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LYAPUNOV STABILITY ANALYSIS OF NETWORKS OF SCALAR CONSERVATION LAWS

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Abstract. It is shown how an entropy-based Lyapunov function can be used for the stability analysis of equilibria in networks of scalar conservation laws. The analysis gives a sufficient stability condition which is weaker than the condition which was previously known in the literature. Various extensions and generalisations are briefly discussed. The approach is illustrated with an application to ramp-metering control of road traffic networks.

1. **Introduction.** Conservation laws are first-order partial differential equations that are commonly used to express the fundamental balance laws that occur in many physical systems and engineering problems when small friction or dissipation effects are neglected (e.g.[4]). Physical networks described by systems of 2x2 conservation laws have been recently considered in the literature. Among others, we may mention for instance Saint-Venant equations for hydraulic networks (e.g.[11],[7]), isothermal Euler equations for gas pipeline networks (e.g.[1]), or Aw-Rascle equations for road traffic networks (e.g.[10], [8]). In this paper, our concern is to analyse the stability (in the sense of Lyapunov) of the steady-states of such networks. However, for the sake of simplicity, we shall restrict ourselves to networks of *scalar* conservation laws. Typical examples include LWR models for road traffic networks (e.g. [9, Chapter 6]), Eulerian flow models for air traffic networks (e.g. [3]) and fluid models for switched-packets networks (e.g. [12]). The case of networks of 2x2 conservation

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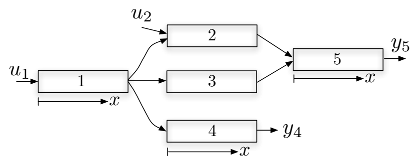

FIGURE 1. Physical network

laws, which is more complex, will be addressed in a future paper [5] and is only briefly commented in the conclusion section of this paper.

We consider physical networks as illustrated in Fig.1. The structure of the network is reminiscent to the structure of so-called compartmental systems that are commonly used for describing dynamic conservative networks (see e.g. [2] and the references therein). The nodes of the network represent physical devices (called "compartments") with dynamics expressed by conservation laws of the form

$$\partial_t \rho_j(t,x) + \partial_x q_j(t,x) = 0,\ t \geqslant 0,\ x \in (0,L),\ j = 1,\dots,n. \tag{1}$$

In these equations, the independent variables are the time t and a space coordinate x over a finite interval $(0,L)$. The dependent variables ρ_j and q_j are called *densities* and *fluxes* respectively. We consider the specific situation where each flux q_j is a static monotonic increasing function of the density ρ_j:

$$q_j(t,x) = \phi_j(\rho_j(t,x)).$$

This relation is supposed to be invertible as

$$\rho_j(t,x) = \phi_j^{-1}(q_j(t,x))$$

in such a way that the system may also be written as a set of so-called "kinematic wave equations"

$$\partial_t q_j(t,x) + c_j(q_j(t,x))\partial_x q_j(t,x) = 0 \tag{2}$$

with

$$c_j(q_j(t,x)) \triangleq \left[\frac{\partial \phi_j^{-1}(q)}{\partial q}(q_j(t,x))\right]^{-1} > 0.$$

The directed arcs $i \to j$ of the network represent instantaneous mass transfers between the compartments. The transfer rate or *flow* from the output of a compartment i to the input of a compartment j is denoted $f_{ij}(t)$. Additional input and output arcs represent interactions with the surroundings: either inflows $u_j(t)$ injected from the outside into some compartments or outflows $y_j(t)$ from some compartments to the outside. Hence, the set of PDEs (2) is subject to boundary conditions of the form:

$$q_j(t,0) = \sum_{i \neq j} f_{ij}(t) + u_j(t), \tag{3a}$$

$$q_j(t,L) = \sum_{k \neq j} f_{jk}(t) + y_j(t) \qquad j = 1,\dots,n. \tag{3b}$$

In equations (2)-(3), only the terms corresponding to actual links of the network are explicitly written. Otherwise stated, all the u_j, y_j and f_{ij} for non existing links do not appear in the equations.

It is assumed here that the flows f_{ij} and y_i are fractions of the outgoing flux $q_i(t, L)$ from compartment i:

$$f_{ij}(t) \triangleq a_{ij} q_i(t, L),\ 0 < a_{ij} \leqslant 1 \quad \text{and} \quad y_i(t) \triangleq a_{io} q_i(t, L),\ 0 < a_{io} \leqslant 1.$$

The conservation of flows then imposes the following obvious constraints:

$$\sum_{j=0}^{n} a_{ij} = 1,\ i = 1, \ldots, n. \tag{4}$$

We define the following vector and matrix notations:

$$\begin{aligned}
\mathbf{q} &\triangleq (q_1, q_2, \ldots, q_n)^T, \\
\mathbf{C}(\mathbf{q}) &\triangleq \operatorname{diag}\left(c_1(q_1), \ldots, c_n(q_n)\right), \\
\mathbf{u} &\triangleq (u_1, u_2, \ldots, u_n)^T, \\
\mathbf{y} &\triangleq (y_1, y_2, \ldots, y_n)^T, \\
\mathbf{A} &= \text{matrix with entries } a_{ji}\ (j = 1, n; i = 1, n), \\
\mathbf{B} &= \text{matrix with entries } a_{io}.
\end{aligned}$$

With these notations, the system (2)-(3) may be written in the following compact form:

$$\partial_t \mathbf{q}(t, x) + \mathbf{C}(\mathbf{q}(t, x)) \partial_x \mathbf{q}(\mathbf{t}, \mathbf{x}) = 0, \tag{5a}$$

$$\mathbf{q}(t, 0) = \mathbf{A}\mathbf{q}(t, L) + \mathbf{u}(t), \tag{5b}$$

$$\mathbf{y}(t) = \mathbf{B}\mathbf{q}(t, L). \tag{5c}$$

The first equation (5a) is an hyperbolic quasi-linear PDE that defines the system state dynamics with state $\mathbf{q}(t, x)$. The second equation defines the boundary conditions of the system, some of them being assignable by the system input $\mathbf{u}(t)$. The third equation can be interpreted as an output equation with system output $\mathbf{y}(t)$ being the set of outflows.

For any constant input $\bar{\mathbf{u}}$, a steady-state (or equilibrium state) of the system is defined as a constant state $\bar{\mathbf{q}}$ which satisfies the state equation (5a) and the boundary condition (5b):

$$(\mathbf{A} - \mathbf{I})\bar{\mathbf{q}} + \bar{\mathbf{u}} = 0.$$

Under the constraints (4) it is readily verified that the matrix $\mathbf{A} - \mathbf{I}$ is a full-rank compartmental matrix. It follows that, for any positive $\bar{\mathbf{u}}$ there exists a unique positive steady-state*:

$$\bar{\mathbf{q}} = -(\mathbf{A} - \mathbf{I})^{-1}\bar{\mathbf{u}}.$$

As we have mentioned above, our goal in this paper is to analyse the Lyapunov stability of the steady-state $\bar{\mathbf{q}}$ of the control system (5). The stability analysis, presented in Section 2.1, relies on an entropy-based Lyapunov function and gives a sufficient stability condition which is weaker than the condition which was previously known in the literature. In order to clarify the presentation and as a matter of example, we examine the stability of the steady-state $\bar{\mathbf{q}}$ when the system is under a linear state feedback control of the form

$$\mathbf{u}(t) = \bar{\mathbf{u}} + \mathbf{G}\left(\mathbf{q}(t, L) - \bar{\mathbf{q}}\right) \tag{6}$$

where the matrix $\mathbf{G}$ is the control gain. Remark that the special case $\mathbf{G} = 0$ is included which means that the analysis is applicable also to the open-loop system

*A discussion of the existence of equilibria in data networks can be found in [12].

(5) with constant input $\bar{\mathbf{u}}$. Some variants and extensions will be briefly presented in Section 2.2. In Section 3, our approach will be illustrated through an example arising from ramp-metering control in road traffic networks.

2. **Lyapunov stability analysis.** Defining the state deviation $\boldsymbol{\xi} \triangleq \mathbf{q} - \bar{\mathbf{q}}$, the Cauchy problem associated to the closed-loop control system (5)-(6) is equivalently written as:

$$\partial_t \boldsymbol{\xi}(t,x) + \boldsymbol{\Lambda}(\boldsymbol{\xi}(t,x))\partial_x \boldsymbol{\xi}(t,x) = 0,\ t \geqslant 0,\ x \in (0,L), \tag{7a}$$

$$\boldsymbol{\xi}(t,0) = \mathbf{K}\boldsymbol{\xi}(t,L),\ t \geqslant 0, \tag{7b}$$

$$\boldsymbol{\xi}(0,x) = \boldsymbol{\xi}_0(x),\ x \in (0,L), \tag{7c}$$

with $\boldsymbol{\Lambda}(\boldsymbol{\xi}) \triangleq \mathbf{C}(\bar{\mathbf{q}} + \boldsymbol{\xi})$ and $\mathbf{K} \triangleq \mathbf{A} + \mathbf{G}$.

In order to analyse the Lyapunov stability of the steady-state $\boldsymbol{\xi} = 0$ of this system, we introduce some norm notations and definitions. For a real vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, $|\mathbf{x}|$ denotes the usual 2-norm: $|\mathbf{x}| \triangleq \sqrt{\mathbf{x}^T\mathbf{x}}$. For a $n \times n$ real matrix $\mathbf{K}$ the induced matrix 2-norm is defined as:

$$\|\mathbf{K}\| \triangleq \max\{|\mathbf{Kx}|, |\mathbf{x}| = 1\}.$$

We denote by $\mathcal{D}_n$ the set of diagonal $n \times n$ real matrices with strictly positive diagonal coefficients. Then we define

$$\rho_o(\mathbf{K}) \triangleq \operatorname{Inf}\left\{\|\Delta\mathbf{K}\Delta^{-1}\|,\ \Delta \in \mathcal{D}_n\right\}. \tag{8}$$

2.1. **Main result.** For the system (7a), for any $\mathbf{P} \in \mathcal{D}_n$, the function $E(\boldsymbol{\xi}) = (1/2)\boldsymbol{\xi}^T\mathbf{P}\boldsymbol{\xi}$ is an entropy for which the corresponding entropy flux is any function $F : \boldsymbol{\xi} \to F(\boldsymbol{\xi})$ such that

$$\frac{\partial F}{\partial \boldsymbol{\xi}} = \boldsymbol{\xi}^T\mathbf{P}\boldsymbol{\Lambda}(\boldsymbol{\xi}).$$

Along the smooth solutions of (7a), this entropy/entropy-flux pair satisfies the equation

$$\partial_t E(\boldsymbol{\xi}(t,x)) + \partial_x F(\boldsymbol{\xi}(t,x)) = 0. \tag{9}$$

For the stability analysis of the steady-state $\boldsymbol{\xi} = 0$ of system (7), we then consider the following Lyapunov function candidate:

$$V = \int_0^L \frac{1}{2}\boldsymbol{\xi}^T(t,x)\mathbf{P}\boldsymbol{\xi}(t,x)dx = \int_0^L E(\boldsymbol{\xi}(t,x))dx. \tag{10}$$

Theorem 1. *If $\rho_o(\mathbf{K}) < 1$, there exists a matrix $\mathbf{P} \in \mathcal{D}_n$ such that the Lyapunov function V is decreasing along the classical solutions of (7) (i.e. $\dot{V} \leqslant 0$) in a neighbourhood of the origin. Consequently the steady-state $\boldsymbol{\xi} = 0$ is stable.*

Proof. The time derivative of V along the solutions of (7) is

$$\begin{aligned}\dot{V} &= \int_0^L \partial_t E(\boldsymbol{\xi}(t,x))dx = -\int_0^L \partial_x F(\boldsymbol{\xi}(t,x))dx \\ &= -\left[F(\boldsymbol{\xi}(t,x))\right]_0^L = -\left[F(\boldsymbol{\xi}(t,L)) - F(\boldsymbol{\xi}(t,0))\right].\end{aligned}$$

Using (7b), we have

$$\dot{V} = -\left[F(\boldsymbol{\xi}(t,L)) - F(\mathbf{K}\boldsymbol{\xi}(t,L))\right] \triangleq -\tilde{F}(\boldsymbol{\xi}(t,L)).$$

Remark that

$$\tilde{F}(0) = 0 \quad \text{ and } \quad \frac{\partial \tilde{F}}{\partial \boldsymbol{\xi}} = -\boldsymbol{\xi}^T\left[\mathbf{P}\boldsymbol{\Lambda}(\boldsymbol{\xi}) - \mathbf{K}^T\mathbf{P}\boldsymbol{\Lambda}(\boldsymbol{\xi})\mathbf{K}\right].$$

Then, for any real positive constant γ, there exists a neighbourhood of the origin such that, for all $\boldsymbol{\xi}(t,L)$ in this neighbourhood:

$$\dot{V} \leqslant -\boldsymbol{\xi}^T(t,L)\left[\mathbf{P}\bar{\boldsymbol{\Lambda}} - \mathbf{K}^T\mathbf{P}\bar{\boldsymbol{\Lambda}}\mathbf{K}\right]\boldsymbol{\xi}(t,L) + \gamma\boldsymbol{\xi}^T(t,L)\boldsymbol{\xi}(t,L)$$

with $\bar{\boldsymbol{\Lambda}} \triangleq \boldsymbol{\Lambda}(0) = \mathbf{C}(\bar{\mathbf{q}}) \in \mathcal{D}_n$.

Since $\rho_o(\mathbf{K}) < 1$ by assumption, there exists $\mathbf{D} \in \mathcal{D}_n$ such that

$$\|\mathbf{D}\mathbf{K}\mathbf{D}^{-1}\| < 1. \tag{12}$$

The matrix $\mathbf{P}$ is then selected such that $\mathbf{P}\bar{\boldsymbol{\Lambda}} = \mathbf{D}^2$. We define $\boldsymbol{\zeta} \triangleq \mathbf{D}\boldsymbol{\xi}$. Then, using inequality (12), we have:

$$\begin{aligned}\boldsymbol{\xi}^T\mathbf{K}^T\mathbf{P}\bar{\boldsymbol{\Lambda}}\mathbf{K}\boldsymbol{\xi} = \boldsymbol{\xi}^T\mathbf{K}^T\mathbf{D}\mathbf{D}\mathbf{K}\boldsymbol{\xi} = \left[\boldsymbol{\zeta}^T\mathbf{D}^{-1}\mathbf{K}^T\mathbf{D}\right]\left[\mathbf{D}\mathbf{K}\mathbf{D}^{-1}\boldsymbol{\zeta}\right] = |\mathbf{D}\mathbf{K}\mathbf{D}^{-1}\boldsymbol{\zeta}|^2 \\ < |\boldsymbol{\zeta}|^2 = \boldsymbol{\xi}^T\mathbf{D}^2\boldsymbol{\xi} = \boldsymbol{\xi}^T\mathbf{P}\bar{\boldsymbol{\Lambda}}\boldsymbol{\xi}.\end{aligned}$$

From this inequality it follows that the quadratic form

$$\boldsymbol{\xi}^T(t,L)\left[\mathbf{P}\bar{\boldsymbol{\Lambda}} - \mathbf{K}^T\mathbf{P}\bar{\boldsymbol{\Lambda}}\mathbf{K}\right]\boldsymbol{\xi}(t,L)$$

is positive definite. Hence, choosing γ sufficiently small ensures that $\dot{V} \leqslant 0$ in a neighbourhood of the origin. □

2.2. **Comments.** Various extensions and generalisations of this analysis can be found in [5] and are briefly summarized in the following comments.

1) The function (10) is not a strict Lyapunov function because its time derivative is not negative definite but only semi negative definite. Hence, from Theorem 1 we have that the steady-state $\bar{\mathbf{q}}$ is stable (in the sense of Lyapunov) but not necessarily *asymptotically* stable. In fact the *exponential* asymptotic stability can be proved by using a more general "strict" Lyapunov function (see also [6]) defined as

$$W = \int_0^L \left(\boldsymbol{\xi}^T\mathbf{P}\boldsymbol{\xi} + \boldsymbol{\zeta}^T\mathbf{Q}\boldsymbol{\zeta} + \boldsymbol{\eta}^T\mathbf{R}\boldsymbol{\eta}\right)e^{-\mu x}dx$$

with

$$\boldsymbol{\zeta} \triangleq \partial_x\boldsymbol{\xi}, \qquad \boldsymbol{\eta} \triangleq \partial_x\boldsymbol{\zeta} = \partial_{xx}\boldsymbol{\xi}$$
$$\mathbf{P},\mathbf{Q},\mathbf{R} \in \mathcal{D}_n, \qquad \mathbf{M} \triangleq \operatorname{diag}\left(e^{-\frac{\mu x}{c_1(\bar{q}_1)}},\ldots,e^{-\frac{\mu x}{c_n(\bar{q}_n)}}\right).$$

With this Lyapunov function, the following Theorem is proved in [5].

Theorem 2. *If $\rho_o(\mathbf{K}) < 1$, there exist $\mu > 0$ and $\mathbf{P},\mathbf{Q},\mathbf{R} \in \mathcal{D}_n$ such that W is a strict Lyapunov function exponentially decreasing along the classical solutions of (7) (i.e. $\dot{W} < -\mu W$) in a neighbourhood of the origin. Consequently the steady-state $\boldsymbol{\xi} = 0$ is exponentially asymptotically stable.*

2) The stability of hyperbolic quasi-linear systems of the form (7) has been previously considered in the literature. In particular, it has been established in [13] that a sufficient stability condition is $\rho_s(|\mathbf{K}|) < 1$ where $|\mathbf{K}|$ is the matrix with entries $|K_{ij}|$ and ρ_s denotes the spectral radius. This result is obtained in a rather tedious way by systematically utilizing explicit estimates of the solutions of system (7). Our Lyapunov approach is much more concise. Furthermore, it is remarkable that it leads to a weaker sufficient stability condition since it can be shown that $\rho_0(\mathbf{K}) \leqslant \rho_s(|\mathbf{K}|)$ (this inequality may be

strict for certain $\mathbf{K}$ matrices, see [5] for details).

3) System (7) and Theorem 1 has been given above with a linear boundary condition $\boldsymbol{\xi}(t,0) = \mathbf{K}\boldsymbol{\xi}(t,L)$. It is rather straightforward that Theorem 1 holds also for non-linear boundary conditions $\boldsymbol{\xi}(t,0) = \mathbf{k}(\boldsymbol{\xi}(t,L))$ with the matrix $\mathbf{K} = \nabla\mathbf{k}(0)$. Such non-linear boundary conditions may result either from non-linear models for the transfer rates f_{ij} of the network or from static non-linear feedback control.

4) The Lyapunov stability analysis can be extended to quasi-linear hyperbolic systems of the form

$$\partial_t \mathbf{q}(t,x) + \mathbf{C}(\mathbf{q}(t,x))\partial_x \mathbf{q}(\mathbf{t},\mathbf{x}) = \mathbf{g}(\mathbf{q}(t,x)) \tag{13}$$

i.e. systems with a non-zero right hand side $\mathbf{g}(\mathbf{q}(t,x)) \triangleq (g_1(q_1),\ldots,g_n(q_n))^T$ representing small perturbations. For such systems, the steady-state solution may depend on x and is defined as the solution $\bar{\mathbf{q}}(x)$ of the ordinary differential equation

$$\mathbf{C}(\bar{\mathbf{q}}(x))\partial_x \bar{\mathbf{q}}(x) = \mathbf{g}(\bar{\mathbf{q}}(x))$$

Then defining the deviation $\boldsymbol{\xi}(t,x) \triangleq \mathbf{q}(t,x) - \bar{\mathbf{q}}(x)$, it is easy to check that the system (13) is equivalently written as

$$\partial_t \boldsymbol{\xi}(t,x) + \boldsymbol{\Lambda}(\boldsymbol{\xi}(t,x))\partial_x \boldsymbol{\xi}(t,x) = \mathbf{h}(\boldsymbol{\xi}(t,x)) \tag{14}$$

with $\boldsymbol{\Lambda}(\boldsymbol{\xi}) \triangleq \mathbf{C}(\bar{\mathbf{q}}+\boldsymbol{\xi})$ and $\mathbf{h}(\boldsymbol{\xi}) \triangleq \mathbf{g}(\bar{\mathbf{q}}+\boldsymbol{\xi}) - \boldsymbol{\Lambda}(\boldsymbol{\xi})\partial_x\bar{\mathbf{q}}$. Remark that $\mathbf{h}(0) = 0$. Then Theorem 2 can be extended to system (14) with boundary conditions (7b) and initial condition (7c) provided $\|\nabla\mathbf{h}(0)\|$ is sufficiently small.

3. Application to ramp-metering control in road traffic networks.

3.1. The LWR model. In the fluid paradigm for road traffic modelling, the traffic state is usually represented by a macroscopic variable $\rho(t,x)$ which represents the density of the vehicles (# veh/km) at time t and at position x along the road. The traffic dynamics are represented by a conservation law

$$\partial_t \rho(t,x) + \partial_x q(t,x) = 0$$

which expresses the conservation of the number of vehicles on a road segment without entries nor exits. In this equation, $q(t,x)$ is the traffic *flux* representing the flow rate of the vehicles at (t,x). By definition, we have $q(t,x) \triangleq \rho(t,x)v(t,x)$ where $v(t,x)$ is the velocity of the vehicles at (t,x). The basic assumption of the so-called LWR model (see e.g. [9, Chapter 3]) is that the drivers intantaneously adapt their speed to the local traffic density, which is expressed by a function $v(t,x) = V(\rho(t,x))$. The LWR traffic model is therefore written as

$$\partial_t \rho(t,x) + \partial_x\left(\rho(t,x)V(\rho(t,x))\right) = 0. \tag{15}$$

In accordance with the physical observations, the velocity-density relation is a monotonic decreasing function ($dV/d\rho < 0$) on the interval $[0,\rho_m]$ (see Fig.2) with:

1. $V(0) = V_m$ the maximal vehicle velocity when the road is (almost) empty;
2. $V(\rho_m) = 0$: the velocity is zero when the density is maximal, the vehicles are stopped and the traffic is totally congested.

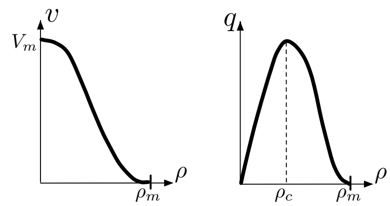

FIGURE 2. Velocity v and flux q viz. density ρ

Then the flux $q(\rho) = \rho V(\rho)$ is a non-monotonic function with $q(0) = 0$ and $q(\rho_m) = 0$ which is maximal at some critical value ρ_c which separates free-flow and traffic-congestion: the traffic is flowing freely when $\rho < \rho_c$ while the traffic is congested when $\rho > \rho_c$ (see Fig.2).

3.2. **Ramp-metering control.** As a matter of example, let us now consider the network of interconnected one-way road segments as depicted in Fig.3. The network

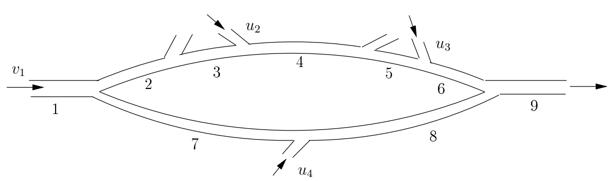

FIGURE 3. A road network

is made up of nine road segments with four entries and three exits. The densities and flows on the road segments are denoted ρ_j and q_j, $j = 1, 9$. The flow rate v_1 is a disturbance input and the flow rates u_2, u_3, u_4 at the three other entries are control inputs.

Our objective is to analyse the stability of this network under a feedback ramp-metering strategy which consists in using traffic lights for modulating the entry flows u_i. The motivation behind such control strategy is that a temporary limitation of the flow entering a highway can prevent the appearance of traffic jams and improve the network efficiency (possibly at the price of temporary queue formation at the ramps). The traffic dynamics are described by a set of LWR models (15):

$$\partial_t \rho_j(t,x) + \partial_x \left(\rho_j(t,x) V(\rho_j(t,x))\right) = 0, \; j = 1, \ldots, 9. \tag{16}$$

Under free-flow conditions, the flows $q_j(\rho_j) = \rho_j V(\rho_j)$ are monotonic increasing functions and the model for the network of Fig.3 is written as a set of kinematic wave equations

$$\partial_t q_j(t,x) + c(q_j(t,x)) \partial_x q_j(t,x) = 0, \; c(q_j) > 0, \; j = 1, \ldots, 9 \tag{17}$$

with the boundary conditions

$$\begin{aligned}
q_1(t,0) &= v_1(t)\\
q_2(t,0) &= \alpha q_1(t,L)\\
q_3(t,0) &= \beta q_2(t,L)\\
q_4(t,0) &= q_3(t,L) + u_2(t)\\
q_5(t,0) &= \gamma q_4(t,L)\\
q_6(t,0) &= q_5(t,L) + u_3(t)\\
q_7(t,0) &= (1-\alpha) q_1(t,L)\\
q_8(t,0) &= q_7(t,L) + u_4(t)\\
q_9(t,0) &= q_6(t,L) + q_8(t,L)
\end{aligned}$$

where α, β, γ are traffic splitting factors at the diverging junction and the two exits of the network. Obviously the set-point for the feedback traffic regulation is selected as a free-flow steady-state $(\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_9)^T$. A linear state-feedback is then defined for the ramp-metering of the three input flows:

$$\begin{aligned}
u_2(t) &= \bar{u}_2 + k_2\,(q_6(t,L) - \bar{q}_6)\\
u_3(t) &= \bar{u}_3 + k_3\,(q_6(t,L) - \bar{q}_6)\\
u_4(t) &= \bar{u}_4 + k_4\,(q_8(t,L) - \bar{q}_8)
\end{aligned}$$

where k_2, k_3, k_4 are tuning control parameters. Defining the control deviations $\xi_i = q_i - \bar{q}_i$, the boundary conditions of the system under the ramp-metering control are:

$$\begin{pmatrix} \xi_1(t,0)\\ \xi_2(t,0)\\ \xi_3(t,0)\\ \xi_4(t,0)\\ \xi_5(t,0)\\ \xi_6(t,0)\\ \xi_7(t,0)\\ \xi_8(t,0)\\ \xi_9(t,0) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & k_2 & 0 & 0 & 0\\ 0 & 0 & 0 & \gamma & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & k_3 & 0 & 0 & 0\\ 1-\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & k_4 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}}_{\mathbf{K}} \begin{pmatrix} \xi_1(t,L)\\ \xi_2(t,L)\\ \xi_3(t,L)\\ \xi_4(t,L)\\ \xi_5(t,L)\\ \xi_6(t,L)\\ \xi_7(t,L)\\ \xi_8(t,L)\\ \xi_9(t,L) \end{pmatrix}.$$

In this example, with parameter values $\alpha = 0.8$, $\beta = 0.9$, $\gamma = 0.8$, the stability condition $\rho_0(\mathbf{K}) < 1$ can be shown to be satisfied if and only if the control parameters are selected such that:

$$0.8|k_2| + |k_3| < 1 \qquad |k_4| < 1.$$

4. **Conclusion.** In this paper, we have presented an entropy-based Lyapunov function which is used for the stability analysis of systems of conservation laws. We have shown that this analysis gives a sufficient stability condition which is weaker than the condition which was previously known in the literature. Our approach has been illustrated through an example arising from ramp-metering control in road traffic networks.

For the sake of simplicity, our paper has been limited to systems of *scalar* conservation laws with characteristic velocities $\lambda_j(\xi_j)$. It must however be mentioned that Theorems 1 and 2 are applicable to more general systems of *vector* conservation laws which can be transformed into the form considered in [13, Chapter 5]

$$\partial_t \xi_j + \lambda_j(\boldsymbol{\xi}) \partial_x \xi_j = 0 \qquad j = 1, \dots, m. \tag{18}$$

where the characteristic velocities $\lambda_j(\boldsymbol{\xi})$ may depend on the full vector $\boldsymbol{\xi}$ (and not only on the sole component ξ_j, see [5] for details). It must be emphasized that this class of systems involves in particular many examples of networks of 2×2 hyperbolic conservation laws

$$\partial_t \begin{pmatrix} \rho_j \\ q_j \end{pmatrix} + \partial_x \begin{pmatrix} q_j \\ f_j(\rho_j, q_j) \end{pmatrix} = 0 \qquad j = 1, \dots, n,$$

where the first equation is a mass conservation law and the second equation is a momentum conservation law. In order to perform the Lyapunov stability analysis, this system is transformed into the form (18) (with $m = 2n$) by using Riemann invariants (the special case of a single 2×2 conservation law is comprehensively treated in [6]).

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