$$

Proof. Since $1\leq\alpha<2$, it suffices to prove the inequality for $\alpha=1$, that can be written as

$$\frac{\sum_{i=0}^k(n-1+k-i)(n-2+k-i)\dots(k-i+1)}{(n+k)(n+k-1)\dots(k+1)}\leq\frac{1}{n}.$$

The proof is then obvious, by induction on n. □

Using (76), (77) and Lemma 4.2, we get, from (75),

$$\left|\frac{\partial^k G_n}{\partial t^k}\right|\leq\left(\frac{\partial^k}{\partial t^k}h(t)^{n+1}\right)^\alpha\frac{C^{n+1}\sqrt{\beta^n(\xi+\beta)^n}}{(n!)^\gamma},$$

thus proving (72).

We have proved that $G\in\mathcal{G}^\alpha(0,T;L^\infty(\mathring{\mathcal{T}}_1))$. To get higher regularity in space, we differentiate in the ξ variable the integral equation (67), obtaining

$$\begin{aligned}G_\xi(t,\xi,\beta) &= \frac{1}{4\epsilon}\int_0^\beta G_t(t,\xi,\sigma)d\sigma d\tau+\frac{1}{4\epsilon}\int_0^\beta A(t,\xi,\sigma)G(t,\xi,\sigma)d\sigma d\tau-\frac{1}{4\epsilon}\int_0^\beta B(t,\xi,\sigma)d\sigma d\tau \\ &+\frac{1}{4\epsilon}\int_0^\beta\int_0^\sigma G(t,\xi+\mu,\sigma-\mu)f(t,\mu+(\xi-\sigma),\mu)\,d\mu d\sigma d\tau-\frac{1}{4\epsilon}A(t,\xi,0)d\tau,\end{aligned}$$

which explicitly defines G_ξ. Next, we differentiate in the β variable the integral equation (67), reaching

$$\begin{aligned}G_\beta(t,\xi,\beta) &= \frac{1}{4\epsilon}\int_\beta^\xi G_t(t,\tau,\beta)d\tau-\frac{1}{4\epsilon}\int_0^\beta G_t(t,\beta,\sigma)d\sigma+\frac{1}{4\epsilon}\int_\beta^\xi A(t,\tau,\beta)G(t,\tau,\beta)d\tau \\ &-\frac{1}{4\epsilon}\int_0^\beta A(t,\beta,\sigma)G(t,\beta,\sigma)d\sigma-\frac{1}{4\epsilon}\int_\beta^\xi B(t,\tau,\beta)d\tau+\frac{1}{4\epsilon}\int_0^\beta B(t,\beta,\sigma)d\sigma \\ &+\frac{1}{4\epsilon}\int_\beta^\xi\int_0^\beta G(t,\tau+\mu,\beta-\mu)f(t,\mu+(\tau-\beta),\mu)\,d\mu d\tau \\ &-\frac{1}{4\epsilon}\int_0^\beta\int_0^\sigma G(t,\beta+\mu,\sigma-\mu)f(t,\mu+(\beta-\sigma),\mu)\,d\mu d\sigma+\frac{1}{4\epsilon}A(t,\beta,0)d\tau-\frac{1}{\epsilon}g_\beta(t,\beta) \\ &+\frac{1}{\epsilon}g(t,\beta)G(t,2\beta,0)+\frac{1}{\epsilon}\int_0^\beta g(t,\sigma)G_\beta(t,\beta+\sigma,\beta-\sigma)\,d\sigma+\frac{1}{\epsilon}\int_0^\beta g(t,\sigma)G_\xi(t,\beta+\sigma,\beta-\sigma)\,d\sigma,\end{aligned}$$

an integral equation for G_β. It can be written as

$$G_\beta(t,\xi,\beta) = \Phi(t,\xi,\beta) + \frac{1}{\epsilon}\int_0^\beta g(t,\sigma) G_\beta\left(t,\beta+\sigma,\beta-\sigma\right) d\sigma.$$

where the function $\Phi(t,\xi,\beta)$ is computed from G, A, B, and g. This equation is solved using a successive approximation scheme as before. We skip the details.

Hence, G_ξ and G_β are well-defined, as long as g_β is well-defined. Iterating this process, higher order derivatives can be computed as long as the coefficients are differentiable. It follows that the regularity of G is determined by the regularity of the coefficients; it is proved by induction that $G \in \mathcal{G}^\alpha(0,T;H^{m+1,\infty}(\mathring{\mathcal{T}}_1))$. This means that G has the same regularity as g, has one more derivative than λ, and two more derivatives than f. Moreover, repeating this argument for all values of m, if the coefficients are smooth in space, then the kernel is smooth in ξ and β.

The second item of Proposition 4.1 is proved similarly, covering the infinite interval $(0,\infty)$ with an infinite number of *uniform* subintervals of the form $[t_0, \frac{R^{1/\alpha}}{2}+t_0)$. We obtain the same bounds for G. Hence a compactness argument is *not* required, as we obtain a uniform, finite bound for G, showing that the successive approximation series is well-defined for all times. □

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