

## GLOBAL STEADY-STATE CONTROLLABILITY OF ONE-DIMENSIONAL SEMILINEAR HEAT EQUATIONS\*

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**Abstract.** We investigate the problem of exact boundary controllability of semilinear one-dimensional heat equations. We prove that it is possible to move from any steady-state to any other by means of a boundary control, provided that both are in the same connected component of the set of steady-states. The proof is based on an effective feedback stabilization procedure, which is implemented.

**Key words.** heat equation, controllability, pole shifting, Lyapunov functional

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### 1. Introduction.

**1.1. Statement of the main result.** Let  $L > 0$  be fixed and  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function of class  $C^2$ . Let us consider the boundary control system

$$(1.1) \quad \begin{cases} \frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} + f(y), \\ y(t, 0) = 0, \quad y(t, L) = u(t), \end{cases}$$

where the state is  $y(t, \cdot) : [0, L] \rightarrow \mathbb{R}$  and the control is  $u(t) \in \mathbb{R}$ .

Concerning the global controllability problem, one of the main results [5] asserts that if  $f$  is globally Lipschitzian, then this control system is approximately globally controllable (see also [11] for exact controllability). When  $f$  is superlinear, the situation is still widely open, in particular because of possible blowing up. Indeed, it is well known that if  $yf(y) > 0$  as  $y \neq 0$ , then blow-up phenomena may occur for the Cauchy problem

$$(1.2) \quad \begin{cases} \frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} + f(y), \\ y(t, 0) = 0, \quad y(t, L) = 0, \\ y(0, x) = y_0(x). \end{cases}$$

For example, if  $f(y) = y^3$ , then for numerous initial data there exists  $T > 0$  such that the unique solution to the previous Cauchy problem is well defined on  $[0, T) \times [0, L]$  and satisfies

$$\lim_{t \rightarrow T} \|y(t, \cdot)\|_{L^\infty(0, L)} = +\infty$$

(see, for instance, [1, 8, 2, 12, 14, 15, 18] and references therein).

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One may ask if, acting on the boundary of  $[0, L]$ , one could avoid the blow-up phenomenon. Actually the answer to this question is negative in general (see [7]; see also [6] for a weaker nonlinearity): for some nonlinear functions  $f$  satisfying

$$|f(y)| \sim |y| \log^p(1 + |y|) \quad \text{as } |y| \rightarrow +\infty,$$

with  $p > 2$ , and for any time  $T > 0$ , there exist initial data which lead to blow-up before time  $T$ , whatever the control function  $u$  is. Notice, however, that if

$$|f(y)| = o\left(|y| \log^{3/2}(1 + |y|)\right) \quad \text{as } |y| \rightarrow +\infty,$$

then the blow-up (which could occur in the absence of control) can be avoided by means of boundary control (see [7]).

Nevertheless, in the first case where the blow-up phenomenon cannot be compensated by means of boundary control, the situation is not completely desperate. In fact, as we shall see in this paper, we can move from any given steady-state to any other belonging to the same connected component of the set of steady-states. More precisely, let us define the notion of steady-state.

DEFINITION 1.1. *A function  $y \in C^2([0, L])$  is a steady-state of the control system (1.1) if*

$$\frac{d^2y}{dx^2} + f(y) = 0, \quad y(0) = 0.$$

We denote by  $\mathcal{S}$  the set of steady-states endowed with the  $C^2$  topology.

Let us also introduce the Banach space

$$(1.3) \quad Y_T = \left\{ y(t, x), (t, x) \in (0, T) \times (0, L) / y \in L^2(0, T, W^{2,2}(0, L)) \right. \\ \left. \text{and } \frac{\partial y}{\partial t} \in L^2((0, T) \times (0, L)) \right\}$$

endowed with the norm

$$\|y\|_{Y_T} = \|y\|_{L^2(0, T, W^{2,2}(0, L))} + \left\| \frac{\partial y}{\partial t} \right\|_{L^2((0, T) \times (0, L))}.$$

Notice that  $Y_T$  is continuously imbedded in  $L^\infty((0, T) \times (0, L))$ .

The main result of the paper is the following.

THEOREM 1.2. *Let  $y_0$  and  $y_1$  be two steady-states belonging to a same connected component of  $\mathcal{S}$ . There exist a time  $T > 0$  and a control function  $u \in L^2(0, T)$  such that the solution  $y(t, x)$  in  $Y_T$  of*

$$(1.4) \quad \begin{cases} \frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} + f(y), \\ y(t, 0) = 0, \quad y(t, L) = u(t), \\ y(0, x) = y_0(x) \end{cases}$$

satisfies  $y(T, \cdot) = y_1(\cdot)$ .

Remark 1.3. In fact, we prove the following result: for all neighborhood  $V$  of  $y_1$  in  $H^1$ -topology, there exists a positive real number  $\varepsilon_0$  such that for all  $\varepsilon \in (0, \varepsilon_0)$

there exists a control function  $u \in H^1(0, 1/\varepsilon)$  such that the solution  $y(t, x)$  in  $Y_T$  of the Cauchy–Dirichlet problem (1.4) satisfies  $y(1/\varepsilon, \cdot) \in V$ .

In the proof of this result, which represents the main part of the paper, we give an explicit construction of the control  $u$  in a *feedback-type form* and also of a *Lyapunov functional*. We stress that the procedure is effective and consists actually of solving a stabilization problem in finite dimension. Indeed, in order to construct  $u$  we need to compute only a finite number of quantities related to a Hilbertian expansion of the solution. The procedure has been implemented numerically, and simulations are presented in the last section of the paper.

*Remark 1.4.* For any  $T > 0$  and  $u \in L^2(0, T)$ , there is at most one solution of (1.4) in the Banach space  $Y_T$ .

*Remark 1.5.* This is a (partial) global exact controllability result. The time needed in our proof is large, but on the other hand there are indeed cases where the time  $T$  of controllability cannot be taken arbitrarily small. For instance, in the case where  $f(y) = -y^3$ , any solution of (1.4) starting from 0 satisfies the inequality

$$\int_0^L (L - x)^4 y(T, x)^2 dx \leq 8LT,$$

and hence, if  $y_0 = 0$ , a minimal time is needed to reach a given  $y_1 \neq 0$ . This result is due to Bamberger [10] (see also [9, Lemma 2.1]).

*Remark 1.6.* In section 3 we prove that if  $y_0$  and  $y_1$  belong to distinct connected components of  $\mathcal{S}$ , then it is actually impossible to move either from  $y_0$  to  $y_1$  or from  $y_1$  to  $y_0$ , whatever the time and the control are. In the same section we also investigate the connectedness of the set  $\mathcal{S}$  of steady-states.

*Remark 1.7.* The result of Theorem 1.2 may be achieved directly by using repeatedly a local exact controllability theorem (see [9, Theorem 4.4] or [11, Theorem 3.3]). Here we present a new controllability strategy based on a feedback stabilization procedure, which is more effective. It is clear also that this approach may be applied to other problems without requiring controllability of the linearized system around an equilibrium (see [3]).

**1.2. The idea of the proof.** The method we shall use to prove Theorem 1.2 stems from classical Lyapunov stability theory together with quasi-static deformation theory. For the sake of simplicity we explain it in finite dimension. Let us consider in  $\mathbb{R}^n$  a general control system of the form

$$(1.5) \quad \dot{y}(t) = g(y(t), u(t)),$$

where  $g : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is of class  $C^1$ ,  $u(t) \in \mathcal{U}$ , and  $\mathcal{U}$  denotes the set of measurable essentially bounded admissible controls. Let  $y_0, y_1 \in \mathbb{R}^n$  be two equilibrium points of system (1.5); that is,

$$g(y_i, u_i) = 0, \quad i = 0, 1,$$

for some  $u_0, u_1 \in \mathbb{R}^m$ . We assume that  $(y_0, u_0)$  and  $(y_1, u_1)$  belong to the same connected component of the zero set of  $g$  in  $\mathbb{R}^n \times \mathbb{R}^m$ . Our aim is to steer the system from  $y_0$  to  $y_1$  in some (large) time  $T > 0$ . The method splits into four steps.

**First step.** Construct a  $C^1$ -path  $(\bar{y}(\tau), \bar{u}(\tau))$ , with  $\tau \in [0, 1]$ , connecting  $(y_0, u_0)$  to  $(y_1, u_1)$  and such that

$$\forall \tau \in [0, 1], \quad g(\bar{y}(\tau), \bar{u}(\tau)) = 0.$$

Of course, this path is not in general a solution of system (1.5), but if  $\varepsilon > 0$  is small enough, then the  $C^1$ -path  $(y^\varepsilon, u^\varepsilon)$

$$\begin{aligned} [0, 1/\varepsilon] &\rightarrow \mathbb{R}^n \times \mathbb{R}^m, \\ t &\mapsto (y^\varepsilon(t), u^\varepsilon(t)) = (\bar{y}(\varepsilon t), \bar{u}(\varepsilon t)) \end{aligned}$$

is “almost” a solution of system (1.5). Indeed,

$$\|\dot{y}^\varepsilon - g(y^\varepsilon, u^\varepsilon)\| = O(\varepsilon).$$

**Second step.** This quasi-static trajectory is not in general stable, and thus has to be stabilized. To this aim, introduce the following change of variable:

$$\begin{aligned} z(t) &= y(t) - y^\varepsilon(t), \\ v(t) &= u(t) - u^\varepsilon(t), \end{aligned}$$

where  $t \in [0, 1/\varepsilon]$ . In the new variables  $z, v$ , the control system writes, at least if  $\|z(t)\| + \|v(t)\|$  is small enough,

$$\dot{z}(t) = A(\varepsilon t)z(t) + B(\varepsilon t)v(t) + O(\|z(t)\|^2 + \|v(t)\|^2 + \varepsilon),$$

where  $t \in [0, 1/\varepsilon]$ , and where

$$A(\tau) = \frac{\partial g}{\partial y}(\bar{y}(\tau), \bar{u}(\tau)) \quad \text{and} \quad B(\tau) = \frac{\partial g}{\partial u}(\bar{y}(\tau), \bar{u}(\tau)),$$

with  $\tau = \varepsilon t \in [0, 1]$ . Therefore we have to stabilize near the origin a *slowly-varying in time* linear control system; we refer to [13] for this classical theory.

**Third step.** Under mild controllability assumptions, namely,

$$\forall \tau \in [0, 1], \quad \text{rank}(B(\tau), A(\tau)B(\tau), \dots, A(\tau)^{n-1}B(\tau)) = n$$

(Kalman condition), it is actually possible to stabilize the system by *pole shifting* and to construct a quadratic Lyapunov function. Notice that this does not work in general if the system is not slowly-varying. So if  $\varepsilon$  is small enough, then using this Lyapunov function we infer that  $y(1/\varepsilon)$  belongs to some prescribed neighborhood of the target  $y_1$ . At this stage, a stabilization result is achieved.

**Fourth step.** If the system (1.5) is *locally controllable* near the point  $y_1$ , we conclude that it is possible to steer the system in finite time from the point  $y(1/\varepsilon)$  to the desired target  $y_1$ . Usually such a local controllability result is achieved by using an implicit function argument, after proving that the linearized system is controllable.

*Remark 1.8.* The use of quasi-static deformation for the controllability of a nonlinear partial differential control system has already been used in [3]. But note that in [3] the quasi-static trajectory  $(y^\varepsilon, u^\varepsilon)$  was stable, so it was not necessary to perform steps 2 and 3.

**2. Proof of the main results.** In order to prove Theorem 1.2 we shall follow exactly the steps described previously.

**2.1. Construction of a path of steady-states.** The following lemma is obvious.

LEMMA 2.1. *Let  $\phi_0, \phi_1 \in \mathcal{S}$ . Then  $\phi_0$  and  $\phi_1$  belong to the same connected component of  $\mathcal{S}$  if and only if, for any real number  $\alpha$  between  $\phi_0'(0)$  and  $\phi_1'(0)$ , the maximal solution of*

$$\frac{d^2 y}{dx^2} + f(y) = 0, \quad y(0) = 0, \quad y'(0) = \alpha,$$

denoted by  $y^\alpha(\cdot)$ , is defined on  $[0, L]$ .

Now let  $y_0$  and  $y_1$  be in the same connected component of  $\mathcal{S}$ . Let us construct in  $\mathcal{S}$  a  $C^1$ -path  $(\bar{y}(\tau, \cdot), \bar{u}(\tau))$ ,  $0 \leq \tau \leq 1$ , joining  $y_0$  to  $y_1$ . For each  $i = 0, 1$ , set

$$\alpha_i = y_i'(0).$$

Then with our previous notation,  $y_i(\cdot) = y^{\alpha_i}(\cdot)$ ,  $i = 0, 1$ . Now set

$$\bar{y}(\tau, x) = y^{(1-\tau)\alpha_0 + \tau\alpha_1}(x) \quad \text{and} \quad \bar{u}(\tau) = \bar{y}(\tau, L),$$

where  $\tau \in [0, 1]$  and  $x \in [0, L]$ . By construction we have

$$\bar{y}(0, \cdot) = y_0(\cdot), \quad \bar{y}(1, \cdot) = y_1(\cdot) \quad \text{and} \quad \bar{u}(0) = \bar{u}(1) = 0,$$

and thus  $(\bar{y}(\tau, \cdot), \bar{u}(\tau))$  is a  $C^1$ -path in  $\mathcal{S}$  connecting  $y_0$  to  $y_1$ .

**2.2. Reduction of the problem.** Let  $\varepsilon > 0$ . We set, for any  $t \in [0, 1/\varepsilon]$  and any  $x \in [0, L]$ ,

$$(2.1) \quad \begin{aligned} z(t, x) &= y(t, x) - \bar{y}(\varepsilon t, x), \\ v(t) &= u(t) - \bar{u}(\varepsilon t). \end{aligned}$$

Then from the definition of  $(\bar{y}, \bar{u})$  we infer that  $z$  satisfies the initial boundary problem

$$(2.2) \quad \begin{cases} z_t = z_{xx} + f'(\bar{y})z + z^2 \int_0^1 (1-s)f''(\bar{y} + sz)ds - \varepsilon \bar{y}_\tau, \\ z(t, 0) = 0, \quad z(t, L) = v(t), \\ z(0, x) = 0. \end{cases}$$

Now, in order to deal with a Dirichlet-type problem instead, we set

$$(2.3) \quad w(t, x) = z(t, x) - \frac{x}{L}v(t),$$

and we suppose that the control  $v$  is derivable. This leads to the equation

$$(2.4) \quad \begin{cases} w_t = w_{xx} + f'(\bar{y})w + \frac{x}{L}f'(\bar{y})v - \frac{x}{L}v' + r(\varepsilon, t, x), \\ w(t, 0) = w(t, L) = 0, \\ w(0, x) = -\frac{x}{L}v(0), \end{cases}$$

where

$$(2.5) \quad r(\varepsilon, t, x) = -\varepsilon \bar{y}_\tau + \left(w + \frac{x}{L}v\right)^2 \int_0^1 (1-s)f''\left(\bar{y} + s\left(w + \frac{x}{L}v\right)\right) ds,$$

and the next step is to prove that there exist  $\varepsilon$  small enough and a pair  $(v, w)$  solution of (2.4) such that  $w(1/\varepsilon, \cdot)$  belongs to some arbitrary neighborhood of 0 in  $H_0^1$ -topology. To achieve this we shall construct an appropriate control function and a Lyapunov functional which stabilizes system (2.4) to 0.

In fact, as we shall see, the control will be chosen in  $H^1(0, 1/\varepsilon)$  and such that  $v(0) = 0$ .

**2.3. Construction of a Lyapunov functional.** This is the most technical part of the work. In order to motivate what follows, let us first notice that if the residual term  $r$  and the control  $v$  were equal to zero, then (2.4) would reduce to

$$\begin{aligned} w_t &= w_{xx} + f'(\bar{y})w, \\ w(t, 0) &= w(t, L) = 0. \end{aligned}$$

This suggests that we introduce the *one-parameter family of linear operators*

$$(2.6) \quad A(\tau) = \Delta + f'(\bar{y}(\tau, \cdot))Id, \quad \tau \in [0, 1],$$

defined on  $H^2(0, L) \cap H_0^1(0, L)$ . Let  $(e_j(\tau, \cdot))_{j \geq 1}$  be a Hilbertian basis of  $L^2(0, L)$  of eigenfunctions of  $A(\tau)$ , such that for each  $j \geq 1$  and each  $\tau \in [0, 1]$ ,

$$e_j(\tau, \cdot) \in H_0^1(0, L) \cap C^2([0, L]),$$

and let  $(\lambda_j(\tau))_{j \geq 1}$  denote the corresponding eigenvalues. A standard application of the minimax principle (see, for instance, [16]) shows that these eigenfunctions and eigenvalues are  $C^1$  functions of  $\tau$ . Moreover, for each  $\tau \in [0, 1]$ ,

$$-\infty < \dots < \lambda_n(\tau) < \dots < \lambda_1(\tau) \quad \text{and} \quad \lambda_n(\tau) \xrightarrow{n \rightarrow +\infty} -\infty.$$

From the continuity of the eigenvalues on  $[0, 1]$ , we can define  $n$  as the maximal number of eigenvalues taking at least a nonnegative value as  $\tau \in [0, 1]$ ; i.e., there exists  $\eta > 0$  such that

$$(2.7) \quad \forall t \in [0, 1/\varepsilon], \quad \forall k > n, \quad \lambda_k(\varepsilon t) < -\eta < 0.$$

*Remark 2.2.* Note that the integer  $n$  can be arbitrarily large. For example, if  $f(y) = y^3$  and if  $y_1'(0) \rightarrow +\infty$ , then  $n \rightarrow +\infty$ .

We also set, for any  $\tau \in [0, 1]$  and  $x \in [0, L]$ ,

$$a(\tau, x) = \frac{x}{L} f'(\bar{y}(\tau, x)) \quad \text{and} \quad b(x) = -\frac{x}{L}.$$

In this notation, system (2.4) leads to

$$(2.8) \quad w_t(t, \cdot) = A(\varepsilon t)w(t, \cdot) + a(\varepsilon t, \cdot)v(t) + b(\cdot)v'(t) + r(\varepsilon, t, \cdot).$$

Any solution  $w(t, \cdot) \in H^2(0, L) \cap H_0^1(0, L)$  of (2.8) can be expanded as a series in the eigenfunctions  $e_j(\varepsilon t, \cdot)$ , convergent in  $H_0^1(0, L)$ :

$$w(t, \cdot) = \sum_{j=1}^{\infty} w_j(t) e_j(\varepsilon t, \cdot).$$

In fact, the  $w_j$ 's depend on  $\varepsilon$  and should be called, for example,  $w_j^\varepsilon$ . For simplicity we omit the index  $\varepsilon$ , and we shall also omit the index  $\varepsilon$  for other functions.

In what follows we are going to move, by means of an appropriate *feedback control*, the  $n$  first eigenvalues of the operator  $A$ , without moving the others, in order to make all eigenvalues negative. This pole shifting process is the first part of the stabilization procedure (see [17, p. 711]).

For any  $\tau \in [0, 1]$ , let  $\pi_1(\tau)$  denote the orthogonal projection onto the subspace of  $L^2(0, L)$  spanned by  $e_1(\tau, \cdot), \dots, e_n(\tau, \cdot)$ , and let

$$(2.9) \quad w^1(t) = \pi_1(\varepsilon t)w(t, \cdot) = \sum_{j=1}^n w_j(t)e_j(\varepsilon t, \cdot).$$

It is clear that for any  $\tau$ , the operators  $\pi_1(\tau)$  and  $A(\tau)$  commute, and moreover, for any  $y \in L^2(0, L)$ , we have

$$\pi_1'(\tau)y = \sum_{j=1}^n \langle y, e_j(\tau, \cdot) \rangle_{L^2(0,L)} \frac{\partial e_j}{\partial \tau}(\tau, \cdot) + \sum_{j=1}^n \left\langle y, \frac{\partial e_j}{\partial \tau}(\tau, \cdot) \right\rangle_{L^2(0,L)} e_j(\tau, \cdot).$$

Hence, derivating (2.9) with respect to  $t$ , we get

$$\sum_{j=1}^n w_j'(t)e_j(\varepsilon t, \cdot) = \pi_1(\varepsilon t)w_t(t, \cdot) + \varepsilon \sum_{j=1}^n \left\langle w(t, \cdot), \frac{\partial e_j}{\partial \tau}(\varepsilon t, \cdot) \right\rangle_{L^2(0,L)} e_j(\varepsilon t, \cdot).$$

On the other hand

$$A(\varepsilon t)w^1(t) = \sum_{j=1}^n \lambda_j(\varepsilon t)w_j(t)e_j(\varepsilon t, \cdot),$$

and thus (2.8) yields

$$(2.10) \quad \begin{aligned} \sum_{j=1}^n w_j'(t)e_j(\varepsilon t, \cdot) &= \sum_{j=1}^n \lambda_j(\varepsilon t)w_j(t)e_j(\varepsilon t, \cdot) + \pi_1(\varepsilon t)a(\varepsilon t, \cdot)v(t) \\ &\quad + \pi_1(\varepsilon t)b(\cdot)v'(t) + r^1(\varepsilon, t, \cdot), \end{aligned}$$

where

$$(2.11) \quad r^1(\varepsilon, t, \cdot) = \pi_1(\varepsilon t)r(\varepsilon, t, \cdot) + \varepsilon \sum_{j=1}^n \left\langle w, \frac{\partial e_j}{\partial \tau}(\varepsilon t, \cdot) \right\rangle_{L^2(0,L)} e_j(\varepsilon t, \cdot).$$

Let us set an upper bound to the residual term  $r^1$ . First, it is not difficult to check that there exists a constant  $C$  such that, if  $|v(t)|$  and  $\|w(t, \cdot)\|_{L^\infty(0,L)}$  are less than 1, then the inequality

$$\|r(\varepsilon, t, \cdot)\|_{L^\infty(0,L)} \leq C(\varepsilon + v(t)^2 + \|w(t, \cdot)\|_{L^\infty(0,L)}^2)$$

holds, where  $r$  is defined by (2.5). Therefore we get easily

$$\|r^1(\varepsilon, t, \cdot)\|_{L^\infty(0,L)} \leq C_1(\varepsilon + v(t)^2 + \|w(t, \cdot)\|_{L^\infty(0,L)}^2).$$

Moreover, since  $H^1(0, L)$  is continuously imbedded in  $C^0([0, L])$ , we can assert that there exists a constant  $C_2$  such that, if  $|v(t)|$  and  $\|w(t, \cdot)\|_{L^\infty(0,L)}$  are less than 1, then

$$(2.12) \quad \|r^1(\varepsilon, t, \cdot)\|_{L^\infty(0,L)} \leq C_2(\varepsilon + v(t)^2 + \|w(t, \cdot)\|_{H_0^1(0,L)}^2).$$

Now projecting (2.10) on each  $e_i, i = 1, \dots, n$ , one comes to

$$(2.13) \quad w_i'(t) = \lambda_i(\varepsilon t)w_i(t) + a_i(\varepsilon t)v(t) + b_i(\varepsilon t)v'(t) + r_i^1(\varepsilon, t), \quad i = 1, \dots, n,$$

where

$$\begin{aligned}
 r_i^1(\varepsilon, t) &= \langle r^1(\varepsilon, t, \cdot), e_i(\varepsilon t, \cdot) \rangle_{L^2(0,L)}, \\
 (2.14) \quad a_i(\varepsilon t) &= \langle a(\varepsilon t, \cdot), e_i(\varepsilon t, \cdot) \rangle_{L^2(0,L)} = \frac{1}{L} \int_0^L x f'(\bar{y}(\varepsilon t, x)) e_i(\varepsilon t, x) dx, \\
 b_i(\varepsilon t) &= \langle b(\cdot), e_i(\varepsilon t, \cdot) \rangle_{L^2(0,L)} = -\frac{1}{L} \int_0^L x e_i(\varepsilon t, x) dx.
 \end{aligned}$$

The  $n$  equations (2.13) form a differential system controlled by  $v, v'$ . Set

$$(2.15) \quad \alpha(t) = v'(t),$$

and consider now  $v(t)$  as a state and  $\alpha(t)$  as a control. Then the former finite dimensional system may be rewritten as

$$(2.16) \quad \begin{cases} v' = \alpha, \\ w'_1 = \lambda_1 w_1 + a_1 v + b_1 \alpha + r_1^1, \\ \vdots \\ w'_n = \lambda_n w_n + a_n v + b_n \alpha + r_n^1. \end{cases}$$

If we introduce the matrix notation

$$X_1(t) = \begin{pmatrix} v(t) \\ w_1(t) \\ \vdots \\ w_n(t) \end{pmatrix}, \quad R_1(\varepsilon, t) = \begin{pmatrix} 0 \\ r_1^1(\varepsilon, t) \\ \vdots \\ r_n^1(\varepsilon, t) \end{pmatrix},$$

$$A_1(\tau) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_1(\tau) & \lambda_1(\tau) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n(\tau) & 0 & \cdots & \lambda_n(\tau) \end{pmatrix}, \quad B_1(\tau) = \begin{pmatrix} 1 \\ b_1(\tau) \\ \vdots \\ b_n(\tau) \end{pmatrix},$$

then equations (2.16) yield the *finite dimensional linear control system*

$$(2.17) \quad X'_1(t) = A_1(\varepsilon t)X_1(t) + B_1(\varepsilon t)\alpha(t) + R_1(\varepsilon, t).$$

Let us now prove the following lemma.

LEMMA 2.3. *For each  $\tau \in [0, 1]$ , the pair  $(A_1(\tau), B_1(\tau))$  satisfies the Kalman condition, i.e.,*

$$(2.18) \quad \text{rank} (B_1(\tau), A_1(\tau)B_1(\tau), \dots, A_1(\tau)^{n-1}B_1(\tau)) = n.$$

*Proof.* Let  $\tau \in [0, 1]$  be fixed. We compute directly

$$(2.19) \quad \det (B_1, A_1 B_1, \dots, A_1^{n-1} B_1) = \prod_{j=1}^n (a_j + \lambda_j b_j) \text{VdM}(\lambda_1, \dots, \lambda_n),$$



where  $\text{VdM}(\lambda_1, \dots, \lambda_n)$  is a Van der Monde determinant and thus is never equal to zero, since the  $\lambda_i(\tau), i = 1, \dots, n$ , are distinct for any  $\tau \in [0, 1]$ . On the other hand, using the fact that each  $e_j(\tau, \cdot)$  is an eigenfunction of  $A(\tau)$  and belongs to  $H_0^1(0, L)$ , we compute

$$\begin{aligned} a_j(\tau) + \lambda_j(\tau)b_j(\tau) &= \frac{1}{L} \int_0^L x (f'(\bar{y}(\tau, x))e_j(\tau, x) - \lambda_j(\tau)e_j(\tau, x)) dx \\ &= -\frac{1}{L} \int_0^L x \frac{\partial^2 e_j}{\partial x^2}(\tau, x) dx \\ &= -\frac{\partial e_j}{\partial x}(\tau, L). \end{aligned}$$

But this quantity is never equal to zero since  $e_j(\tau, L) = 0$  and  $e_j(\tau, \cdot)$  is a nontrivial solution of a linear second-order scalar differential equation. Therefore the determinant (2.19) is never equal to zero and we are done.  $\square$

It is a standard fact that the Kalman condition (2.18) implies a *pole shifting* result, and we get the following corollary (see [13]).

**COROLLARY 2.4.** *For each  $\tau \in [0, 1]$ , there exist scalars  $k_0(\tau), \dots, k_n(\tau)$  such that, if we denote*

$$K_1(\tau) = (k_0(\tau), \dots, k_n(\tau)),$$

*then the matrix  $A_1(\tau) + B_1(\tau)K_1(\tau)$  admits  $-1$  as an eigenvalue with order  $n + 1$ .*

*Moreover, there exists a  $C^1$  application  $\tau \mapsto P(\tau)$  on  $[0, 1]$ , where  $P(\tau)$  is an  $(n + 1) \times (n + 1)$  symmetric positive definite matrix, such that the identity*

$$(2.20) \quad P(\tau) (A_1(\tau) + B_1(\tau)K_1(\tau)) + {}^t(A_1(\tau) + B_1(\tau)K_1(\tau)) P(\tau) = -I$$

*holds for any  $\tau \in [0, 1]$ .*

We are now able to construct a control Lyapunov functional in order to stabilize system (2.8). Leave  $c > 0$  to be chosen later. For any  $t \in [0, 1/\varepsilon]$ ,  $v \in \mathbb{R}$ , and  $w \in H^2(0, L) \cap H_0^1(0, L)$ , we set

$$(2.21) \quad V(t, v, w) = c {}^tX_1(t)P(\varepsilon t)X_1(t) - \frac{1}{2} \langle w, A(\varepsilon t)w \rangle_{L^2(0, L)},$$

where  $X_1(t)$  denotes the matrix vector in  $\mathbb{R}^{n+1}$ ,

$$X_1(t) = \begin{pmatrix} v \\ w_1(t) \\ \vdots \\ w_n(t) \end{pmatrix},$$

and

$$w_i(t) = \langle w, e_i(\varepsilon t, \cdot) \rangle_{L^2(0, L)}.$$

In particular, we have

$$(2.22) \quad V(t, v, w) = c {}^tX_1(t)P(\varepsilon t)X_1(t) - \frac{1}{2} \sum_{j=1}^{\infty} \lambda_j(\varepsilon t)w_j(t)^2.$$

In what follows we will repeatedly use the equivalence of norms in finite dimension. The following notation will thus be useful.

NOTATION. Let  $\Lambda$  be a set and let  $\Delta = \{(\varepsilon, t) / 0 < \varepsilon \leq 1, 0 \leq t \leq 1/\varepsilon\}$ . Let  $F_1, F_2$  be two real functions defined on  $\Delta \times \Lambda$ . The notation  $F_1 \lesssim F_2$  means that  $F_2 \geq 0$  and that there exists a positive constant  $C$  such that

$$F_1(\varepsilon, t, \lambda) \leq CF_2(\varepsilon, t, \lambda) \quad \forall (\varepsilon, t) \in \Delta, \quad \forall \lambda \in \Lambda.$$

We say that  $F_1 \sim F_2$  if both  $F_1 \lesssim F_2$  and  $F_2 \lesssim F_1$ . Moreover, if  $F_3$  is a real function defined on  $\Delta \times \Lambda$  and if  $\theta \in [0, +\infty)$ , the notation  $F_1 \lesssim F_2$  for  $F_3 \leq \theta$  means that  $F_2 \geq 0$  and that there exists a positive constant  $C$  such that

$$\forall (\varepsilon, t) \in \Delta, \quad \forall \lambda \in \Lambda, \quad (F_3(\varepsilon, t, \lambda) \leq \theta) \Rightarrow (F_1(\varepsilon, t, \lambda) \leq CF_2(\varepsilon, t, \lambda)).$$

For simplicity, when the set  $\Lambda$  is clear from the context it will not be given explicitly.

Let  $\|\cdot\|_2$  denote the euclidean norm in  $\mathbb{R}^{n+1}$ . Since  $P(\tau)$  is symmetric positive definite, we can write (with  $\Lambda = \mathbb{R} \times (H^2(0, L) \cap H_0^1(0, L))$ )

$${}^tX_1(t)P(\varepsilon t)X_1(t) \sim \|X_1(t)\|_2^2 = v^2 + \sum_{j=1}^n w_j(t)^2.$$

From (2.7) we know that, except the  $n$  first ones, the eigenvalues of  $A$  are all negative, less than  $-\eta < 0$ . By continuity the  $n$  first eigenvalues are bounded as  $\tau \in [0, 1]$  and thus we can assert that, if  $c$  is large enough in the definition of  $V$ , then

$$(2.23) \quad V(t, v, w) \sim \|X_1(t)\|_2^2 - \sum_{j=n+1}^{\infty} \lambda_j(\varepsilon t)w_j(t)^2,$$

where  $t \in [0, 1/\varepsilon]$ . In particular,  $V(t, \dots)$  is positive definite. Let us further prove the following lemma.

LEMMA 2.5. *The equivalence*

$$(2.24) \quad V(t, v, w) \sim v^2 + \|w\|_{H_0^1(0, L)}^2$$

holds with  $\Lambda = \{(v, w) / v \in \mathbb{R}, w \in H^2(0, L) \cap H_0^1(0, L)\}$ , where  $\|w\|_{H_0^1(0, L)} = \|w_x\|_{L^2(0, L)}$ . Moreover,

$$(2.25) \quad V(t, v, w) \lesssim \|X_1(t)\|_2^2 + \|Aw\|_{L^2(0, L)}^2.$$

*Proof.* Any  $w \in H^2(0, L) \cap H_0^1(0, L)$  can be expanded as a series in the eigenfunctions of  $A(\varepsilon t)$ , convergent in  $H_0^1(0, L)$ ,

$$w(\cdot) = \sum_i w_i(t)e_i(\varepsilon t, \cdot).$$

Hence

$$\|w\|_{H_0^1(0, L)}^2 = \sum_{i, j} w_i(t)w_j(t) \int_0^L e_{ix}(\varepsilon t, x)e_{jx}(\varepsilon t, x)dx.$$

Integrating by parts and using the definition of  $e_j$ , we compute

$$\int_0^L e_{ix}(\varepsilon t, x)e_{j_x}(\varepsilon t, x)dx = \int_0^L f'(\bar{y}(\varepsilon t, x))e_i(\varepsilon t, x)e_j(\varepsilon t, x)dx - \lambda_j \delta_{ij},$$

and thus

$$\|w\|_{H_0^1(0,L)}^2 = \int_0^L f'(\bar{y}(\varepsilon t, x))w(x)^2 dx - \sum_{j=1}^{\infty} \lambda_j(\varepsilon t)w_j(t)^2.$$

Therefore, since  $f'(\bar{y})$  is bounded on  $[0, 1/\varepsilon] \times [0, L]$  uniformly in  $\varepsilon \in (0, 1]$ ,

$$\|w\|_{H_0^1(0,L)}^2 \lesssim \|w\|_{L^2(0,L)}^2 - \sum_{j=n+1}^{\infty} \lambda_j(\varepsilon t)w_j(t)^2 \lesssim V(t, v, w).$$

Conversely, we have

$$\begin{aligned} - \sum_{j=n+1}^{\infty} \lambda_j(\varepsilon t)w_j(t)^2 &= \|w\|_{H_0^1(0,L)}^2 - \int_0^L f'(\bar{y}(\varepsilon t, x))w(x)^2 dx + \sum_{j=1}^n \lambda_j(\varepsilon t)w_j(t)^2 \\ &\lesssim \|w\|_{H_0^1(0,L)}^2 + \|w\|_{L^2(0,L)}^2 \\ &\lesssim \|w\|_{H_0^1(0,L)}^2, \end{aligned}$$

and using (2.23) we conclude easily that (2.24) holds.

On the other hand, notice that

$$\|w\|_{H_0^1(0,L)}^2 \lesssim \sum_{j=n+1}^{\infty} |\lambda_j|w_j^2 + \sum_{j=1}^n w_j^2.$$

Therefore, using (2.7),

$$\|w\|_{H_0^1(0,L)}^2 \lesssim \sum_{j=1}^n w_j^2 + \sum_{j=1}^{\infty} \lambda_j^2 w_j^2 = \sum_{j=1}^n w_j^2 + \|Aw\|_{L^2(0,L)}^2,$$

and hence the estimate (2.25) follows.  $\square$

Let now  $(v(t), w(t, \cdot))$  denote a solution of (2.8) in which we choose the control in the feedback form suggested from Corollary 2.4, namely,

$$\alpha(t) = K_1(\varepsilon t)X_1(t),$$

such that  $v(0) = 0$  and  $w(0, \cdot) = 0$ , i.e.,  $(v(t), w(t, \cdot))$  satisfies

$$(2.26) \quad w_t = Aw + av + bK_1(\varepsilon t)X_1(t) + r, \quad v'(t) = K_1(\varepsilon t)X_1(t),$$

$$(2.27) \quad v(0) = 0, \quad w(0, x) = 0.$$

We set

$$V_1(t) = V(t, v(t), w(t, \cdot)) = c^t X_1(t)P(\varepsilon t)X_1(t) - \frac{1}{2} \langle w(t, \cdot), A(\varepsilon t)w(t, \cdot) \rangle_{L^2(0,L)}.$$

Let us compute  $V_1'(t)$  and state a differential inequality satisfied by  $V_1$ . We have

$$(2.28) \quad \begin{aligned} V_1'(t) &= c \left( {}^tX_1'(t)P(\varepsilon t)X_1(t) + {}^tX_1(t)P(\varepsilon t)X_1'(t) \right) + \varepsilon c {}^tX_1(t)P'(\varepsilon t)X_1(t) \\ &\quad - \frac{1}{2} \langle w_t(t, \cdot), A(\varepsilon t)w(t, \cdot) \rangle_{L^2(0,L)} - \frac{1}{2} \langle w(t, \cdot), A(\varepsilon t)w_t(t, \cdot) \rangle_{L^2(0,L)} \\ &\quad - \frac{1}{2} \varepsilon \langle w(t, \cdot), A'(\varepsilon t)w(t, \cdot) \rangle_{L^2(0,L)}. \end{aligned}$$

Note the following facts:

- From (2.17) and (2.20), we infer

$${}^tX_1'PX_1 + {}^tX_1PX_1' = -\|X_1\|_2^2 + {}^tR_1PX_1 + {}^tX_1PR_1.$$

- The operator  $A$  is self-adjoint in  $L^2$ , hence

$$\langle w, Aw_t \rangle_{L^2(0,L)} = \langle Aw, w_t \rangle_{L^2(0,L)}.$$

- Equation (2.26) leads to

$$\begin{aligned} \langle Aw, w_t \rangle_{L^2(0,L)} &= \langle Aw, Aw + av + bK_1X_1 + r \rangle_{L^2(0,L)} \\ &= \|Aw\|_{L^2(0,L)} + \langle Aw, a \rangle_{L^2(0,L)}v \\ &\quad + \langle Aw, b \rangle_{L^2(0,L)}K_1X_1 + \langle Aw, r \rangle_{L^2(0,L)}. \end{aligned}$$

- From the definition of  $A(\tau)$ , we have

$$A'(\tau) = f''(\bar{y}(\tau, \cdot))\bar{y}_\tau(\tau, \cdot)Id,$$

and thus

$$\langle w(t, \cdot), A'(\varepsilon t)w(t, \cdot) \rangle_{L^2(0,L)} = \langle w(t, \cdot), f''(\bar{y}(\varepsilon t, \cdot))\bar{y}_\tau(\varepsilon t, \cdot)w(t, \cdot) \rangle_{L^2(0,L)}.$$

Therefore, turning back to (2.28),

$$(2.29) \quad \begin{aligned} V_1' &= -c \|X_1\|_2^2 - \|Aw\|_{L^2(0,L)}^2 - \langle Aw, a \rangle_{L^2(0,L)}v - \langle Aw, b \rangle_{L^2(0,L)}K_1X_1 \\ &\quad - \langle Aw, r \rangle_{L^2(0,L)} + \varepsilon c {}^tX_1P'X_1 + c ({}^tR_1PX_1 + {}^tX_1PR_1) \\ &\quad - \frac{1}{2} \varepsilon \langle w, f''(\bar{y})\bar{y}_\tau w \rangle_{L^2(0,L)}. \end{aligned}$$

Let us set an upper bound for the terms of the second line of (2.29), as follows:

- From Corollary 2.4, the application  $\tau \mapsto P'(\tau)$  is bounded on  $[0, 1]$ , hence

$$|\varepsilon c {}^tX_1P'X_1| \lesssim \varepsilon \|X_1\|_2^2 \lesssim \varepsilon V_1.$$

- Inequality (2.12) yields, for  $|v(t)| + \|w(t, \cdot)\|_{L^\infty(0,L)} \leq 1$ ,

$$\|R_1(\varepsilon, t)\|_{L^\infty(0,L)} \lesssim \varepsilon + v(t)^2 + \|w(t, \cdot)\|_{H_0^1(0,L)}^2,$$

and thus, still for  $|v(t)| + \|w(t, \cdot)\|_{L^\infty(0,L)} \leq 1$ ,

$$\begin{aligned} {}^tR_1PX_1 + {}^tX_1PR_1 &\lesssim \|X_1\|_2 \left( \varepsilon + v^2 + \|w\|_{H_0^1(0,L)}^2 \right) \\ &\lesssim \sqrt{V_1} (\varepsilon + V_1) = \varepsilon \sqrt{V_1} + V_1^{3/2}. \end{aligned}$$

- Since  $f$  is of class  $C^2$ , we can assert

$$\varepsilon \left| \frac{1}{2} \langle w, f''(\bar{y}) \bar{y}_\tau w \rangle_{L^2(0,L)} \right| \lesssim \varepsilon \|w\|_{L^2(0,L)}^2 \lesssim \varepsilon \|w\|_{H_0^1(0,L)}^2 \lesssim \varepsilon V_1.$$

- The term  $\langle Aw, r \rangle_{L^2(0,L)}$  is the most difficult to handle. Using (2.5), write

$$\begin{aligned} \langle Aw, r \rangle_{L^2(0,L)} &= \left\langle Aw, -\varepsilon \bar{y}_\tau + \left(w + \frac{x}{L} v\right)^2 \int_0^1 (1-s) f''\left(\bar{y} + s\left(w + \frac{x}{L} v\right)\right) ds \right\rangle_{L^2(0,L)} \\ &= \left\langle Aw, \left(w + \frac{x}{L} v\right)^2 \int_0^1 (1-s) f''\left(\bar{y} + s\left(w + \frac{x}{L} v\right)\right) ds \right\rangle_{L^2(0,L)} \\ &\quad - \varepsilon \langle Aw, \bar{y}_\tau \rangle_{L^2(0,L)}. \end{aligned}$$

We clearly have

$$|\varepsilon \langle Aw, \bar{y}_\tau \rangle_{L^2(0,L)}| \lesssim \varepsilon \|Aw\|_{L^2(0,L)}.$$

Let us now deal with the integral term. First of all, using the continuous imbedding of  $H_0^1$  in  $C^0$ , we estimate

$$\begin{aligned} \left\| \left(w(t, x) + \frac{x}{L} v(t)\right)^2 \right\|_{L^\infty(0,L)} &\lesssim \|w(t, \cdot)\|_{L^\infty(0,L)}^2 + v(t)^2 \\ &\lesssim \|w(t, \cdot)\|_{H_0^1(0,L)}^2 + \|X_1(t)\|_2^2 \\ &\lesssim V_1(t) \end{aligned}$$

since  $V_1 \sim \|w\|_{H_0^1(0,L)}^2 + \|X_1\|_2^2$ . For  $|v(t)| + \|w(t, \cdot)\|_{L^\infty(0,L)} \leq 1$ , one has

$$\left\langle Aw, \left(w + \frac{x}{L} v\right)^2 \int_0^1 (1-s) f''\left(\bar{y} + s\left(w + \frac{x}{L} v\right)\right) ds \right\rangle_{L^2(0,L)} \lesssim \|Aw\|_{L^2(0,L)} V_1,$$

and we arrive at the estimate

$$|\langle Aw, r \rangle_{L^2(0,L)}| \lesssim \varepsilon \|Aw\|_{L^2(0,L)} + \|Aw\|_{L^2(0,L)} V_1$$

for  $|v(t)| + \|w(t, \cdot)\|_{L^\infty(0,L)} \leq 1$ .

Let us also estimate the terms of the principal part of (2.29). We clearly have

$$|\langle Aw, a \rangle_{L^2(0,L)} v| \leq \frac{1}{4} \|Aw\|_{L^2(0,L)}^2 + \|a\|_{L^2(0,L)}^2 \|X_1\|_2^2$$

and

$$|\langle Aw, b \rangle_{L^2(0,L)} K_1 X_1| \leq \frac{1}{4} \|Aw\|_{L^2(0,L)}^2 + M \|X_1\|_2^2,$$

where

$$M = \|b\|_{L^2(0,L)} \max \left\{ \sum_{i=0}^n k_i(\tau)^2 / \tau \in [0, 1] \right\}.$$

Hence, concerning the principal part of (2.29), we first get

$$\begin{aligned}
 & -c\|X_1\|_2^2 - \|Aw\|_{L^2(0,L)}^2 - \langle Aw, a \rangle_{L^2(0,L)}v - \langle Aw, b \rangle_{L^2(0,L)}K_1X_1 \\
 & \leq -c_1\|X_1\|_2^2 - \frac{1}{2}\|Aw\|_{L^2(0,L)}^2,
 \end{aligned}$$

where  $c_1 = c - \|a\|_{L^2(0,L)}^2 - M$ . We choose  $c$  so that  $c_1 > 0$ .

The previous estimates and (2.29) now yield, for  $|v(t)| + \|w(t, \cdot)\|_{L^\infty(0,L)} \leq 1$ ,

(2.30)

$$V_1' + \|X_1\|_2^2 + \|Aw\|_{L^2(0,L)}^2 \lesssim \varepsilon\sqrt{V_1} + V_1^{3/2} + \varepsilon\|Aw\|_{L^2(0,L)} + V_1\|Aw\|_{L^2(0,L)}.$$

Note that, for every  $\theta \in (0, +\infty)$ ,

$$\begin{aligned}
 \varepsilon\sqrt{V_1} & \leq \frac{\theta}{2}V_1 + \frac{1}{2\theta}\varepsilon^2, \\
 \varepsilon\|Aw\|_{L^2(0,L)} & \leq \frac{\theta}{2}\|Aw\|_{L^2(0,L)}^2 + \frac{1}{2\theta}\varepsilon^2, \\
 V_1\|Aw\|_{L^2(0,L)} & \leq \frac{\theta}{2}\|Aw\|_{L^2(0,L)}^2 + \frac{1}{2\theta}V_1^2.
 \end{aligned}$$

Hence, taking  $\theta > 0$  small enough, we get, using (2.25) and (2.30), the existence of  $\sigma > 0$  and  $\rho \in (0, \sigma]$  such that, for every  $\varepsilon \in (0, 1]$  and for every  $t \in [0, 1/\varepsilon]$  such that  $V_1(t) \leq \rho$ ,

$$V_1'(t) \leq \sigma\varepsilon^2.$$

Hence, since  $V_1(0) = 0$ , we get, if  $\varepsilon \in (0, \rho/\sigma]$ , that

$$V_1(t) \leq \sigma\varepsilon \quad \forall t \in [0, 1/\varepsilon]$$

and, in particular,

$$V_1\left(\frac{1}{\varepsilon}\right) \leq \sigma\varepsilon.$$

Coming back to definitions (2.1) and (2.3), we have proved

$$(2.31) \quad \left\| y\left(\frac{1}{\varepsilon}, \cdot\right) - y_1(\cdot) \right\|_{H^1(0,L)} \leq \gamma\varepsilon,$$

where  $y_1(\cdot) = \bar{y}(1, \cdot)$  is the final target and  $\gamma$  is a positive constant which does not depend on  $\varepsilon \in (0, \rho/\sigma]$ . This concludes the third step, and thus the proof of the stabilization part of Theorem 1.2 (see Remark 1.3).

**2.4. End of the proof.** The last step consists of solving a local exact controllability result: from the previous section,  $y(\frac{1}{\varepsilon}, \cdot)$  belongs to an arbitrarily small neighborhood of  $y_1(\cdot)$  in  $H^1$ -topology if  $\varepsilon$  is small enough, and our aim is now to construct a trajectory  $q(t, x)$  solution of the control system steering  $y(\frac{1}{\varepsilon}, \cdot)$  to  $y_1(\cdot)$  in some time  $T > 0$  (for instance,  $T = 1$ ), i.e.,

$$\begin{cases} q_t = q_{xx} + f(q), \\ q(t, 0) = 0, \quad q(t, L) = u(t), \\ q(0, x) = y\left(\frac{1}{\varepsilon}, x\right), \quad q(T, x) = y_1(x). \end{cases}$$

Existence of such a solution  $q$  is given by [11, Theorem 3.3]. Actually, in [11] the function  $f$  is assumed to be globally Lipschitzian, but the local result we need here follows readily from the proofs and the estimates contained in this paper. Indeed, let  $T > 0$  and let  $\tilde{f}$  be a globally Lipschitzian mapping such that

$$(2.32) \quad \tilde{f}(s) = f(s) \quad \forall s \in [-\|y_1\|_{L^\infty} - 1, \|y_1\|_{L^\infty} + 1].$$

From the proof of [11, Theorem 3.3], we get the existence of  $\mu > 0$  such that there exists  $z \in Y_T$  satisfying

$$\begin{cases} z_t = z_{xx} + \tilde{f}(z + y_1) - \tilde{f}(y_1), \\ z(t, 0) = 0, \\ z(0, x) = y\left(\frac{1}{\varepsilon}, x\right) - y_1(x), \quad z(T, x) = 0 \end{cases}$$

and the estimate

$$(2.33) \quad \|z\|_{Y_T} \leq \mu \left\| y\left(\frac{1}{\varepsilon}, \cdot\right) - y_1(\cdot) \right\|_{H^1(0,L)},$$

which leads, with  $q = z + \tilde{y}_1$ , to

$$\begin{cases} q_t = q_{xx} + \tilde{f}(q), \\ q(t, 0) = 0, \\ q(0, x) = y\left(\frac{1}{\varepsilon}, x\right), \quad q(T, x) = y_1(x) \end{cases}$$

and

$$(2.34) \quad \|q - \tilde{y}_1\|_{Y_T} \leq \mu \left\| y\left(\frac{1}{\varepsilon}, \cdot\right) - y_1(\cdot) \right\|_{H^1(0,L)},$$

where  $\tilde{y}_1(t, x) := y_1(x)$ . From (2.33) and (2.34), we get

$$(2.35) \quad \|q - \tilde{y}_1\|_{L^\infty((0,T) \times (0,L))} \leq 1$$

for  $\|y(1/\varepsilon, \cdot) - y_1(\cdot)\|_{H^1(0,L)}$  small enough. From (2.32) and (2.35), we infer that  $\tilde{f}(q) = f(q)$ , which ends the proof.

**3. Controllability versus connectedness.** Let us first give some sufficient conditions ensuring the connectedness of  $\mathcal{S}$ .

PROPOSITION 3.1. *In each of the following cases, the set of steady-states  $\mathcal{S}$  is connected:*

- *The function  $F$ , defined as*

$$F(y) = \int_0^y f(s) ds,$$

*satisfies the asymptotic condition*

$$F(y) \xrightarrow{|y| \rightarrow +\infty} +\infty.$$

- For any  $\alpha > 0$ , the indefinite integral

$$\int \frac{dy}{\sqrt{\alpha - F(y)}}$$

diverges in  $-\infty$  and in  $+\infty$  (if it makes sense).

- The function  $f$  is odd, i.e., for any  $y \in \mathbb{R}$ ,

$$f(-y) = -f(y).$$

*Remark 3.2.* Notice that, contrary to the two first cases of the proposition, in the third case blow-up phenomena may occur. Nevertheless, the set of steady-states is connected.

On the other hand, we have the following result.

**PROPOSITION 3.3.** *If  $y_0$  and  $y_1$  belong to distinct connected components of  $\mathcal{S}$ , then it is not possible to move either from  $y_0$  to  $y_1$ , or from  $y_1$  to  $y_0$ , whatever the control  $u \in L^2(0, T)$  and the time  $T$  are.*

*Remark 3.4.* If  $y_0$  and  $y_1$  are both periodic, then they are in the same connected component.

In order to prove these two propositions, let us first note some general facts about the maximal solutions of the scalar differential equation

$$(3.1) \quad y''(x) + f(y(x)) = 0, \quad y(0) = 0.$$

**LEMMA 3.5.**

- Any solution of (3.1) satisfies on its maximal interval of definition the conservation law

$$(3.2) \quad y'(x)^2 + 2F(y(x)) = y'(0)^2.$$

- Any solution of (3.1) such that  $y'$  vanishes at least at two distinct points is actually periodic.
- The phase portrait in the plane  $(y, y')$  of the associated differential system

$$y' = z, \quad z' = -f(y),$$

is symmetric with respect to the  $y$  axis, and moreover, all singular points of the system are located on this axis.

The proof of these facts is obvious. Now the key lemma for proving Propositions 3.1 and 3.3 is the following.

**LEMMA 3.6.** *Let  $y_0$  and  $y_1$  be two steady-states, extended on their maximal interval of definition as solutions of (3.1), belonging to distinct connected components of  $\mathcal{S}$ , such that  $y'_0(0) < y'_1(0)$ . Then there exists an  $l \in (0, L]$  and  $\bar{y} \in C^2([0, l])$  solution of (3.1) such that either*

$$(3.3) \quad \bar{y}(x) \xrightarrow{x \rightarrow l} +\infty$$

or

$$(3.4) \quad \bar{y}(x) \xrightarrow{x \rightarrow l} -\infty.$$

In the first case we have, moreover (see Figure 3.1), the following:



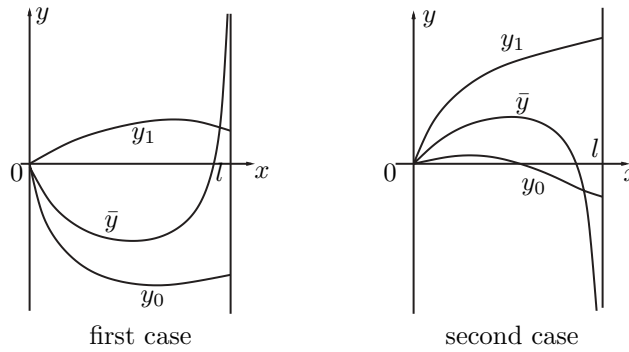


FIG. 3.1. Existence of an explosive solution.

1.  $y_0(x) < \bar{y}(x)$  for any  $x \in [0, l]$ .
2.  $y'_0(0) < \bar{y}'(0) < y'_1(0)$ ,  $|y'_1(0)| < |\bar{y}'(0)|$ , and  $\bar{y}'(0) < 0$ .
3.  $\#\{x \in [0, l] / \bar{y}(x) = y_1(x)\} = 1$ .
4.  $y_0$  is not periodic, and  $y_1(x)$  does not tend to  $-\infty$  as  $x$  tends to  $b$ , where  $(a, b)$  denotes the maximal interval of definition of  $y_1$ .

In the second case we have the symmetric situation (see Figure 3.1), as follows:

1.  $y_1(x) > \bar{y}(x)$  for any  $x \in [0, l]$ .
2.  $y'_0(0) < \bar{y}'(0) < y'_1(0)$ ,  $|y'_1(0)| < |\bar{y}'(0)|$ , and  $\bar{y}'(0) > 0$ .
3.  $\#\{x \in [0, l] / \bar{y}(x) = y_0(x)\} = 1$ .
4.  $y_1$  is not periodic, and  $y_0(x)$  does not tend to  $+\infty$  as  $x$  tends to  $b$ , where  $(a, b)$  denotes the maximal interval of the definition of  $y_0$ .

*Proof of Lemma 3.6.* Clearly one of the two cases (3.3) and (3.4) occurs, with, moreover,  $y'_0(0) < \bar{y}'(0) < y'_1(0)$ . Assume we are in the first case, and let us first prove the second of the four properties claimed in that case. To proceed, we have to distinguish between the following three possibilities:

- $y_1$  is monotonic on  $[0, l]$ . In this case the conservation law (3.2) immediately implies that  $|y'_1(0)| < |\bar{y}'(0)|$ .
- $y_1$  is not monotonic on  $[0, l]$  and is not periodic on its maximal interval  $(a, b)$ . That is,  $y'_1$  vanishes exactly once on  $(a, b)$ . The only nonobvious case occurs when  $y'_1(0) > 0$ . But then, since the phase portrait is symmetric with respect to the  $y$  axis, either  $y_1(x)$  tends to  $-\infty$  as  $x$  tends to  $a$  and  $b$  or  $y_1(x)$  tends to a finite limit which corresponds to a singular point on the phase portrait (see Figure 3.2). In both cases it is clear on the phase portrait that  $\bar{y}(x)$  may tend to  $+\infty$  as  $x$  tends to  $l$  only if  $|y'_1(0)| < |\bar{y}'(0)|$ .
- $y_1$  is periodic (i.e.,  $y'_1$  vanishes at least two times). Again in this case the phase portrait immediately implies the desired inequality.

Before proving that  $\bar{y}'(0) < 0$ , let us prove the third point. The only nonobvious case occurs when  $y_1$  is periodic and  $\bar{y}$  is not monotonic on  $[0, l]$ . Notice that  $\bar{y}'$  vanishes only once (if not,  $\bar{y}$  would be periodic), and thus it decreases on an interval  $[0, x_0]$  and increases on  $[x_0, l]$ . Now, on the one hand, the function  $y_1$  cannot intersect  $\bar{y}$  on the interval  $[0, x_0]$ , for this would contradict the conservation law (3.2). On the other hand, if  $y_1$  would intersect  $\bar{y}$  more than once on the interval  $[x_0, l]$ , then there would be at least three intersections, and again this leads to a contradiction with (3.2). This proves the third point.

Now the inequality  $\bar{y}'(0) < 0$  is an obvious consequence of (3.2).

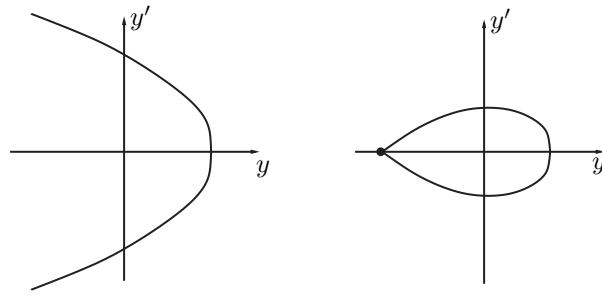


FIG. 3.2. Behavior of  $(y_1(x), y'_1(x))$  in the phase plane.

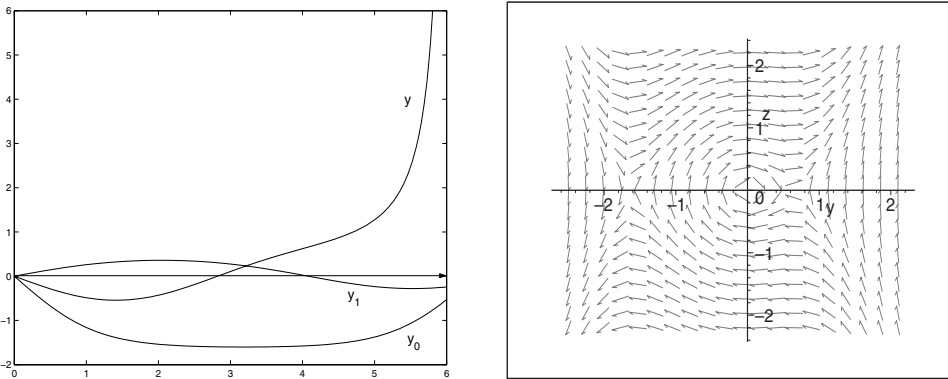


FIG. 3.3. An example where  $y_0$  and  $y_1$  are not in the same connected component of  $\mathcal{S}$ .

Let us now prove that  $y_0 < \bar{y}$  on  $[0, l)$ . The same reasoning as above shows that  $\bar{y}$  intersects  $y_0$  at most once (notice that  $y'_0$  cannot vanish more than once). But such an intersection would contradict the fact that  $\bar{y}(x)$  tends to  $+\infty$  as  $x$  tends to  $l$ .

Finally, the last point of the lemma is proved by observing the phase portrait.  $\square$

*Proof of Proposition 3.3.* Proposition 3.3 follows from Lemma 3.6. Indeed, let us assume, for example, that we are in the first case of the lemma. Then for any  $T > 0$  and  $u \in L^2([0, T])$ , the solution  $y$  of the control system (1.4) satisfies, as long as defined, the inequality

$$y(t, x) \leq \bar{y}(x)$$

(see [4] for this application of the classical maximum principle to similar control problems). In particular,  $y(T, \cdot) \neq y_1(\cdot)$ .  $\square$

Finally let us prove Proposition 3.1. The only difficult case is to prove that if  $f$  is odd, then the set  $\mathcal{S}$  is connected. In this case the conservation law (3.2) implies that the phase portrait is symmetric with respect to the  $y$  axis and the  $y'$  axis. As a consequence, any solution of (3.1) such that  $y'$  vanishes at least once is necessarily periodic.

Now from Lemma 3.6 we know that if  $y_0$  and  $y_1$  are not in the same connected component, then there exists an explosive solution  $\bar{y}$  of (3.1) such that  $\bar{y}'$  vanishes at least once. Hence,  $\bar{y}$  must be periodic, and we get a contradiction.

*Example 3.7.* An example where the situation of Proposition 3.3 and Lemma 3.6 occurs is given by

$$f(y) = y - y^2 - y^3.$$

The graph of  $y_0, y_1$ , an explosive  $y$ , and the phase portrait are drawn on Figure 3.3.

**4. Numerical simulations.** In this section we present numerical simulations using Matlab for the nonlinear function  $f(y) = y^3$ . Let  $L = 1$ ; the set  $\mathcal{S}$  of steady-states consists of all solutions of class  $C^2$  on  $[0, 1]$  such that

$$(4.1) \quad y''(x) + y(x)^3 = 0, \quad y(0) = 0.$$

It follows from Proposition 3.1 that this set is connected. Let  $y_0$  be identically zero, and let  $y_1$  denote the solution of (4.1) vanishing at 0, 1/2, and 1, and having no other zero on  $[0, 1]$  (see Figure 4.1).

For all  $\tau \in [0, 1]$ , we define the function  $\bar{y}(\tau, \cdot)$  on  $[0, 1]$  as the solution of (4.1) such that

$$\frac{\partial \bar{y}}{\partial x}(\tau, 0) = \tau y_1'(0)$$

and we set  $\bar{u}(\tau) = \bar{y}(\tau, 1)$ . We then introduce on  $H^2(0, 1) \cap H_0^1(0, 1)$  the one-parameter family of linear operators

$$A(\tau) = \Delta + 3\bar{y}(\tau, \cdot)^2 Id, \quad \tau \in [0, 1].$$

For  $\tau = 0$ , we have  $A(0) = \Delta$ , and the eigenvalues and eigenvectors write

$$\lambda_i(0) = -i^2\pi^2, \quad e_i(0, x) = \sqrt{2} \sin k\pi x.$$

Then, solving by homotopy as  $\tau \in [0, 1]$  boundary value problems, we compute numerically, using a standard finite difference code implemented in Matlab, the first eigenvalues and associated eigenvectors. In the present example, numerical experiments show that only the two first eigenvalues may take positive values as  $\tau \in [0, 1]$ . In other words, with the notations of section 2.3, one has  $n = 2$ . Then we achieve a

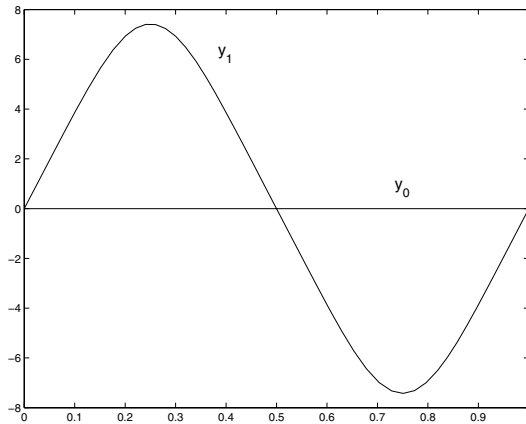


FIG. 4.1. Definition of the steady-states  $y_0$  and  $y_1$ .

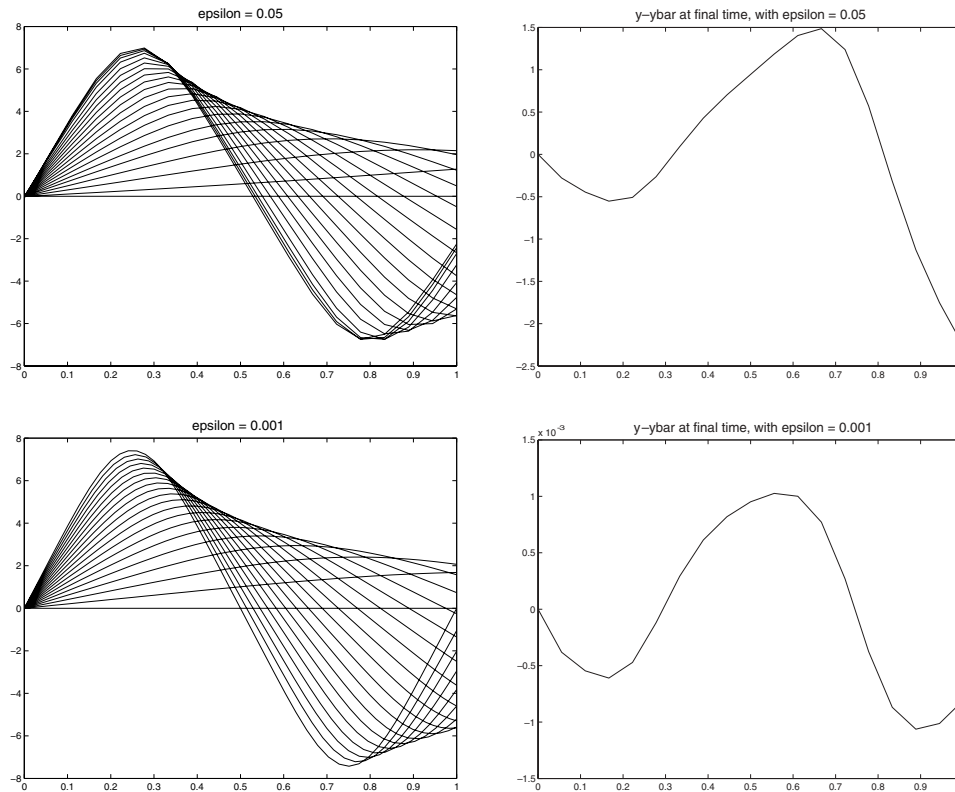


FIG. 4.2. Simulations results for  $y(t, \cdot)$ , where  $t \in [0, 1/\varepsilon]$ .

pole placement on the finite dimensional system (2.16) by applying an LQR algorithm (see [13]). Notice that the finite dimensional system corresponding to these two first modes is very unstable: numerically, one has  $\lambda_1(1) \simeq 89.743$  and  $\lambda_2(1) \simeq 82.518$ .

Results are drawn on Figure 4.2 for  $\varepsilon = 0.05$  and  $\varepsilon = 0.001$ . Notice that if  $\varepsilon$  is too large, then the solution blows up.

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