Computational and Applied Mathematics
Vol. 21, N. 1, pp. 101–134, 2002

Stabilization of an overhead crane with a variable length flexible cable*

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Abstract. This paper deals with the asymptotic stabilization of an hybrid PDE-ODE system which describes an overhead crane with a variable length flexible cable. Stabilizing boundary feedback laws have previously been developed (see e.g. [2, 1, 3, 9]) in the case of a cable with fixed length.

Mathematical subject classification: 93C20, 93D15, 35L50.

Key words: boundary control, asymptotic stabilization, flexible cable, Lyapunov approach.

1 Introduction

In previous papers (e.g. [2, 1, 3, 9]) several boundary control laws have been developed for stabilizing hybrid PDE-ODE systems describing a motorized platform moving on an horizontal bench equipped with a winch around which a cable is wound. In those papers the cable was supposed to have a fixed length so that the only control considered was the force acting on the platform or the velocity of the platform. In this paper we propose to consider also the torque applied to the winch to wind or unwind the cable.

In Section 2 we shall compute a model of the overhead crane with a variable length flexible cable, leading to an hybrid PDE-ODE system with a moving domain. We shall then present a boundary feedback law making the natural

*Supported by a grant from PRC-GDR ''Automatique'' of the CNRS and the MST.

energy function bounded, at least **formally**. The torque applied to the winch will only depend on the length L of the cable and its time-derivatives, and it will make the cable length exponentially converge to its set value L_c. Concerning the platform, for simplicity, we shall first consider its velocity as a control variable since we may use **backstepping** approaches (see e.g. [6] for this notion), as was done e.g. in [3], to derive a stabilizing control for the cascaded system.

This ''decoupled structure'' of the control strategy makes the closed-loop PDE linear, but non-autonomous, since it depends on $L - L_c$, $\dot{L}$, $\ddot{L}$ which can be seen as exogeneous variables exponentially converging to zero.

Therefore, in Section 3, we shall consider a change of variable to transform the closed-loop system into the more classical framework of an hybrid PDE-ODE system with fixed domain. Then, using a theorem due to Kato [7] about equations of hyperbolic type, we shall study the well-posedness of the closed-loop system.

Finally, using a stronger norm and making some estimations, we shall prove in Section 4 our main result concerning the asymptotic stabilization of the closed-loop PDE-ODE system and conclude.

2 Modeling of the overhead crane with variable length flexible cable and formal computation of a boundary control law

2.1 *Modeling of the system*

The physical system is described in Figure 1, where X_p denotes the position of the platform, L the length of the unrolled part of the cable, θ the angular displacement of the cable at the connection point to the platform, $\hat{y}(s,t)$ the horizontal displacement at time t of the point the curvilinear abscissa along the cable of which is s; $\hat{y}_s(s,t)$ is the angular inclination of the cable at s with respect to the vertical (in fact $\hat{y}_s(0,t) = \theta$), $\hat{T}(s,t)$ is the tension along the cable, u_1 the force applied to the platform and u_2 the torque applied to the winch.

In the sequel, ρ will denote the mass per unit length of the cable, M the mass of the platform and the winch, m the load mass, J the moment of inertia of the winch, R its radius and L_0 the total length of the cable.

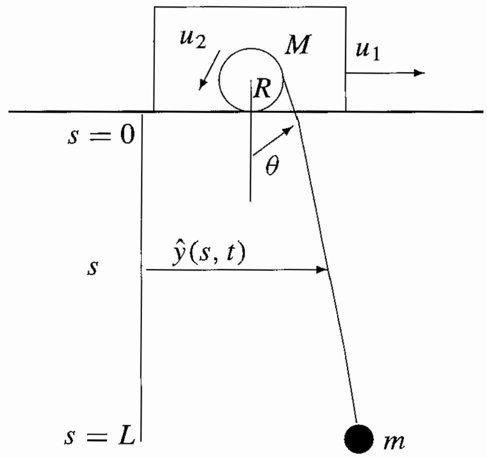

Figure 1 – The overhead crane with variable length flexible cable.

We make the following assumptions

A1: The cable is completely flexible and non-stretching.

A2: Transversal and angular displacements are small.

Dynamical equations of the platform and the winch. We consider the system made of the platform and the winch. We denote by M_e the mass of the part of the cable which is wound on the winch

$$M_e = \rho(L_0 - L). \tag{1}$$

Writing the fundamental principle of mechanics for the platform and the winch gives:

- For the platform and the part of the cable wound on the winch, using **A2**,

$$(M + M_e)\ddot{X}_p = u_1 + (\hat{T}(0,t) - \rho\dot{L}^2)\theta. \tag{2}$$

Remark 1. Due to assumption **A2**, $\sin\theta$ in equation (2) has been replaced by $\hat{y}_s(0,t)$.

- For the winch

$$\left(\frac{J}{R^2} + M_e\right)\ddot{L} = -\frac{u_2}{R} + \hat{T}(0, t). \tag{3}$$

Computation of the tension along the cable. For equations (2) and (3) to be explicit, we have to compute an expression of the tension $\hat{T}(s, t)$ along the cable. In fact, the tension can be seen as a Lagrange multiplier associated with the non stretching constraint (see assumption **A1**) and is in general difficult to obtain explicitly (see e.g. [4, 12, 13,14]).

Due to assumption **A2**, we will take for $\hat{T}(s, t)$ the tension when the cable is at the vertical rest position. Writing the fundamental law of dynamics for the unwound part of the cable from point P (see Figure 2) leads to

$$(m + \rho(L - s))\ddot{L} = (m + \rho(L - s))g - \hat{T},$$

which gives

$$\hat{T}(s, t) = (m + \rho(L - s))(g - \ddot{L}). \tag{4}$$

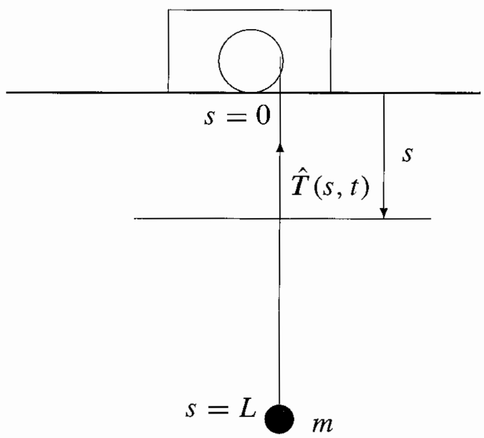

Figure 2 – Computation of the tension.

As in [2, 1, 3, 9] the tension is of course a function of the curvilinear abscissa but, now, also a function of time since it is a function of L and $\ddot{L}$.

Dynamical equation of the unrolled part of the cable. Let us now consider the transversal dynamics of the unrolled part of the cable under the assumption of small displacements (see **A2**). Writing the fundamental principle of dynamics for a small segment of cable, we obtain (see e.g. [4,12, 13, 14])

$$\rho \frac{d^2}{dt^2} \hat{y}(s,t) = (\hat{T}(s,t)\hat{y}_s)_s,$$

where $\hat{T}(s,t)$ is the tension given by (4). Noticing that $\frac{d}{dt} f(s,t) = f_t(s,t) + \dot{L} f_s(s,t)$, (since $\dot{s} = \dot{L}$), the dynamical equation of the cable can be finally written as:

$$\rho \hat{y}_{tt} = (\hat{T}(s,t)\hat{y}_s)_s - 2\rho \dot{L} \hat{y}_{st} - \rho \dot{L}^2 \hat{y}_{ss} - \rho \ddot{L} \hat{y}_s. \tag{5}$$

Remark 2. In the case of a fixed length (i.e. $\dot{L} = 0$), we retrieve the dynamical equations for the heavy flexible cable considered in [2, 1, 3, 9].

Dynamical equation of the load mass. Writing the dynamical equation of the load mass in projection on the horizontal axis gives

$$m \frac{d^2}{dt^2} \hat{y}(L,t) = -\hat{T}(L,t)\hat{y}_s(L,t),$$

which can be finally rewritten as:

$$m(\hat{y}_{tt}(L,t) + 2\dot{L}\hat{y}_{st}(L,t) + \ddot{L}\hat{y}_s(L,t) + \dot{L}^2\hat{y}_{ss}(L,t)) = -\hat{T}(L,t)\hat{y}_s(L,t), \tag{6}$$

with $\hat{T}(L,t)$ given by (4).

Change of variables. In fact, to make computations easier, we apply the following change of variables

$$y(x,t) = y(L-s,t) = \hat{y}(s,t), \text{ with } x = L - s. \tag{7}$$

Therefore we have

$$\begin{cases} y_x(x,t) = -\hat{y}_s(s,t), \\ y_{xx}(x,t) = \hat{y}_{ss}(s,t), \\ y_t(x,t) = \hat{y}_t(s,t) + \dot{L}\hat{y}_s(s,t), \\ y_{tt}(x,t) = \hat{y}_{tt}(s,t) + 2\dot{L}\hat{y}_{st}(s,t) + \ddot{L}\hat{y}_s(s,t) + \dot{L}^2\hat{y}_{ss}(s,t). \end{cases} \tag{8}$$

Then, equation (5) takes the following simple form

$$\rho y_{tt} = (Ty_x)_x, \tag{9}$$

with the corresponding expression of the tension

$$T(x,t) = \hat{T}(s,t) = (m + \rho x)(g - \ddot{L}). \tag{10}$$

Similarly, we have for the load mass equation

$$my_{tt}(0,t) = m(g - \ddot{L})y_x(0,t) \tag{11}$$

since when $s = L$, $x = 0$. Moreover, let us recall that the angle θ between the vertical axis and the cable at the connection point to the platform is given by

$$\theta = \hat{y}_s(0,t) = -y_x(L,t) \tag{12}$$

and that

$$y(L,t) = X_p(t). \tag{13}$$

2.2 *Formal computation of a boundary control law*

As explained in the introduction, we shall first define the torque control u_2 to make L exponentially converge to the set value L_c and $\dot{L}$ to zero. But in order to have a well-posed Cauchy problem for the partial differential equation (9) one needs to satisfy some constraints on $(\ddot{L}, \dot{L}, L)$ that we are now going to specify. Let $\xi = y_t$, $\chi = y_x$. Then (9) gives

$$\frac{\partial}{\partial t}\begin{pmatrix} \xi \\ \chi \end{pmatrix} + \begin{pmatrix} 0 & -\frac{T}{\rho} \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \xi_x \\ \chi_x \end{pmatrix} + \begin{pmatrix} 0 & -\frac{T_x}{\rho} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \xi \\ \chi \end{pmatrix} = 0. \tag{14}$$

This system is strictly hyperbolic if and only if

$$T(s,t) > 0, \ \ \forall s \in [0, L], \ \ \forall t,$$

which by (10) is equivalent to

$$g > \ddot{L}(t), \ \ \forall t. \tag{15}$$

In particular, since we want to have $\ddot{L}(t) \to 0$ as $t \to +\infty$, system (14) will be hyperbolic for t large enough. So, in order to have a well-posed Cauchy problem for the PDE part (9), it is natural to impose (15) for all t. Since the partial differential equation (9) is now hyperbolic, in order to have a well-posed Cauchy problem for this PDE part (9), it is also natural to impose that the boundaries $x = 0$ and $x = L(t)$ should not be characteristic, which leads respectively to

$$T(0,t) \neq 0, \ \ \forall t, \tag{16}$$

$$T(L,t) \neq \rho \dot{L}(t)^2, \ \ \forall t. \tag{17}$$

Inequality (15) already implies (16). Since we want to have $\dot{L}(t) \to 0$ and $L(t) \to L_c$ as $t \to +\infty$, we expect that, for t large enough,

$$T(L,t) > \rho \dot{L}(t)^2. \tag{18}$$

Hence, in order to have (17), we have to impose (18) for every t, that is using (10)

$$\ddot{L}(t) < g - \frac{\rho}{m + \rho L(t)} \dot{L}(t)^2, \ \ \forall t. \tag{19}$$

Notice that (19) implies (15), and that we of course also want $L > 0$. Hence one considers the control system

$$\begin{cases} \dot{L} = v, \\ \dot{v} = U_2, \end{cases} \tag{20}$$

where the state (L, v), and the control U_2 are such that $(L, v, U_2) \in \mathcal{D}$, with

$$\mathcal{D} = \left\{ (L, v, U_2); \ L > 0 \ \text{ and } \ U_2 < g - \frac{\rho}{m + \rho L} v^2 \right\}. \tag{21}$$

From (3), the relation between u_2 and U_2 is

$$u_2 = R(T(L,t) - (J/R^2 + M_e)U_2), \tag{22}$$

with T given by (10). One first needs to check if, (L_0, v_0) being given, it is possible to asymptotically steer (for control system (20)) this state to $(L_c, 0)$ with a locally bounded measurable open loop control $U_2 : [0, +\infty) \to \mathbb{R}$ so that

$$(L(t), v(t), U_2(t)) \in \mathcal{D}, \text{ for almost every } t \in [0, +\infty), \tag{23}$$

$$L(t) > 0, \quad \forall t \in [0, +\infty). \tag{24}$$

The condition for the existence of such an open loop is given in the following lemma.

Lemma 1. *Such an open loop control U_2 exists if and only if $L_0 > 0$ and*

$$v_0 > \bar{v}(L_0),$$

where $\bar{v} : [0, +\infty) \mapsto (-\infty, 0]$ is defined by

$$\bar{v}(L) = -\sqrt{\frac{2g}{3\rho}}\frac{1}{m+\rho L}\sqrt{(m+\rho L)^3 - m^3}.$$

Proof.

- **"Only if"** Let $\phi : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ defined by

$$\phi(L,v) = \frac{\rho}{g}v^2(m+\rho L)^2 + \frac{2}{3}m^3 - \frac{2}{3}(m+\rho L)^3. \tag{25}$$

One has

$$\dot{\phi} := \frac{\partial \phi}{\partial L}v + \frac{\partial \phi}{\partial v}U_2 = \frac{2\rho}{g}v(m+\rho L)\left[(m+\rho L)(U_2 - g) + \rho v^2\right]. \tag{26}$$

Hence

$$\dot{\phi} \geqslant 0, \quad \forall (L, v, U_2) \in \mathcal{D} \text{ such that } v \leqslant 0. \tag{27}$$

Moreover

$$(\phi(L, v) \geqslant 0 \text{ and } v = 0) \Rightarrow (L \leqslant 0). \tag{28}$$

From (27) and (28), one gets that, for every trajectory $t \in [0, +\infty) \rightarrow (L(t), v(t), U_2(t))$ of the control system (20) such that (23) and (24) hold and such that

$$\phi(L(0), v(0)) \geqslant 0 \text{ and } v(0) \leqslant 0,$$

one has

$$\phi(L(t), v(t)) \geqslant 0 \text{ and } v(t) \leqslant 0, \ \forall t \geqslant 0. \tag{29}$$

But (28) and (29) imply that one cannot have

$$(L(t), v(t)) \rightarrow (L_c, 0) \text{ as } t \rightarrow +\infty.$$

- **"if" part**. This part follows from Proposition 1 below, where we give feedback laws $(L, v) \rightarrow U_2(L, v)$ defined on

$$\Omega = \left\{(L, v) \in \mathbb{R}^2;\ L > 0 \text{ and } v > \bar{v}(L)\right\}$$

such that

$$U_2(L, v) < g - \frac{\rho}{m + \rho L} v^2, \tag{30}$$

$$(L_c, 0) \text{ is globally asymptotically stable in } \Omega \tag{31}$$

and this ends the proof. □

We now turn to the construction of feedback laws $(L, v) \rightarrow U_2(L, v)$ defined on Ω such that (30) and (31) hold. Let $U_2 : \bar{\Omega} \mapsto \mathbb{R}$ be such that

$$U_2 \in C^\infty(\bar{\Omega}), \tag{32}$$

$$\frac{\partial U_2}{\partial v} < 0 \text{ in } \Omega, \tag{33}$$

$$U_2(L, \bar{v}(L)) = g - \frac{\rho}{m + \rho L} \bar{v}(L)^2, \ \ \forall L \in [0, +\infty), \tag{34}$$

$$\exists \delta > 0 \text{ such that } U_2(L, 0) \leqslant -\delta(L - L_c), \ \ \forall L \in [L_c, +\infty), \tag{35}$$

$$U_2(L, v) < g - \frac{\rho}{m + \rho L} v^2, \ \ \forall (L, v) \in \Omega, \tag{36}$$

$$U_2(L, 0) > 0, \ \ \forall L \in [0, L_c). \tag{37}$$

Since

$$g - \frac{\rho}{m+\rho L}\bar{v}(L)^2 = \frac{g}{3} + \frac{2gm^3}{3(m+\rho L)^3} > \frac{g}{3},$$

there are such U_2.

One has the following proposition.

Proposition 1. *For every* $U_2 : \bar{\Omega}\setminus\{(0, v);\ v \in \mathbb{R}\} \mapsto \mathbb{R}$ *satisfying (32) to (37),* $(L_c, 0)$ *is globally exponentially asymptotically stable in* Ω*, i.e.*

(i) $(L_c, 0)$ *is (locally) asymptotically stable for the closed-loop system*

$$\begin{cases} \dot{L} = v, \\ \dot{v} = U_2(L, v). \end{cases} \tag{38}$$

(ii) *There exists* $\nu > 0$ *such that, for every (maximal) solution of the closed-loop system (38) such that* $(L(0), v(0)) \in \Omega$*, one has*

$$(L(t), v(t)) \in \Omega, \ \ \forall t \in [0, +\infty), \tag{39}$$

$$(|L(t) - L_c| + |v(t)|)e^{\nu t} \to 0 \text{ as } t \to +\infty. \tag{40}$$

Proof. By (35) and (37)

$$U_2(L_c, 0) = 0, \tag{41}$$

$$\frac{\partial U_2}{\partial L}(L_c, 0) < 0. \tag{42}$$

Since, by (33), $\partial U_2/\partial v(L_c, 0) < 0$, it follows from (41) and (42) that $(L_c, 0)$ is (locally) exponentially asymptotically stable for the closed-loop system (38).

Let

$$\Omega_1 = \{(L, v) \in \Omega;\ v > 0,\ L \geqslant L_c\},$$
$$\Omega_2 = \{(L, v) \in \Omega;\ v \leqslant 0,\ L > L_c\},$$
$$\Omega_3 = \{(L, v) \in \Omega;\ v < 0,\ L \leqslant L_c\},$$
$$\Omega_4 = \{(L, v) \in \Omega;\ v \geqslant 0,\ L < L_c\}.$$

The proof will rely on the following two steps.

Step 1. Let $t \to (L(t), v(t))$ be a trajectory of the closed-loop system (38). We argue by contradiction and assume that $(L(t), v(t))$ does not tend to zero as t goes to infinity. We then prove that if for some time t_i we are in Ω_i, $i \in \{1, 2, 3, 4\}$, then for some time $t_{i+1} > t_i$ we are in Ω_{i+1} with the convention $\Omega_5 = \Omega_1$.

Step 2. To conclude, we have to check that 0 is a globally asymptotically stable point for the Poincaré map associated with the manifold $\{(L_c, v)\ ;\ v \geqslant 0\}$. More precisely, let $(L(t), v(t))$ be a trajectory of the closed-loop system (38) such that $L(0) = L_c$ and $v(0) > 0$. From step 1, if (ii) does not hold, there exists a smallest time instant $\tau > 0$ such that $L(\tau) = L_c$ and $v(\tau) \geq 0$. We then have to prove that $v(\tau) < v(0)$.

Step 1. Let us start with the study of the closed-loop system (38) in Ω_1. One has

$$\dot{L} > 0 \text{ in } \Omega_1. \tag{43}$$

From (33) and (35) one gets

$$U_2 < -\delta(L - L_c) \text{ in } \Omega_1,$$

which implies that

$$\overbrace{\delta(L - L_c)^2 + v^2}^{\cdot} < 0 \text{ in } \Omega_1. \tag{44}$$

Let $t \to (L(t), v(t))$ be a maximal solution of the closed-loop system (38) such that, for some time t_1, $(L(t_1), v(t_1)) \in \Omega_1$. From (43) and (44) one gets the existence of $t_2 > t_1$ such that

$$L(t_2) > L_c,\ v(t_2) = 0,$$
$$(L(t), v(t)) \in \Omega_1,\ \ \forall t \in [t_1, t_2).$$

Let us now study the closed-loop system (38) in Ω_2. From (35) one gets that

$$U_2(L, 0) < 0,\ \ \forall L > L_c. \tag{45}$$

Let $t \to (L(t), v(t))$ be a maximal solution of the closed-loop system (38) such that, for some time t_2, $(L(t_2), v(t_2)) \in \Omega_2$. From (25), (27) and (34) one gets that, as long as (L, v) stays in Ω_2 and for $t \geqslant t_2$,

$$v(t) > \bar{v}(L(t)). \tag{46}$$

Since

$$\dot{L} < 0 \text{ in } \Omega_2 \backslash \{(L, 0);\ L \in \mathbb{R}\},$$

one gets from (45) and (46) that, if

$$\lim_{t \to +\infty} |L(t) - L_c| + |v(t)| = 0 \tag{47}$$

does not hold, there exists $t_3 > t_2$ such that

$$L(t_3) = L_c,\ v(t_3) < 0,$$
$$(L(t), v(t)) \in \Omega_2,\quad \forall t \in [t_2, t_3).$$

Let us continue with the study of the closed-loop system (38) in Ω_3. One has

$$\dot{L} < 0 \text{ in } \Omega_3. \tag{48}$$

Let $t \to (L(t), v(t))$ be a maximal solution of the closed-loop system (38) such that, for some time t_3, $(L(t_3), v(t_3)) \in \Omega_3$. Again one has (46) for $t \geqslant t_3$ as long as (L, v) stays in Ω_3. From (37), (46) and (48) one gets the existence of $t_4 > t_3$ such that

$$0 < L(t_4) < L_c,\ v(t_4) = 0,$$
$$(L(t), v(t)) \in \Omega_3,\quad \forall t \in [t_3, t_4).$$

Finally we study the closed-loop system (38) in Ω_4. From (36), one has

$$U_2 \leqslant g \text{ in } \Omega_4. \tag{49}$$

Let $t \to (L(t), v(t))$ be a maximal solution of the closed-loop system (38) such that, for some time t_4, $(L(t_4), v(t_4)) \in \Omega_4$. From (37) and (49), one gets that, if (47) does not hold, there exists $t_5 > t_4$ such that

$$L(t_5) = L_c,\ v(t_5) > 0,$$
$$(L(t), v(t)) \in \Omega_4,\quad \forall t \in [t_4, t_5).$$

Step 2. In order to end the proof of Proposition 1, there only remains to check that if $t \in [t_1, t_5] \to (L(t), v(t)) \in \Omega$ is a solution of the closed-loop system (38) such that, for some (t_2, t_3, t_4) with $t_1 < t_2 < t_3 < t_4 < t_5$,

$$
\begin{gathered}
L(t_1) = L_c,\ v(t_1) > 0, \\
L(t_2) > L_c,\ v(t_2) = 0, \\
L(t_3) = L_c,\ v(t_3) < 0, \\
L(t_4) < L_c,\ v(t_4) = 0, \\
L(t_5) = L_c,\ v(t_5) > 0, \\
(L(t), v(t)) \in \Omega_1,\ \ \forall t \in [t_1, t_2), \\
(L(t), v(t)) \in \Omega_2,\ \ \forall t \in [t_2, t_3), \\
(L(t), v(t)) \in \Omega_3,\ \ \forall t \in [t_3, t_4), \\
(L(t), v(t)) \in \Omega_4,\ \ \forall t \in [t_4, t_5),
\end{gathered}
$$

one has

$$v(t_5) < v(t_1). \tag{50}$$

In order to prove (50), one uses a method due to Bendixson (see e.g. [5, p. 44]). Let us denote by X the vector field in Ω

$$X = v\frac{\partial}{\partial L} + U_2(L, v)\frac{\partial}{\partial v}$$

and let $\mathcal{O}$ be the open bounded subset of $\mathbb{R}^2$ such that

$$\partial\mathcal{O} = \{(L(t), v(t));\ t \in [t_1, t_5]\} \cup \{(L_c, s);\ s \in [v(t_1), v(t_5)] \cup [v(t_5), v(t_1)]\}.$$

Then the Stokes formula gives

$$\int_{\mathcal{O}} \frac{\partial U_2}{\partial v} = \int_{\mathcal{O}} \operatorname{div} X = \int_{\partial\mathcal{O}} X \cdot n = \int_{v(t_1)}^{v(t_5)} s\,ds. \tag{51}$$

But, by (33), the left hand side of (51) is strictly negative, which with (51) gives (50) and therefore ends the proof of Proposition 1. □

From now on $U_2 : \bar{\Omega} \mapsto \mathbb{R}$ is a fixed function satisfying (32) to (37). Let us now define the boundary control law for the platform. To compute a stabilizing boundary feedback law, we will consider the following Lyapunov function candidate

$$E = \frac{1}{2}\int_0^{L(t)} (\rho y_t^2 + T(x,t)y_x^2)dx + \frac{1}{2}my_t^2(0,t) + K\frac{X_p^2}{2}, \tag{52}$$

where K is a strictly positive real number.

Remark 3. To steer the platform to a nonzero set position x_c the term $KX_p^2/2$ should be, of course, replaced by the term $K(X_p - x_c)^2/2$.

Recall that by (39) and (36)

$$T(x,t) > 0, \quad \forall t \in [0,+\infty), \quad \forall x \in [0, L(t)].$$

In fact, given a (maximal) solution $t \to (L(t), v(t))$ of the closed-loop system (38) such that $(L(0), v(0)) \in \Omega$, it follows from (36), (41), (39) and (40) that there exists $\delta > 0$ such that

$$T(x,t) > \delta, \quad \forall t \in [0,+\infty), \quad \forall x \in [0, L(t)].$$

Using (9), (10), (11), the expression of θ given by (12) and noticing by (13) that

$$y_t(L,t) = \dot{X}_p + \dot{L}\theta,$$

a formal computation of the time derivative of E gives

$$\dot{E} = KX_p\dot{X}_p + B_1 + B_2 + B_3, \tag{53}$$

with

$$B_1(t) = -(m + \rho L)(g - \ddot{L})\theta(\dot{X}_p + \dot{L}\theta), \tag{54}$$

$$\begin{aligned} B_2(t) &= \frac{\dot{L}}{2}\Big(\rho(\dot{X}_p + \dot{L}\theta)^2 + (m+\rho L)(g - \ddot{L})\theta^2\Big), \\ B_3(t) &= \frac{1}{2}\int_0^L T_t(x,t)y_x^2 dx. \end{aligned} \tag{55}$$

In order to motivate the choice of our feedback law, let us write it in the following form

$$\dot{X}_p(t) = \alpha X_p(t) + \beta\theta, \tag{56}$$

where α and β are functions of L and $\dot{L}$ only (and so do not depend on y). Then $\dot{E}$ is a polynomial of degree 2 in θ. Since one does not have any a priori information on θ, in order to have an upper bound on $\dot{E}$, we need the coefficient P of θ^2 in $\dot{E}$ to be negative. One easily gets

$$P := -(m+\rho L)(g-\ddot{L})(\beta+\frac{\dot{L}}{2}) + \rho\frac{\dot{L}}{2}(\beta+\dot{L})^2. \tag{57}$$

Let us define β as follows

$$\beta = \sqrt{\frac{1}{\rho}\left[(g-\ddot{L})(m+\rho L) - \dot{L}^2\rho\right]}. \tag{58}$$

By (36) and (38), β is well defined. One has

$$P = -\rho\beta^3 < 0. \tag{59}$$

Let us now sum up the action of our boundary feedback laws U_2 and $\dot{X}_p$ in the following proposition.

Proposition 2. *Let β be given by (58) and let*

$$\alpha = -\frac{K\beta}{a} \text{ with } a = (m+\rho L)(g-\ddot{L}). \tag{60}$$

Let (L, v) be a solution of (20) with $(L(0), v(0)) \in \Omega$ and let (X_p, y) be a smooth (i.e. of class C^2) solution of (9), (11), (13), (56). Then

i) *$(L - L_c)$, $\dot{L}$ and $\ddot{L}$ exponentially converge to zero;*

ii) *there are constants $C_1 > 0$ and $C_2 > 0$, depending on $(L(0), v(0))$ but not on (X_p, y), such that the following inequalities are satisfied*

$$\dot{E} \leqslant C_1(|\dot{L}| + |L - L_c|)E, \tag{61}$$

$$E(t) \leqslant C_2 E(0) \text{ and } \int_0^{+\infty} (|\dot{L}| + |L-L_c|)E\,dt < +\infty. \tag{62}$$

Proof. The fact that $L - L_c$, $\dot{L}$ and $\ddot{L}$ exponentially converge to zero results from Proposition 1.

Concerning point ii), we know that the time derivative of $E(t)$ is a polynomial of degree 2 in θ with coefficient P made negative (see (59)) by a suitable choice (58) of coefficient β in the feedback (56). Let us now compute $\dot{E}_M$, the maximum value of $\dot{E}$, obtained for an angle θ such that

$$\frac{\partial \dot{E}}{\partial \theta} = 0.$$

After some computations, and for α given by (60), we can write

$$\begin{aligned} \dot{E}_M &= X_p^2 \dot{L} \left[\frac{K\rho}{a} \dot{L} + \rho\alpha \left(\rho\alpha \dot{L} \left(\beta + \dot{L}\right)^2 + 4K\beta \left(\beta + \dot{L}\right) + 2\rho\alpha\beta^3 \right) \right] \\ &\quad - \frac{\dddot{L}}{2} \int_0^L (m + \rho L) y_x^2 dx. \end{aligned} \tag{63}$$

Using (38) leads to

$$\dot{E}_M \leqslant C_1(|\ \dot{L}\ | + |\ L - L_c\ |)E$$

for some constant C_1 depending on $(L(0), v(0))$ but not on (X_p, y). Then inequalities (62) hold by applying Gronwall's lemma 5 in Appendix 1. This ends the proof of the proposition. □

Remark 4. From (22) and (56), we can see that the boundary feedback laws $\dot{X}_p$ and u_2 only depend on L, $\dot{L}$, $X_p = y(L, t)$ and $\theta = -y_x(L, t)$, which are quite easy to measure.

3 Well-posedness of the closed-loop system. Existence of solutions

3.1 Well-posedness of the closed-loop system

In this section $(L, v) : [0, +\infty) \mapsto \Omega$ denotes a solution of the closed-loop system (38) and we study the well-posedness of the following Cauchy problem: $y^0 : [0, L(0)] \mapsto \mathbb{R}$, $y^1 : [0, L(0)] \mapsto \mathbb{R}$ being given, find $y : \{(x, t);\ 0 \leqslant x \leqslant$

$L(t),\ 0 \leqslant t\} \mapsto \mathbb{R}$ satisfying the closed-loop system (see (9), (11), (12), (13) and (56))

$$\begin{cases} \rho y_{tt} = (Ty_x)_x,\ x \in [0, L(t)],\ t \geqslant 0, \\ y_{tt}(0,t) = (g - \ddot{L})y_x(0,t),\ t \geqslant 0, \\ y_t(L,t) = \alpha y(L,t) - (\beta + \dot{L})y_x(L,t),\ t \geqslant 0, \end{cases} \tag{64}$$

and the initial conditions

$$y(x,0) = y^0(x),\ x \in [0, L(0)], \tag{65}$$

$$y_t(x,0) = y^1(x),\ x \in [0, L(0)]. \tag{66}$$

To apply classical results for PDE systems to prove the well-posedness of this Cauchy problem we first transform system (64) into an evolutionary PDE system on a *fixed* domain. For that purpose we introduce the following change of variable

$$\begin{cases} x = L\sigma \\ \tilde{y}(\sigma,t) = y(x,t) = y(L\sigma,t)\,,\ \sigma \in [0,1], \end{cases} \tag{67}$$

where the space variable σ is now in the *fixed* interval [0, 1]. Using the following obvious relations

$$\begin{cases} \tilde{y}_\sigma(\sigma,t) = Ly_x(L\sigma,t), \\ \tilde{y}_{\sigma\sigma}(\sigma,t) = L^2 y_{xx}(L\sigma,t), \\ \tilde{y}_t(\sigma,t) = y_t(L\sigma,t) + \dot{L}\sigma y_x(L\sigma,t), \\ \tilde{y}_{tt}(\sigma,t) = y_{tt}(L\sigma,t) + 2\dot{L}\sigma y_{xt}(L\sigma,t) \\ \qquad + \ddot{L}\sigma y_x(L\sigma,t) + \dot{L}^2\sigma^2 y_{xx}(L\sigma,t) \text{ with} \\ y_{xt}(L\sigma,t) = \dfrac{\tilde{y}_{\sigma t}}{L} - \dfrac{\dot{L}}{L^2}\tilde{y}_\sigma - \dfrac{\dot{L}\sigma}{L^2}\tilde{y}_{\sigma\sigma}, \end{cases} \tag{68}$$

one sees that (64) is equivalent to

$$\begin{cases} \rho\tilde{y}_{tt} = 2\rho\dfrac{\dot{L}\sigma}{L}\tilde{y}_{\sigma t} - \rho\left[\dfrac{2\dot{L}^2\sigma}{L^2} - \dfrac{\ddot{L}\sigma}{L} - \dfrac{g - \ddot{L}}{L}\right]\tilde{y}_\sigma \\ \qquad + \left[\dfrac{(\tilde{T} - \rho\dot{L}^2\sigma^2)}{L^2}\right]\tilde{y}_{\sigma\sigma},\ \sigma \in [0,1],\ t \geqslant 0, \\ \tilde{y}_{tt}(0,t) = \dfrac{(g - \ddot{L})}{L}\tilde{y}_\sigma(0,t),\ t \geqslant 0, \\ \tilde{y}_t(1,t) = \alpha\tilde{y}(1,t) - \dfrac{\beta}{L}\tilde{y}_\sigma(1,t),\ t \geqslant 0, \end{cases} \tag{69}$$

where

$$\tilde{T}(\sigma, t) = (m + \rho L\sigma)(g - \ddot{L}).$$

Moreover, the energy function defined by (52) can be rewritten as:

$$\begin{aligned} E &= \frac{1}{2}\int_0^1 \left(\rho(\tilde{y}_t - \frac{\dot{L}\sigma}{L}\tilde{y}_\sigma(\sigma, t))^2 + \tilde{T}(\sigma, t)\frac{\tilde{y}_\sigma^2}{L^2}\right) L d\sigma \\ &\quad + \frac{1}{2}m\tilde{y}_t^2(0, t) + K\frac{\tilde{y}(1, t)^2}{2}. \end{aligned} \tag{70}$$

Let us now write system (69) in an operator form. We introduce the new variables

$$z = \tilde{y}_t, \; b = \tilde{y}_t(0, t),$$

and the ''state'' space

$$X = \{w = (\tilde{y}, z, b) \in H^1(0, 1) \times L^2(0, 1) \times \mathbb{R}\}. \tag{71}$$

Let us recall that X is a Hilbert space for the scalar product

$$\langle w, \hat{w}\rangle_X = \int_0^1 (\tilde{y}_\sigma \hat{y}_\sigma + z\hat{z})d\sigma + \tilde{y}(1)\hat{y}(1) + b\hat{b},$$

with $w = (\tilde{y}, z, b)$ and $\hat{w} = (\hat{y}, \hat{z}, \hat{b})$. With this new variable w, system (69) becomes:

$$\dot{w} + A(t)w = 0, \tag{72}$$

where the *time-varying* operator $A(t)$ is defined, for every $t \in [0, +\infty)$, by

$$\begin{aligned} Dom(A(t)) &= \Big\{(\tilde{y}, z, b) \in H^2(0, 1) \times H^1(0, 1) \times \mathbb{R}; \\ &\qquad z(0) = b, z(1) = \alpha(t)\tilde{y}(1) - \frac{\beta(t)}{L(t)}\tilde{y}_\sigma(1)\Big\}, \end{aligned} \tag{73}$$

$$A(t)w = \begin{pmatrix} -z \\ -2\frac{\dot{L}\sigma}{L}z_\sigma + \left[\frac{2\dot{L}^2\sigma}{L^2} - \frac{\ddot{L}\sigma}{L} - \frac{g-\ddot{L}}{L}\right]\tilde{y}_\sigma - \left[\frac{(\tilde{T}-\rho\dot{L}^2\sigma^2)}{\rho L^2}\right]\tilde{y}_{\sigma\sigma} \\ -\frac{(g-\ddot{L})}{L}\tilde{y}_\sigma(0) \end{pmatrix}. \tag{74}$$

We are going to prove the well-posedness of the Cauchy problem

$$\dot{w} + A(t)w = 0,\ t > 0,\ \text{and}\ w(0) = w^0 \tag{75}$$

where w^0 is given in X by applying [7, Theorem 4.1] (one could use different methods, as for example those given in [8, Chapter IV]). In order to apply [7, Theorem 4.1], one performs a new change of variables. Let $y^\sharp$, $z^\sharp$, and $b^\sharp$ defined by

$$y^\sharp(\sigma, t) = \tilde{y}(\sigma, t), \tag{76}$$

$$z^\sharp(\sigma, t) = z(\sigma, t) - \sigma\alpha(t)\tilde{y}(\sigma, t) + \frac{\sigma\beta(t)}{L(t)}\tilde{y}_\sigma(\sigma, t), \tag{77}$$

$$b^\sharp(t) = b(t). \tag{78}$$

Let $w^\sharp = (y^\sharp, z^\sharp, b^\sharp)$. Then (72) is equivalent to

$$\dot{w}^\sharp + A^\sharp(t)w^\sharp = 0, \tag{79}$$

where the *time-varying* operator $A^\sharp$ is defined, for every $t \in [0, +\infty)$, by

$$\begin{aligned} Dom(A^\sharp(t)) = Y := \Big\{ &(y^\sharp, z^\sharp, b^\sharp) \in H^2(0,1) \times H^1(0,1) \times \mathbb{R}; \\ &z^\sharp(0) = b^\sharp,\ z^\sharp(1) = 0 \Big\}, \end{aligned} \tag{80}$$

$$A^\sharp(t)w^\sharp = \begin{pmatrix} -z^\sharp - \sigma\alpha y^\sharp + \frac{\sigma\beta}{L} y^\sharp_\sigma \\ A^\sharp_2 \\ -\frac{(g-\ddot{L})}{L} y^\sharp_\sigma(0) \end{pmatrix}, \tag{81}$$

with

$$\begin{aligned} A^\sharp_2 = &-2\frac{\dot{L}\sigma}{L}\left(z^\sharp + \sigma\alpha y^\sharp - \frac{\sigma\beta}{L} y^\sharp_\sigma\right)_\sigma + \left(\frac{2\dot{L}^2\sigma}{L^2} - \frac{\ddot{L}\sigma}{L} - \frac{g-\ddot{L}}{L}\right) y^\sharp_\sigma \\ &- \left(\frac{\tilde{T} - \rho\dot{L}^2\sigma^2}{\rho L^2}\right) y^\sharp_{\sigma\sigma} + \sigma\Bigg[\dot{\alpha} y^\sharp - \frac{(\dot{\beta}L - \beta\dot{L})}{L^2} y^\sharp_\sigma \\ &+ \alpha\left(z^\sharp + \sigma\alpha y^\sharp - \frac{\sigma\beta}{L} y^\sharp_\sigma\right) - \frac{\beta}{L}\left(z^\sharp + \sigma\alpha y^\sharp - \frac{\sigma\beta}{L} y^\sharp_\sigma\right)_\sigma\Bigg]. \end{aligned}$$

Let us now check that the assumptions of [7, Theorem 4.1] are satisfied. Let us first emphasize that $Dom(A^\sharp(t)) = Y$ does not depend on t and let us recall that Y is a Hilbert space for the scalar product

$$\langle w^\sharp, \hat{w}^\sharp \rangle_Y = \int_0^1 (y^\sharp_{\sigma\sigma}\hat{y}^\sharp_{\sigma\sigma} + z^\sharp_\sigma \hat{z}^\sharp_\sigma)d\sigma + \langle w^\sharp, \hat{w}^\sharp \rangle_X$$

with $w = (y^\sharp, z^\sharp, b^\sharp)$ and $\hat{w} = (\hat{y}^\sharp, \hat{z}^\sharp, \hat{b}^\sharp)$. With this new variable $w^\sharp$, the energy function defined by (52) (see also (70)) is

$$E = a(t, w^\sharp, w^\sharp),$$

where $a(t, \cdot, \cdot)$ is the symmetric quadratic form defined on X by

$$\begin{aligned} a(t, w^\sharp, \hat{w}^\sharp) &= \frac{1}{2}\int_0^1 \Bigg[\rho\left(z^\sharp + \sigma\alpha y^\sharp - \frac{\sigma\beta + \sigma\dot{L}}{L} y^\sharp_\sigma\right) \\ &\quad \left(\hat{z}^\sharp + \sigma\alpha\hat{y}^\sharp - \frac{\sigma\beta + \sigma\dot{L}}{L}\hat{y}^\sharp_\sigma\right) + \tilde{T}\frac{y^\sharp_\sigma \hat{y}^\sharp_\sigma}{L^2}\Bigg] L d\sigma \\ &\quad + \frac{1}{2} m z^\sharp(0,t)\hat{z}^\sharp(0,t) + \frac{K}{2} y^\sharp(1,t)\hat{y}^\sharp(1,t), \end{aligned} \tag{82}$$

with $w^\sharp = (y^\sharp, z^\sharp, b^\sharp)$ and $\hat{w}^\sharp = (\hat{y}^\sharp, \hat{z}^\sharp, \hat{b}^\sharp)$. Note that (61) gives

$$\begin{aligned} &\frac{\partial a}{\partial t}(t, w^\sharp, w^\sharp) - 2a(t, A^\sharp(t)w^\sharp, w^\sharp) \\ &\leqslant C_1(|\dot{L}| + |L - L_c|)a(t, w^\sharp, w^\sharp), \ \forall (t, w^\sharp) \in [0, +\infty) \times Y. \end{aligned} \tag{83}$$

But there exists a positive constant C_3 (depending on $L(0)$ and $\dot{L}(0)$) such that

$$\left|\frac{\partial a}{\partial t}(t, w^\sharp, w^\sharp)\right| \leqslant C_3 a(t, w^\sharp, w^\sharp), \ \ \forall (t, w^\sharp) \in [0, +\infty) \times X, \tag{84}$$

which, with (83), gives the existence of a positive constant C_4 (depending on $L(0)$ and $\dot{L}(0)$) such that

$$a(t, A^\sharp(t)w^\sharp, w^\sharp) \geqslant -C_4 a(t, w^\sharp, w^\sharp), \ \forall (t, w^\sharp) \in [0, +\infty) \times Y. \tag{85}$$

Let us point out that there exists also a positive constant C_5 (depending on $L(0)$ and $\dot{L}(0)$) such that

$$\begin{aligned} C_5^{-1}\langle w^\sharp, w^\sharp \rangle_X \leqslant \ a(t, w^\sharp, w^\sharp) \leqslant C_5 \langle w^\sharp, w^\sharp \rangle_X, \\ \forall \ (t, w^\sharp) \in [0, +\infty) \times X. \end{aligned} \tag{86}$$

Let X_t be the space X with scalar product

$$\langle w^\sharp, \hat{w}^\sharp \rangle_{X_t} = a(t, w^\sharp, \hat{w}^\sharp).$$

By (86), the norm $\| \cdot \|_t$ induced by this scalar product is equivalent to the norm $\| \cdot \|$ associated with the original scalar product $\langle \cdot, \cdot \rangle_X$. Straightforward arguments on elliptic equations show the existence of a constant $C_6 > C_4$ (depending on $L(0)$ and $\dot{L}(0)$) such that the range of $C_6 I + A(t)$ is X for every $t \geqslant 0$. Hence, by (85) and the Lumer-Phillips theorem (see e.g. [11, Chapter 1, Theorem 4.3]) applied to $C_4 I + A(t)$ and the Hilbert space X_t we have that, with the notations of [7, p. 243],

$$A^\sharp(t) \in G(X_t, 1, C_4), \ \ \forall t \in [0, +\infty). \tag{87}$$

Inequalities (86), (87) and [7, Proposition 3.4] imply that, with Definition 3.1 of [7], for every $T \in (0, +\infty)$, $\{A(t);\ t \in [0, T]\}$ is stable. Let us also point out that there exists a positive constant C_7 (depending on $L(0)$ and $\dot{L}(0)$) such that

$$C_7^{-1} \langle w^\sharp, w^\sharp \rangle_Y \leqslant a(t, A^\sharp(t) w^\sharp, A^\sharp(t) w^\sharp) \leqslant C_7 \langle w^\sharp, w^\sharp \rangle_Y,$$
$$\forall\, (t, w^\sharp) \in [0, +\infty) \times Y,$$

which with (85) also implies that (ii) of [7, Theorem 4.1] holds (with, with the notations of [7], $\tilde{M} = C_7$ and $\tilde{\beta} = C_4$). Clearly (iii) of [7, Theorem 4.1] also holds. Hence we can apply [7, Theorem 4.1]. Letting $T \to +\infty$ in this theorem and denoting by $\mathcal{L}(X)$ the set of linear continuous maps from X into X equipped with the usual topology, we get the following proposition.

Proposition 3. *There exists a unique continuous map $W^\sharp : \Delta := \{(t, s); 0 \leqslant s \leqslant t < +\infty\} \mapsto \mathcal{L}(X)$ such that*

$$W^\sharp(s, s) = Identity,$$
$$\forall T,\ \exists M > 0, \exists \eta > 0 \ s.t.\ |W^\sharp(t, s)|_{\mathcal{L}(X)} \leqslant M e^{\eta(t-s)},\ \ \forall (t, s) \in \Delta \ s.t.\ t \leqslant T,$$
$$W^\sharp(t, s) W^\sharp(s, r) = W^\sharp(t, r), r \leqslant s \leqslant t,$$
$$(\partial_t^+ W^\sharp(t, s) w^\sharp)_{t=s} = -A^\sharp(s) w^\sharp,\ w^\sharp \in Y,\ s \geqslant 0,$$
$$\partial_s W^\sharp(t, s) w^\sharp = W^\sharp(t, s) A^\sharp(s) w^\sharp,\ w^\sharp \in Y,\ (t, s) \in \Delta,$$

where the derivatives are in the strong sense in X (and where ∂_s means right derivative if $s = 0$ and left derivative if $s = t$).

Coming back to the w variable, we define $W : \Delta := \{(t, s); 0 \leqslant s \leqslant t < +\infty\} \mapsto \mathcal{L}(X)$ by

$$W(t, s)w = S(t)W^{\sharp}(t, s)S(s)^{-1}w$$

where, for every $t \geqslant 0$, $S(t) \in \mathcal{L}(X)$ is defined by

$$S(t)(y^{\sharp}, z^{\sharp}, b^{\sharp}) = (y^{\sharp}, z, b^{\sharp}),$$

with

$$z(\sigma, t) = z^{\sharp}(\sigma, t) + \sigma\alpha(t)y^{\sharp}(\sigma, t) - \frac{\sigma\beta(t)}{L(t)}y^{\sharp}_{\sigma}(\sigma, t).$$

Then the (weak) solution of (75) is $w = W(t, 0)w^0$. Similarly, coming back to the variable y, we get from this w a function y that we call the (weak) solution to (64) satisfying the initial conditions (65)-(66) with $(y^0, y^1) \in H^1(0, L(0)) \times L^2(0, L(0))$.

Under some regularity and compatibility assumptions on y^0 and y^1, y is a solution of class C^2 of (64) which satisfies (65)-(66) in the usual sense.

Before stating the next proposition, let us point out that if y is a solution of (64) satisfying (93) to (95), substracting equation (9) at $x = 0$ from equation (11) gives, using (10):

$$y_{xx}(0, t) = 0. \tag{88}$$

Let us now prove the following proposition.

Proposition 4. *Let us assume that y^0, y^1 are such that*

$$y^0 \in H^3(0, L(0)) \text{ and } y^1 \in H^2(0, L(0)), \tag{89}$$

$$y^0_{xx}(0) = 0, \tag{90}$$

$$y^1(L) = \alpha(0)y^0(L) - (\beta + \dot{L})(0)y^0_x(L), \tag{91}$$

$$\begin{aligned}\left(\frac{(m + \rho L)(g - \ddot{L})}{\rho} + \dot{L}^2 + \beta\dot{L}\right)(0)y^0_{xx}(L) + \left[(\dot{\beta} - \alpha\dot{L})(0) + g\right]y^0_x(L)\\ -\dot{\alpha}(0)y^0(L) + (\beta + 2\dot{L})(0)y^1_x(L) - \alpha(0)y^1(L) = 0.\end{aligned} \tag{92}$$

Then

$$y \in C^0([0, +\infty); H^3(0, L(t))), \tag{93}$$

$$y_t \in C^0([0, +\infty); H^2(0, L(t))), \tag{94}$$

$$y_{tt} \in C^0([0, +\infty); H^1(0, L(t))). \tag{95}$$

In particular y is of class C^2 on $\{(x, t);\ 0 \leqslant x \leqslant L(t),\ 0 \leqslant t\}$. Moreover y satisfies (64), (65) and (66) in the usual sense.

Remark 5. Of course, by (93) to (95), we mean that

$$\tilde{y} \in C^0([0, +\infty); H^3(0, 1)),$$
$$\tilde{y}_t \in C^0([0, +\infty); H^2(0, 1)),$$
$$\tilde{y}_{tt} \in C^0([0, +\infty); H^1(0, 1)).$$

Proof of Proposition 4. One first points out that (93) to (95) are equivalent to

$$w_0^\sharp \in Y, \tag{96}$$

$$A^\sharp(0) w_0^\sharp \in Y. \tag{97}$$

Let $w^\sharp(t) = W^\sharp(t, 0) w_0^\sharp$. From (96) and [7, Theorem 6.1 and Remark 6.2] one gets that

$$w^\sharp \in C^1([0, +\infty); X) \cap C^0([0, +\infty); Y). \tag{98}$$

Let $r \in C^0([0, +\infty); Y)$ be defined by

$$r(t) = A(t) w^\sharp(t).$$

Then

$$r(t) = W^\sharp(t, 0) r(0) + \int_0^t W^\sharp(t, s) \dot{A}(s) w^\sharp(s) ds. \tag{99}$$

By (97), $r(0) \in Y$. Moreover $s \to \dot{A}(s) w^\sharp(s)$ is in $\in C^0([0, +\infty); X)$. Hence, by [7, Theorem 6.1, Remark 6.2 and Theorem 7.2]

$$r \in C^1([0, +\infty); X) \cap C^0([0, +\infty); Y), \tag{100}$$

which implies that

$$y^\sharp \in C^0([0, +\infty); H^3(0, 1)), \tag{101}$$

$$y_t^\sharp \in C^0([0, +\infty); H^2(0, 1)), \tag{102}$$

$$y_{tt}^\sharp \in C^0([0, +\infty); H^1(0, 1)). \tag{103}$$

Hence (93) to (95) hold. One then easily checks that (64), (65) and (66) hold in the usual sense, this ends the proof. □

Remark 6. Instead of Kato's theorem, one could probably use a variational approach developed by J.-L. Lions (see [10, Chapter VIII]).

4 Strong global asymptotic stability

It follows from the previous sections (see in particular Propositions 1 and 2) that $(L = L_c, \dot{L} = 0, y = 0)$ is a stable equilibrium point for the closed-loop system (20)-(64) in the following sense: for every $\epsilon_0 > 0$, there exists $\epsilon_1 > 0$ such that every (maximal) solution of the closed-loop system (L, v, y) (20)-(64) such that

$$|L(0) - L_c| + |v(0)| + |y(\cdot, 0)|_{H^1(0,L(0))} + |y_t(\cdot, 0)|_{L^2(0,L(0))} < \epsilon_1$$

is defined for all time and satisfies

$$|L(t) - L_c| + |v(t)| + |y(\cdot, t)|_{H^1(0,L(t))} + |y_t(\cdot, t)|_{L^2(0,L(t))} < \epsilon_0, \quad \forall t \geqslant 0.$$

The goal of this section is to prove that, if

$$K < 2(m + \rho L_c)g, \tag{104}$$

then $(L = L_c, \dot{L} = 0, y = 0)$ is a global attractor in the following sense: every (maximal) solution (L, v, y) of the closed-loop system (20)-(64) such that

$$(L(0), v(0)) \in \Omega, \tag{105}$$

$$(y(\cdot, 0), y_t(\cdot, 0)) \in H^1(0, L(0)) \times L^2(0, L(0)), \tag{106}$$

which, by the previous sections, is defined for every $t \geqslant 0$, satisfies, as $t \to +\infty$,

$$|L(t) - L_c| + |v(t)| \to 0, \tag{107}$$

$$|y(\cdot, t)|_{H^1(0,L(t))} + |y_t(\cdot, t)|_{L^2(0,L(t))} \to 0. \tag{108}$$

Property (107) follows from (40). Property (108) is a consequence of the following proposition

Proposition 5. *Assume that (104) holds. Then, for every (maximal) solution (L, v, y) of the closed-loop system (20)-(64) satisfying (105) and (106),*

$$\lim_{t \to +\infty} E(t) = 0. \tag{109}$$

From now on we assume that (104) holds. We first prove Proposition 5 when y is smooth (subsection 4.1) and then get the general case by density (subsection 4.2).

4.1 The smooth case

In this subsection we assume (L, v) is a solution of the closed-loop system (20) such that (105) holds. We also assume that y^0 and y^1 satisfy (89) to (92). Then, by Proposition 4, the (classical) solution y of the Cauchy problem (64), (65), (66) is such that (93), (94) and (95) hold.

The goal of this subsection is to prove the following proposition

Proposition 6. *For every solution y of (64) satisfying (93) to (95), one has (109).*

The main ingredient for the proof of this theorem is an estimate on the following stronger norm

$$E_1 = \frac{1}{2} \int_0^{L(t)} \left(\frac{((Ty_x)_x)^2}{\rho} + Ty_{xt}^2\right) dx + m(g - \ddot{L})^2 \frac{y_x^2(0, t)}{2}, \tag{110}$$

given in the following lemma.

Lemma 2. *For every solution y of (64) satisfying (93) to (95), one has*

$$E_1 \in L^\infty(0, +\infty). \tag{111}$$

To prove Lemma 2, we have first to prove the following one.

Lemma 3.

$$\int_0^{+\infty} \dot{X}_p^2 \, dt < +\infty.$$

Proof. Using (60) and replacing X_p by its value given by (56), equation (53) can be rewritten as:

$$\dot{E} = \dot{X}_p^2\left(\frac{K}{\alpha} + \frac{\rho\dot{L}}{2}\right) + \frac{\dot{L}}{2}\theta^2\left(-a + \rho\dot{L}^2\right) + \rho\dot{L}^2\dot{X}_p\theta + B_3,$$

where the expression of B_3 is given in (55).

We apply the Cauchy-Schwarz inequality to the product $\dot{X}_p\theta$. Moreover, from (56), we have

$$\frac{\theta^2}{2} \leqslant \frac{1}{\beta^2}\left(\dot{X}_p^2 + \alpha^2 X_p^2\right).$$

Therefore, the previous equality on $\dot{E}$ implies

$$\dot{E} \leqslant -\lambda_0 \dot{X}_p^2 + \epsilon(t) X_p^2 + B_3(t), \text{ with}$$

$$\begin{cases} \lambda_0 = -\frac{K}{\alpha} - \rho\frac{\dot{L}}{2} - \rho\frac{\dot{L}^2}{2} - \frac{|\dot{L}||\rho\dot{L}+\rho\dot{L}^2-a|}{\beta^2}, \\ \epsilon(t) = \frac{\alpha^2}{\beta^2} \mid \dot{L} \mid\mid \rho\dot{L} + \rho\dot{L}^2 - a \mid . \end{cases}$$

But we have

$$\epsilon(t) X_p^2 + B_3(t) \leqslant h(t) E,$$

where h is a positive function in $L^1(0, +\infty)$, since $\epsilon(t)$ and $T_t(x, t)$ (and therefore $B_3(t)$) "have terms in $L^{(3)}$ as a factor".

From Proposition 2, we know that $\mid \dot{L} \mid$ exponentially converges to zero. Therefore, there exists a time $t_1 \geqslant 0$ and $\eta > 0$ such that

$$\lambda_0 > \eta, \quad \forall t \geqslant t_1.$$

Hence

$$\eta \dot{X}_p^2 \leqslant -\dot{E} + h(t)E, \ \ \forall t \geqslant t_1.$$

Integrating this inequality from t_1 to $+\infty$ leads to

$$\eta \int_{t_1}^{+\infty} \dot{X}_p^2 dt \leqslant E(t_1) - E(+\infty) + \int_{t_1}^{+\infty} h(t)E(t)dt,$$

which is bounded from Proposition 2 and Lemma 5 in Appendix 1, and this allows to conclude. □

Our last lemma to prove Lemma 2 is the following one.

Lemma 4. *There exists a time $t_2 \geqslant 0$, (depending only on $(L(0), v(0))$) such that the following inequality is satisfied for the time derivative of the norm E_1*

$$\dot{E}_1 \leqslant f(t)E_1 + f_1(t)\,, \ \ \forall t \geqslant t_2, \tag{112}$$

where $f(t)$ and $f_1(t)$ are L^1 positive time functions defined on $[t_2, +\infty)$.

Proof of Lemma 4. Computing $\dot{E}_1$ we obtain

$$\begin{aligned} \dot{E}_1 \ &= \int_0^L \frac{(Ty_x)_x}{\rho}(T_{xt}y_x + T_x y_{xt} + T_t y_{xx} + T y_{xxt}) + \frac{T_t}{2} y_{xt}^2 + T y_{xt} y_{xtt}\, dx \\ &\quad + \frac{1}{2}\dot{L}\left(\frac{((Ty_x)_x)^2(L,t)}{\rho} + T(L,t)y_{xt}^2(L,t)\right) \\ &\quad + m(g-\ddot{L})^2 y_x(0,t)y_{xt}(0,t) - \frac{1}{2}m(g-\ddot{L})L^{(3)}y_x^2(0,t). \end{aligned}$$

Due to the expression (110) of E_1, the terms

$$\int_0^L \left[(Ty_x)_x(T_{xt}y_x + T_t y_{xx}) + \frac{T_t}{2}y_{xt}^2\right] dx$$

and

$$m(g-\ddot{L})L^{(3)}y_x^2(0,t)$$

are all smaller than $|l(t)|E_1$, $l(t)$ exponentially converging to 0 since $\dot{L}$ or higher time derivatives of L are factors of these terms.

There remains to study

$$R = R_1 + \frac{1}{2}\dot{L}\left(\frac{((Ty_x)_x)^2(L,t)}{\rho} + T(L,t)y_{xt}^2(L,t)\right)$$
$$+ m(g - \ddot{L})^2 y_x(0,t)y_{xt}(0,t),$$

where

$$R_1 = \int_0^L \left[\frac{(Ty_x)_x}{\rho}(T_x y_{xt} + T y_{xxt}) + T y_{xt} y_{xtt}\right] dx.$$

Differentiating (9) with respect to x and using (10) and (88) give

$$y_{xtt} = \frac{2T_x y_{xx} + T y_{xxx}}{\rho}.$$

Replacing y_{xtt} by this expression in R_1 and integrating by parts leads, after some computations, to

$$R_1 = \frac{1}{\rho}\left[y_{xt}T(Ty_x)_x\right]_0^L.$$

Consequently, we have to study the following boundary terms

$$\frac{1}{\rho}\left[y_{xt}T(Ty_x)_x\right]_0^L + \frac{1}{2}\dot{L}\left(\frac{((Ty_x)_x)^2(L,t)}{\rho} + T(L,t)y_{xt}^2(L,t)\right)$$
$$+ m(g - \ddot{L})^2 y_x(0,t)y_{xt}(0,t).$$

But

$$\frac{T(0,t)T_x(0,t)}{\rho} = m(g - \ddot{L})^2$$

and using (88), one sees that terms in $x = 0$ simplify. Finally, there remains to study terms in $x = L$, which are gathered in the following quantity Z:

$$Z = \frac{T(L,t)y_{xt}(L,t)(Ty_x)_x(L,t)}{\rho} + \frac{1}{2}\dot{L}\left(\frac{((Ty_x)_x)^2(L,t)}{\rho} + T(L,t)y_{xt}^2(L,t)\right).$$

But, the position X_p of the platform is given by $X_p = \hat{y}(s = 0, t) = y(L,t)$. Therefore

$$y_{tt}(L,t) = \ddot{X}_p - 2\dot{L}y_{xt}(L,t) - \dot{L}^2 y_{xx}(L,t) - \ddot{L}y_x(L,t). \tag{113}$$

On the other hand, equation (9) at $x = L$ gives

$$y_{tt}(L,t) = \frac{(Ty_{xx})(L,t)}{\rho} + (g - \ddot{L})y_x(L,t). \tag{114}$$

Moreover, using the expression of the feedback (56) and (12) leads to

$$\begin{aligned} \ddot{X}_p &= \alpha\dot{X}_p + \beta\dot{\theta} + \dot{\alpha}X_p + \dot{\beta}\theta \\ &= \alpha\dot{X}_p - \beta(y_{xt} + \dot{L}y_{xx})(L,t) + \dot{\alpha}X_p - \dot{\beta}y_x(L,t). \end{aligned} \tag{115}$$

Then, (114) and (113) (with $\ddot{X}_p$ replaced by (115)) constitute a system of 2 equations with the two unknowns $y_{tt}(L,t)$ and $y_{xx}(L,t)$. When solving this system and replacing $y_{tt}(L,t)$ and $y_{xx}(L,t)$ by their respective values in the expression of Z, it can be easily shown that Z is a second order polynomial in X_p, $\dot{X}_p$, $y_x(L,t)$ and $y_{xt}(L,t)$.

Moreover, if we consider the asymptotic situation $\dot{L} = \ddot{L} = 0$, Z takes the asymptotic value Z^∞ given by

$$Z^\infty = -\beta T(L,t)y_{xt}^2(L,t) + \alpha T(L,t)\dot{X}_p y_{xt}(L,t).$$

Applying the Cauchy-Schwarz inequality to the product $\dot{X}_p y_{xt}(L,t)$ and from (60) we obtain

$$Z^\infty \leqslant -T(L,t)y_{xt}^2(L,t)(\beta - \frac{K\beta}{2a}) + \frac{T(L,t)K\beta}{2a}\dot{X}_p^2.$$

By (60) and (104) there exists $t_3 > t_2$ such that $K < 2a$ for every $t \in [t_3, +\infty)$, which implies

$$Z^\infty \leq \frac{T(L,t)K\beta}{2a}\dot{X}_p^2, \quad \forall t \in [t_3, +\infty).$$

Then, there exists a time $t_2 \geqslant 0$ such that inequality (112) is satisfied with

$$f(t) = |l(t)| \text{ and } f_1(t) = \frac{T(L,t)K\beta}{2a}\dot{X}_p^2.$$

The function f is in $L^1(0,+\infty)$ since it exponentially converges to 0 and f_1 is in $L^1(0,+\infty)$ from Lemma 3, which concludes the proof. □

From inequality (112) and Lemma 5 in Appendix 1, the conclusion of Lemma 2 follows. □

Finally, let us prove Proposition 6. It follows from Lemma 2 that

$$\tilde{y} \in L^\infty([0,+\infty); H^2(0,1)), \tag{116}$$

$$\tilde{y}_t \in L^\infty([0,+\infty); H^1(0,1)), \tag{117}$$

$$\tilde{y}_{tt} \in L^\infty([0,+\infty); L^2(0,1)), \tag{118}$$

where $\tilde{y}$ is defined from y by (67). Moreover, by (61) and (62) there exists $e^\infty \in [0,+\infty)$ such that

$$\lim_{t\to+\infty} E(t) = e^\infty. \tag{119}$$

Let $(t_n \geqslant 0;\ n \in \mathbb{N})$ be an increasing sequence tending to $+\infty$ as n tends to $+\infty$ and let $\tilde{y}^n(\sigma,t) = \tilde{y}(\sigma, t+t_n)$. By (116) to (119) one gets that, for any $n_0 \geqslant 0$ and with $Q_{n_0} = [0,1]\times[-t_{n_0}, +\infty)$,

- the set $\left\{\tilde{y}^n_{|Q_{n_0}};\ n \geqslant n_0\right\}$ is bounded in $L^\infty(Q_{n_0})$,
- the sets $\left\{(\tilde{y}^n_t)_{|Q_{n_0}}\right\}$ and $\left\{(\tilde{y}^n_x)_{|Q_{n_0}}\right\}$ with $n \geqslant n_0$ are uniformly equicontinuous on Q_{n_0}.

Hence, by Ascoli's theorem, there exists a subsequence of the sequence $(t_n \geqslant 0;\ n \in \mathbb{N})$, that we still denote $(t_n \geqslant 0;\ n \in \mathbb{N})$, and $\tilde{y}^\infty \in C^1([0,1]\times\mathbb{R})$ such that

$$\tilde{y}^\infty \in L^\infty(\mathbb{R}; H^2(0,1)), \tag{120}$$

$$\tilde{y}^\infty_t \in L^\infty(\mathbb{R}; H^1(0,1)), \tag{121}$$

$$\tilde{y}^\infty_{tt} \in L^\infty(\mathbb{R}; L^2(0,1)), \tag{122}$$

$$\tilde{y}^n \to \tilde{y}^\infty \text{ in } C^1_{loc}([0,1]\times\mathbb{R}) \text{ as } n \to +\infty, \tag{123}$$

$$\tilde{E}^\infty(t) = e^\infty, \quad \forall t \in \mathbb{R}, \tag{124}$$

where, in (124), $\tilde{E}^\infty(t)$ denotes the energy of $\tilde{y}^\infty(\cdot,t)$ with $L = L_c$ and $\dot{L} = 0$, i.e. (see (70))

$$\begin{aligned} E^\infty(t) &= \frac{1}{2}\int_0^1 \left[\rho(\tilde{y}^\infty_t)^2 + (m+\rho L_c\sigma)\frac{(\tilde{y}^\infty_\sigma)^2}{L_c^2}\right] L_c d\sigma \\ &+ \frac{1}{2} m(\tilde{y}^\infty_t(0,t))^2 + K\frac{(\tilde{y}^\infty(1,t))^2}{2}. \end{aligned} \tag{125}$$

Letting $n \to +\infty$ in the first equation of (69), one gets, using (123),

$$\tilde{y}_{tt}^{\infty} - \left(\frac{m + \rho L_c \sigma}{\rho L_c^2} g \tilde{y}_{\sigma}^{\infty}\right)_{\sigma} = 0. \tag{126}$$

Let us now give the boundary conditions satified by $\tilde{y}^{\infty}$. From the second equation of (69), one gets

$$\tilde{y}_{tt}^{n}(0,t) = \frac{g - \ddot{L}(t + t_n)}{L(t + t_n)} \tilde{y}_{\sigma}^{n}(0,t),$$

which, with (123), gives as $n \to +\infty$,

$$\tilde{y}_{tt}^{\infty}(0,\cdot) = \frac{g}{L_c} \tilde{y}_{\sigma}^{\infty}(0,\cdot) \text{ in } \mathcal{D}'(\mathbb{R}). \tag{127}$$

From the third equation of (69), one gets

$$\tilde{y}_{t}^{n}(1,t) = \alpha(t + t_n)\tilde{y}^{n}(1,t) - \frac{\beta(t + t_n)}{L(t + t_n)} \tilde{y}_{\sigma}^{n}(1,t),$$

which, with (58), (60) and (123), gives as $n \to +\infty$,

$$\tilde{y}_{t}^{\infty}(1,t) = -\frac{K}{\sqrt{\rho g(m + \rho L_c)}} \tilde{y}^{\infty}(1,t) - \frac{\sqrt{g(m + \rho L_c)}}{L_c\sqrt{\rho}} \tilde{y}_{\sigma}^{\infty}(1,t). \tag{128}$$

Note that (126), (127) and (128) are the equations corresponding to the closed-loop system when the length L of the cable is constant. Hence it follows from [2, 1, 9] that

$$\int_0^1 (\tilde{y}_t^{\infty}(\sigma,t))^2 + (\tilde{y}_{\sigma}^{\infty}(\sigma,t))^2 d\sigma \to 0 \text{ as } t \to +\infty,$$

which, with (124), gives $e^{\infty} = 0$ and ends the proof of Proposition 6. □

4.2 The general case

Let $y : \{(x,t);\ x \in [0, L(t)],\ 0 \leqslant t\} \to \mathbb{R}$ be a weak solution of (64) and let $\epsilon > 0$. Then there exist $\bar{y}^0$ and $\bar{y}^1$ satisfying (89) to (92) such that

$$|y(\cdot,0) - \bar{y}^0|_{H^1(0,L(0))} + |y_t(\cdot,0) - \bar{y}^1|_{L^2(0,L(0))} \leqslant \epsilon. \tag{129}$$

Let $\bar{y} : \{(x,t);\ x \in [0, L(t)], 0 \leqslant t\} \to \mathbb{R}$ be the (classical) solution of (64) such that

$$\bar{y}(\cdot,0) = \bar{y}^0 \text{ and } \bar{y}_t(\cdot,0) = \bar{y}^1.$$

Let us denote by $\bar{E}(t)$ the energy at time t of $\bar{y}(\cdot,t)$, i.e.

$$\bar{E}(t) = \frac{1}{2}\int_0^{L(t)} \left(\rho\bar{y}_t^2 + T(x,t)\bar{y}_x^2\right)dx + \frac{1}{2}m\bar{y}_t^2(0,t) + K\frac{\bar{y}(L(t),t)^2}{2}.$$

It follows from Proposition 6 that

$$\lim_{t\to+\infty} \bar{E}(t) = 0. \tag{130}$$

But $y - \bar{y}$ is a solution of (64). Hence, by (62),

$$\Delta(t) \leqslant C_2\Delta(0), \tag{131}$$

where $\Delta(t)$ is the energy at time t of $y(\cdot,t) - \bar{y}(\cdot,t)$. But there exists $C_8 > 0$, depending on $(L(0), \dot{L}(0))$ but not on ϵ, such that

$$\Delta(0) \leqslant C_8\left(|y(\cdot,0) - \bar{y}^0|_{H^1(0,L(0))} + |y_t(\cdot,0) - \bar{y}^1|_{L^2(0,L(0))}\right)^2. \tag{132}$$

Let $E(t)$ be the energy at time t of $y(\cdot,t)$. From (129), (130) and (132), one gets

$$\limsup_{t\to+\infty} E(t) \leqslant C_2C_8\epsilon^2, \tag{133}$$

which gives (109). □

5 Conclusion

We have obtained a boundary controller, asymptotically stabilizing the overhead crane with a variable length flexible cable. For simplicity, we have considered the velocity of the platform as the control variable. One could use a backstepping approach, as was done e.g. [3], to derive a stabilizing control for the original cascaded system.

6 Appendix 1

Lemma 5 (Gronwall's lemma). *Let* $E \in C^0([0,+\infty);\ [0,+\infty))$, $a \in L^1(0,+\infty;\ [0,+\infty))$ *and* $b \in L^1(0,+\infty;\ [0,+\infty))$ *be such that*

$$\dot{E} \leqslant b(t) + a(t)E. \tag{134}$$

Then $E \in L^\infty(0,+\infty)$ *and* $g \in L^\infty(0,+\infty)$, *where* $g(t) = \int_0^t (b(\tau) + a(\tau)E(\tau))d\tau$.

Proof. Using (134) we can write

$$E(t) \leqslant E(0) + g(t). \tag{135}$$

All that remains to be proved is that $g(t)$ is bounded. Noticing that $\dot{g}(t) = b(t) + a(t)E(t)$, the previous inequality can be rewritten as follows

$$\dot{g} - ga \leqslant aE(0) + b$$

and multiplying this last inequality by $e^{-\int_0^t a(\tau)d\tau}$ leads to

$$\frac{d}{dt}\left(ge^{-\int_0^t a(\tau)d\tau}\right) \leqslant (aE(0)+b)e^{-\int_0^t a(\tau)d\tau}.$$

Integrating this inequality from 0 to t and multiplying by $e^{\int_0^t a(\tau)d\tau}$ gives, since $g(0) = 0$,

$$g(t) \leqslant (e^{\int_0^t a(\tau)d\tau}) \int_0^t (a(\tau)E(0) + b(\tau))e^{-\int_0^\tau a(s)ds}d\tau,$$

which means that g is bounded and then, by (135), the same holds for E. This ends the proof. □

7 Acknowledgments

The authors are really indebted to Michel Amestoy, Professor of mechanics at the École des Mines de Paris, for his precious collaboration concerning the obtention of the dynamical equations of the system and also to an anonymous referee for his useful suggestions.

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