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ON CONDITIONS THAT PREVENT STEADY–STATE CONTROLLABILITY OF CERTAIN LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we investigate the connections between controllability properties of distributed systems and existence of non zero entire functions subject to restrictions on their growth and on their sets of zeros. Exploiting these connections, we first show that, for generic bounded open domains in dimension $n \geq 2$, the steady–state controllability for the heat equation with boundary controls dependent only on time, does not hold. In a second step, we study a model of a water tank whose dynamics is given by a wave equation on a two-dimensional bounded open domain. We provide a condition which prevents steady–state controllability of such a system, where the control acts on the boundary and is only dependent on time. Using that condition, we prove that the steady–state controllability does not hold for generic tank shapes.

1. Introduction.

We consider the steady–state controllability in finite time for control systems given by some partial differential equations. In this paper (and as it will be clear from the statements of the results), "steady–state" refers to independence with respect to the state variable, i.e. steady–states are simply constant functions (of the state variable). Moreover, the control strategies considered here are *only time dependent*. For certain control systems modeled by a partial differential equation, we investigate whether given two arbitrary steady states of this control system, one can steer the first steady state to the second one in finite time by means of a suitable (only time-dependent) control. We refer to such a property as the *steady-state controllability* for the corresponding control system. That class of problems has been introduced by N. Petit and P. Rouchon in [21] for a control system modeling a water tank. The control problem they addressed consists of steering in finite time the tank from one steady state to another one, using as a

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control the acceleration of the tank (this leads to two boundary controls which are only dependent on time). The dynamics is given by a linear wave equation on a bounded open set of $\mathbb{R}^2$, as detailed below. They solved positively the problem in the case where the tank is either a disc or a rectangle. For more general tank shapes, they assert this problem of steady–state controllability as open. In the same spirit, P. Rouchon, in [22], considered the steady–state controllability for the heat equation on an open, bounded and non empty subset Ω of $\mathbb{R}^n$, where the control is only dependent on time and acts on the boundary, i.e. $y(t,x) = u(t)$ on $\partial\Omega$ with the usual notations. It is well-known (see [2, Theorem IV.2.7, page 187] and [6, Theorem 2.2]) that there is a negative answer to the null controllability in finite time of this control system, that is there are states which cannot be steered to zero in finite time. Moreover, stronger negative results, showing that, in fact, very few states can be steered to zero in finite time for the heat equation were obtained by S. Micu and E. Zuazua in [16] for the case where the domain Ω is a half-space (see also [17] for a fractional order parabolic equation). In [22], P. Rouchon raised the following question: is it possible to steer the special initial data $y_0 \equiv 1$ to zero in finite time? We use (R) to denote that particular control problem. P. Rouchon shows that (R) has a solution if $n = 1$ or if Ω is a ball in $\mathbb{R}^n$ and asks what is the answer for general open subsets in $\mathbb{R}^n$ with $n \geq 2$.

The first result of the present paper is the characterization of a property on Ω, denoted (A), which is an obstruction to a positive answer to the steady–state controllability for the heat equation with boundary controls that depend only on time. Property (A) is expressed in terms of the averages on Ω of the eigenfunctions of the Laplace–Dirichlet operator. We show that property (A) holds for generic open subsets $\Omega \in \mathbb{R}^n$, $n \geq 2$, of class C^3. Therefore, for generic domains Ω, question (R) has a negative answer. Finally, in the case where Ω is a parallelepiped, we show that, again, property (A) holds, and thus, even if the domain is not regular, $y_0 \equiv 1$ cannot be steered to zero in finite time.

The second result concerns the water tank control problem. We again characterize a property (B) on the shape of the tank, expressed in terms of averages on the boundary of the tank of the eigenfunctions of the Laplace–Neumann operator, which turns out to be an obstruction to the steady–state controllability of the associated control system. The shape of a tank is an open, bounded, connected and non empty subset Ω of $\mathbb{R}^2$. We also show that property (B) holds for generic tank shapes of class C^3.

The strategy we adopt consists in performing a Laplace transform with respect to the time t. The steady–state controllability issue for both control systems is now translated into a (more or less) equivalent problem of complex analysis, namely, the existence of a non-zero holomorphic function $f : \mathbb{C} \to \mathbb{C}$ such that:

- (a)**:** $|f(s)| \leq C_0 e^{C_0 \max\{0, Re(s)\}}$;
- (b)**:** For every distinct eigenvalue λ_i of $-\Delta$, either $f(\lambda_i) = 0$ (for the heat control equation) or $f(\lambda_i)$ belongs to a special one dimensional linear subspace of $\mathbb{C}^2$ (for the water tank control system).

Here, C_0 is a positive constant independent of $s \in \mathbb{C}$ and $-\Delta$ denotes either the Laplace–Dirichlet or the Laplace–Neumann operator. Condition (a) is a consequence of the fact that controllability must be achieved in *finite* time and, thus, simply results from the Paley-Wiener theorem. Condition (b) is the "infinite-dimensional" version of a standard fact of linear system theory: $\lambda_i \in \sigma(-\Delta)$ is a pole of the transfer function associated to the control system under consideration, which is (almost) nothing else but the resolvent of $-\Delta$. For these two control systems, it turns out that null–controllability in finite time would imply the existence of a non-zero entire function subject to restrictions so strong that only the zero function would fulfill them. A contradiction is reached, and so the statements for the conditions preventing steady-state controllability do hold true.

We conclude this introduction with a brief description of the arguments establishing our genericity results. We first show that the theses to be proved are equivalent to the non-existence of particular solutions of over-determined eigenvalue problems. We then argue by contradiction, assuming that such solutions indeed exist. Note that the non-existence argument corresponding to property (B) is rather lengthy .

2. **Heat equation with particular initial data.** Let Ω be an open, bounded and non empty subset of $\mathbb{R}^n$, with $n \geq 2$. For $y_0 \in L^2(\Omega)$ and $T > 0$, consider the heat equation

$$\begin{cases} y_t(t,x) - \Delta y(t,x) = 0, & \text{if } (t,x) \in (0,T) \times \Omega, \\ y(0,x) = y_0(x), & \text{if } x \in \Omega, \\ y(t,x) = u(t), & \text{if } x \in \partial\Omega, \end{cases} \tag{1}$$

where $u \in L^2(0,T)$ is the control. Let us first recall classical results about weak solutions to the Cauchy problem (1). Let $y_0 \in L^2(\Omega)$, $T > 0$ and $u \in L^2(0,T)$. A weak solution to the Cauchy problem (1) is a function $y \in C^0([0,T]; L^2(\Omega))$ such that, for every $\tau \in [0,T]$ and every $\theta \in C^1([0,T]; L^2(\Omega)) \cap C^0([0,T]; H_0^1(\Omega))$ with

$$\theta_t + \Delta\theta = 0 \text{ in } C^0([0,T]; H^{-1}(\Omega)), \tag{2}$$

one has

$$\int_\Omega y(\tau,x)\theta(\tau,x)dx - \int_\Omega y_0(x)\theta(0,x)dx = \int_0^\tau u(t)\left(\int_\Omega \theta_t(t,x)dx\right)dt. \tag{3}$$

Of course, every $y \in C^1([0,T]; L^2(\Omega)) \cap C^0([0,T]; H^1(\Omega))$, which is a classical solution to (1) is a weak solution to (1). It is also well known that, for every $y_0 \in L^2(\Omega)$, $T > 0$ and $u \in L^2(0,T)$, there exists one and only one weak solution y to (1). That unique y will be called the solution to the Cauchy problem (1).

The problem of null controllability associated to (1) goes as follows. Given $y_0 \in L^2(\Omega)$, does there exist $T > 0$ and $u \in L^2(0,T)$ such that the solution to the Cauchy problem (1) satisfies $y(T,\cdot) = 0$? The answer to that question is negative, as shown by H. Fattorini in [6, Theorem 2.2] and by S. Avdonin and S. Ivanov in [2, Theorem IV.2.7, page 187]; see also the articles [16, 17] by S. Micu and E. Zuazua, for even stronger negative results for similar questions.

In this section, we look at a particular y_0, namely $y_0 \equiv 1$, and want to see if it is possible to steer that special y_0 to 0 in finite time, that is, again, does there exists $T > 0$ and a control $u \in L^2(0,T)$ such that the solution y of the Cauchy problem (1) satisfies $y(T,\cdot) = 0$? Of course, a positive answer to that question is equivalent to the steady–state controllability, i.e. given two constant functions $y_0 \equiv C_0$, $y_1 \equiv C_1$, does there exist $T > 0$ and $u \in L^2(0,T)$ such that the solution y of (1) satisfies $y(0,\cdot) = y_0$ and $y(T,\cdot) = y_1$? As mentioned in the introduction, P. Rouchon showed in [22] that the steady–state controllability holds for $n = 1$ or if Ω is a ball in $\mathbb{R}^n$ and asks what is the answer for general open subsets of $\mathbb{R}^n$, $n \geq 2$.

We use $-\Delta_\Omega^D$ to denote the Laplace–Dirichlet operator defined next,

$$\mathcal{D}\left(-\Delta_\Omega^D\right) := \{v \in H_0^1(\Omega); \Delta v \in L^2(\Omega)\},$$
$$-\Delta_\Omega^D v := -\Delta v,\ \forall v \in \mathcal{D}\left(-\Delta_\Omega^D\right).$$

Let us introduce the definition of Property (A), which will turn out to be an obstruction for steering $y_0 \equiv 1$ to 0 in finite time.

Definition 1. The open set Ω has the property (A) if there exists a sequence $(r_k)_{k\in\mathbb{N}^*}$ of distinct eigenvalues of $-\Delta_\Omega^D$ such that

(i) One has

$$\sum_{k=1}^{\infty} \frac{1}{r_k} = \infty, \tag{4}$$

(ii) For every $k \in \mathbb{N}^*$, there exists an eigenfunction w of the operator $-\Delta_\Omega^D$ corresponding to the eigenvalue r_k such that

$$\int_\Omega w dx \neq 0. \tag{5}$$

We are now able to state the main result of this section.

Theorem 1. *Let Ω be a bounded, open and non empty subset of $\mathbb{R}^n$, $n \geq 2$. If Ω has the property (A), then one cannot steer $y_0 \equiv 1$ to 0 in finite time.*

Proof. Assume that property (A) holds for a bounded, open and non empty subset $\Omega \subset \mathbb{R}^n$, $n \geq 2$. We suppose by contradiction that there exist $T > 0$ and $u \in L^2(0,T)$ such that the solution y of the Cauchy problem (1) with

$$y_0 \equiv 1, \tag{6}$$

satisfies

$$y(T,\cdot) = 0. \tag{7}$$

Let λ be an eigenvalue of $-\Delta_\Omega^D$ and w be an eigenfunction associated to λ. Consider $\theta \in C^\infty([0,T]; H_0^1(\Omega))$ defined by

$$\theta(t,x) := e^{\lambda t} w(x).$$

Then θ satisfies (2). Hence, using (3) with $\tau := T$, (6) and (7), one gets

$$B(\lambda) \int_\Omega w dx = 0, \tag{8}$$

where $B : \mathbb{C} \to \mathbb{C}$ is defined by

$$B(s) := 1 + s \int_0^T u(t) e^{st} dt. \tag{9}$$

Since property (A) holds for Ω, it results that B vanishes on a sequence $(r_k)_{k\in\mathbb{N}^*}$ of distinct positive real numbers satisfying (4). By the easy part of the Paley-Wiener theorem, the function B is holomorphic on $\mathbb{C}$ and there exists $C > 0$ such that

$$|B(s)| \leq C e^{T \max\{0, Re(s)\}},\ \forall s \in \mathbb{C}. \tag{10}$$

We then apply the following lemma.

Lemma 1. *Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that, for some $C > 0$,*

$$|f(s)| \leq Ce^{C|Re(s)|}, \, \forall s \in \mathbb{C}.$$

Let us assume that there exists a sequence $(r_k)_{k\geq 1}$ of distinct positive real numbers such that (4) holds and

$$f(r_k) = 0, \forall k \geq 1. \tag{11}$$

Then, f is identically equal to 0*.*

Lemma 1 is a consequence of a much more general theorem due to Cartwright and Levinson; see [12, Theorem 1, p. 127]. Applying Lemma 1 with $f := B$, we conclude that B is identically equal to zero. That contradicts the fact that $B(0) = 1$. □

2.1. **Steady–state controllability does not hold generically with respect to the domain Ω.** In this section, we want to prove that condition (A) holds for generic bounded open subsets Ω (and therefore, by Theorem 1, for such generic sets Ω, one cannot steer $y_0 \equiv 1$ to 0 in finite time).

We use here notations and results of [1, 9, 26]. Let $\mathcal{R}(\mathbb{R}^n)$ be the set of all non empty bounded open subsets Ω of class C^3. To state the result, one needs to define a topology on $\mathcal{R}(\mathbb{R}^n)$. We follow a construction closely related to that proposed by R. Hamilton in [9, pages 86-87]. For $\Omega \in \mathcal{R}(\mathbb{R}^n)$, let $\xi \in C^3(\partial\Omega; \mathbb{R}^n)$ be such that

$$\xi(x) \cdot \nu(x) > 0, \, \forall x \in \partial\Omega, \tag{12}$$

where $\nu \in C^2(\partial\Omega, \mathbb{R}^n)$ denotes the outward normal to Ω.

Let $\varepsilon_0 > 0$ be small enough so that the two following properties hold.

(i) For every x in $\mathbb{R}^n$ such that $\text{dist}(x, \partial\Omega) < \varepsilon_0$, there exists a unique $\pi(x) \in \partial\Omega$ such that $x - \pi(x)$ is parallel to $\xi(\pi(x))$.

(ii) The map $x \mapsto \pi(x)$ is of class C^3 on the open set $\{x \in \mathbb{R}^n; \, \text{dist}(x, \partial\Omega) < \varepsilon_0\}$.

Let $\varepsilon > 0$ and $\eta \in C^3(\partial\Omega)$ be such that

$$|\eta|_{C^3(\partial\Omega)} < \varepsilon. \tag{13}$$

Define

$$\Omega_\eta := \{x \in \Omega; \, \text{dist}(x, \partial\Omega) \geq \varepsilon_0\} \cup \{x \in \mathbb{R}^n; \, \text{dist}(x, \partial\Omega) < \varepsilon_0 \\ \text{and } (x - \pi(x)) \cdot \xi(\pi(x)) < \eta(\pi(x))\}.$$

There exists $\varepsilon_1 > 0$ such that, for every $\eta \in C^3(\partial\Omega)$ with $|\eta|_{C^3(\partial\Omega)} < \varepsilon_1$, Ω_η is a bounded subset of $\mathbb{R}^n$ of class C^3. Let $\mathcal{V}(\varepsilon)$ be the set of all the Ω_η with $\eta \in C^3(\partial\Omega)$ satisfying (13). We define a topology on $\mathcal{R}(\mathbb{R}^n)$ by considering the sets $\mathcal{V}(\varepsilon)$, with $\varepsilon \in (0, \epsilon_1)$, as a base of neighborhoods of Ω, i.e. every neighborhood of Ω in $\mathcal{R}(\mathbb{R}^n)$ contains some $\mathcal{V}(\varepsilon)$ for $\varepsilon \in (0, \epsilon_1)$ small enough. (One easily checks that this topology is independent of the choice of ξ and ε_1.) Recall that a topological space is a Baire space if any residual set, i.e. any intersection of denumerable open dense subsets, is dense. Since, for every Ω in $\mathcal{R}(\mathbb{R}^n)$, $C^3(\partial\Omega)$ is a Baire space, it follows from our definition of the topology on $\mathcal{R}(\mathbb{R}^n)$ that $\mathcal{R}(\mathbb{R}^n)$ is also a Baire space. (Proceeding as in [9, 4.4.7], one can also prove that $\mathcal{R}(\mathbb{R}^n)$ with our topology is a C^0-manifold modeled on the Banach spaces $C^3(\partial\Omega)$ with $\Omega \in \mathcal{R}(\mathbb{R}^n)$. But we do not need that property.)

Let us recall that that a property (P) holds for generic $\Omega \in \mathcal{R}(\mathbb{R}^n)$ if there exists a residual subset $\tilde{D} \subset \mathcal{R}(\mathbb{R}^n)$ such that property (P) holds for every $\Omega \in \tilde{D}$.

Remark 1. The use of the transverse vector field ξ is needed to parameterize the variations of a domain Ω. For that purpose, a simpler choice would have been $\xi := \nu$ but there is a serious difficulty in doing so. Indeed, for an arbitrary Ω of class C^3, ν is, in general, only of class C^2. More generally, there is always a non zero difference between the regularity of Ω and that of its outward normal vector field ν. To overcome that phenomenon of loss of derivative, one could apply a Nash-Moser type of result in order to get the necessary amount of surjectivity, which is clearly needed at some point of the argument. To avoid all that machinery, we followed, instead, the strategy advocated by D. Bresch and J. Simon in [5], consisting in choosing the transverse vector field ξ, as defined in (12), which has the *same* regularity as the domain Ω.

In this section, we prove the following result.

Theorem 2. *Condition (A) holds for generic* $\Omega \in \mathcal{R}(\mathbb{R}^n)$.

Proof of Theorem 2. The strategy of proof is standard and goes as follows (cf. [1]). Let $\mathcal{G} \subset \mathcal{R}(\mathbb{R}^n)$ be the set of $\Omega \in \mathcal{R}(\mathbb{R}^n)$ such that

(a)**:** all eigenvalues of $-\Delta_\Omega^D$ are simple,
(b)**:** $\int_\Omega w dx \neq 0$, for every non zero eigenfunction w of $-\Delta_\Omega^D$.

Similarly, for every positive integer l, the set $\mathcal{S}_l \subset \mathcal{R}(\mathbb{R}^n)$ (respectively $\mathcal{G}_l \subset \mathcal{R}(\mathbb{R}^n)$) of open sets $\Omega \in \mathcal{R}(\mathbb{R}^n)$ is defined such that property (a) (respectively, and property (b)) holds at least for the first l eigenvalues of $-\Delta_\Omega^D$. Clearly, $\mathcal{G}$ is the countable intersection of the $\mathcal{G}_l$'s.

We show next that $\mathcal{G}$ is residual, which implies Theorem 2. Indeed, if property (a) holds for $-\Delta_\Omega^D$, then, by applying the Weyl formula for $-\Delta_\Omega^D$ (cf. [23, Theorem 15.2, p.124]), one deduces that $\lambda_k \sim_{k\to\infty} C(\Omega)k^{2/n}$, where $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \lambda_{j+1} < \cdots$ is the ordered sequence of the eigenvalues of the Laplace-Dirichlet operator $-\Delta_\Omega^D$. Therefore, property (A) holds.

For $l \geq 0$, $\mathcal{S}_0 = \mathcal{G}_0 := \mathcal{R}(\mathbb{R}^n)$, $\mathcal{G}_l \subset \mathcal{S}_l$, $\mathcal{S}_{l+1} \subset \mathcal{S}_l$ and $\mathcal{S} := \cap_{l\geq 0}\mathcal{S}_l$ and, similarly, $\mathcal{G}_{l+1} \subset \mathcal{G}_l$ and $\mathcal{G} = \cap_{l\geq 0}\mathcal{G}_l$. Moreover, for $l \geq 0$, it is clear that the sets $\mathcal{S}_l$ and $\mathcal{G}_l$ are open in $\mathcal{R}(\mathbb{R}^n)$ (see [1]). To show that $\mathcal{G}$ is residual, amounts to establish the next lemma.

Lemma 2. *For every* $l \geq 0$, $\mathcal{G}_{l+1}$ *is dense in* $\mathcal{G}_l$.

Proof of Lemma 2. First, recall that, for every $l \geq 0$, $\mathcal{S}_l$ is dense in $\mathcal{R}(\mathbb{R}^n)$ (see [26]).

We follow the lines of the argument of Theorem 2 in [1]. Let $\Omega \in \mathcal{G}_l$. It is sufficient to exhibit $\Omega' \in \mathcal{G}_{l+1}$, arbitrarily close to Ω. Since $\mathcal{S}_{l+1}$ is dense, it is enough to establish the previous fact for $\Omega \in \mathcal{G}_l \cap \mathcal{S}_{l+1}$. Let $(\mu_k)_{k\in\mathbb{N}^*}$ be the ordered sequence of the eigenvalues of the Laplace-Dirichlet operator $-\Delta_\Omega^D$ *repeated according to their multiplicity.* We have

$$\mu_1 < \mu_2 < \cdots \mu_l < \mu_{l+1} < \mu_{l+2} \leq \mu_{l+3} \leq \cdots .$$

Let w_{l+1} be an eigenfunction of $-\Delta_\Omega^D$ for the eigenvalue μ_{l+1}. If $\int_\Omega w_{l+1} dx \neq 0$, then $\Omega \in \mathcal{G}_{l+1}$. Otherwise, we may assume that

$$\int_\Omega w_{l+1} dx = 0, \tag{14}$$

and we simply use μ and w to denote μ_{l+1} and w_{l+1}. Let $\xi \in C^3(\partial\Omega; \mathbb{R}^n)$ be such that (12) holds and let $\varepsilon_0 > 0$ be as above (see (i) and (ii) in this subsection). Set

$$\varepsilon_0' := \text{ Min}\,\{\xi(\pi(x)) \cdot (\pi(x) - x);\, x \in \Omega,\, \text{dist}(x, \partial\Omega) = \varepsilon_0/2\} > 0.$$

Let $\rho \in C^\infty(\mathbb{R}; [0, 1])$ be such that

$$\rho = 1 \text{ on a neighborhood of } (-\infty, 0],$$
$$\rho = 0 \text{ on a neighborhood of } [\varepsilon_0', +\infty).$$

We use $C^3_\varepsilon(\partial\Omega)$ to denote the set of $\eta \in C^3(\partial\Omega)$ such that $|\eta|_{C^3(\partial\Omega)} < \varepsilon$. For $\eta \in C^3_\varepsilon(\partial\Omega)$, we consider $h_\eta : \overline{\Omega} \to \mathbb{R}^n$ defined by

$$h_\eta(x) := x,$$

for every $x \in \Omega$ with $\text{dist}(x, \partial\Omega) \geq \varepsilon_0/2$ and

$$h_\eta(x) := x + \eta(\pi(x))\,(1 - \rho\,(\varepsilon_0' - \xi(\pi(x)) \cdot (\pi(x) - x)))\,\xi(\pi(x)),$$

for every $x \in \overline{\Omega}$ with $\text{dist}(x, \partial\Omega) \leq \varepsilon_0/2$. We now fix $\varepsilon \in (0, \varepsilon_0)$ small enough so that, for every $\eta \in C^3_\varepsilon(\partial\Omega)$, h_η is a diffeomorphism of class C^3 from $\overline{\Omega}$ into $\overline{\Omega_\eta}$. Let $P : H^2(\Omega) \to H^2(\mathbb{R}^n)$ be a linear continuous map such that

$$P(v) = v \text{ in } \Omega.$$

For $\eta \in C^3_\varepsilon(\partial\Omega)$, let $Q_\eta : H^2(\mathbb{R}^n) \to H^1_0(\Omega_\eta) \cap H^2(\Omega_\eta)$, $\phi \mapsto \psi$, be defined by

$$-\Delta\psi = -\Delta\phi \text{ in } L^2(\Omega_\eta),$$
$$\psi = 0 \text{ on } \partial\Omega_\eta.$$

Consider

$$E := \left\{(v, \eta) \in H^2(\Omega) \times C^3_\varepsilon(\partial\Omega);\, v(x) + \eta(x)\frac{\partial w}{\partial\xi}(x) = 0,\, \forall x \in \partial\Omega\right\},$$

and the following map

$$\begin{aligned} \Phi : E \times \mathbb{R} &\to L^2(\Omega) \times \mathbb{R} \\ ((v, \eta), \chi) &\mapsto \Big(((-\Delta - \chi)(Q_\eta(P(v)))) \circ h_\eta, \textstyle\int_{\Omega_\eta} Q_\eta(P(v))dx\Big). \end{aligned}$$

One has $\Phi((w, 0), \mu) = (0, 0)$ and Lemma 2 holds if Φ is locally onto at $((w, 0), \mu)$. The map Φ is of class C^1 and one has

$$\Phi'((w, 0), \mu)((v, \eta), \chi) = (-\Delta v - \mu v - \chi w, \int_\Omega v dx),$$

for every $(v, \eta) \in H^2(\Omega) \times C^3(\partial\Omega)$ such that

$$v(x) + \eta(x)\frac{\partial w}{\partial\xi}(x) = 0,\, \forall x \in \partial\Omega.$$

Using the Fredholm alternative (recall that the eigenvalue μ is assumed to be simple), one easily checks that, for every $f \in L^2(\Omega)$ and every $\eta \in C^3(\partial\Omega)$, there exists one and only one $(v, \chi) \in H^2(\Omega) \times \mathbb{R}$ such that

$$-\Delta v - \mu v - \chi w = f, \tag{15}$$

$$\int_\Omega vw dx = 0, \tag{16}$$

$$v(x) + \eta(x)\frac{\partial w}{\partial\xi}(x) = 0,\, \forall x \in \partial\Omega. \tag{17}$$

For $f = 0$, let us denote by (v_η, χ_η) the corresponding unique solution. We next prove that

$$\text{there exists } \eta_0 \in C^3(\partial\Omega) \text{ such that } \int_\Omega v_{\eta_0} dx \neq 0. \tag{18}$$

To compute $\int_\Omega v_\eta dx$ in terms of η, we consider the unique solution of the inhomogeneous Dirichlet problem given by

$$\begin{cases} (-\Delta - \mu)S = 1, & \text{in } \Omega, \\ S = 0, & \text{on } \partial\Omega, \\ \int_\Omega Swdx = 0. \end{cases} \tag{19}$$

Since $\int_\Omega wdx = 0$ and the eigenvalue μ is simple, the Fredholm alternative tells us that such an S exists (and is unique). By applying Stokes' formula, one gets, using in particular (15), (16), (17) and (19),

$$\int_\Omega v_\eta dx = \int_\Omega \Big((-\Delta - \mu)S\Big) v_\eta dx = \int_{\partial\Omega} \eta \frac{\partial S}{\partial \nu} \frac{\partial w}{\partial \nu} d\sigma. \tag{20}$$

Let us assume that (18) does not hold. Then, the right hand side of (20) should be equal to zero for every $\eta \in C^3(\partial\Omega)$ and, therefore,

$$\frac{\partial S}{\partial \nu} \frac{\partial w}{\partial \nu} \equiv 0.$$

By the Holmgren uniqueness theorem (see e.g. [25, Proposition 4.3, p. 433]), since w is a non zero eigenfunction of $-\Delta_\Omega^D$, $\partial w / \partial \nu$ cannot be equal to zero on any nonempty open subset of $\partial\Omega$. Therefore, for the previous equation to hold, it results that

$$\frac{\partial S}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{21}$$

The following lemma tells us that (21) cannot hold true (and, therefore, yields (18)).

Lemma 3. *With the notations above, there is no solution to the following overdetermined eigenvalue problem*

$$\begin{cases} (-\Delta - \mu)S = 1, & \text{in } \Omega, \\ S = 0, & \text{on } \partial\Omega, \\ \frac{\partial S}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases} \tag{22}$$

Proof of Lemma 3. The result is classical but we provide a simple proof for sake of completeness. We argue by contradiction. By differentiating (22), it follows that ∇S is solution of the following partial differential system,

$$\begin{cases} (-\Delta - \mu)\nabla S = 0, & \text{in } \Omega, \\ \nabla S = 0, & \text{on } \partial\Omega. \end{cases} \tag{23}$$

It implies that there exists a non zero constant vector $a \in \mathbb{R}^n$ such that

$$\nabla S = wa, \text{ in } \overline{\Omega}.$$

Indeed, for every constant vector $z \in \mathbb{R}^n$, $w_z := \nabla S \cdot z$ is solution of the Dirichlet problem

$$\begin{cases} (-\Delta - \mu)v = 0, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases}$$

Since μ is a simple eigenvalue and w_z is linear in z, there exists $a \in \mathbb{R}^n$ such that $w_z = (a \cdot z)w$ for all $z \in \mathbb{R}^n$. Finally, a is non zero since S is non constant (see (19)).

Modulo a rotation, one can choose $a = \|a\|(1, \cdots, 0)^{\mathrm{tr}}$. Then, S is only function of the variable x_1, with $S = 0$ on $\partial\Omega$. That clearly implies that $S = 0$ on the intersection of $\overline{\Omega}$ with any hyperplane defined by x_1 constant. Therefore $S = 0$ on $\overline{\Omega}$, contradicting $(-\Delta - \mu)S = 1$. □

Finally let

$$E_0 := \left\{(v, \tau) \in H^2(\Omega) \times (-1, 1);\ v(x) + \tau\eta_0(x)\frac{\partial w}{\partial \xi}(x) = 0,\ \forall x \in \partial\Omega\right\},$$

and let $\Phi_0 : E_0 \times \mathbb{R} \to L^2(\Omega) \times \mathbb{R}$ be defined by

$$\Phi_0((v, \tau), \chi) := \Phi((v, \tau\eta_0), \chi).$$

Then, Φ_0 is of class C^1, $\Phi_0((w, 0), \mu) = 0$ and $\Phi_0'((w, 0), \mu)$ is onto. Hence, since $E_0 \times \mathbb{R}$ is an open set of the *Hilbert* space

$$\left(\left\{(v, \tau) \in H^2(\Omega) \times \mathbb{R};\ v(x) + \tau\eta_0(x)\frac{\partial w}{\partial \nu}(x) = 0,\ \forall x \in \partial\Omega\right\}\right) \times \mathbb{R},$$

we get that Φ_0 is locally onto at $((w, 0), \mu)$ and therefore Lemma 2 holds.

Remark 2. In order to prove Theorem 2, one can alternatively use, as we shall do in order to prove Theorem 5, the tools of shape differentiation presented in [15] or [24]. These tools have been previously used by J. Ortega and E. Zuazua to show the simplicity of, on the one hand, the eigenvalues of a plate equation ([18]) and, on the other hand, those of the Stokes' system in two space dimensions ([19]) for generic domains. Note that the situation in [18, 19] is much more complicated than ours since, in [18, 19], one cannot apply the Holmgren uniqueness theorem for every Ω. Note also that the simplicity of the eigenvalues has already been used in a control problem by J.-L. Lions and E. Zuazua in [14], but in order to get a controllability result instead of a non-controllability one, as in here.

2.2. **Open rectangles.** We consider the domain $\Omega = (0, a) \times (0, b) \subset \mathbb{R}^2$, where a and b are strictly positive real numbers. Our goal is to show the following theorem.

Theorem 3. *The initial state $y_0 \equiv 1$ cannot be steered to zero in finite time.*

The eigenfunctions and eigenvalues of $-\Delta_\Omega^D$ are respectively

$$u(x_1, x_2) = K\sin(k_1\pi x_1/a)\sin(k_2\pi x_2/b),$$

and

$$\lambda = \frac{k_1^2\pi^2}{a^2} + \frac{k_2^2\pi^2}{b^2},$$

where $K \in \mathbb{R}$ and $(k_1, k_2) \in \mathbb{N}^2 \setminus \{(0, 0)\}$. Note that

$$\int_\Omega \sin(k_1\pi x_1/a)\sin(k_2\pi x_2/b)dx_1dx_2 \neq 0,$$

if and only if both k_1 and k_2 are odd.

Define $m := a^2/b^2$. Let Σ_0 be the set of eigenvalues of $-\Delta_\Omega^D$ such that there exists a corresponding eigenfunction w satisfying

$$\int_\Omega w dx \neq 0.$$

Then $\lambda \in \Sigma_0$ if and only if there exist two odd positive integers k_1 and k_2 such that

$$\lambda = \frac{\pi^2}{a^2}(k_1^2 + mk_2^2).$$

Therefore, if $N(R)$ denotes the number of $\lambda \in \Sigma_0$ less than or equal to $R > 0$, then $N(R) = N_0(Ra^2/\pi^2)$ where N_0 is the counting function of the set

$$\mathcal{N}_0(R) := \{Y \leq R \mid Y = k_1^2 + mk_2^2 \text{ with } k_1, k_2 \text{ odd integers } \}, \tag{24}$$

i.e. $N_0(R) := \#\mathcal{N}_0(R)$. Theorem 3 is a consequence of the following lemma.

Lemma 4. *With the notations above, there exists $C > 0$ such that, for R large enough,*

$$N_0(R) \geq C\frac{R}{\ln(R)}. \tag{25}$$

Assuming the conclusion of the lemma, let us finish the proof of Theorem 3. We order the real numbers in Σ_0 to get a strictly increasing sequence $(\tilde{\lambda}_n)_{n\geq 1}$. Then, it is clear that $N(\tilde{\lambda}_n) = n$, which implies, by Lemma 4, that there exists $C' > 0$ such that, for n large enough,

$$\tilde{\lambda}_n \leq C' n \ln(n).$$

Therefore, Ω has property (A) and consequently, by Theorem 1, $y_0 \equiv 1$ cannot be steered to 0 in finite time.

It remains to show Lemma 4. Let us first assume that m is irrational. Then, for every $(k_1, k_2, m_1, m_2) \in \mathbb{N}^4$,

$$\left(k_1^2 + mk_2^2 = l_1^2 + ml_2^2\right) \Rightarrow (k_1 = l_1 \text{ and } k_2 = l_2). \tag{26}$$

Therefore, for $R > 0$, $N_0(R)$ is equal to the number of pairs (k_1, k_2) of odd integers such that $k_1^2 + mk_2^2 \leq R$. As one easily checks, there exists $\delta > 0$ depending only on m such that

$$\#\left\{(k_1, k_2) \in \mathbb{N}^2;\ k_1 \text{ and } k_2 \text{ are odd, } k_1^2 + mk_2^2 \leq R\right\} \geq \delta R,\ \forall R \geq 1. \tag{27}$$

Then, equation (25) holds.

We now assume that $m = r/q$, where r, q are positive integers with $g.c.d.(r, q) = 1$. By reducing to the same denominator, we have that $N_0(R) = N_1(qR)$ with $N_1(R) := \#\mathcal{N}_1(R)$, where

$$\mathcal{N}_1(R) := \{Y \leq R \mid Y = qk_1^2 + rk_2^2 \text{ with } k_1, k_2 \text{ odd integers}\}. \tag{28}$$

By possibly exchanging q and r, we may assume that q is odd. We will actually use the asymptotic behavior of another counting function, namely $P(R) := \#\mathcal{P}(R)$, where

$$\mathcal{P}(R) := \{3 \leq p \leq R \mid p \text{ prime and } p = qk_1^2 + rk_2^2 \text{ for some } (k_1, k_2) \in \mathbb{N}^2\}.$$

Recall that there exists $C_m > 0$ only depending on m such that

$$P(R) \sim_{R\to\infty} C_m \frac{R}{\ln(R)}, \tag{29}$$

see [11]. For $R > 0$ large enough, let $\mathcal{S}(R)$ be the set of integers $Y \leq q(r+q)R$ such that, either $Y = p$ or $Y = q(r+q)p$, where $p \in [3, R]$ is a prime number with $p = qk_1^2 + rk_2^2$ for some $(k_1, k_2) \in \mathbb{N}^2$. Finally define the map $i : \mathcal{P}(R) \to \mathcal{S}(R)$ as follows. For $p \in \mathcal{P}(R)$, then $i(p) = p$ if there exist two odd integers k_1 and k_2 such that $p = qk_1^2 + rk_2^2$. Otherwise $i(p) = q(r+q)p$. It is obvious that i is an injection.

We claim that the image of $\mathcal{P}(R)$ by i is a subset of $\mathcal{N}_1\Big(q(r+q)R\Big)$. From the definition of i, this simply amounts to show that, for a prime $p = qk_1^2 + rk_2^2 \leq R$

with k_1 and k_2 integers having different parity, then $q(r+q)p \in \mathcal{N}_1\Big(q(p+q)R\Big)$. The latter simply results from the classical identity

$$q(r+q)(qk_1^2 + rk_2^2) = q(qk_1 - rk_2)^2 + r[q(k_1+k_2)]^2. \tag{30}$$

Indeed, since q is odd, then $q(k_1+k_2)$ is, and since $qk_1 - rk_2$ has the same parity as $qk_1^2 + rk_2^2 = p$, it is also odd. We deduce that

$$N_1\Big(q(r+q)R\Big) \geq P(R), \tag{31}$$

which implies that (25) follows from (29).

Remark 3. It is possible to obtain the first term of the asymptotic expansion of $N(R)$ as R tends to infinity, of the type $C_m R/\ln(R)^{1/2}$, where $C_m > 0$. For instance, in the case where $m = 1$, that corresponds to a famous result due to E. Landau.

Remark 4. Theorem 3 obviously extends to any parallelepiped in any dimension $n \geq 2$.

3. **Steady-state controllability for a water tank.** Let us consider the controllability problem for a tank containing a fluid. As in [21], we consider an open, bounded and connected subset Ω of $\mathbb{R}^2$, which corresponds to the shape of the tank. The mathematical description of this problem is given by the position D in $\mathbb{R}^2$ of the tank and by the height $h(t,x)$ of the fluid with respect to an equilibrium position. The control system is modeled by

$$\begin{cases} \ddot{D}(t) = u(t), & \text{if } t \in (0,T), \\ h_{tt}(t,x) = \Delta h(t,x), & \text{if } (t,x) \in (0,T) \times \Omega, \\ \frac{\partial h}{\partial \nu}(t,x) = -u(t) \cdot \nu(x), & \text{if } (t,x) \in (0,T) \times \partial\Omega. \end{cases} \tag{32}$$

where the control $u(t) \in \mathbb{R}^2$. Here $\nu(x)$ denotes again the outward unit normal vector at $x \in \partial\Omega$. The steady–state control problem is the following one. Let D_0 and D_1 be two arbitrary points in $\mathbb{R}^2$, does there exist $T > 0$ and $u : [0,T] \to \mathbb{R}^2$ such that the solution $D : [0,T] \to \mathbb{R}^2$, $h : [0,T] \times \Omega \to \mathbb{R}$ of (32) with

$$h(0,\cdot) = 0, h_t(0,\cdot) = 0,\ D(0) = D_0,\ \dot{D}(0) = 0, \tag{33}$$

satisfies

$$D(T) = D_1,\ \dot{D}(T) = 0,\ h(T,\cdot) = h_t(T,\cdot) = 0? \tag{34}$$

In [21], N. Petit and P. Rouchon proved that, if the shape Ω of the tank is a rectangle or a circle, then there is a solution to this controllability problem. When Ω has a general form, they assert the problem is open. Here, in the spirit of the first part of this paper, we propose a necessary condition for that steady–state controllability concerning the behavior of eigenvalues and eigenfunctions of a Neumann problem.

Let us fix $\Omega \subseteq \mathbb{R}^2$ a bounded, open and connected subset of $\mathbb{R}^2$ of class C^2 or a convex polygon. Let us first recall some classical results about the weak solution of the following Cauchy problem

$$\begin{cases} \ddot{D}(t) = u(t), & \text{if } t \in (0,T), \\ \dot{D}(0) = s_0, & \\ D(0) = D_0, & \\ h_{tt}(t,x) = \Delta h(t,x), & \text{if } (t,x) \in (0,T)\times\Omega, \\ \frac{\partial h}{\partial \nu}(t,x) = -u(t)\cdot\nu(x), & \text{if } (t,x) \in (0,T)\times\partial\Omega, \\ h(0,x) = h_0(x), & \text{if } x \in \Omega, \\ h_t(0,x) = v_0(x), & \text{if } x \in \Omega. \end{cases} \tag{35}$$

Define

$$H := \{h \in L^2(\Omega); \int_\Omega h dx = 0\},$$
$$V := \{h \in H^1(\Omega); \int_\Omega h dx = 0\},$$

and let V' be the dual space of $V \subset H$. Let $D_0 \in \mathbb{R}^2$, $s_0 \in \mathbb{R}^2$, $(h_0, v_0) \in H \times V'$, $T > 0$ and $u \in L^2(0,T;\mathbb{R}^2)$. A weak solution to the Cauchy problem (35) is a couple (D,h) such that

$$D \in H^2(0,T;\mathbb{R}^2),\ D(0) = D_0, \dot{D}(0) = s_0,\ \ddot{D} = u \in L^2(0,T), \tag{36}$$

$$h \in C^0([0,T];H) \cap C^1([0,T];V'), \tag{37}$$

and such that, for every $\tau \in [0,T]$ and for every

$$\theta \in C^0([0,T];H^2(\Omega)) \cap C^1([0,T];H^1(\Omega)) \cap C^2([0,T];L^2(\Omega))$$

satisfying

$$\theta_{tt} = \Delta\theta, \text{ in } C^0([0,T];L^2(\Omega)), \tag{38}$$

$$\frac{\partial\theta}{\partial\nu} = 0, \text{ in } C^0([0,T];H^{1/2}(\partial\Omega)), \tag{39}$$

one has

$$-\int_0^\tau \int_{\partial\Omega} \theta(t,x)u(t)\cdot\nu(x)d\sigma(x)dt + \langle v_0, \theta(0,\cdot)\rangle_{V',V} - \int_\Omega h_0(x)\theta_t(0,x)dx$$
$$= \langle h_t(\tau,\cdot), \theta(\tau,\cdot)\rangle_{V',V} - \int_\Omega h(\tau,x)\theta_t(\tau,x)dx. \tag{40}$$

Of course, for every $D \in H^2(0,T)$ and every

$$h \in C^0([0,T];H^2(\Omega)) \cap C^1([0,T];H^1(\Omega)) \cap C^2([0,T];L^2(\Omega)),$$

if (D,h) is a classical solution to (35), then it is also a weak solution to (35). Moreover, it is well known that, for every $(D_0,s_0) \in \mathbb{R}^2\times\mathbb{R}^2$, $(h_0,v_0) \in H\times V'$, $T > 0$ and $u \in L^2(0,T;\mathbb{R}^2)$, there exists one and only one weak solution (D,h) to (35). This unique (D,h) will be called the solution to the Cauchy problem (35).

We say that the control system (32) is steady–state controllable if, for every $(D_0,D_1) \in \mathbb{R}^2\times\mathbb{R}^2$, there exist $T > 0$ and $u \in L^2(0,T;\mathbb{R}^2)$ with $u(0) = 0$ such that the solution to the Cauchy problem (35), with $h_0 = v_0 = 0$, $s_0 = 0$, satisfies (34).

Consider the Laplace–Neumann operator $-\Delta_\Omega^N$ defined as follows:

$$\mathcal{D}\left(-\Delta_\Omega^N\right) := \left\{v \in H^2(\Omega); \frac{\partial v}{\partial\nu} = 0 \text{ on } \partial\Omega\right\},$$
$$-\Delta_\Omega^N v = -\,\Delta v,\ \forall v \in \mathcal{D}\left(-\Delta_\Omega^N\right).$$

3.1. **A condition that prevents steady-state controllability.** We next introduce property (B) which turns out to prevent steady–state controllability in finite time.

Definition 2. The open set Ω has the property (B) if there exists a sequence $(\lambda_k)_{k\in\mathbb{N}^*}$ of distinct eigenvalues of $-\Delta_\Omega^N$ such that

(i) There exist $\rho \in (0,2)$ and $C>0$ such that

$$\lambda_k \le Ck^\rho,\ \forall k \ge 1, \tag{41}$$

(ii) For every $k \in \mathbb{N}^*$, there exists an eigenfunction w_k for the eigenvalue λ_k and the operator $-\Delta_\Omega^N$ such that

$$\int_{\partial\Omega} w_k \nu d\sigma \neq 0. \tag{42}$$

We are now able to state the main result of this paragraph.

Theorem 4. *If Ω has property (B), then the control system (32) is not steady–state controllable.*

Remark 5. The previous theorem is, in a sense, optimal. Indeed, if Ω is equal to the disc or the rectangle, then steady-state controllability holds true and condition (B) too, except for (i), where ρ is equal to 2.

Proof. Let us first consider $u \in L^2(0,T;\mathbb{R}^2)$ and let (D,h) be the solution of the Cauchy problem (35), with

$$D_0 := 0,\ s_0 := 0,\ h_0 := 0,\ v_0 := 0. \tag{43}$$

We assume that

$$h(T,\cdot) = 0,\ h_t(T,\cdot) = 0. \tag{44}$$

Let λ be an eigenvalue of $-\Delta_\Omega^N$ and w be an eigenfunction associated to the eigenvalue λ. Let $\theta \in C^\infty([0,T];H^2(\Omega))$ be defined by

$$\theta(t,x) = e^{i\sqrt{\lambda}t} w(x).$$

Then θ satisfies (38) and (39). Hence, using (40) with $\tau = T$, (43) and (44), one gets

$$C(i\sqrt{\lambda}) \cdot \int_{\partial\Omega} w(x)\nu(x)d\sigma(x) = 0, \tag{45}$$

where $C : \mathbb{C} \to \mathbb{C}^2$ is the holomorphic function defined by

$$C(s) := \int_0^T u(t)e^{st}dt. \tag{46}$$

The proof of Theorem 4 goes now by contradiction. We suppose that property (B) holds and assume, by contradiction, that the control system (32) is steady–state controllable. Then, for every $q \in \mathbb{R}^2$, there exists $u(t) \in L^2(0,T;\mathbb{R}^2)$ such that the solution (D,h) to the Cauchy problem (35), with $D_0 := 0$, $s_0 := 0$, $h_0 := 0$ and $v_0 := 0$, satisfies

$$h(T,\cdot) = 0,\ h_t(T,\cdot) = 0,\ D(T) = q, \dot{D}(T) = 0. \tag{47}$$

We use u^1, D^1 and u^2, D^2 to denote u, D, for $q := (1,0)^{\mathrm{tr}}$ and $q := (0,1)^{\mathrm{tr}}$ respectively. Similarly, $C^1 := (C_1^1, C_2^1)^{\mathrm{tr}}$ and $C^2 := (C_1^2, C_2^2)^{\mathrm{tr}}$ are defined by (see (46))

$$C^1(s) := \int_0^T u^1(t)e^{st}dt,\ C^2(s) := \int_0^T u^2(t)e^{st}dt.$$

We will derive a contradiction from the existence of both u^1 and u^2. Let $(\lambda_k)_{k\in\mathbb{N}^*}$ and $(w_k)_{k\in\mathbb{N}^*}$ be as in Definition 2. For $k \in \mathbb{N}^*$, define

$$t_k := \sqrt{\lambda_k},\ v_k := \int_{\partial\Omega} w_k(x)\nu(x)d\sigma(x) \in \mathbb{R}^2 \setminus \{0\}.$$

By (45), we have, for $j = 1, 2$ and $k \geq 1$,

$$C^j(it_k) \cdot v_k = 0. \tag{48}$$

For every $k \geq 1$, v_k is a non zero vector of $\mathbb{C}^2$. Therefore $C^1(it_k)$ and $C^2(it_k)$ must be collinear. We deduce that, for every $k \geq 1$,

$$C_1^1(it_k)C_2^2(it_k) - C_2^1(it_k)C_1^2(it_k) = 0. \tag{49}$$

Introducing the holomorphic function $G : \mathbb{C} \to \mathbb{C}$, $G(s) := C_1^1(s)C_2^2(s) - C_2^1(s)C_1^2(s)$, we reformulate (49) as

$$G(it_k) = 0. \tag{50}$$

Let us recall the following classical result.

Lemma 5. *Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that*

$$\exists C_0 > 0,\ \textit{such that},\ \forall s \in \mathbb{C},\ |f(s)| \leq C_0 e^{C_0|s|}. \tag{51}$$

Assume that $f \neq 0$. Let $n : [0, +\infty) \to \mathbb{N}$ be defined by

$$n(R) := \#\{s \in \mathbb{C};\ f(s) = 0 \text{ and } |s| \leq R\}.$$

Then,

$$\exists C_1 > 0, \forall R \in (1, +\infty),\ \int_1^R \frac{n(t)}{t}dt \leq C_1 R. \tag{52}$$

(This lemma follows easily from the Jensen formula, see e.g. [12, Lecture 2, section 2.3, p. 10-11].) We apply this lemma with $f := G$. By the easy part of the Paley-Wiener theorem, G is a holomorphic function which satisfies (51). By (41) and (50), (52) does not hold. Hence, by Lemma 5, $G = 0$. On the other hand, we compute $G(s)$ for s small enough. Simple integrations by parts yield, for $j, l = 1, 2$ and $s \in \mathbb{C}^2$,

$$C_l^j(s) = -sD_l^j(T)e^{sT} + s^2 \int_0^T D_l^j(t)e^{st}dt. \tag{53}$$

As s goes to 0, the previous equation can be written $C_l^j(s) = -sD_l^j(T)e^{sT} + O(s^2)$ and then,

$$G(s) = s^2e^{2sT}\Big(D_1^1(T)D_2^2(T) - D_2^1(T)D_1^2(T)\Big) + O(s^3) = s^2e^{2sT} + O(s^3),$$

which implies that $G(s) \neq 0$ for s small enough but nonzero. That contradicts the fact that G is the zero function. Theorem 4 is proved. □

Remark 6. The control system (32) is the linearized control system of a control system modeled by the Saint-Venant equations (see [21]). For a tank in dimension 1 (i.e. with $\Omega = (0, L) \subset \mathbb{R}$) it is proved in [4] -see also [21]- that the linearized control system is steady state controllable but is not locally controllable. However, the non linear terms give the local controllability for a tank in dimension 1; see [3]. It would be interesting to know if this is also the case in dimension 2.

3.2. **Genericity of condition** (B)**.** In this section, we prove that condition (B) holds generically for tank shapes of class C^3, and therefore by Theorem 4, for such generic tank shapes Ω, steady-state controllability for a water-tank does not hold.

We use here notations and results of [24]. Let $\mathcal{S}_3$ be the set of all open, bounded, connected subsets $\Omega \subset \mathbb{R}^2$ of class C^3. The topology on $\mathcal{S}_3$ is defined as follows ([24, p. 7]).

Let $C^3_b(\mathbb{R}^2;\mathbb{R}^2)$ be the space of functions $u : \mathbb{R}^2 \to \mathbb{R}^2$ of class C^3 such that

$$\|u\|_3 := \mathrm{Sup}\{|\partial^\alpha u(x)|\,;\, x \in \mathbb{R}^2,\ \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2 \text{ with } \alpha_1 + \alpha_2 \le 3\} < +\infty.$$

Then $C^3_b(\mathbb{R}^2;\mathbb{R}^2)$ equipped with the norm $\|\cdot\|_3$ is a Banach space. For $\Omega \in \mathcal{S}_3$ and $u \in C^3_b(\mathbb{R}^2;\mathbb{R}^2)$, let $\Omega + u := (Id + u)(\Omega)$ be the subset of points $y \in \mathbb{R}^2$ such that $y = x + u(x)$ for some $x \in \Omega$ and $\partial\Omega + u := (Id + u)(\partial\Omega)$ its boundary. By simple topological arguments, one easily gets that, for $u \in C^3_b(\mathbb{R}^2;\mathbb{R}^2)$ small enough, $\Omega + u$ belongs to $\mathcal{S}_3$.

For $\varepsilon > 0$, let $\mathcal{V}(\varepsilon)$ be the sets of all $\Omega + u$ with $u \in C^3_b(\mathbb{R}^2;\mathbb{R}^2)$ and $\|u\|_3 < \varepsilon$. The topology on $\mathcal{S}_3$ is defined by taking the sets $\mathcal{V}(\varepsilon)$ with ε small enough as a base of neighborhoods of Ω. Then $\mathcal{S}_3$ is a Baire space.

Theorem 5. *Condition (B) holds for generic $\Omega \in \mathcal{S}_3$.*

Proof. The strategy is entirely similar to that of the argument of Theorem 4. Let $\mathcal{H} \subset \mathcal{S}_3$ be the set of $\Omega \in \mathcal{S}_3$ such that

(a)**:** all eigenvalues of $-\Delta^N_\Omega$ and $-\Delta^D_\Omega$ are simple;
(b)**:** $\int_{\partial\Omega} w\nu dx \neq 0$, for every non zero eigenfunction w of $-\Delta^N_\Omega$ corresponding to a nonzero eigenvalue.

For $l \ge 1$, define the set $\mathcal{K}_l \in \mathcal{S}_3$ (respectively $\mathcal{H}_l \in \mathcal{S}_3$) of open sets $\Omega \in \mathcal{S}_3$ such that property (a) (respectively, and property (b)) holds at least for the first l eigenvalues of $-\Delta^D_\Omega$ and $-\Delta^N_\Omega$. Then, $\mathcal{H}$ is the countable intersection of the $\mathcal{H}_l$'s.

We show next that $\mathcal{H}$ is residual, which implies Theorem 5. Similarly to the proof of Theorem 4, that follows by using a Weyl formula for $-\Delta^N_\Omega$, namely $\lambda_k \sim_{k\to\infty} C(\Omega)k$, (cf. [10, formula (10.2.40), page 505]).

For $l \ge 0$, $\mathcal{K}_0 = \mathcal{H}_0 := \mathcal{S}_3$, $\mathcal{H}_l \subset \mathcal{K}_l$, $\mathcal{K}_{l+1} \subset \mathcal{K}_l$ and $\mathcal{K} := \cap_{l\ge 0}\mathcal{K}_l$ and, similarly, $\mathcal{H}_{l+1} \subset \mathcal{H}_l$ and $\mathcal{H} = \cap_{l\ge 0}\mathcal{H}_l$. Moreover, for $l \ge 0$, it is clear that the sets $\mathcal{K}_l$ and $\mathcal{H}_l$ are open in $\mathcal{S}_3$.

We first need to establish the next result.

Lemma 6. *For every $l \ge 0$, $\mathcal{K}_{l+1}$ is dense in $\mathcal{S}_3$.*

Proof of Lemma 6. Following the standard strategy, one must prove that $\mathcal{K}_{l+1}$ is dense in $\mathcal{K}_l$ for $l \ge 1$. Let $\Omega \in \mathcal{K}_l$ and $\lambda_{l+1} := \lambda$ is the $(l+1)-$th eigenvalue of $-\Delta^N_\Omega$ with multiplicity $h \ge 2$. Since $\mathcal{K}_l$ is open and property (a) on page 6 is generic, we may also assume that the eigenvalues of $-\Delta^D_\Omega$ are simple.

By [20, Theorem 4], we have the following proposition.

Proposition 1. *There exist h continuous functions with values in $\mathbb{R}$, $u \mapsto \lambda_i(u)$ and h functions $u \mapsto w_i(u)$, with values in $H^3(\Omega + u)$, $l+1 \leq i \leq l+h$, defined in a neighborhood U of $u = 0$ in $C_b^3(\mathbb{R}^2;\mathbb{R}^2)$, such that the following properties hold:*

- (i)**:** *$\lambda_i(0) = \lambda$, $l+1 \leq i \leq l+h$;*
- (ii)**:** *for every $u \in U$, $\lambda_i(u)$ is an eigenvalue of $-\Delta^N_{\Omega+u}$ with associated eigenfunction $w_i(u)$;*
- (iii)**:** *for every $u \in U$, the set $\{w_{l+1}(u), \cdots, w_{l+h}(u)\}$ is orthonormal in $L^2(\Omega + u)$;*
- (iv)**:** *for each open interval $I \subset \mathbb{R}$, such that the intersection of I with the sequence $(\lambda_i)_{i\geq 0}$ contains only the eigenvalue λ, there exists a neighborhood $U_I \subset U$ such that there exist exactly h eigenvalues (counting the multiplicity) $\lambda_i(u)$, $l+1 \leq i \leq l+h$ of $-\Delta^N_{\Omega+u}$ contained in I;*
- (v)**:** *For each $u \in C_b^3(\mathbb{R}^2;\mathbb{R}^2)$ and $l+1 \leq i \leq l+h$, the map $t \in \mathbb{R} \mapsto (\lambda_i(tu), w_i(tu) \circ (Id + tu)) \in \mathbb{R} \times H^3(\Omega)$ is analytic in a neighborhood of $t = 0$.*

Let us denote by $\lambda_i'(u)$ the value of the directional derivative of λ_i at 0 in the direction of u. In order to compute $\lambda_i'(u)$, we use classical results on shape differentiation, as those one can find in [15, 24]. We use notations and results of [24]. Note that one cannot apply directly the statements of [24] since the map $u \in C_b^3(\mathbb{R}^2;\mathbb{R}^2) \mapsto (\lambda_i(u), w_i(u) \circ (Id + u)) \in \mathbb{R} \times H^2(\Omega)$ is not differentiable at 0. But this map has a directional derivative at 0 in every direction, which is sufficient to adapt the proofs given in [24]. So, when we say that we apply a specific statement of [24], we should say that we apply the proof given in [24] of the specific statement in order to get the desired statement for directional derivatives. By [24, Teorema 2.13 p. 49] and (v) of Proposition 1, for every $i \in \{l+1, \ldots, l+h\}$, there exists $w_i'(u) \in H^2(\Omega)$ such that, for every open subset ω of Ω such that $\overline{\omega} \subset \Omega$,

$$\frac{1}{t}\left(w_i(tu)|_\omega - w_i(0)|_\omega\right) \to w_i'(u)|_\omega \text{ in } H^2(\omega) \text{ as } t \to 0. \tag{54}$$

We have, for every $t \in \mathbb{R}$ such that $|t|$ is small enough,

$$\Delta w_i(tu) + \lambda_i(tu) w_i(tu) = 0 \text{ in } \Omega + tu, \tag{55}$$

$$\frac{\partial w_i(tu)}{\partial \nu} = 0 \text{ on } \partial(\Omega + tu), \tag{56}$$

$$\int_{\Omega+tu} w_i(tu)^2 dx = 1. \tag{57}$$

From (54) and (55), we get

$$\Delta w_i'(u) + \lambda_i'(u) w_i(0) + \lambda_i w_i'(u) = 0 \text{ in } \Omega, \tag{58}$$

By (57) and [24, Teorema 4.14 p. 103],

$$\frac{\partial w_i'(u)}{\partial \nu} + (u \cdot \nu)\frac{\partial^2 w_i(0)}{\partial \nu^2} - \frac{\partial w_i(0)}{\partial \tau}\frac{\partial (u \cdot \nu)}{\partial \tau} = 0 \text{ on } \partial\Omega, \tag{59}$$

where $\tau(x) \in \mathbb{R}^2$ denotes the tangent unit vector to $\partial\Omega$ at $x \in \partial\Omega$ such that $(\nu(x), \tau(x))$ is a direct basis of $\mathbb{R}^2$ and where

$$\frac{\partial^2}{\partial \nu^2} := \sum_{p,q=1}^{2} \nu_p \nu_q \frac{\partial^2}{\partial x_p \partial x_q}.$$

By [24, Teorema 2.21 p. 62-63, formula (2.72)'], the evaluation at $t = 0$ of the derivative with respect to t of the left hand side of (57) is equal to

$$2\int_\Omega w_i(0)w_i'(u)dx + 2\int_{\partial\Omega}(u\cdot\nu)w_i^2(0)d\sigma.$$

Hence, by (57), one gets

$$\int_\Omega w_i(0)w_i'(u)dx + \int_{\partial\Omega}(u\cdot\nu)w_i(0)^2 d\sigma = 0. \tag{60}$$

We multiply (58) by $w_i(0)$ and integrate on Ω. Then, straightforward integrations by parts, together with (55), (56), (59) and (60), give

$$\begin{aligned}\lambda_i'(u) &= -\int_\Omega w_i(0)(\Delta+\lambda)w_i'(u)dx\\ &= -\int_{\partial\Omega} w_i(0)\frac{\partial w_i'(u)}{\partial\nu}d\sigma\\ &= \int_{\partial\Omega}(u\cdot\nu)\left[\left(\frac{\partial w_i(0)}{\partial\tau}\right)^2 - \lambda w_i^2(0)\right]d\sigma.\end{aligned} \tag{61}$$

As pointed out by Albert [1], in order to prove the lemma, it is enough, for every pair of distinct indices (i_1, i_2) in $\{l+1,\cdots,l+h\}$, to exhibit $u\in C_b^3(\mathbb{R}^2,\mathbb{R}^2)$ arbitrary small such that $\lambda_{i_1}'(u)\neq\lambda_{i_2}'(u)$.

We argue by contradiction. For every u small enough and any pair of distinct indices (i_1, i_2) in $\{l+1,\cdots,l+h\}$, one has

$$\lambda_{i_1}(u)=\lambda_{i_2}(u)=:\lambda(u) \text{ and } \lambda_{i_1}'(u)=\lambda_{i_2}'(u)=:\lambda'(u).$$

For $\alpha\in\mathbb{R}$, set $w_\alpha(u) := \cos(\alpha)w_{i_1}(u)+\sin(\alpha)w_{i_2}(u)$ and $\lambda_\alpha(u)=\lambda(u)$ the corresponding eigenvalue. Clearly $w_\alpha(u)$ is a normalized eigenfunction of $-\Delta_{\Omega+u}^N$ associated to $\lambda(u)$. Set $\lambda_\alpha'(u)$ to be the directional derivative of λ_α in the direction u. The same computations as above give

$$\lambda_\alpha'(u)=\lambda'(u)+2\cos(\alpha)\sin(\alpha)\int_{\partial\Omega}(u.\nu)[\frac{\partial w_{i_1}(0)}{\partial\tau}\frac{\partial w_{i_2}(0)}{\partial\tau}-\lambda w_{i_1}(0)w_{i_2}(0)]d\sigma.$$

But, for every $\alpha\in\mathbb{R}$, $\lambda_\alpha'(u)=\lambda'(u)$. Hence

$$\int_{\partial\Omega}(u.\nu)[\frac{\partial w_{i_1}(0)}{\partial\tau}\frac{\partial w_{i_2}(0)}{\partial\tau}-\lambda w_{i_1}(0)w_{i_2}(0)]d\sigma=0.$$

The previous equality must hold for every $u\in C_b^3(\mathbb{R}^2;\mathbb{R}^2)$. Therefore, on $\partial\Omega$,

$$\frac{\partial w_{i_1}(0)}{\partial\tau}\frac{\partial w_{i_2}(0)}{\partial\tau}-\lambda w_{i_1}(0)w_{i_2}(0)=0. \tag{62}$$

We first assume that the above equality implies that $w_{i_1}(0)$ is equal to zero on $\partial\Omega$. By the Holmgren uniqueness theorem, this is not possible. A contradiction is reached and Lemma 6 is proved.

It remains to show that (62) implies that $w_{i_1}(0)=0$ on $\partial\Omega$. The set $\partial\Omega$ is the finite union of regular Jordan curves $\Gamma_1,\cdots,\Gamma_m$. It is enough to show that, for every such a curve, say Γ_k, (62) on Γ_k implies $w_{i_1}(0)=0$ on Γ_k. For notational ease, we set $w_{i_1}(0):=f$ and $w_{i_2}(0):=w$. On Γ_k we have

$$\lambda wf-\frac{\partial w}{\partial\tau}\frac{\partial f}{\partial\tau}=0. \tag{63}$$

Modifying λ if necessary (but still a positive constant), equation (63) simply reduces to

$$\lambda w f = w' f' \tag{64}$$

where w and f are C^1 functions on the unit circle S^1. The conclusion follows using the next lemma.

Lemma 7. *With the above notations, then $f \equiv 0$.*

Proof of Lemma 7. Indeed, consider $H := f\frac{\lambda w^2}{2}$. Note that H is of class C^1 and, by (64),

$$H = \frac{ww'f'}{2}, \tag{65}$$

$$H' = f'\Big((w')^2 + \frac{\lambda}{2}w^2\Big). \tag{66}$$

We first claim that H must be identically equal to zero. Reasoning by contradiction, there exists some $\theta_0 \in S^1$ such that $H'(\theta_0) = 0$ and $H(\theta_0) \neq 0$. By (66), one has $f'(\theta_0) = 0$ or $w'(\theta_0) = w(\theta_0) = 0$, since $\lambda > 0$. Both equalities contradict $H(\theta_0) \neq 0$ (see (65)). We get that $fw \equiv 0$. By the Holmgren uniqueness theorem, w cannot be equal to zero on any nonempty open subset of $\partial\Omega$ since w is a non zero eigenfunction of $-\Delta_\Omega^N$. We conclude that $f \equiv 0$ on S^1. □

Showing that $\mathcal{H}$ is residual amounts to establish the next lemma.

Lemma 8. *For every $l \geq 0$, $\mathcal{H}_{l+1}$ is dense in $\mathcal{S}_3$.*

Proof of Lemma 8. The argument follows the lines of that of Lemma 2. Since $\mathcal{H}_1$ $(= \mathcal{K}_1)$ is dense in $\mathcal{S}_3$, we may assume that $l \geq 1$.

Let $\Omega \in \mathcal{H}_l$. It is sufficient to exhibit $\Omega' \in \mathcal{H}_{l+1}$, arbitrarily close to Ω. Since $\mathcal{K}_{l+1}$ is dense, it is enough to establish the previous fact for $\Omega \in \mathcal{H}_l \cap \mathcal{K}_{l+1}$. Let $(\mu_k)_{k\in\mathbb{N}^*}$ be the ordered sequence of the eigenvalues of $-\Delta_\Omega^D$ repeated according to their multiplicity, and similarly, let $(\lambda_k)_{k\in\mathbb{N}^*}$ be the ordered sequence of the eigenvalues of $-\Delta_\Omega^N$ repeated according to their multiplicity. We have

$$\mu_1 < \mu_2 < \cdots \mu_l < \mu_{l+1} < \mu_{l+2} \leq \mu_{l+3} \leq \cdots,$$

and

$$0 = \lambda_1 < \lambda_2 < \cdots \lambda_l < \lambda_{l+1} < \lambda_{l+2} \leq \lambda_{l+3} \leq \cdots.$$

If $\int_{\partial\Omega} w_{l+1}\nu dx \neq 0$ then $\Omega \in \mathcal{H}_{l+1}$. Therefore, we may assume that

$$\int_{\partial\Omega} w_{l+1}\nu d\sigma = 0, \tag{67}$$

and we simply use λ, $\lambda(u)$, w and $w(u)$ to denote λ_{l+1}, $\lambda_{l+1}(u)$, w_{l+1} and $w_{l+1}(u)$. Note that since λ is a simple eigenvalue of $-\Delta_\Omega^N$, there exists a neighborhood $\mathcal{U}$ of the zero function in $C_b^3(\mathbb{R}^2;\mathbb{R}^2)$ such that the maps

$$u \in \mathcal{U} \mapsto w(u) \circ (Id + u) \in H^3(\Omega) \ \text{ and } u \in \mathcal{U} \mapsto \lambda(u) \in \mathbb{R}$$

are well-defined and analytic in $\mathcal{U}$. We argue by contradiction: we assume that

$$\int_{\partial\Omega+u} w(u)\nu d\sigma = 0, \, \forall u \in \mathcal{U}, \tag{68}$$

at least if $\mathcal{U}$ is a small enough neighborhood of the zero function in $C_b^3(\mathbb{R}^2;\mathbb{R}^2)$. Integrations by parts and (68) give

$$\int_{\Omega+u} \nabla w(u)dx = 0. \tag{69}$$

By [24, Teorema 2.21 p. 62-63, formula (2.72)'], the derivative of the left hand side of (68) at 0 in the direction of $u \in C_b^3(\mathbb{R}^2;\mathbb{R}^2)$ is

$$\int_{\Omega} \nabla(w'(u))dx + \int_{\partial\Omega} (u\cdot\nu)\nabla w d\sigma.$$

Hence, with also (57), (69) and integrations by parts,

$$\int_{\partial\Omega} (w'(u)\nu + (u\cdot\nu)\nabla w)d\sigma = 0,\ \forall u \in C_b^3(\mathbb{R}^2;\mathbb{R}^2). \tag{70}$$

In order to express $\int_{\partial\Omega} w'(u)\nu d\sigma$ explicitly in terms of u, we consider the unique solution of the inhomogeneous Neumann problem given by

$$\begin{cases} (-\Delta-\lambda)S = 0, & \text{in } \Omega,\\ \frac{\partial S}{\partial\nu} = \nu, & \text{on } \partial\Omega,\\ \int_\Omega Swdx = 0. \end{cases} \tag{71}$$

Since $\int_{\partial\Omega} w\nu d\sigma = 0$ and the eigenvalue λ is simple, the Fredholm alternative tells us that such a S exists and is unique. Then, for every $u \in \mathcal{U}$, one has, by using (58), (59) and (71), applying Stokes' formula and performing integrations by parts,

$$\int_{\partial\Omega} w'(u)\nu d\sigma = \int_{\partial\Omega} w'(u)\frac{\partial S}{\partial\nu}d\sigma = \int_{\partial\Omega}(u\cdot\nu)\Big(\lambda wS - \frac{\partial w}{\partial\tau}\frac{\partial S}{\partial\tau}\Big)d\sigma$$

which, with (70), leads to

$$\int_{\partial\Omega}(u\cdot\nu)\left[\lambda wS - \frac{\partial w}{\partial\tau}\frac{\partial S}{\partial\tau} + \frac{\partial w}{\partial\tau}\tau\right]d\sigma = 0,\ \forall u \in \mathcal{U}. \tag{72}$$

Hence, on $\partial\Omega$,

$$\lambda wS - \frac{\partial w}{\partial\tau}\Big(\frac{\partial S}{\partial\tau} - \tau\Big) = 0. \tag{73}$$

Of course, by (68), one can replace in (73) w by $w(u)$ and Ω by $\Omega + u$, that is

$$\lambda(u)w(u)S(u) - \frac{\partial w(u)}{\partial\tau(u)}\Big(\frac{\partial S(u)}{\partial\tau(u)} - \tau(u)\Big) = 0,\ \forall u \in \mathcal{U}, \tag{74}$$

on $\partial(\Omega+u)$, where $S(u)$ is the unique solution to the inhomogeneous Neumann problem

$$(-\Delta - \lambda(u))S(u) = 0,\ \text{in } \Omega+u, \tag{75}$$

$$\frac{\partial S(u)}{\partial\nu(u)} = \nu(u),\ \text{on } \partial(\Omega+u), \tag{76}$$

$$\int_{\Omega+u} S(u)w(u)dx = 0, \tag{77}$$

$\nu(u)$ denoting the outward normal to $\Omega+u$ on $\partial(\Omega+u)$, and where $\tau(u)(x) \in \mathbb{R}^2$ denotes the tangent unit vector to $\partial(\Omega+u)$ at $x \in \partial(\Omega+u)$ so that $(\nu(u)(x), \tau(u)(x))$ is an orthonormal basis of $\mathbb{R}^2$. Moreover, we may assume that $\nu(\cdot)$ is actually the extension of the normal field $\nu := \nu(0)$ to Ω which is defined in [24], page 86. Recall that the map $u \mapsto \nu(u)$, defined on $\mathcal{U}$, is differentiable at zero and $\frac{\partial\nu(0)}{\partial\nu} = 0$ on $\partial\Omega$.

We use $\nu'(u)$ to denote the evaluation at $u \in \mathcal{U}$ of the differential of $\nu(\cdot)$ at zero, which is equal to

$$\nu'(u) = -\frac{\partial(u\cdot\nu)}{\partial\tau}\tau,$$

on $\partial\Omega$, where $u\cdot\nu$ again denotes the normal component of u.

A proof similar to the proof of (v) in Proposition 1 tell us that the map $u \in \mathcal{U} \mapsto S(u)\circ(Id+u) \in H^3(\Omega)$ is of class C^1. Hence, by [24, Teorema 2.13 p. 49] again, there exists $S'(u) \in H^2(\Omega)$ such that, for every open subset ω of Ω such that $\overline{\omega} \subset \Omega$,

$$\frac{1}{t}\left(S(tu)|_\omega - S(0)|_\omega\right) \to S'(u)|_\omega \text{ in } H^2(\omega) \text{ as } t \to 0. \tag{78}$$

Using (75) and (78), we get that

$$\lambda'(u)S + (\Delta+\lambda)S'(u) = 0 \text{ in } \Omega. \tag{79}$$

In addition, we get that $S'(u)$ satisfies

$$\frac{\partial S'(u)}{\partial\nu} = \frac{\partial(u\cdot\nu)}{\partial\tau}\frac{\partial S}{\partial\tau} - \frac{\partial(u\cdot\nu)}{\partial\tau}\tau - (u\cdot\nu)\frac{\partial^2 S}{\partial\nu^2} \text{ on } \partial\Omega, \tag{80}$$

as a consequence of [24, Teorema 2.16, p. 54] applied to the map $u \mapsto \nabla S(u)\cdot\nu(u) - \nu(u)$.

Finally, we derive another relation verified by $S'(u)$. Similarly to (60), the derivative with respect to t at $t=0$ of

$$\int_{\Omega+tu} S(tu)w(tu)dx = 0,$$

yields, by using [24, Teorema 2.21 p. 62-63, formula (2.72)'],

$$\int_\Omega S'(u)wdx + \int_\Omega Sw'(u)dx + \int_{\partial\Omega}(u\cdot\nu)Swd\sigma = 0. \tag{81}$$

At this stage, we are not able to prove that the existence of $S'(u)$ satisfying (79) and (80), for every u belonging to some open neighborhood in $C_b^3(\mathbb{R}^2;\mathbb{R}^2)$ of the zero function, yields a contradiction. The next step would be to consider second-order domain variations. The required theory for doing so is actually developed in [24] but we chose another way to complete the argument of Lemma 8. It proceeds as follows.

First recall that $\mathcal{H}_l \cap \mathcal{K}_{l+1}$ is open and dense in $C_b^3(\mathbb{R}^2;\mathbb{R}^2)$. Let $\mathcal{R}_{l+1}$ be the union of the domains Ω of $\mathcal{H}_l \cap \mathcal{K}_{l+1}$ so that Eq. (73) does not hold true (in which, of course, λ and w denote respectively λ_{l+1} and w_{l+1}, and S is defined in (71)). We claim that the denseness of $\mathcal{R}_{l+1}$ in Ω of $\mathcal{H}_l \cap \mathcal{K}_{l+1}$ would complete the proof of Lemma 8. Indeed, take Ω in $\mathcal{H}_l \cap \mathcal{K}_{l+1}$. If (67) does not hold, the vector field S is not even defined and, then, $\Omega \in \mathcal{R}_{l+1}$. Consider next Ω in $\mathcal{H}_l \cap \mathcal{K}_{l+1}$ so that (67) holds. If $\mathcal{R}_{l+1}$ is dense in Ω of $\mathcal{H}_l \cap \mathcal{K}_{l+1}$, then, clearly, for any neighborhood $\mathcal{U}$ of the zero function in $C_b^3(\mathbb{R}^2;\mathbb{R}^2)$, Eq. (68) cannot hold. Therefore, there exist domains Ω' arbitrarily close to Ω so that (67) does not hold for Ω', meaning exactly that condition (B) is generic.

It remains to show that $\mathcal{R}_{l+1}$ is dense in $\mathcal{H}_l \cap \mathcal{K}_{l+1}$. We have to prove that there does not exist a domain $\Omega \in \mathcal{H}_l \cap \mathcal{K}_{l+1}$ where (67) holds, and a neighborhood $\mathcal{U}$ of the zero function in $C_b^3(\mathbb{R}^2;\mathbb{R}^2)$ so that, for every $u \in \mathcal{U}$, Eq. (68) and thus Eq. (74) hold true.

As usual, we argue by contradiction and first consider the directional derivative of (74) with respect to the variation $u \in \mathcal{U}$. Applying [24, Teorema 2.16, p. 54] to the map

$$u \mapsto \lambda(u)w(u)S(u) - \nabla w(u)\nabla S(u) + \nabla w(u),$$

we get that, on $\partial\Omega$,

$$\lambda'(u)wS+\lambda w'(u)S+\lambda wS'(u)-\nabla w'(u)\nabla S-\nabla w\nabla S'(u)+\nabla w'(u) = -(u\cdot\nu)G(w,S), \tag{82}$$

where $G(w,S) := \frac{\partial}{\partial\nu}\Big(\lambda wS - \nabla w\nabla S + \nabla w\Big)$. Note that the previous equation involves the functions $w'(u)$ and $S'(u)$ defined respectively in Eqs. (58)-(59) and (79)-(80).

We next sketch the remaining part of the proof and then proceed later on with the details. For notational ease, we set $\varphi := u\cdot\nu$, which is defined on $\partial\Omega$. Note that the dependence of $\lambda'(u)$, $w'(u)$ and $S'(u)$ with respect to u is only through φ. Therefore, for the rest of the paper, there will be a slight ambiguity when we will refer to a variation φ only defined on $\partial\Omega$, giving rise to $\lambda'(u)$, $w'(u)$ and $S'(u)$, with $\varphi = u\cdot\nu$. Note that, at the present stage of the proof, the variation φ may be taken in $C^2(\partial\Omega)$. Indeed, fix $\varphi \in C^2(\partial\Omega)$. By a standard partition of unity argument on Ω, one can construct a C^2-extension $\tilde{\nu}$ of the outward normal vector field ν such that $\tilde{\nu}$ is defined on an open neighborhood N of $\partial\Omega$ and compactly supported. On the other hand, by using $\tilde{\nu}$, one can get a C^2-extension $\tilde{\varphi}$ of φ, defined on N and compactly supported. Set finally $u := \tilde{\varphi}\tilde{\nu}$. Then u clearly belongs to $C^2(\Omega)$ and, thanks to the fact that $C^3(\Omega)$ is dense in $C^2(\Omega)$, the global variation u is admissible.

The key point now consists of rewriting Eq. (82) as follows,

$$\nabla w'(u)\nabla S+\nabla w\nabla S'(u)-\nabla w'(u) = \lambda'(u)wS+\lambda w'(u)S+\lambda wS'(u)+\varphi G(w,S), \tag{83}$$

and, then, to observe that there is a "gap of regularity" between the left-hand side and the right-hand side of (83). Indeed, the left-hand side of (83) should be at most as regular as $\frac{d\varphi}{d\sigma}$ (σ denotes the arc-length of $\partial\Omega$) whereas the right-hand side of (83) should have at least the regularity of φ (recall that $\lambda'(u)$ is scalar). Since both sides are equal, some information should be gained on the left-hand side of (83) by considering variations, as irregular as possible and, which can be "localized" at any arbitrary point of $\partial\Omega$. By the latter, we mean that we would like, in a first step, to consider variations exhibiting just one jump of discontinuity at an arbitrary point $a \in \partial\Omega$, so that, for all the quantities involved in (83), an "irregular part" only occurs at the point a. If we are then able to compute exactly that "irregular part" (at a), we could infer that it has to be equal to zero by using the equality in (83).

The rest of the paper intends to render the above observation completely rigorous. The first task consists of defining an appropriate extension for the variations space.

Recall that the set of admissible φ contains an open neighborhood of the zero function in $C^2(\partial\Omega;\mathbb{R})$. We would like actually to handle variations φ belonging to $L^2(\partial\Omega)$. For that purpose, we rely on [13, Théorème 7.4 , p.202]. In order to state completely that theorem, we must introduce some functional spaces.

We first recall standard definitions on Sobolev spaces and distributions over Ω (see, for instance p. 3 of [13]). If m is a positive integer, we use $H^m(\Omega)$ to denote the Sobolev space of order m on Ω defined by

$$H^m(\Omega) = \{u \mid D^\alpha u \in L^2(\Omega), |\alpha| \le m\},$$

where $D^\alpha = \frac{\partial^{\alpha_1+\alpha_2}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}$, $\alpha = (\alpha_1, \alpha_2)$ and $|\alpha| := \alpha_1 + \alpha_2$. Here the differential operators D^α are defined in the sense of distributions in Ω. We also use $\mathcal{D}(\Omega)$ to denote the set of smooth functions on Ω with compact support and endow it with the (standard) inductive topology as defined in [13, Eq. (1.5), p. 4]. Let $\mathcal{D}'(\Omega)$ be the dual of $\mathcal{D}(\Omega)$ endowed with the strong dual topology; it is called the space of distributions on Ω.

Let $\rho : \overline{\Omega} \to \mathbb{R}^+$ be a $C^2(\overline{\Omega})$ function, positive on Ω and equal to the distance function to $\partial\Omega$ ($\rho(x) = d(x, \partial\Omega)$) for $d(x, \partial\Omega)$ small enough. Such a function ρ exists as (essentially) noted in [13], p. 183.

According to [13, Définition 6.1, p. 183], set

$$\Xi^2(\Omega) := \{u \in L^2(\Omega) \,|\, \rho^{|\alpha|} D^\alpha u \in L^2(\Omega),\, |\alpha| \leq 2\}, \tag{84}$$

equipped with the norm $\|u\|_{\Xi^2(\Omega)} = \sum_{|\alpha|\leq 2} \|\rho^{|\alpha|} D^\alpha u\|_{L^2(\Omega)}$. Then, $\Xi^2(\Omega)$ is a Hilbert space so that $H^2(\Omega) \subset \Xi^2(\Omega) \subset L^2(\Omega)$ and $\mathcal{D}(\Omega)$ is dense in $\Xi^2(\Omega)$ ([13, Proposition 6.2, p. 183]). Therefore, the dual space of $\Xi^2(\Omega)$ is itself a space of distributions and we set $\Xi^{-2}(\Omega) := \left(\Xi^2(\Omega)\right)'$. In fact, the exponent 2 in $\Xi^2(\Omega), \Xi^{-2}(\Omega)$ can be replaced by any positive real number $s > 0$ with the same conclusions (see Eqs. 6.20 and 6.21, p. 184, [13]).

Next, for $s \in [0, 2)$, define $D^s(\Omega)$ as the space of functions $u \in H^s(\Omega)$ such that $(\Delta_\Omega^N + \lambda)u$ belongs to $\Xi^{s-2}(\Omega)$, (see bottom of p. 199 and Eq. 6.24, p. 185, of [13]). By [13, Théorème 7.2 , p.200], we have the following interpolation result

$$D^{1/2}(\Omega) = [H^2(\Omega), D^0(\Omega)]_{3/4},$$

where the definition of an intermediary space $[\cdot,\cdot]_\theta$, $\theta \in [0, 1]$, is given in [13, Définition 2.1, p. 12].

We can now apply [13, Théorème 7.4 , p.202] and get the following proposition.

Proposition 2. *For every $\varphi \in L^2(\Omega)$, there exists a unique solution*

$$(w'(u), S'(u), \lambda'(u)) \in D^{1/2}(\Omega) \times \left(D^{1/2}(\Omega)\right)^2 \times \mathbb{R}$$

for the equations (58), (59), (60) and (79), (80), (81), so that (61) and (83) hold true. By solution, we mean that

- **(i):** *Eqs. (58) and (79) are verified in the distributional sense in Ω;*
- **(ii):** *Eqs. (59) and (80) are verified in the sense of the trace theorem* [13, Théorème 7.3, p.201]*;*
- **(iii):** *The mapping $\varphi \mapsto (w'(u), S'(u), \lambda'(u))$ is continuous;*
- **(iv):** *Eq. (83) holds in $\left(H^{-1}(\partial\Omega)\right)^2$, in the sense of the trace theorem* [13, Théorème 7.3, p.201].

Proof of Proposition 2: We first need to be precise with the regularity for w and S as defined respectively as a normalized eigenfunction of Δ_Ω^N and by (71). Applying [8, Theorem 6.31 , p.128] and the remark which immediately follows it, one gets that both w and S belong to the Hölder spaces $C^{2,\alpha}(\overline{\Omega})$, for every $\alpha \in (0, 1)$.

It is clear that all the hypotheses to apply [13, Théorème 7.3, p.201] are verified and thus we can apply them with $s = 1/2$, $m = m_0 = 1$, $A = \Delta + \lambda$, $B_0 = \frac{\partial}{\partial\nu}$.

We first have to check that the right-hand sides of Eqs. (58) and (79) belong to $\Xi^{-3/2}$, which is true since $C^2(\overline{\Omega}) \subset H^2(\Omega)$ and

$$H^{3/2}(\Omega) \subset \Xi^{3/2}(\Omega) \subset L^2(\Omega) \subset \Xi^{-3/2},$$

where the last two inclusions are consequences of respectively [13, Eq. (6.19), p.183] and [13, Eq. (6.22), p.184].

Items (i), (ii) and (iii) follow by essentially translating the contents of [13, Théorème 7.3,p.201] and [13, Théorème 7.4 , p.202] with the notations of our problem. The uniqueness part of the proposition is a consequence of Eqs. (60) and (81) and the fact that the kernel of the operator $\Delta_\Omega^N + \lambda$ is generated by w (since λ is a simple eigenvalue of $-\Delta_\Omega^N$).

We now turn to an argument for item (iv). As a consequence of items (i) and (ii), we first notice that, for every $\varphi \in L^2(\partial\Omega)$, every term in both sides of the equality (83) belong to $\big(H^{-1}(\partial\Omega)\big)^2$. Then, recall that, for every $\varphi \in C^2(\partial\Omega)$, (83) holds pointwise on $\partial\Omega$. Therefore, to establish the result, it is enough to prove that, for every sequence $(\varphi_n)_{n\geq 0}$ of $L^2(\partial\Omega)$ converging (in $L^2(\partial\Omega)$) to φ and so that (83) holds in $\big(H^{-1}(\partial\Omega)\big)^2$ for every φ_n, then (83) holds in $\big(H^{-1}(\partial\Omega)\big)^2$ for φ. By using (iii) and standard arguments, we conclude. □

Having at hand the previous proposition, we next choose appropriate variations φ to deduce additional pointwise information on $\frac{\partial w}{\partial \tau}$ and $\frac{\partial S}{\partial \tau}$. More precisely, we will prove that

Lemma 9. *With the above notations, we have on $\partial\Omega$,*

$$\frac{\partial w}{\partial \tau}\big(\frac{\partial S}{\partial \tau} - \tau\big) = 0. \tag{85}$$

Proof of Lemma 9. We fix a point $a \in \partial\Omega$ and set

$$w_t := \frac{\partial w}{\partial \tau}(a).$$

Up to a translation and an orthogonal transformation of Ω, we may assume that $a = 0 \in \mathbb{R}^2$ and $\nu(0) = (0,1)^{\mathrm{tr}}$ (thus $\tau(0) = (-1,0)^{\mathrm{tr}}$). Let Γ_0 be the connected component of $\partial\Omega$ containing 0. Let σ be the arc-length of a point of Γ_0 such that $\sigma = 0$ at 0 and $\sigma \in [-L/2, L/2)$. Choose the variation φ so that $\varphi(\sigma) = 1$ if $\sigma \in [0, L/4]$, $\varphi(\sigma) = 0$ if $\sigma \in [-L/4, 0)$ and smooth on $\Gamma_0 \setminus \{0\}$.

Consider the fundamental solution of Laplace's equation, i.e., the function $\psi : \Omega \to \mathbb{R}$ defined by

$$\psi(x,y) = -\frac{1}{2\pi}\ln(x^2+y^2).$$

Then, we have the following lemma, whose proof is deferred to the end of the paper.

Lemma 10. *The function ψ belongs to $D^{1/2}(\Omega)$ and $\Delta\psi = 0$ in the distributional sense in Ω. Moreover, there exist two functions T_t, T_n in $L^2(\partial\Omega)$ so that, on $\partial\Omega$,*

$$\frac{\partial \psi}{\partial \nu} = \delta_0 + T_n, \tag{86}$$

$$\frac{\partial \psi}{\partial \tau} = -\frac{1}{\pi} p.v.(\frac{1}{\sigma})\xi(\sigma) + T_t, \tag{87}$$

where δ_0 denotes the Dirac measure at 0, $p.v.(\frac{1}{\sigma})$ denotes the principal value of the function $\frac{1}{\sigma}$ and ξ is a smooth function compactly supported in $[-3L/8, 3L/8]$ so that $\xi \equiv 1$ on $[-L/4, L/4]$.

We proceed with the proof of Lemma 9 assuming that Lemma 10 holds true.

Consider the function $\tilde{w}'(u) := w'(u) - w_t\psi$. Then, clearly,

$$\Delta\tilde{w}'(u) + \lambda\tilde{w}'(u) = -\lambda'(u)w - \lambda w_t\psi,$$

belongs to $H^{1/2}(\Omega) \subset \Xi^{-1/2}(\Omega)$. Moreover,

$$\frac{\partial \varphi}{\partial \tau} = \delta_0 + \phi_R,$$

where ϕ_R is a periodic smooth function on $[-L/2, L/2]$. Then,

$$\frac{\partial \tilde{w}'(u)}{\partial \nu} \in L^2(\partial\Omega), \tag{88}$$

thanks to (59) and (86). Applying [13, Théorème 7.4 , p.202] with $s = 3/2$ and the other data unchanged ($m = m_0 = 1$, $A = \Delta + \lambda$, $B_0 = \frac{\partial}{\partial \nu}$), we get that $\tilde{w}'(u)$ belongs to $D^{3/2}(\Omega)$. With $\tilde{w}'(u)$ at hand, we apply [13, Théorème 7.3 , p.201] with the same data as before except that B_0 stands now for the Dirichlet boundary condition on $\partial\Omega$. We deduce that the trace of $\tilde{w}'(u)$ to $\partial\Omega$ belongs to $H^1(\partial\Omega)$, implying that

$$\frac{\partial \tilde{w}'(u)}{\partial \tau} \in L^2(\partial\Omega). \tag{89}$$

From (87) and (89), it follows that

$$\frac{\partial w'(u)}{\partial \tau} + \frac{1}{\pi} w_t \xi(\sigma) p.v.(\frac{1}{\sigma}) \in L^2(\Gamma_0). \tag{90}$$

Set

$$S_t := \frac{\partial S}{\partial \tau}(0,0).$$

Proceeding similarly with $S'(u)$, we obtain that

$$\frac{\partial S'(u)}{\partial \tau} + \frac{1}{\pi}\xi(\sigma) p.v.(\frac{1}{\sigma})\big(S_t - \tau(0,0)\big) \in \big(L^2(\Gamma_0)\big)^2. \tag{91}$$

Plugging (90) and (91) in (83), we deduce that

$$w_t\big(S_t - \tau(0,0)\big)\xi(\sigma) p.v.(\frac{1}{\sigma}) \in \big(L^2(\Gamma_0)\big)^2.$$

Since $\xi(\sigma) p.v.(\frac{1}{\sigma})$ does not belong to $L^2(\Gamma_0)$, then (85) holds, i.e, Lemma 9 is proved. □

End of the proof of Lemma 8. By (85), then (73) reduces to

$$\lambda w S \equiv 0, \text{ on } \partial\Omega.$$

If $S(x_0) \neq 0$ for some $x_0 \in \partial\Omega$, then w is equal to zero on an open subset of $\partial\Omega$, which is impossible by the Holmgren uniqueness theorem. Therefore, $S = 0$ on $\partial\Omega$.

The following lemma tells us that an S subject to (71) and (73) does not exist. In that case, (68) cannot hold true, which in turn, yields the existence of domains $\Omega(u)$ arbitrary close to Ω, for which

$$\int_{\partial\Omega} w_{l+1}(u) \nu d\sigma \neq 0, \tag{92}$$

i.e., $\mathcal{R}_{l+1}$ is dense in $\mathcal{H}_l \cap \mathcal{K}_{l+1}$.

Lemma 11. *With the above notations, there is no solution to the following overdetermined eigenvalue problem*

$$\begin{cases} (-\Delta - \lambda)S = 0, & \text{in } \Omega, \\ \frac{\partial S}{\partial \nu} = \nu, & \text{on } \partial\Omega, \\ S = 0, & \text{on } \partial\Omega. \end{cases} \tag{93}$$

Proof of Lemma 11. For every vector $a \in \mathbb{R}^2$, $S \cdot a$ is solution of the Dirichlet problem

$$\begin{cases} (-\Delta - \lambda)y = 0, & \text{in } \Omega, \\ y = 0, & \text{on } \partial\Omega. \end{cases} \tag{94}$$

Since $S \neq 0$, then there exists $a \in \mathbb{R}^2$ such that $S \cdot a$ is a nonzero solution of (94). Therefore λ is also an eigenvalue of $-\Delta_\Omega^D$. As in the proof of Lemma 3, one gets that there exists a nonzero vector $a \in \mathbb{R}^2$ such that $S = az$, where z is a nonzero eigenfunction of $-\Delta_\Omega^D$ associated to λ. Up to a rotation, one concludes that $S = (\|a\|z, 0)^{\mathrm{tr}}$, and then ν is constant on each connected component of $\partial\Omega$ and equal to $(1,0)^{\mathrm{tr}}$ or $-(1,0)^{\mathrm{tr}}$. This yields a contradiction with the boundedness of Ω.

Proof of Lemma 10. We first prove $\Delta\psi = 0$, in the distributional sense in Ω, together with (86). It amounts to show the existence of a function $T_n \in L^2(\partial\Omega)$ such that, for every function $\Phi \in C^2(\overline{\Omega})$ Stokes' formula holds true, i.e.

$$I := \int_\Omega \psi\Delta\Phi dx - \int_{\partial\Omega} \psi\frac{\partial\Phi}{\partial\nu} d\sigma = -\phi(0) - \int_{\partial\Omega} T_n \Phi d\sigma. \tag{95}$$

For $\varepsilon > 0$, consider the function ψ_ε defined on $\mathbb{R}^2$ by

$$\psi_\varepsilon(x,y) = -\frac{1}{2\pi}\ln(x^2+y^2+\varepsilon^2).$$

Note that

$$\|\nabla\psi_\varepsilon\| \le C\frac{|x|+|y|}{x^2+y^2+\varepsilon^2}, \tag{96}$$

where C is a positive constant independent of $(x,y) \in \mathbb{R}^2$, and

$$\Delta\psi_\varepsilon = \frac{\varepsilon^2}{\pi(x^2+y^2+\varepsilon^2)^2}. \tag{97}$$

Set

$$I_\varepsilon := \int_\Omega \psi_\varepsilon \Delta\Phi dx - \int_{\partial\Omega} \psi_\varepsilon \frac{\partial\Phi}{\partial\nu} d\sigma.$$

Let us first prove that I_ε tends to I as ε tends to zero.

The proof consists of computing the limit, as ε tends to zero, of several integrals over Ω or $\partial\Omega$. Let Ω_0 be the bounded connected component of Ω, with boundary Γ_0, the connected component of $\partial\Omega$ containing $(0,0)$. From (96) and (97), it is clear that possible difficulties regarding the convergence of the integrals only occur in the neighborhood of the point $(0,0)$. It is then immediate that

$$\lim_{\varepsilon\to 0}\int_{\Omega\setminus\Omega_0} \psi_\varepsilon\Delta\Phi dx - \int_{\partial\Omega\setminus\Gamma_0} \psi_\varepsilon\frac{\partial\Phi}{\partial\nu} d\sigma = \int_{\Omega\setminus\Omega_0} \Delta\Phi dx - \int_{\partial\Omega\setminus\Gamma_0} \psi\frac{\partial\Phi}{\partial\nu} d\sigma.$$

It remains to prove a similar result for Ω_0 and Γ_0. In the sequel, it is therefore enough to assume that Ω is simply connected, i.e., Ω reduces to Ω_0.

It is also clearly enough to prove that the two integrals

$$I_\varepsilon^1 := \int_\Omega |(\psi-\psi_\varepsilon)(x,y)| dxdy \text{ and } I_\varepsilon^2 := \int_{\partial\Omega} |(\psi-\psi_\varepsilon)(\sigma)| d\sigma$$

tend to zero as ε tends to zero. For $(x,y) \in \Omega$,

$$|(\psi-\psi_\varepsilon)(x,y)| = \frac{1}{2\pi}\ln(1+\frac{\varepsilon^2}{x^2+y^2}).$$

In both integrals, we perform the change of variable $(x', y') = \frac{1}{\varepsilon}(x, y)$ and denote by Ω_ε and $\partial\Omega_\varepsilon$ respectively the domain $\frac{1}{\varepsilon}\Omega$ and its boundary. Then, we get

$$I_\varepsilon^1 \leq C_1\varepsilon^2 \int_{\Omega_\varepsilon} \ln(1 + \frac{1}{x^2 + y^2})dxdy \text{ and } I_\varepsilon^2 \leq C_2\varepsilon \int_{\partial\Omega_\varepsilon} \ln(1 + \frac{1}{x^2 + y^2})d\sigma,$$

where C_1, C_2 are positive constants independent on ε. Since Ω_ε is contained in an open ball of radius R_0/ε, with R_0 independent on ε, it is easy to see that

$$I_\varepsilon^1 \leq C_1'\varepsilon^2 \int_0^{R_0/\varepsilon} \ln(1 + \frac{1}{r^2})rdr \text{ and } I_\varepsilon^2 \leq C_2'\varepsilon \int_0^{R_0/\varepsilon} \ln(1 + \frac{1}{\sigma^2})d\sigma,$$

where C_1', C_2' are positive constants independent on ε. For I_ε^1, we used polar coordinates and for I_ε^2, we simply noticed that that $\lim_{\sigma\to 0^+} \frac{\sigma + x(\sigma)}{\sigma} = 0$ (see below eq. (98)). Since $\lim_{r\to 0} r\ln(1 + \frac{1}{r^2}) = 0$ and $r\ln(1 + \frac{1}{r^2}) \sim_{r\to\infty} 1/r$, we get that $\int_0^{R_0/\varepsilon} \ln(1 + \frac{1}{r^2})rdr$ is equivalent to $\ln(R_0/\varepsilon)$ as ε tends to zero. Therefore $\lim_{\varepsilon\mapsto 0} I_\varepsilon^1 = 0$. As for I_ε^2, the same conclusion holds by noticing that $\sigma \to \ln(1+\frac{1}{\sigma^2})$ is integrable at zero and infinity.

We next compute the limit of I_ε as ε tends to zero. Applying Stokes' formula in $\overline{\Omega}$ for ψ_ε, we get

$$I_\varepsilon = \int_\Omega \Phi\Delta\psi_\varepsilon - \int_{\partial\Omega} \Phi\frac{\partial\psi_\varepsilon}{\partial\nu}d\sigma.$$

Set $I_\varepsilon := J_\varepsilon - K_\varepsilon$, where

$$J_\varepsilon := \int_\Omega \Phi\Delta\psi_\varepsilon dx \text{ and } K_\varepsilon := \int_{\partial\Omega} \Phi\frac{\partial\psi_\varepsilon}{\partial\nu}d\sigma.$$

For the convergence of K_ε to $\int_{\partial\Omega} T_n\Phi d\sigma$, it is enough to show that $\|\frac{\partial\psi_\varepsilon}{\partial\nu}\|_{L^2(\partial\Omega)}$ is bounded by a positive constant independent of ε. We actually show that $\frac{\partial\psi_\varepsilon}{\partial\nu}$ is bounded over $\partial\Omega$ in L^∞-norm, uniformly with respect to ε small enough. It is clear that it amounts to do the job in a neighborhood of zero, since it is only there where $x^2 + y^2$ may be dominated by ε^2 in the denominator of ψ_ε.

The key point is to get the Taylor expansion of the arc-length σ in a neighborhood of zero. Such a neighborhood only depends on the geometry of Ω near zero and not on ε. After standard computations, it follows that there exists a neighborhood $\mathcal{N}_0$ of 0 ($\in \mathbb{R}$) such that, we have, for $\sigma \in \mathcal{N}_0$,

$$x(\sigma) = -\sigma + O(\sigma^3), \tag{98}$$

$$y(\sigma) = \kappa x^2(\sigma) + 0(\sigma^3), \tag{99}$$

$$\nu(\sigma) = \nu(0) + x\kappa\tau(0) + O(\sigma^3), \tag{100}$$

$$\tau(\sigma) = \tau(0) - x\kappa\nu(0) + O(\sigma^2), \tag{101}$$

where κ is the geodesic curvature of $\partial\Omega$ at zero and $O(\sigma^j)$, $j = 2, 3$, are functions of σ dominated by σ^j, $j = 2, 3$, uniformly over $\mathcal{N}_0$. From the equation

$$\frac{\partial\psi_\varepsilon}{\partial\nu} = \nabla\psi_\varepsilon(\sigma) \cdot \nu(\sigma),$$

we get, by using (96) and (100), that

$$\frac{\partial\psi_\varepsilon}{\partial\nu} = \frac{\partial\psi_\varepsilon}{\partial y} - \kappa x\frac{\partial\psi_\varepsilon}{\partial x} + g(x, y),$$

where $g(x,y)$ is bounded above by the product of x^2+y^2 and $\|\nabla\psi_\varepsilon\|$. It is easy to see that each term of the right-hand side of the previous equation is bounded above over $\mathcal{N}_0$, uniformly with respect to ε. For instance, from (99), we have

$$|\frac{\partial\psi_\varepsilon}{\partial y}| \le C\frac{|y|}{x^2+y^2+\varepsilon^2} \le C\frac{x^2}{x^2+y^2+\varepsilon^2} \le C.$$

It remains to compute the limit, as ε tends to zero, of J_ε. We have

$$J_\varepsilon = -\frac{2}{\pi}\int_\Omega \frac{\varepsilon^2}{(x^2+y^2+\varepsilon^2)^2}\Phi(x,y)dxdy.$$

Performing the change of variable $(x',y') = \frac{1}{\varepsilon}(x,y)$, we get

$$J_\varepsilon = -\frac{2}{\pi}\int_{\Omega_\varepsilon} \frac{1}{(x^2+y^2+1)^2}\Phi(\varepsilon x,\varepsilon y)dxdy.$$

Set $J_\varepsilon := -\frac{2}{\pi}(\Phi(0,0)J_\varepsilon^1 + J_\varepsilon^2)$, where

$$J_\varepsilon^1 = \int_{\Omega_\varepsilon} \frac{1}{(x^2+y^2+1)^2}dxdy, \tag{102}$$

$$J_\varepsilon^2 = \int_{\Omega_\varepsilon} \frac{1}{(x^2+y^2+1)^2}\big(\Phi(\varepsilon x,\varepsilon y)-\Phi(0,0)\big)dxdy. \tag{103}$$

We first prove that $\lim_{\varepsilon\to 0}|J_\varepsilon^2| = 0$. We use $B_R(x_0)$ to denote the ball of center x_0 and radius $R \ge 0$. For every $\varepsilon > 0$ small enough, fix $R(\varepsilon)$ such that $\lim_{\varepsilon\to 0} R(\varepsilon) = \infty$, and if

$$\rho_\varepsilon := \max_{B_{R(\varepsilon)}(0)} |\Phi(\varepsilon x,\varepsilon y)-\Phi(0,0)|,$$

then $\lim_{\varepsilon\to 0}\rho(\varepsilon) = 0$. Set $J(x,y) := \frac{|\Phi(\varepsilon x,\varepsilon y)-\Phi(0,0)|}{(x^2+y^2+1)^2}$. Then, we have

$$|J_\varepsilon^2| \le \int_{\Omega\cap B_{R(\varepsilon)}(0)} J(x,y)dxdy + \int_{\Omega\setminus B_{R(\varepsilon)}(0)} J(x,y)dxdy. \tag{104}$$

The first integral J_1^2 in the above right-hand side is smaller than

$$\rho(\varepsilon)\int_{B_{R(\varepsilon)}(0)} \frac{1}{(x^2+y^2+1)^2}dxdy.$$

By taking polar coordinates, one can see that J_1^2 is itself smaller than $\pi\rho(\varepsilon)$.

Set $M_\Phi := \max_{(x,y)\in\Omega}|\Phi(x,y)-\Phi(0,0)|$. For J_2^2, the second integral in the right-hand side of (104), we have

$$J_2^2 \le M_\Phi\int_{\Omega\setminus B_{R(\varepsilon)}(0)} \frac{1}{(x^2+y^2+1)^2}dxdy \le \int_{\mathbb{R}^2\setminus B_{R(\varepsilon)}(0)} \frac{1}{(x^2+y^2+1)^2}dxdy,$$

the last integral being equal to $\frac{\pi M_\Phi}{1+R_\varepsilon^2}$. Gathering the previous estimates, one gets that $\lim_{\varepsilon\to 0}|J_\varepsilon^2| = 0$.

We next prove that $\lim_{\varepsilon\to 0} J_\varepsilon^1 = \frac{\pi}{2}$, i.e.,

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \frac{1}{(x^2+y^2+1)^2}dxdy = \frac{\pi}{2}.$$

Let $\Omega^+ := \Omega\cap H^+$ and $\Omega^- := \Omega\cap H^-$, where H^+ (H^-) is the closed upper (closed lower) half-plane. Because of the geometric assumptions on Ω, then $H^+\setminus\Omega$ contains a closed ball $B^+ := B_{r^+}(0,r^+)$, $r^+ > 0$. On the other hand, $\overline{\Omega}\cap H^-$ contains a closed ball $B^- := B_{r^-}$ with center $(0,-r^-)$, $r^- > 0$. Note that both B^+ and B^- contain zero on their respective boundaries. After performing the dilation by $1/\varepsilon$, then $\Omega_\varepsilon^+ := \Omega_\varepsilon \cap H^+$ lies in $H^+ \backslash B_\varepsilon^+$, with $B_\varepsilon^+ := B_{r^+/\varepsilon}(0, r^+/\varepsilon)$, and $\Omega_\varepsilon^- := \Omega_\varepsilon \cap H^-$ contains $B_\varepsilon^- := B_{r^-/\varepsilon}(0, r^-/\varepsilon)$. Setting $J_\varepsilon^1 := J_\varepsilon^{1,+} + J_\varepsilon^{1,-}$ with

$$J_\varepsilon^{1,+} = \int_{\Omega_\varepsilon^+} \frac{1}{(x^2+y^2+1)^2} dxdy,$$

and

$$J_\varepsilon^{1,-} = \int_{\Omega_\varepsilon^-} \frac{1}{(x^2+y^2+1)^2} dxdy,$$

we deduce that

$$J_\varepsilon^{1,+} \leq \int_{H^+ \backslash B_\varepsilon^+} \frac{1}{(x^2+y^2+1)^2} dxdy, \tag{105}$$

and

$$\int_{B_\varepsilon^-} \frac{1}{(x^2+y^2+1)^2} dxdy \leq J_\varepsilon^{1,-} \leq \int_{H^-} \frac{1}{(x^2+y^2+1)^2} dxdy. \tag{106}$$

By taking polar coordinates, one can see that the right-hand side of (105) is equal to

$$2 \int_0^{\pi/2} \frac{d\theta}{1 + (r^+ \sin(\theta)/\varepsilon)^2}.$$

Since $\sin(\theta) \geq 2\theta/\pi$ on $[0, \pi/2]$, we end up with $J_\varepsilon^{1,+} \leq C\varepsilon$, for some positive constant independent of ε. Therefore, $\lim_{\varepsilon \to 0} J_\varepsilon^{1,+} = 0$.

In (106), we take polar coordinates in the left and right integrals. The left one is equal to

$$\pi/2 - 1/2 \int_0^{\pi/2} \frac{d\theta}{1 + (r^+ \sin(\theta)/\varepsilon)^2},$$

and the right integral is equal to $\pi/2$. By letting ε tend to zero, we obtain that $J_\varepsilon^{1,+}$ tends to $\pi/2$.

Finally, we turn to the proof of (87). Let ξ be a smooth periodic function on Γ_0 such that $\xi(\sigma) = 1$ if $|\sigma| \leq L/4$ and $\xi(\sigma) = 0$ if $|\sigma| \geq 3L/8$. Consider the function $k : \Gamma_0 \setminus \{0\} \to \mathbb{R}$ defined by

$$k(\sigma) = \psi(x(\sigma), y(\sigma)) + \frac{1}{\pi} \ln(|\sigma|)\xi(\sigma).$$

Let us prove that k can be extended by continuity at $\sigma = 0$ as a C^1 function on $\partial\Omega$. It is clear that the only possible problem occurs at $\sigma = 0$. For σ small enough and different from zero, we have

$$k(\sigma) = \frac{2}{\pi} \ln(\frac{\sigma^2}{x^2(\sigma) + y^2(\sigma)}),$$

implying that k tends to zero as σ tends to zero. Therefore, k can be extended by continuity at $\sigma = 0$. We next remark that, for $x \neq 0$, then

$$|\frac{x}{x^2+y^2} - \frac{1}{x}| = \frac{y^2}{|x|(x^2+y^2)}.$$

On $\mathcal{N}_0$, $\frac{y^2}{|x|(x^2+y^2)} \leq 2\kappa^2|\sigma|$. Writing $k'(\sigma)$ on $\mathcal{N}_0 \setminus \{0\}$ as

$$k'(\sigma) = \frac{\partial\psi}{\partial\tau} + \frac{1}{\pi\sigma} = (\tau(\sigma) - \tau(0)) \cdot \nabla\psi + \frac{\partial\psi}{\partial x} + \frac{1}{\pi\sigma},$$

and by using (99) and (101), we have, for $\sigma \in \mathcal{N}_0$ and $\sigma \neq 0$,

$$|\frac{\partial\psi}{\partial\tau} + \frac{1}{\pi\sigma}| \leq C|\sigma|,$$

for some positive constant C independent of $\sigma \in \mathcal{N}_0$. ¿From the previous equations, we deduce that k' tends to zero as σ goes to zero. Therefore k is a C^1 function on $\partial\Omega$, implying that k' is continuous on $\partial\Omega$.

Using finally that $p.v.(\frac{1}{\sigma})$ is the distributional derivative of the locally integrable function over $\mathbb{R}$, $\sigma \mapsto \ln(|\sigma|)$ (see for instance [7, Ex. 6, p. 328]), we immediately get (87). □

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