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ON THE NULL ASYMPTOTIC STABILIZATION OF THE TWO-DIMENSIONAL INCOMPRESSIBLE EULER EQUATIONS IN A SIMPLY CONNECTED DOMAIN*

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Abstract. We study the asymptotic stabilization of the origin for the two-dimensional (2-D) Euler equation of incompressible inviscid fluid in a bounded domain. We assume that the controls act on an arbitrarily small nonempty open subset of the boundary. We prove the null global asymptotic stabilizability by means of explicit feedback laws if the domain is connected and simply connected.

Key words. inviscid fluid, stabilization, nonlinear control

AMS subject classifications. 76C99, 93B52, 93C20, 93D15

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1. Introduction.

In previous papers [2, 3] we have considered the problem of the controllability of the two-dimensional (2-D) Euler equation of incompressible inviscid fluid in a bounded domain. In particular, we have proved that, if the controls act on an arbitrarily small open subset of the boundary which meets every connected component of this boundary, then the 2-D Euler equation is exactly controllable. This result has been extended recently by Glass to the three-dimensional (3-D) Euler equation in [9, 10].

For linear control systems, the exact controllability implies the asymptotic stabilizability by means of feedback laws. This is well known for linear control systems of finite dimension and, by Slemrod [22], J.-L. Lions [17], Lasiecka–Triggiani [16], and Komornik [15], it also holds in infinite dimension in very general cases. But, as pointed out by Sussmann in [24], Sontag–Sussman in [23], and Brockett in [1], this is no longer true for *nonlinear* control systems, even of finite dimension. Let us also notice that, as in the counterexample of [1], the linearized control system of the Euler equation around the origin is not controllable.

Therefore it is natural to ask what is the situation for the asymptotic stabilizability of the origin for the 2-D Euler equation of incompressible inviscid fluid in a bounded domain when the controls act on an arbitrarily small open subset of the boundary which meets any connected component of this boundary. In this paper we prove the null global asymptotic stabilizability by means of feedback laws if the domain is simply connected.

Our paper is organized as follows.

- In section 2, we give explicit feedback laws which globally asymptotically stabilize the origin and state our main results.
- In sections 3 and 4, we give the proofs of our main results.

2. Explicit stabilizing feedbacks.

Let Ω be a nonempty bounded connected and simply connected subset of $\mathbb{R}^2$ of class C^∞ and let γ be a nonempty open subset of the boundary $\partial\Omega$ of Ω. This set γ is the location of the control. Let y be the velocity

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field of the inviscid fluid contained in Ω. We assume that the fluid is incompressible, so that

$$\text{div } y = 0. \tag{2.1}$$

Since Ω is simply connected, y is completely characterized by $\omega := \text{curl } y$ and $y \cdot n$ on $\partial\Omega$, where n denotes the unit outward normal to $\partial\Omega$. For the problem of controllability, one does not really need to specify the control and the state: one considers the "Euler control system" as an underdetermined system by requiring $y \cdot n = 0$ on $\partial\Omega \setminus \gamma$ instead of $y \cdot n = 0$ on $\partial\Omega$ as for the uncontrolled usual Euler equation. For the stabilization problem, one needs to specify more precisely the control and the state. In this paper the state is ω. For the control there are at least two natural possibilities:

(a) The control is $y \cdot n$ on γ and the time derivative $\partial\omega/\partial t$ of the vorticity at the points of γ where $y \cdot n < 0$, i.e., at the points where the fluid enters into the domain Ω.

(b) The control is $y \cdot n$ on γ and the vorticity ω at the points where $y \cdot n < 0$.

Let us point out that, by (2.1), in both cases $y \cdot n$ has to satisfy $\int_{\partial\Omega} y \cdot n = 0$.

2.1. Case where one controls the time derivative of the vorticity of the incoming flow. In this subsection we concentrate on case (a); for case (b), see subsection 2.2.

Let us give our stabilizing feedback laws. Let $g \in C^\infty(\partial\Omega)$ be such that

$$\text{Support } g \subset \gamma, \tag{2.2}$$

$$\gamma_+ := \{g > 0\} \text{ and } \gamma_- := \{g < 0\} \text{ are connected,} \tag{2.3}$$

$$g \neq 0, \tag{2.4}$$

$$\overline{\gamma_+} \cap \overline{\gamma_-} = \emptyset, \tag{2.5}$$

$$\int_{\partial\Omega} g = 0. \tag{2.6}$$

For any compact set K of $\mathbb{R}^q$ and any $f \in C^0(K; \mathbb{R}^m)$, we denote

$$|f|_{0,K} = \text{Max } \{|f(x)|\,; x \in K\}.$$

For simplicity, we write $|f|_0$ instead of $|f|_{0,\overline{\Omega}}$. Our stabilizing feedback laws are

$$y \cdot n = M\, |\omega|_0\, g \text{ on } \gamma,$$
$$\frac{\partial\omega}{\partial t} = -M\, |\omega|_0\, \omega \text{ on } \gamma_- \text{ if } |\omega|_0 \neq 0,$$

where $M > 0$ is large enough. With these feedback laws, a function $\omega : I \times \overline{\Omega} \to \mathbb{R}$, where I is an interval, is a solution of the closed loop system Σ if

$$\frac{\partial\omega}{\partial t} + \text{ div } (\omega y) = 0 \text{ in } \overset{\circ}{I} \times \Omega, \tag{2.7}$$

$$\text{div } y = 0 \text{ in } \overset{\circ}{I} \times \Omega, \tag{2.8}$$

$$\text{curl } y = \omega \text{ in } \overset{\circ}{I} \times \Omega, \tag{2.9}$$

$$y(t) \cdot n = M\, |\omega(t)|_0\, g \text{ on } \partial\Omega\, \forall t \in I, \tag{2.10}$$

$$\frac{\partial\omega}{\partial t} = -M\, |\omega(t)|_0\, \omega \text{ on } \{t;\, \omega(t) \neq 0\} \times \gamma_-, \tag{2.11}$$

where, for $t \in \Omega$, $\omega(t) : \overline{\Omega} \to \mathbb{R}$ and $y(t) : \overline{\Omega} \to \mathbb{R}^2$ are defined by requiring $\omega(t)(x) = \omega(t,x), y(t)(x) = y(t,x), \forall x \in \overline{\Omega}$. More precisely, the definition of a solution of system Σ is as follows.

DEFINITION 2.1. *Let I be an interval. A function $\omega : I \to C^0(\overline{\Omega})$ is a solution of system Σ if*

(i) $\omega \in C^0(I; C^0(\overline{\Omega}))(\cong C^0(I \times \overline{\Omega}))$,

(ii) *for $y \in C^0(I \times \overline{\Omega}; \mathbb{R}^2)$ defined by requiring* (2.8) *and* (2.9) *in the sense of distributions and* (2.10)*, one has* (2.7) *in the sense of distributions,*

(iii) *in the sense of distributions on the open manifold $\{t \in \overset{\circ}{I}; \omega(t) \neq 0\} \times \gamma_-$ one has $\partial\omega/\partial t = -M\, |\omega(t)|_0\, \omega$.*

Our first theorem says that, for M large enough, the Cauchy problem for system Σ has at least one solution defined on $[0, +\infty)$ for any initial data in $C^0(\overline{\Omega})$. More precisely one has the following.

THEOREM 2.2. *There exists $M_0 > 0$ such that, for any $M \geqslant M_0$, the following two properties hold:*

(i) *For any $\omega_0 \in C^0(\overline{\Omega})$, there exists a solution of system Σ defined on $[0, +\infty)$ such that $\omega(0) = \omega_0$.*

(ii) *Any maximal solution of system Σ defined at time* 0 *is defined on* $[0, +\infty)$ *(at least).*

REMARK 2.3. (a) *In this theorem, property* (i) *is in fact implied by property* (ii) *and Zorn's lemma. We state* (i) *in order to emphasize the existence of a solution to the Cauchy problem for system Σ.* (b) *We do not know if the solution to the Cauchy problem is unique for positive time. (For negative time, one does not have uniqueness since there are solutions ω of system Σ defined on $[0, +\infty)$ such that $\omega(0) \neq 0$ and $\omega(T) = 0$ for $T \in [0, +\infty)$ large enough.) But let us emphasize that, already for control systems in finite dimension, one considers feedback laws which are merely continuous; with these feedback laws, the Cauchy problem for the closed loop system may have many solutions. It turns out that this lack of uniqueness is not a real problem. Indeed, in finite dimension at least, if a point is asymptotically stable for a continuous vector field, then there exists, as in the case of regular vector fields, a (smooth) strict Lyapunov function. This result is due to Kurzweil* [13]. *It is tempting to conjecture that a similar result holds in infinite dimension under reasonable assumptions. The existence of this Lyapunov function ensures some robustness to perturbations. It is precisely this robustness which makes the interest of feedback laws compared to open loop controls. We will see that, for our feedback laws, there exists also a strict Lyapunov—see Proposition* 3.6 *below—and therefore our feedback laws provide some kind of robustness.*

Our next theorem shows that, at least for M large enough, our feedback laws globally and strongly asymptotically stabilize the origin in $C^0(\overline{\Omega})$ for system Σ.

THEOREM 2.4. *There exists a positive constant $M_1 \geqslant M_0$ such that, for any $\varepsilon \in (0, 1]$, any $M \geqslant M_1/\varepsilon$, and any maximal solution ω of system Σ defined at time* 0,

$$|\omega(t)|_0 \leqslant \operatorname{Min} \left\{ |\omega(0)|_0, \frac{\varepsilon}{t} \right\} \forall t > 0. \tag{2.12}$$

REMARK 2.5. *Due to the term $|\omega(t)|_0$ appearing in* (2.10) *and in* (2.11) *our feedback laws do not depend only on the value of ω on γ. Let us point out that there is no asymptotically stabilizing feedback law depending only on the value of ω on γ such that the origin is asymptotically stable for the closed loop system. In fact, given a nonempty open subset Ω_0 of Ω, there is no feedback law which does not depend on the values of ω on Ω_0. This phenomenon is due to the existence of "phantom vortices":*

there are smooth stationary solutions $\bar{y} : \overline{\Omega} \to \mathbb{R}^2$ *of the* 2*-D Euler equations such that* Support $\bar{y} \subset \Omega_0$ *and* $\bar{\omega} :=$ curl $\bar{y} \neq 0$*; see, e.g.,* [20]. *Then* $\omega(t) = \bar{\omega}$ *is a solution of the closed loop system if the feedback law does not depend on the values of* ω *on* Ω_0 *and vanishes for* $\omega = 0$.

REMARK 2.6. *Let us emphasize that* (2.12) *implies that*

$$|\omega(t)|_0 \leqslant \varepsilon \quad \forall t \geqslant 1, \tag{2.13}$$

for any maximal solution ω *of system* Σ *defined at time 0 (whatever* $\omega(0)$ *is). It would be interesting to know if one could have a similar result for the* 2*-D Navier–Stokes equations of viscous incompressible flows, that is, if given* $\varepsilon > 0$*, does there exist a feedback law such that* (2.13) *holds for any solution of the closed loop Navier–Stokes control system? Note that* $y = 0$ *on* γ *is a feedback which leads to asymptotic stabilization of the null solution of the Navier–Stokes control system. But this feedback does not have the required property. For recent results on the controllability of the Navier–Stokes control system, see the papers by Imanuvilov and Fursikov* [6, 7, 8] *and the paper by Imanuvilov* [12] *as well as* [4, 5].

2.2. Case where one controls the vorticity of the incoming flow. In this subsection one does no longer control the time-derivative of the vorticity of the incoming flow but the vorticity of the incoming flow itself. Therefore the control is $y \cdot n$ on γ with the constraint $\int_{\partial\Omega} y \cdot n = 0$ (as above) and the vorticity ω at the points where $y \cdot n < 0$. Of course in this new situation one cannot take the state ω in $C^0(\overline{\Omega})$: if the state ω is in $C^0(\overline{\Omega})$, then it will determine part of the control, namely, the vorticity of the incoming flow. It is therefore natural to consider the state ω as being in $L^\infty(\Omega)$.

For any measurable subset B of $\mathbb{R}^q$ and any $f \in L^\infty(B; \mathbb{R}^m)$, we denote by $|f|_{\infty,B}$ the essential supremum of f on B. For simplicity, we write $|f|_\infty$ instead of $|f|_{\infty,\overline{\Omega}}$. Our stabilizing feedback laws are

$$y \cdot n = M\, |\omega|_\infty\, g \text{ on } \gamma, \tag{2.14}$$

$$\omega = 0 \text{ on } \gamma_- \text{ if } |\omega|_\infty \neq 0, \tag{2.15}$$

where, again, $M > 0$ is large enough. With these feedback laws, a function $\omega : I \times \overline{\Omega} \to \mathbb{R}$, where I is an interval, is a solution of the closed loop system that we call Σ_1, if one has (2.7), (2.8), (2.9) and

$$y(t) \cdot n = M\, |\omega(t)|_\infty\, g \text{ on } \partial\Omega \text{ for almost every } t \in I, \tag{2.16}$$

$$\omega = 0 \text{ on } \{t;\, \omega(t) \neq 0\} \times \gamma_-. \tag{2.17}$$

Since $\omega(t)$ is only in $L^\infty(\Omega)$, the meaning of (2.17) has to be specified. As usual, (2.17) has to be understood in a "weak sense," which is obtained by multiplying (2.7) by suitable smooth test functions, integrating on $I \times \Omega$, and performing integration by parts. More precisely, the definition of a solution of system Σ_1 is as follows.

DEFINITION 2.7. *Let* I *be an interval. A function* $\omega : I \to L^\infty(\Omega)$ *is a solution of system* Σ_1 *if*

(i) $\omega \in C^0(I; H^{-1}(\Omega))$,
(ii) $\omega \in L^\infty_{loc}(I; L^\infty(\Omega)) \cong L^\infty_{loc}(I \times \overline{\Omega})$,
(iii) *for any* $\varphi \in C^1(I \times \overline{\Omega})$ *with compact support such that*

$$\text{Support } \varphi \subset \left(\overset{\circ}{I} \times \Omega\right) \cup \left(\{t \in \overset{\circ}{I};\, |\omega(t)|_\infty > 0\} \times \gamma_-\right), \tag{2.18}$$

one has

$$\int_{I\times\Omega}\left(\omega\frac{\partial\varphi}{\partial t}+\omega(y\cdot\nabla)\varphi\right)=0, \tag{2.19}$$

where $y\in L^\infty\left(I;C^0\left(\overline{\Omega}\right)\right)$ *is defined by requiring* (2.16) *and, in the sense of distributions on* $\overset{\circ}{I}\times\Omega$, (2.8) *and* (2.9).

Our first theorem says that, for M large enough, the Cauchy problem for system Σ_1 has at least one solution defined on $[0,+\infty)$ for any initial data in $L^\infty(\Omega)$. More precisely, one has the following.

THEOREM 2.8. *There exists* $M_2\geqslant 0$ *such that, for any* $M\geqslant M_2$, *the following two properties hold.*

(i) *For any* $\omega_0\in L^\infty(\Omega)$, *there exists a solution of system* Σ_1 *defined on* $[0,+\infty)$ *such that* $\omega(0)=\omega_0$.

(ii) *Any maximal solution of system* Σ_1 *defined at time* 0 *is defined on* $[0,+\infty)$ *(at least).*

Our next theorem, which is analogous to Theorem 2.4, tells us that, at least for M large enough, feedbacks laws (2.14) and (2.15) globally and strongly asymptotically stabilize the origin in $L^\infty(\Omega)$ for system Σ_1.

THEOREM 2.9. *There exists a positive constant* $M_3>0$ *such that, for any* $\epsilon\in(0,1]$, *any* $M\geqslant M_3/\varepsilon$, *and any maximal solution of system* Σ_1 *defined at time* 0, *one has*

$$|\omega(t)|_\infty\leqslant\ \mathrm{Min}\ \left\{|\omega(0)|_\infty,\frac{\varepsilon}{t}\right\}\ \forall t>0.$$

The proof of Theorem 2.9 is very similar to the proof of Theorem 2.4 and therefore is omitted. The end of this paper is organized as follows.

- In section 3, we prove Theorems 2.2 and 2.4.
- In section 4, we prove Theorem 2.8.

3. Proof of Theorems 2.2 and 2.4.

3.1. Proof of Theorem 2.2. For a compact subset K and a function $y\in C^0(K;\mathbb{R}^2)$, we let

$$q_K(y):=|y|_0+\sup\{|y(x)-y(x')|/r(|x-x'|);(x,x')\in K^2,x\neq x'\},$$

where

$$r(s)=s+s\ln(1/s)\ \ \forall s\in(0,1),\ \text{and}\ r(s)=s\ \ \forall s\geqslant 1. \tag{3.1}$$

For simplicity, we write q instead of $q_{\overline{\Omega}}$. For technical reasons, it is useful to extend $y\in C^0(\overline{\Omega})$ outside $\overline{\Omega}$. Let $R>0$ be such that

$$\overline{\Omega}\subset B_{R/2}:=\{x\in\mathbb{R}^2;|x|<R/2\}.$$

Let $B_R:=\{x\in\mathbb{R}^2;|x|<R\}$. Let $\mathcal{P}:C^0(\overline{\Omega};\mathbb{R}^2)\to C^0(\overline{B_R};\mathbb{R}^2)$ be a continuous linear map such that

$$\mathcal{P}(y)(x)=y(x)\ \ \forall x\in\overline{\Omega}\ \ \forall y\in C^0(\overline{\Omega};\mathbb{R}^2), \tag{3.2}$$

$$\text{Support}\ \mathcal{P}(y)\subset\overline{B_{R/2}}\ \ \forall y\in C^0(\overline{\Omega};\mathbb{R}^2), \tag{3.3}$$

and such that, for some $C_0>0$,

$$q_{\overline{B_R}}(\mathcal{P}\left(y\right))\leqslant C_0q(y)\,\forall y\in C^0(\overline{\Omega};\mathbb{R}^2). \tag{3.4}$$

Let us recall the following important theorem, due to Wolibner [25] and Yudovich [26] (see also [14, Lemma 2.6]).

THEOREM 3.1 (see Wolibner [25] and Yudovich [26]). *Let T be a positive real number and let $y \in L^\infty\left((0,T);C^0\left(\overline{B_R};\mathbb{R}^2\right)\right)$ be such that*

(3.5)
$$y(t,x) := y(t)(x) = 0 \text{ for almost everywhere (a.e.) } (t,x) \in (0,T) \times \left(B_R \setminus B_{R/2}\right),$$

and, for some constant $K \in (0,+\infty)$,

$$q_{\overline{B_R}}(y(t)) \leqslant K \text{ for a.e. } t \in (0,T). \tag{3.6}$$

Then there exists one and only one map $\Phi^y \in C^0([0,T]\times[0,T]\times\overline{B_R};\overline{B_R})$, $(t,s,x) \to \Phi^y(t,s,x)$ such that

$$\Phi^y(t,s,x) = x + \int_s^t y(t',\Phi^y(t',s,x))dt' \,\forall (t,s,x) \in [0,T]\times[0,T]\times\overline{B_R}.$$

Moreover there exist two constants $C_1 = C_1(K,R,T) > 0$ and $\delta = \delta(K,R,T) > 0$, depending on K, R, T, such that, for any $(x,x') \in \overline{B_R}^2$, for any $(t,t',s,s') \in [0,T]^4$, and for any $y \in L^\infty\left((0,T);C^0\left(\overline{B_R};\mathbb{R}^2\right)\right)$ satisfying (3.5) and (3.6),

$$|\Phi^y(t',s',x') - \Phi^y(t,s,x)| \leqslant C_1(|s'-s|^\delta + |t'-t|^\delta + |x'-x|^\delta). \tag{3.7}$$

Our proof of Theorem 2.2 is divided in two parts.

- We first prove the existence of a solution to the Cauchy problem for small positive time.
- Then we prove that any maximal solution to Σ defined at time 0 is defined on $[0,+\infty)$.

So let us first start with the proof of the following proposition.

PROPOSITION 3.2. *There exists $M_0 > 0$ such that, for any $M \geqslant M_0$ and for any $\omega_0 \in C^0(\overline{\Omega})$, there exists $T > 0$ and a solution of system Σ defined on $[0,T]$ such that $\omega(0) = \omega_0$.*

Of course if $\omega_0 = 0$ one can take arbitrary $T > 0$ and choose $\omega = 0$. Therefore we may assume that $\omega_0 \neq 0$. Then there exists a point x^0 in Ω such that

$$\omega_0(x^0) \neq 0. \tag{3.8}$$

Let $M > 0$. Let $C_2 > 0$ (depending on M) be such that

$$|y|_0 \leqslant C_2 \tag{3.9}$$

for any $y \in C^0(\overline{\Omega};\mathbb{R}^2)$ such that

$$|\text{curl } y|_0 \leqslant |\omega_0|_0, \text{ div } y = 0,$$
$$|y\cdot n| \leqslant M|\omega_0|_0|g|_{0,\partial\Omega} \text{ on } \partial\Omega.$$

Let $\rho > 0$ be such that

$$B(x^0,\rho) := \{x \in \mathbb{R}^2; |x - x^0| < \rho\} \subset \Omega \tag{3.10}$$

and let

$$T = \rho/C_2. \tag{3.11}$$

Let us denote by $| \ |_{H^{-1}(\Omega)}$ one of the usual norm of the Sobolev space $H^{-1}(\Omega)$. Let $C_3 > 0$ be such that, for any $f \in L^\infty(\Omega;\mathbb{R}^2)$,

$$|\text{div } f|_{H^{-1}(\Omega)} \leqslant C_3 \, |f|_{L^\infty(\Omega)} \, . \tag{3.12}$$

Let also $C_4 > 0$ be such that, for any divergence free $f \in C^0(\overline{\Omega};\mathbb{R}^2)$ with a bounded curl,

$$|f|_0 \leqslant C_4 \left(| \text{ curl } f|_{L^\infty(\Omega)} + |f \cdot n|_{0,\partial\Omega} \right). \tag{3.13}$$

We are going to construct a solution $\omega \in C^0([0,T] \times \overline{\Omega})$ of system Σ satisfying the initial condition $\omega(0) = \omega_0$ as a fixed point of a map $F : X \to X$, where X is the set of functions $\omega \in C^0([0,T] \times \overline{\Omega})$ such that

$$\omega(0) = \omega_0, \tag{3.14}$$

$$t \in [0,T] \to |\omega(t)|_0 \text{ is nonincreasing,} \tag{3.15}$$

$$\left| \frac{\partial \omega}{\partial t} \right|_{L^\infty((0,T);H^{-1}(\Omega))} \leqslant C_3 C_4 |\omega_0|_0^2 \, (M|g|_{0,\partial\Omega} + 1) \, . \tag{3.16}$$

Note that X is a closed convex subset of $C^0([0,T] \times \overline{\Omega})$ equipped with the sup-norm $| \ |_{0,[0,T]\times\overline{\Omega}}$. Let us define F. For $\omega \in X$, let us define $\tilde{y}_\omega \in C^0([0,T] \times \overline{\Omega};\mathbb{R}^2)$ by requiring

$$\text{div } \tilde{y}_\omega = 0 \text{ in } (0,T) \times \Omega, \tag{3.17}$$

$$\text{curl } \tilde{y}_\omega = \omega \text{ in } (0,T) \times \Omega, \tag{3.18}$$

$$\tilde{y}_\omega(t,\cdot) \cdot n = M \text{ Max } \{|\omega(t)|_0, |\omega_0(x^0)|\} g \text{ on } \partial\Omega \ \forall t \in [0,T]. \tag{3.19}$$

Note that, by (3.15), (3.14), (3.17), (3.18), (3.19), and a theorem due to Wolibner [25] (see also [14, Lemma 1.4]), there exists a constant C_5 such that

$$q(\tilde{y}_\omega(t,\cdot)) \leqslant C_5 \, \forall t \in [0,T] \ \forall \omega \in X. \tag{3.20}$$

Let $y_\omega \in C^0([0,T] \times \overline{B_R};\mathbb{R}^2)$ be defined by

$$y_\omega(t,\cdot) = \mathcal{P}(\tilde{y}_\omega(t,\cdot)) \ \forall t \in [0,T]. \tag{3.21}$$

By (3.4) and (3.20),

$$q_{\overline{B_R}}(y_\omega(t,\cdot) \leqslant C_0 C_5 \ \ \forall \omega \in X. \tag{3.22}$$

In particular, by the Wolibner–Yudovich theorem (Theorem 3.1), there exists a flow Φ^{y_ω} associated with y_ω. For any interval I containing 0 such that $0 = \text{Min} I$ and for any $y \in L^\infty_{\text{loc}}\left(I; C^0\left(\overline{B_R};\mathbb{R}^2\right)\right)$ satisfying (3.5) and such that $q_{\overline{B_R}}(y) \in L^\infty_{\text{loc}}(I)$, let us define $s_y : I \times \overline{\Omega} \to I$ by

$$s_y(t,x) = \text{ Max } \{t' \in [0,t]; \, \Phi^y(t',t,x) \in \overline{\gamma_-}\}, \tag{3.23}$$

with the convention Max $\emptyset = 0$. Let us also define $a_y : I \times \overline{\Omega} \to \overline{\Omega}$ by

$$a_y(t,x) = \Phi^y(s_y(t,x),t,x). \tag{3.24}$$

With these notations, we can now define our map $F(\omega) : [0,T] \times \overline{\Omega} \to \mathbb{R}$ by

$$F(\omega)(t,x) = \omega_0 \left(a_{y_\omega} (t,x)\right) \exp \left(-M \int_0^{s_{y_\omega}(t,x)} |\omega(t')|_0 \, dt' \right). \tag{3.25}$$

It follows from our construction of F (recall also (3.17)) that, in the sense of distributions,

$$\frac{\partial F(\omega)}{\partial t} + \text{ div } (F(\omega)y_\omega) = 0 \text{ in } (0,T)\times\Omega. \tag{3.26}$$

Indeed, let $\bar{t}\in(0,T)$ and $\bar{x}\in\Omega$. Then, for (t,x) close enough to $(\bar{t},\bar{x})$, one has

$$s_{y_\omega}(t,\Phi^{y_\omega}(t,\bar{t},x)) = s_{y_\omega}(\bar{t},x),$$
$$a_{y_\omega}(t,\Phi^{y_\omega}(t,\bar{t},x)) = a_{y_\omega}(\bar{t},x),$$

which imply that

$$F(\omega)(t,\Phi^{y_\omega}(t,\bar{t},x)) = F(\omega)(\bar{t},x). \tag{3.27}$$

But (3.27), together with (3.2), (3.17), (3.21), and standard smoothing procedures, gives (3.26) in the sense of distributions.

Let us now check that if $\omega\in X$ is a fixed point of F, then ω is a solution of system Σ which, by (3.14), satisfies the Cauchy initial data $\omega(0)=\omega_0$. Indeed, let ω be a fixed point of F and let $y=\tilde{y}_\omega$. Then, from (3.17) and (3.18), we get (2.8) and (2.9). From (3.26), we get (2.7). Let us also point out that, by (3.23),

$$s_{y_\omega}(t,x) = t \ \ \forall t\in[0,T]\,\forall x\in\gamma_-. \tag{3.28}$$

From (3.24), (3.25), and (3.28), one gets

$$\omega(t,x) = F(\omega)(t,x) = \omega_0(x)\exp\left(-M\int_0^t |\omega(t')|_0\,dt'\right)\ \forall (t,x)\in[0,T]\times\gamma_-,$$

which implies (2.11). It remains only to verify that (2.10) holds. By (3.19), it suffices to check that

$$\left|\omega_0(x^0)\right| \leqslant |\omega(t)|_0 \ \forall t\in[0,T]. \tag{3.29}$$

From the definition of C_2, (3.2), (3.14), (3.15), (3.17), (3.18), (3.19), and (3.21), one gets that $|y_\omega(t)|_0\leqslant C_2$ for any $t\in[0,T]$, which, with (3.10) and (3.11), gives

$$\Phi^{y_\omega}(t,0,x^0)\in\Omega\ \ \forall t\in[0,T].$$

Therefore $s_{y_\omega}(t,\Phi^{y_\omega}(t,0,x_0))=0$ for any $t\in[0,T]$, which implies that

$$F(\omega)(t,\Phi^{y_\omega}(t,0,x^0)) = \omega_0(x^0)\,\forall t\in[0,T]. \tag{3.30}$$

From (3.30), one has

$$(|\omega(t)|_0 =)\,|F(\omega(t))|_0 \geqslant \left|\omega_0(x^0)\right|\ \ \forall t\in[0,T].$$

Therefore (3.29) holds and the fixed point ω of F is indeed a solution of system Σ.

By the Leray–Schauder fixed point theorem, in order to prove the existence of a fixed point to F, it suffices to check that

$$F(X)\subset X, \tag{3.31}$$

$$F \text{ is continuous}, \tag{3.32}$$

$$F(X) \text{ is relatively compact in } C^0([0,T]\times\overline{\Omega}). \tag{3.33}$$

Let us first check (3.31). Let $\omega\in X$. It follows directly from (3.25) that

$$|F(\omega)|_{L^\infty(\overline{\Omega}\times[0,T])}\leqslant|\omega_0|_0\,.$$

Clearly

$$F(\omega)(0) = \omega(0) = \omega_0. \tag{3.34}$$

Let us check that

$$t \in [0,T] \to |F(\omega(t))|_0 \text{ is nonincreasing.} \tag{3.35}$$

Let $0 \leqslant t_1 \leqslant t_2 \leqslant T$ and let $x \in \overline{\Omega}$. If $s_{y_\omega}(t_2,x) \leqslant t_1$, one has

$$F(\omega)(t_2,x) = F(\omega)\left(t_1, \Phi^{y_\omega}(t_1,t_2,x)\right).$$

If $s_{y_\omega}(t_2,x) > t_1$, one has

$$F(\omega)(t_2,x) = F(\omega)\left(a_{y_\omega}(t_2,x), t_1\right) \exp - \left(M \int_{t_1}^{s_{y_\omega}(t_2,x)} |\omega(t')|_0 dt' \right).$$

In both cases

$$|F(\omega)(t_2,x)| \leqslant |F(\omega)(t_1)|_0,$$

which shows (3.35). From (3.12), (3.13), (3.17), (3.18), (3.19), (3.26), (3.34), and (3.35), one gets that

$$\left| \frac{\partial F(\omega)}{\partial t} \right|_{L^\infty((0,T);H^{-1}(\Omega))} \leqslant C_3 C_4 |\omega_0|_0^2 \left(M|g|_{0,\partial\Omega} + 1 \right).$$

Therefore, in order to prove (3.31), it suffices to check that $F(\omega)$ is continuous on $[0,T] \times \overline{\Omega}$. From the continuity of Φ^{y_ω} and the definition (3.23) of s_{y_ω}, it is clear that

$$s_{y_\omega} \text{ is upper semicontinuous on } [0,T] \times \overline{\Omega}. \tag{3.36}$$

Since the continuity of s_{y_ω} implies the continuity of F, it remains only to check that

$$s_{y_\omega} \text{ is lower semicontinuous on } [0,T] \times \overline{\Omega}. \tag{3.37}$$

In order to prove this lower semicontinuity, let us assume that the following lemma, proved in Appendix B, holds.

LEMMA 3.3. *There exists $M_0 > 0$ such that, for any $T > 0$ and for any $y \in L^\infty\left((0,T); C^0\left(\overline{B_R}; \mathbb{R}^2\right)\right)$ satisfying* (3.5), (3.6) *for some $K > 0$, and for some function $\alpha \in L^\infty\left((0,T); (0,+\infty)\right)$,*

$$y(t,\cdot) \cdot n = \alpha(t) g \text{ on } \partial\Omega \text{ for a.e. } t \in (0,T), \tag{3.38}$$

$$M_0 |\text{curl } y(t)|_{L^\infty(\Omega)} \leqslant \alpha(t) \text{ for a.e. } t \in (0,T), \tag{3.39}$$

$$\text{div } y = 0 \text{ in } (0,T) \times \Omega \tag{3.40}$$

for any $(\tilde{t},\tilde{x}) \in (0,T] \times \overline{\gamma_-}$ and for any $\nu \in (0,\tilde{t})$, there exists $t \in (\tilde{t}-\nu, \tilde{t})$, such that

$$\Phi^y\left(t, \tilde{t}, \tilde{x}\right) \notin \overline{\Omega}.$$

Let us also point out that, by the definition of γ_-, (3.2), (3.19), (3.21), and (3.22), for any $\tilde{x} \in \overline{\Omega}$, if for $0 \leqslant t' \leqslant t \leqslant T$ $\Phi^{y_\omega}(t',t,\tilde{x})$ is not in $\overline{\Omega}$, then

$$\exists t'' \in [t',t] \text{ such that } \Phi^{y_\omega}(t',t,\tilde{x}) \in \overline{\gamma_-}. \tag{3.41}$$

This is indeed clear if y_ω is smooth enough, for example, locally Lipschitz with respect to x. The case where y is not smooth follows from the main ingredient, due to Wolibner [25], to prove the uniqueness of Φ^y in Theorem 3.1. Let us briefly sketch the proof. Let $W \in C^\infty(\mathbb{R}^2)$ be such that $\{W=0\}=\partial\Omega$, $W>0$ in Ω, $W<0$ in $\mathbb{R}^2\setminus\overline{\Omega}$ and ∇W does not vanish on $\partial\Omega$. Let $w(s)=W(\Phi^{y_\omega}(s,t,\tilde{x}))$. If (3.41) does not hold one easily sees, using (3.2), (3.19), (3.21), and (3.22), that there exists $C_6>0$ such that

$$\frac{\mathrm{d}w}{\mathrm{d}s} \leqslant C_6 r(|w|) \text{ on } [t',t], \tag{3.42}$$

where r is defined in (3.1). Since $w(t')<0\leqslant w(t)$, there exists $t_1\in(t',t]$ such that $w(t_1)=0$. Then, using (3.42) and the fact that $\int_0^1 /r(s)ds=+\infty$, one gets that $w\geqslant 0$ on $[t',t_1]$, which is in contradiction with the fact that $w(t')<0$. Hence we have (3.41). Using (3.2), (3.17), (3.18), (3.19), (3.21), and (3.22), one easily sees that $y=y_\omega$ satisfies the assumptions of Lemma 3.3 if $M\geqslant M_0$. From now on we assume that $M\geqslant M_0$ with M_0 as in Lemma 3.3. From the continuity of Φ^{y_ω}, the definition (3.23) of s_{y_ω} and Lemma 3.3, one easily sees that (3.37) holds. Hence we have (3.31). The continuity of F can be proved with the same type of arguments used to prove the continuity of $F(\omega)$. We omit the proof.

Let us now turn to the proof of (3.33). By Wolibner–Yudovich's theorem (Theorem 3.1), (3.22), (3.24), (3.25), and Ascoli's theorem, it suffices to check that

$$s_{y_\omega} \text{ is relatively compact in } C^0([0,T]\times\overline{\Omega}). \tag{3.43}$$

Let $(\omega_k;\, k\in\mathbb{N})$ be a sequence of functions in X. We want to prove the existence of a subsequence of the $(\omega_k;\, k\in\mathbb{N})$ converging in $C^0([0,T]\times\overline{\Omega})$. By Ascoli's theorem and Wolibner–Yudovich's theorem, (Theorem 3.1), the set $\{\Phi^{y_{\omega_k}};\, k\in\mathbb{N}\}$ is relatively compact in $C^0([0,T]\times[0,T]\times\overline{B_R};\overline{B_R})$. Hence, without loss of generality, we may assume the existence of $\Phi\in C^0([0,T]\times[0,T]\times\overline{B_R};\overline{B_R})$ such that the sequence $(\Phi^{y_{\omega_k}};\, k\in\mathbb{N})$ is converging to Φ in $C^0([0,T]\times[0,T]\times\overline{B_R};\overline{B_R})$. Associated with Φ is the function $s:[0,T]\times\overline{\Omega}:\to\mathbb{R}$ defined by (see (3.23))

$$s(t,x)=\text{ Max }\{t'\in[0,t];\, \Phi(t',t,x)\in\overline{\gamma_-}\}.$$

(Let us recall the convention Max $\emptyset$ =0.) Let $(t_k;\, k\in\mathbb{N})$ be a sequence of real numbers in $[0,T]$ converging to some $\bar{t}$ as k goes to $+\infty$. Let $(x_k;\, k\in\mathbb{N})$ be a sequence of points in $\overline{\Omega}$ converging to some $\bar{x}$ as k goes to $+\infty$. It is clear that

$$s(\bar{t},\bar{x})\geqslant \limsup_{k\to+\infty} s_{y_{\omega_k}}(t_k,x_k). \tag{3.44}$$

Therefore in order to prove (3.43) it suffices to check that

$$s(\bar{t},\bar{x})\leqslant \liminf_{k\to+\infty} s_{y_{\omega_k}}(t_k,x_k). \tag{3.45}$$

Indeed, from (3.44) and (3.45), one gets that the sequence $(s_{y_{\omega_k}};\, k\in\mathbb{N})$ converges uniformly to s on $[0,T]\times\overline{\Omega}$ as $k\to+\infty$. Let us again point out that Φ has the following property: for any $x\in\overline{\Omega}$, if for $0\leqslant t'\leqslant t\leqslant T$ $\Phi(t',t,x)$ is not in $\overline{\Omega}$, then there exists $t''\in[t',t]$ such that $\Phi(t'',t,x)\in\overline{\gamma_-}$. Indeed, it follows from the fact that the $\Phi^{y_{\omega_k}}$ have this property (see above) and converge to Φ in $C^0([0,T]\times[0,T]\times\overline{B_R};\overline{B_R})$ as k goes to $+\infty$. From this property and the convergence of the $\Phi^{y_{\omega_k}}$ to Φ in $C^0([0,T]\times[0,T]\times\overline{B_R};\overline{B_R})$ one easily sees, as above, that (3.45) holds if Φ satisfies the following property:

$$\forall 0<\nu<t\leqslant T\, \forall x\in\overline{\gamma_-},\ \Phi([t-\nu,t],t,x)\not\subset\overline{\Omega}. \tag{3.46}$$

Let us prove this property. Let $\theta \in C^\infty(\overline{\Omega})$ be defined by

$$\Delta\theta = 0 \text{ in } \overline{\Omega}, \tag{3.47}$$

$$\frac{\partial\theta}{\partial n} = g \text{ on } \partial\Omega. \tag{3.48}$$

Let us point out that the existence of θ follows from (2.6). Let us write

$$\tilde{y}_{\omega_k} = \alpha_k(t)\nabla\theta + \tilde{z}_k,$$

where $\alpha_k \in C^0([0,T])$ and $\tilde{z}_k \in C^0([0,T] \times \overline{\Omega}; \mathbb{R}^2)$ are defined by

$$\alpha_k(t) = M \text{ Max } \{|\omega_k(t)|_0, |\omega_0(x^0)|\} \,\forall t \in [0,T], \tag{3.49}$$

$$\text{div } \tilde{z}_k = 0 \text{ in } (0,T) \times \Omega, \tag{3.50}$$

$$\text{curl } \tilde{z}_k = \omega_k \text{ in } (0,T) \times \Omega, \tag{3.51}$$

$$\tilde{z}_k \cdot n = 0 \text{ on } [0,T] \times \partial\Omega. \tag{3.52}$$

Using (3.14) and (3.15) for $\omega = \omega_k$ and using (3.49), one gets

$$M|\omega_0(x^0)| \leqslant \alpha_k(t) \leqslant M|\omega_0|_0, \tag{3.53}$$

$$t \in [0,T] \to \alpha_k(t) \text{ is nonincreasing.} \tag{3.54}$$

From (3.53) and (3.54), we get the existence of $\alpha \in L^\infty(0,T)$ such that, extracting subsequences if necessary, one has

$$0 < M|\omega_0(x^0)| \leqslant \alpha(t) \text{ for a.e. } t \in (0,T), \tag{3.55}$$

$$\alpha_k(t) \to \alpha(t) \text{ as } k \to +\infty \text{ for a.e. } t \in (0,T). \tag{3.56}$$

Let us now turn to the sequence $(\tilde{z}_k; k \in \mathbb{N})$. Let us fix $r \in (2,+\infty)$. Using (3.14), (3.15), and (3.16) with $\omega = \omega_k$, using also (3.50), (3.51), and (3.52), one gets that

$$\tilde{z}_k \text{ is bounded in } C^0([0,T]; W^{1,r}(\Omega;\mathbb{R}^2)), \tag{3.57}$$

$$\frac{\partial\tilde{z}_k}{\partial t} \text{ is bounded in } L^\infty([0,T]; H^{-1}(\Omega;\mathbb{R}^2)). \tag{3.58}$$

Then, by a compactness lemma due to P.-L. Lions [18, Lemma C1, Appendix C] (take, with the notations of [18], $X = W^{1,r}(\Omega;\mathbb{R}^2)$ and $Y = H^{-1}(\Omega;\mathbb{R}^2)$) and by the Rellich–Kondrakov theorem, (3.57) and (3.58) imply that the sequence $(\tilde{z}_k; k \in \mathbb{N})$ is relatively compact in $C^0([0,T] \times \overline{\Omega}; \mathbb{R}^2)$. Hence, extracting subsequences if necessary, we may assume the existence of $\tilde{z} \in C^0([0,T] \times \overline{\Omega}; \mathbb{R}^2)$ such that $\tilde{z}_k$ tends to $\tilde{z}$ in $C^0([0,T] \times \overline{\Omega}; \mathbb{R}^2)$ as k tends to $+\infty$. Let $y : (0,T) \times B_R \to \mathbb{R}^2$ be defined by

$$y(t,\cdot) = \alpha(t)\mathcal{P}(\nabla\theta) + \mathcal{P}(\tilde{z}(t,\cdot)) \,\forall t \in (0,T). \tag{3.59}$$

Let us check that, if $M \geqslant M_0$, y satisfies the assumptions of Lemma 3.3. By (3.3) and (3.59), one has (3.5). It follows easily from (3.22) that

$$q_{\overline{B_R}}(y_\omega(t,\cdot) \leqslant C_0 C_5 \text{ for a.e. } t \in (0,T),$$

which implies (3.5). Passing to the limit in (3.52), one gets $z \cdot n = 0$, which with (3.59) gives (3.38). Finally, if $M \geqslant M_0$, (3.39) follows easily from the fact that

$$|\text{curl } y(t)|_{L^\infty(\Omega)} \leqslant \liminf_{k\to+\infty} |\text{curl } y_k(t)|_{L^\infty(\Omega)}.$$

Hence y satisfies the assumptions of Lemma 3.3, and, by this lemma, we have (3.46) if one has

$$\Phi = \Phi^y. \tag{3.60}$$

But, by the definition of $\Phi^{y_{\omega_k}}$,

$$\Phi^{y_{\omega_k}}(t,s,x) = x + \int_s^t y_{\omega_k}(t', \Phi^{y_{\omega_k}}(t',s,x))dt'. \tag{3.61}$$

Letting $k \to +\infty$ in (3.61), we get, by the dominated convergence theorem,

$$\Phi(t,s,x) = x + \int_s^t y(t', \Phi^{y_{\omega_k}}(t',s,x))dt',$$

for any $(t,s,x) \in [0,T] \times [0,T] \times \overline{B_R}$, which implies (3.60). This ends the proof of Proposition 3.2.

Let us now prove the next proposition.

PROPOSITION 3.4. *For any $M \geqslant M_0$, any maximal solution of system Σ defined at time 0 is defined on $[0,+\infty)$ (at least).*

Let us recall that $\omega : I \to C^0(\overline{\Omega})$ is a *maximal* solution of system Σ if, for any interval J containing I but not equal to I, there exists no solution of system Σ which is equal to ω on the interval I. The existence of a maximal solution follows, as usual, from Zorn's lemma; hence Theorem 2.2 is a corollary of Propositions 3.2 and 3.4.

Until the end of this paper, for any $\omega \in L^\infty(\Omega)$, one defines $\tilde{y}_\omega \in C^0(\overline{\Omega})$ and $y_\omega \in C^0(\overline{B_R})$ by requiring

$$\begin{gathered}\text{div } \tilde{y}_\omega = 0 \text{ and curl } \tilde{y}_\omega = \omega \text{ in } \Omega,\\ \tilde{y}_\omega \cdot n = M|\omega|_{L^\infty(\Omega)} g \text{ on } \partial\Omega,\\ y_\omega = \mathcal{P}(\tilde{y}_\omega).\end{gathered}$$

Of course if, for some interval I, ω is a map from I into $L^\infty(\Omega)$, the above conditions specified at any time in I, give maps $\tilde{y}_\omega : I \to C^0(\overline{\Omega})$ and $y_\omega : I \to C^0(\overline{B_R})$. With these notations, let us start the proof of Proposition 3.4 by the following simple observation.

LEMMA 3.5. *Let $T > 0$ and let $\omega \in C^0([0,T]; C^0(\overline{\Omega}))$ be a solution of system Σ. Then, for any $t \in [0,T]$ and for any $x \in \overline{\Omega}$,*

$$\omega(t,x) = \omega(0, a_{y_\omega}(t,x)) \exp\left(-M \int_0^{s_{y_\omega}(t,x)} |\omega(t')|_0 dt'\right). \tag{3.62}$$

In particular,

$$t \to |\omega(t)|_0 \text{ is nonincreasing.} \tag{3.63}$$

Indeed, for $t \in (0,T]$ and $x \in \overline{\Omega}$, let $\omega^* : [0,t] \to \mathbb{R}$ be defined by $\omega^*(t') = \omega(t', \Phi^{y_\omega}(t',t,x))$. Using (2.7), (2.8), and standard smoothing procedures, one gets that, in the sense of distributions,

$$\frac{\mathrm{d}\omega^*}{\mathrm{d}t'} = 0 \text{ on } (s_{y_\omega}(t,x), t).$$

In particular,

$$\omega(s_{y_\omega}(t,x), a_{y_\omega}(t,x)) = \omega^*(s_{y_\omega}(t,x)) = \omega^*(t) = \omega(t,x). \tag{3.64}$$

If $s_{y_\omega}(t,x)=0$, this gives (3.62). Let us study the case where $0<s_{y_\omega}(t,x)$. It follows directly from (2.11) that, in the sense of distributions,

$$\frac{\partial \omega}{\partial t'}(t',a) = -M|\omega(t')|_0\omega(t',a) \text{ on } \{t'\in(0,T)\}\,\forall a\in\gamma_-.$$

In particular,

$$\omega^*(s_{y_\omega}(t,x)) = \omega(0,a_{y_\omega}(t,x))\exp\left(-M\int_0^{s_{y_\omega}(t,x)}|\omega(t')|_0 dt'\right),$$

which, with (3.64), gives again (3.62). Property (3.63) follows from (3.62) (see the proof of (3.35)) or note that, if $0\leqslant t_1\leqslant T$, $t\in[0,T-t_1]\to\omega(t+t_1)$ is a solution of system Σ and apply, for $t=t_2-t_1\in[0,T-t_1]$, (3.62) to this solution.

Let $\omega\in C^0(I\times\overline{\Omega})\cong C^0(I;C^0(\overline{\Omega}))$ be a maximal solution to Σ such that I is an interval containing 0. Let $T=\operatorname{Sup} I$. We want to prove that $T=+\infty$. Let us assume that $T<+\infty$. From Proposition 3.2, it follows that $T>0$ and $T\notin I$. Therefore, in order to get a contradiction with the maximal property of ω, it suffices to check that

$$\omega(t) \text{ converges in } C^0(\overline{\Omega}) \text{ as } t\to T^-. \tag{3.65}$$

Indeed, if (3.65) holds, then $\bar\omega : I\cup\{T\}\to C^0(\overline{\Omega})$ defined by $\bar\omega=\omega$ on I and $\bar\omega(T)=\lim_{t\to T}\omega(t)$ is also a solution to system Σ.

If $|\omega(t)|_0\to 0$ as $t\to T^-$, (3.65) holds. Therefore, by (3.63), we may assume that, for some $\eta>0$,

$$|\omega(t)|_0\geqslant\eta\ \forall t\in[0,T). \tag{3.66}$$

As above, using, in particular, Lemma 3.3 and (3.66), one gets that s_{y_ω} is continuous on $[0,T]\times\overline{\Omega}$. Therefore $F(\omega)$ defined by (3.25) is continuous on $[0,T]\times\overline{\Omega}$. But, by Lemma 3.5, this function is equal to ω on $[0,T)\times\overline{\Omega}$. This proves (3.65) and therefore ends the proof of Theorem 2.2.

3.2. Proof of Theorem 2.4. Let $V: C^0(\overline{\Omega})\to[0,+\infty)$ be defined by

$$V(\omega)=|\omega\exp(-\theta)|_0,$$

where $\theta\in C^\infty(\overline{\Omega})$ satisfies (3.47) and (3.48). Theorem 2.4 is an easy consequence of the following proposition.

PROPOSITION 3.6. *There exists $M_4\geqslant M_0$ and $\mu>0$ such that, for any $M\geqslant M_4$ and any solution $\omega:[0,+\infty)\to C^0(\overline{\Omega})$ of system Σ, one has, for any $t\in[0,+\infty)$,*

$$[-\infty,0]\ni\dot V(t):=\frac{d}{dt^+}V(\omega(t))\leqslant -\mu M V^2(\omega(t)), \tag{3.67}$$

where $d/dt^+V(\omega(t)):=\lim_{\varepsilon\to0^+}(V(\omega(t+\varepsilon))-V(\omega(t)))/\varepsilon$.

Let us check that this proposition indeed implies Theorem 2.4. Let $\omega:[0,+\infty)\to C^0(\overline{\Omega})$ be a solution of system Σ with $\omega(0)\neq0$ and $M\geqslant M_4$. Integrating (3.67), one gets

$$V(\omega(t))\leqslant\frac{V(\omega(0))}{1+\mu M t V(\omega(0))}\ \forall t\geqslant 0.$$

In particular,

$$V(\omega(t))\leqslant\frac{1}{\mu M t}\ \forall t>0. \tag{3.68}$$

But

$$|\omega(t)|_0 \leqslant |\exp(\theta)|_0 V(\omega(t))\,\forall t \geqslant 0,$$

which, with (3.68), gives

$$|\omega(t)|_0 \leqslant \frac{|\exp(\theta)|_0}{\mu M t}\,\forall t > 0.$$

Since, by (3.63), $|\omega(t)|_0 \leqslant |\omega(0)|_0$ for any $t \geqslant 0$, we get (2.12) if one takes $M \geqslant |\exp(\theta)|_0/(\mu\varepsilon)$. Therefore Theorem 2.4 holds with

$$M_1 = \text{ Max } \{M_4, |\exp(\theta)|_0/\mu\}.$$

Let us now turn to the proof of Proposition 3.6. Clearly, since system Σ is autonomous, it suffices to check that (3.67) holds for $t = 0$. Let us also assume that the following lemma, which is proved in Appendix A, holds.

LEMMA 3.7. *For any x in $\overline{\Omega}$,*

$$\nabla\theta(x) \neq 0. \tag{3.69}$$

From this lemma and standard elliptic estimates, we get the existence of $\mu_0 \in (0, 1]$ such that

$$\nabla\theta \cdot (\nabla\theta + z) \geqslant \mu_0 \text{ on } \overline{\Omega} \tag{3.70}$$

for any $z \in C^0(\overline{\Omega}; \mathbb{R}^2)$ such that div $z = 0$, $|\text{curl } z|_0 \leqslant \mu_0$, and $z \cdot n = 0$ on $\partial\Omega$. For $t \geqslant 0$, let $x(t) \in \overline{\Omega}$ be such that

$$V(\omega(t)) = |\omega(t, x(t))| \exp(-\theta(x(t))).$$

For simplicity, we assume that $\omega(t, x(t)) \geqslant 0$; the case where $\omega(t, x(t)) < 0$ can be treated in a similar way. We have

$$V(\omega(t)) - V(\omega(0)) \leqslant \kappa(t), \tag{3.71}$$

with

$$\begin{aligned} \kappa(t) &= \omega(t, x(t)) \exp(-\theta(x(t))) - \omega(0, a_{y_\omega}(t, x(t))) \exp(-\theta(a_{y_\omega}(t, x(t)))) \\ &= V(\omega(t)) \left(1 - \exp\left(\theta(x(t)) - \theta(a_{y_\omega}(t, x(t))) + M \int_0^{s_{y_\omega}(t,x(t))} |\omega(t')|_0 dt'\right)\right). \end{aligned} \tag{3.72}$$

We choose $M_4 = \text{ Max } \{1/\mu_0, M_0\}$ and take any $M \geqslant M_4$. Let us again decompose y_ω in the following way:

$$y_\omega = M|\omega(t)|_0(\nabla\theta + z), \tag{3.73}$$

with div $z = 0$, curl $z = \omega/(M|\omega(t)|_0)$ ($= 0$ if $\omega(t) = 0$), and $z \cdot n = 0$ on $\partial\Omega$. For any x in $\overline{\Omega}$ and for any s in $[s_{y_\omega}(t, x), t]$, one has

$$\frac{\partial}{\partial s}\left(\theta\left(\Phi^{y_\omega}(s, t, x)\right)\right) = \nabla\theta(\Phi^{y_\omega}(s, t, x)) y_\omega\left(s, \Phi^{y_\omega}(s, t, x)\right),$$

which, with (3.70), (3.73), gives

$$\frac{\partial}{\partial s}\left(\theta\left(\Phi^{y_\omega}(s, t, x)\right)\right) \geqslant \mu_0 M|\omega(t)|_0.$$

In particular,

$$\theta(x(t)) - \theta\left(a_{y_\omega}(t, x(t))\right) \geqslant \mu_0 M|\omega(t)|_0 (t - s_{y_\omega}(t, x(t))),$$

which, with (3.63) and (3.72), implies that

$$\begin{aligned}\kappa(t) &\leqslant MV(\omega(t))|\omega(t)|_0 \left(-\mu_0 \left(t - s_{y_\omega}\left(t, x\left(t\right)\right)\right) - s_{y_\omega}\left(t, x\left(t\right)\right)\right)\\ &\leqslant -\mu_0 MV(\omega(t))|\omega(t)|_0 t\\ &\leqslant -\mu_0 M \text{ Min } \{\exp(\theta(x));\, x \in \overline{\Omega}\} V(\omega(t))^2 t\end{aligned}$$

Hence, one gets Proposition 3.6 by taking $\mu = \mu_0 \text{ Min } \{\exp(\theta(x));\, x \in \overline{\Omega}\}$.

4. Proof of Theorem 2.8. Let us first prove (i) of Theorem 2.8. We are going to deduce (i) of Theorem 2.8 from (the proof of) Theorem 2.2. Roughly speaking the idea is that, if we replace (2.11) with

$$\frac{\partial \omega}{\partial t} = -kM\left|\omega(t)\right|_0 \omega \text{ on } \{t;\, \omega(t) \neq 0\} \times \gamma_-,$$

then, as $k \to +\infty$, the solutions of the Cauchy problem for system Σ converge to a solution of the Cauchy problem for system Σ_1.

Reversing time in the proof of Lemma 3.3, one easily gets the following.

LEMMA 4.1. *There exists $M_5 \geqslant M_0$ such that, for any $T > 0$ and for any $y \in L^\infty\left((0, T); C^0\left(\overline{B_R}; \mathbb{R}^2\right)\right)$ satisfying* (3.5), (3.6) *for some $K > 0$, and for some function $\alpha \in L^\infty\left((0, T); (0, +\infty)\right)$,*

$$y(t, \cdot) \cdot n = \alpha(t) g \text{ on } \partial\Omega \text{ for a.e. } t \in (0, T), \tag{4.1}$$

$$M_5 |\text{ curl } y(t)|_{L^\infty(\Omega)} \leqslant \alpha(t) \text{ for a.e. } t \in (0, T), \tag{4.2}$$

$$\text{div y} = 0 \text{ in } (0, T) \times \Omega \tag{4.3}$$

for any $(\tilde{t}, \tilde{x}) \in [0, T) \times \overline{\gamma_+}$ and for any $\nu \in (0, T - \tilde{t})$, there exists $t \in (\tilde{t}, \tilde{t} + \nu)$, such that

$$\Phi^y\left(t, \tilde{t}, \tilde{x}\right) \notin \overline{\Omega}.$$

We choose $M_2 = \text{ Max } \{M_0, M_5\}$, where M_0 is defined in the proof of Lemma 3.3; see (B.3). Let $M \geqslant M_2$. Let $\omega_0 \in L^\infty(\Omega)$. There exists a sequence $(\omega_{0,k};\, k \in \mathbb{N}^*)$ of functions in $C^0(\overline{\Omega})$ such that

$$|\omega_{0,k}(x)| \leqslant |\omega_0|_{\infty, \Omega \cap B(x, 1/k)} \;\forall k \in \mathbb{N}^* \,\forall x \in \overline{\Omega}, \tag{4.4}$$

$$\omega_{0,k}(x) \underset{k \to +\infty}{\longrightarrow} \omega_0(x) \text{ for a.e. } x \in \Omega. \tag{4.5}$$

By (the proof of) Theorem 2.2, there exists a solution $\omega^k \in C^0([0, \infty); C^0(\overline{\Omega}))$ of system Σ, with (2.11) replaced by

$$\frac{\partial \omega_k}{\partial t} = -kM\left|\omega_k(t)\right|_0 \omega_k \text{ on } \{t;\, \omega_k(t) \neq 0\} \times \gamma_-, \tag{4.6}$$

such that $\omega_k(0) = \omega_{0,k}$. Let $\tilde{y}_k = \tilde{y}_{\omega_k} \in C^0([0, +\infty) \times \overline{\Omega}; \mathbb{R}^2)$ and let $y_k = y_{\omega_k} \in C^0([0, +\infty) \times \overline{B_R}; \mathbb{R}^2)$. Let $\alpha_k \in C^0([0, +\infty))$ and $\tilde{z}_k \in C^0([0, +\infty) \times \overline{\Omega}; \mathbb{R}^2)$ be defined by

$$\alpha_k(t) = M|\omega_k(t)|_0 \,\forall t \in [0, +\infty), \tag{4.7}$$

$$\text{div } \tilde{z}_k = 0, \quad \text{curl } \tilde{z}_k = \omega_k \text{ in } (0, +\infty) \times \Omega,$$

$$\tilde{z}_k \cdot n = 0 \text{ on } [0, +\infty) \times \partial\Omega.$$

One has $\tilde{y}_k = \alpha_k \nabla\theta + \tilde{z}_k$. As in the proof of (3.43), extracting subsequences if necessary, one gets the existence of ω in $L^\infty((0,+\infty);L^\infty(\Omega))\cap C^0([0,+\infty);H^{-1}(\Omega))$, of a nonincreasing function α in $L^\infty((0,+\infty),[0,+\infty))$ and of $\tilde{z}$ in $C^0([0,+\infty)\times\overline{\Omega})$ with $t\to q(\tilde{z}(t,\cdot))\in L^\infty_{\mathrm{loc}}(0,+\infty)$ such that, for any $T\in[0,+\infty)$, one has, as $k\to+\infty$

$$\omega_k \to \omega \text{ in } C^0([0,T];H^{-1}(\Omega)), \tag{4.8}$$

$$\omega_k \rightharpoonup \omega \text{ in } \sigma(L^1((0,+\infty)\times\Omega), L^\infty((0,+\infty)\times\Omega)), \tag{4.9}$$

$$\tilde{z}_k \to \tilde{z} \text{ in } C^0([0,T]\times\overline{\Omega};\mathbb{R}^2), \tag{4.10}$$

$$\alpha_k(t)\to\alpha(t) \text{ for a.e. } t\in(0,+\infty). \tag{4.11}$$

Let $\tilde{y}(t)=\alpha(t)\nabla\theta+\tilde{z}$. One has (2.8) and (2.9) for $y=\tilde{y}$ and $I=[0,T]$. Let $y\in L^\infty([0,+\infty);C^0(\overline{B_R};\mathbb{R}^2))$ be defined by $y(t,\cdot)=\mathcal{P}(\tilde{y}(t,\cdot))$. Let $\varphi\in C^1((0,+\infty)\times\overline{\Omega})$ with compact support such that

$$\text{Support } \varphi\subset((0,+\infty)\times\Omega)\cup(\{t\in(0,+\infty)\,;\,\alpha(t)>0\}\times\gamma_-)\,. \tag{4.12}$$

Let us check that (2.19) holds. Since α_k, $k\in\mathbb{N}^*$, and α are nonincreasing, one gets from (4.11) and (4.12) that, for some $k_0\in\mathbb{N}^*$, one has, for any $k\geqslant k_0$,

$$\text{Support } \varphi\subset((0,+\infty)\times\Omega)\times(\{t\in(0,+\infty);\,|\omega_k(t)|_0>0\}\times\gamma_-).$$

Hence, for any $k\geqslant k_0$,

$$\int_{(0,+\infty)\times\Omega}\left(\omega_k\frac{\partial\varphi}{\partial t}+\omega(\tilde{y}_k\cdot\nabla)\varphi\right)=\int_{(0,+\infty)\times\Omega}\alpha_k\omega_k g\varphi. \tag{4.13}$$

Let $t_0>0$ be such that

$$\alpha(t_0)>0, \tag{4.14}$$

$$\text{Support } \varphi\subset[0,t_0]\times(\Omega\cup\gamma_-). \tag{4.15}$$

Since α_k, $k\in\mathbb{N}^*$, and α are nonincreasing, it follows again from (4.12) that there exists a positive integer $k_1\geqslant k_0$ such that, for any $k\geqslant k_1$ and any $t\in[0,t_0]$, $\alpha_k(t)\geqslant\alpha(t_0)/2$. Hence, by (4.4) and (4.6), one has, for any $(t,x)\in[0,t_0]\times\gamma_-$,

$$|\omega_k(t,x)|\leqslant|\omega|_\infty\exp\left(-\frac{k\alpha(t_0)t}{2}\right). \tag{4.16}$$

Hence, letting k go to $+\infty$ in (4.13), and using (4.14), (4.15), and (4.16), one gets (2.19).

It remains to check that

$$M|\omega(t)|_\infty=\alpha(t) \text{ for a.e. } t\in(0,+\infty). \tag{4.17}$$

By (4.7), (4.8), and (4.11),

$$M|\omega(t)|_\infty\leqslant\alpha(t) \text{ for a.e. } t\in(0,+\infty).$$

Hence, in order to prove (4.17), it suffices to check that, if for some $0<t_1<t_2$, for some $\chi>0$, and for some $k_2>0$,

$$\alpha_k(t_2)\geqslant\chi\,\forall k\geqslant k_2, \tag{4.18}$$

then

$$M|\omega(t_1)|_\infty\geqslant\chi. \tag{4.19}$$

Let $x_k \in \overline{\Omega}$ be such that

$$M|\omega_k(t_2, x_k)| = \alpha_k(t_2). \tag{4.20}$$

Still extracting subsequences if necessary, we may assume that, for some $x_\infty \in \overline{\Omega}$,

$$x_k \to x_\infty \text{ as } k \to \infty. \tag{4.21}$$

Let $\bar{x}_\infty = \Phi^y(0, t_2, x_\infty)$. Let us assume for the moment that the following lemma holds.

LEMMA 4.2. *There exists $\rho > 0$ and $k_3 \geqslant k_2$ such that, for any $k \geqslant k_3$ and for any $t \in [0, t_1]$,*

$$\Phi^{y_k}(t, 0, \bar{x}) \in \overline{\Omega}\, \forall \bar{x} \in \overline{\Omega} \cap B(\bar{x}_\infty, \rho). \tag{4.22}$$

Letting $k \to +\infty$ in (4.22), one gets

$$\Phi^y(t_1, 0, \overline{\Omega} \cap B(\bar{x}_\infty, \rho)) \subset \overline{\Omega}. \tag{4.23}$$

From Lemma 3.3 and (4.22), it follows that, for any $\bar{x} \in \overline{\Omega} \cap B(\bar{x}_\infty, \rho)$ and any $k \geqslant k_3$,

$$s_{\omega_k}(t_1, \Phi^{y_k}(t_1, 0, \bar{x})) = 0,$$

so that, by (3.62),

$$\omega_k(t_1, \Phi^{y_k}(t_1, 0, \bar{x})) = \omega_0(\bar{x}). \tag{4.24}$$

Let us point out that (4.24) implies that

$$\omega(t_1, x) = \omega_0(\Phi^y(0, t_1, x)) \text{ for a.e. } x \in \Phi^y(t_1, 0, \overline{\Omega} \cap B(\bar{x}_\infty, \rho)). \tag{4.25}$$

Indeed, let us first notice that, since $\Phi^y(t, 0, \cdot)$ is a homeomorphism of $\overline{B_R}$—its inverse is $\Phi^y(0, t, \cdot)$—it follows from (4.23) and the invariance of domain theorem (see, e.g., [19, Theorem 3.3.2]) that

$$\Phi^y(t, 0, \Omega \cap B(\bar{x}_\infty, \rho)) \subset \Omega\, \forall t \in [0, t_1]. \tag{4.26}$$

Let $\psi \in C^\infty(\overline{\Omega})$, the support of which is included in the open subset $\Phi^y(t_1, 0, \Omega \cap B(\bar{x}_\infty, \rho)) \subset \Omega$. Then, by (4.8),

$$< \omega(t_1, x), \psi >_{H^{-1}(\Omega), H_0^1(\Omega)} = \lim_{k \to +\infty} I_k, \tag{4.27}$$

with

$$I_k = \int_{\Phi^y(t_1, 0, \Omega \cap B(\bar{x}_\infty, \rho))} \omega_k(t_1, x)\psi(x) dx.$$

Since $\operatorname{div} \tilde{y}_k = 0$, one gets, using also (4.22) and (4.24), the existence of $k_4 \geqslant k_3$ such that, for any $k \geqslant k_4$,

$$\begin{aligned} I_k &= \int_{\Phi^{y_k}(t, 0, \Omega \cap B(\bar{x}_\infty, \rho))} \omega_k(t_1, x)\psi(x) dx \\ &= \int_{\Omega \cap B(\bar{x}_\infty, \rho)} \omega_k(t_1, \Phi^{y_k}(t_1, 0, \bar{x})\psi(\Phi^{y_k}(t_1, 0, \bar{x})) d\bar{x} \\ &= \int_{\Omega \cap B(\bar{x}_\infty, \rho)} \omega_0(\bar{x})\psi(\Phi^{y_k}(t_1, 0, \bar{x})) d\bar{x}. \end{aligned}$$

Note that, by (4.26) and the fact that div $y = 0$ in $(0, T) \times \Omega$, $\Phi^y(t_1, 0, \cdot)$ restricted to $\Omega \cap B(\bar{x}_\infty, \rho)$ preserves the Lebesgue measure. Hence

$$\lim_{k \to +\infty} I_k = \int_{\Omega \cap B(\bar{x}_\infty, \rho)} \omega_0(\bar{x}) \psi(\Phi^y(t_1, 0, \bar{x})) d\bar{x}$$
$$= \int_{\Phi^y(t_1, 0, \Omega \cap B(\bar{x}_\infty, \rho))} \omega_0(\Phi^y(0, t_1, x)) \psi(x) dx,$$

which, with (4.27), implies (4.25). From Lemma 3.5, with kM instead of M in (3.62), (4.18), and (4.20), one has

$$\chi \leqslant M|\omega_k(t_2, x_k)| \leqslant M|\omega_{0,k}(\Phi^{y_k}(s_{y_k}(t_2, x_k), t_2, x_k)| \times \left(\exp\left(-kM\chi s_{y_k}(t_2, x_k)\right)\right). \tag{4.28}$$

In particular, by (4.4),

$$s_{y_k}(t_2, x_k) \to 0 \text{ as } k \to +\infty,$$

which, with (4.4), (4.18), (4.20), and (4.28), implies that

$$M|\omega_0|_{\infty, \Omega \cap B(\bar{x}_\infty, \rho)} \geqslant \chi. \tag{4.29}$$

This inequality, together with (4.25) and the fact that $\Phi^y(t_1, 0, \cdot)$ restricted to $\Omega \cap B(\bar{x}_\infty, \rho)$ preserves the Lebesgue measure, implies that

$$M|\omega(t_1)|_{\infty, \Phi^y(t_1, 0, \Omega \cap B(\bar{x}_\infty, \rho))} \geqslant \chi, \tag{4.30}$$

which gives (4.19).

It remains to prove Lemma 4.2. Let us first point out that

$$\Phi^y(t, t_2, x_\infty) \in \overline{\Omega} \, \forall t \in [0, t_2]. \tag{4.31}$$

Indeed, if (4.31) does not hold, there exists $t_3 \in (0, t_2]$ such that $\Phi^y(t_3, t_2, x_\infty) \notin \overline{\Omega}$. Then, for some $k_5 > 0$,

$$s_{y_k}(t_2, x_k) \geqslant t_3/2 \, \forall k \geqslant k_5,$$

which, with (4.4) and (4.6), implies that

$$\chi \leqslant M|\omega_k(t_2, x_k)| \leqslant M|\omega_0|_\infty \exp\left(-kM\chi t_3/2\right). \tag{4.32}$$

Letting k go to $+\infty$ in (4.32), we get a contradiction. This ends the proof of (4.31).

One easily checks that y satisfies the assumption of Lemma 4.1. Since $\Phi^y(t, t_2, x_\infty) = \Phi(t, 0, \bar{x}_\infty)$, Lemma 4.1 and (4.31) give that

$$\Phi^y(t, 0, \bar{x}_\infty) \in \overline{\Omega} \setminus \overline{\gamma_+} \, \forall t \in [0, t_1],$$

which implies Lemma 4.2.

Let us now prove (ii) of Theorem 2.8. One needs the following lemma, which is analogous to Lemma 3.5 but requires a different proof.

Lemma 4.3. *Let $T > 0$, let $\omega \in C^0([0, T]; H^{-1}(\Omega)) \cap L^\infty([0, T]; L^\infty(\Omega))$ be a solution of system Σ_1, and let $t \in [0, T]$. Then the closed set $S(t) := \{x \in \omega;\ a_{y_\omega}(t, x) \in \partial\Omega$ and $s_{y_\omega}(t, x) = 0\}$ has measure 0 and, for a.e. $x \in \Omega$,*

$$\omega(t, x) = \omega(0, \Phi^{y_\omega}(0, t, x)) \text{ if } s_{y_\omega}(t, x) = 0, \tag{4.33}$$

$$\omega(t, x) = 0 \text{ if } s_{y_\omega}(t, x) > 0, \tag{4.34}$$

where s_{y_ω} *is defined by* (3.23)*—recall the convention* Max $\emptyset = 0$*. In particular,* (3.62) *again holds (for a.e.* $(t,x) \in (0,T) \times \Omega$*).*

With this lemma, which is proved at the end of this section, one gets easily (ii) of Theorem 2.8. Indeed, let ω be a maximal solution of system Σ_1 defined on an interval I containing 0 and let us assume that $T := \text{ Sup } I < +\infty$. Again, it follows from (i) of Theorem 2.8 that

$$T \notin I. \tag{4.35}$$

Using (3.62), which holds by Lemma 4.3, one gets that $|\omega(t)|_\infty \leqslant |\omega(0)|_\infty$ for any t in $[0,T)$. Then, using the fact that $\partial\omega/\partial t = -\text{ div }(\omega y)$, one gets that $\omega \in H^1((0,T); H^{-1}(\Omega))$. Hence

$$\omega(t) \text{ converges in } H^{-1}(\Omega) \text{ as } t \to T^-. \tag{4.36}$$

Then $\bar\omega : I \cup \{T\} \to C^0(\overline{\Omega})$ defined by $\bar\omega = \omega$ on I and $\bar\omega(T) = \lim_{t\to T} \omega(t)$ is also a solution of system Σ_1, a contradiction with the maximal property of ω and (4.35).

Finally, we briefly sketch the proof of Lemma 4.3. Let us first check that $S(t)$ has measure 0. One has

$$S(t) \subset \{\Phi^{y_\omega}(t,0,\bar x);\, \bar x \in \gamma_-,\, \Phi^{y_\omega}(s,0,\bar x) \in \Omega \ \forall s \in (0,T]\}.$$

Hence, since div $y_\omega = 0$ in $(0,T) \times \Omega$, $S(t)$ has measure 0.

Let us now prove (4.33). Let $U = \{(t,x) \in (0,T) \times \Omega;\, a_{y_\omega}(t,x) \in \Omega\}$. Let us consider the *linear* hyperbolic equation that we call $\mathcal{L}$, where $f : U \to \mathbb{R}$ is the unknown,

$$\frac{\partial f}{\partial t} + \text{ div }(f y_\omega) = 0 \text{ in } U, \tag{4.37}$$

$$f(0,\cdot) = \omega(0,\cdot) \text{ on } \{0\} \times \Omega, \tag{4.38}$$

where (4.37) and (4.38) have to be understood in a weak sense. More precisely a function $f : U \to \mathbb{R}$ is a solution of $\mathcal{L}$ if $f \in L^\infty(U)$ and is such that, for any $\psi \in C^1_0(U \cup (\{0\} \times \Omega))$,

$$\int_U f\left(\frac{\partial\psi}{\partial t} + (y_\omega \cdot \nabla)\,\psi\right) = -\int_\Omega \omega(0,\cdot)\psi(0,\cdot).$$

Clearly ω is a solution of $\mathcal{L}$. Moreover $\bar\omega : U \to \mathbb{R}$, defined by

$$\bar\omega(t,x) = \omega(0, \Phi^{y_\omega}(0,t,x))\, \forall (t,x) \in U,$$

is also a solution of $\mathcal{L}$. Hence, in order to prove (4.33), it suffices to check that $\mathcal{L}$ has a unique solution. When y_ω is of class C^1, this is a classical result due to Oleĭnik [21]; see also [11, Theorem 2.2.1]. When $y_\omega \in L^\infty\left([0,T]; C^0\left(\overline{B_R}; \mathbb{R}^2\right)\right)$—satisfying (3.5)—and such that $t \to q_{\overline{B_R}}(y(t,\cdot)) \in L^\infty(0,T)$ one needs a (very slight) modification of the proof since, for example, (2.2.10) in [11] is no longer true. Let us briefly describe the modification. By standard smoothing procedures, one can construct, for $\varepsilon \in (0,1]$, $y^\varepsilon \in C^1([0,T] \times \overline{B_R}; \overline{B_R})$ be such that, for some $C_7 > 0$,

$$|y^\varepsilon - y_\omega|_{0,[0,T]\times\overline{B_R}} \leqslant C_7 \varepsilon^{3/4} \ \forall \varepsilon \in (0,1], \tag{4.39}$$

$$\text{Support } y^\varepsilon \subset [0,T] \times \overline{B_R/2} \ \forall \varepsilon \in (0,1], \tag{4.40}$$

$$|\nabla y^\varepsilon|_{0,[0,T]\times\overline{B_R}} \leqslant C_7 \varepsilon^{-1/4} \ \forall \varepsilon \in (0,1]. \tag{4.41}$$

Let $\varphi \in C^1([0,T] \times \overline{B_R})$, the support of which is included in U. Let, for $\varepsilon \in (0,1]$, $\psi^\varepsilon \in C^1([0,T] \times \overline{B_R})$ be defined by

$$\psi^\varepsilon(t,x) = -\int_t^T \varphi(s, \Phi^{y^\varepsilon}(s,t,x))ds.$$

One has

$$\psi^\varepsilon(T,\cdot) = 0,$$
$$\frac{\partial \psi^\varepsilon}{\partial t} + (y^\varepsilon \cdot \nabla)\psi^\varepsilon = \varphi,$$

and one easily checks that, at least for ε small enough, Support $\psi^\varepsilon \subset U \cup (\{0\} \times \Omega)$. Therefore, at least for ε small enough,

$$\int_U (\omega - \bar{\omega})\varphi = \int_U (\omega - \bar{\omega})((y^\varepsilon - y_\omega) \cdot \nabla)\psi^\varepsilon. \tag{4.42}$$

But, with (4.41), one gets the existence of C_8 such that

$$|\nabla \Phi^{y^\varepsilon}|_{0,[0,T]\times\overline{B_R}\times\overline{B_R}} \leqslant C_8 \varepsilon^{-1/4} \ \forall \varepsilon \in (0,1].$$

This gives the existence of C_9 such that

$$|\nabla \psi^\varepsilon|_{0,[0,T]\times\overline{B_R}} \leqslant C_9 \varepsilon^{-1/4} \ \forall \varepsilon \in (0,1],$$

which, with (4.39), implies that

$$\lim_{\varepsilon \to 0} |((y^\varepsilon - y_\omega) \cdot \nabla)\psi^\varepsilon|_{0,[0,T]\times\overline{B_R}} = 0. \tag{4.43}$$

From (4.42) and (4.43), one gets

$$\int_U (\omega - \bar{\omega})\varphi = 0,$$

and therefore $\omega = \bar{\omega}$ on U. The proof of (4.34) is similar to the proof of (4.33): consider $U' = \{(t,x) \in (0,T) \times \Omega;\, s_{y_\omega}(t,x) > 0\}$ and the linear hyperbolic equation $\mathcal{L}'$ $(\partial f/\partial t) +$ div $(f y_\omega) = 0$ on U' with, instead of (4.38), the boundary condition $f = 0$ on $(0,T) \times \gamma_-$; as above, one shows that this equation $\mathcal{L}'$ has a unique solution; but ω and 0 are solutions of $\mathcal{L}'$ on U', which proves (4.34). We omit the details.

Appendix A. Proof of Lemma 3.7. Since θ is harmonic on $\overline{\Omega}$, which is simply connected, it admits a harmonic conjugate $\psi \in C^\infty(\overline{\Omega})$. One has, with $x = (x_1, x_2)$,

$$\frac{\partial \psi}{\partial x_1} = \frac{\partial \theta}{\partial x_2},\ \frac{\partial \psi}{\partial x_2} = -\frac{\partial \theta}{\partial x_1}. \tag{A.1}$$

Let $\tau \in C^\infty(\partial\Omega; \mathbb{R}^2)$ be the unit tangent vector field on $\partial\Omega$ such that (τ, n) is a direct basis of $\mathbb{R}^2$ at any point of $\partial\Omega$. From (A.1) one gets

$$\frac{\partial \psi}{\partial \tau} = \frac{\partial \theta}{\partial n} \text{ on } \partial\Omega, \tag{A.2}$$

$$\frac{\partial \psi}{\partial n} = -\frac{\partial \theta}{\partial \tau} \text{ on } \partial\Omega. \tag{A.3}$$

By (2.3), the closed set $\partial\Omega \setminus (\gamma_+ \cup \gamma_-)$ has two connected components, that we call Γ_+ and Γ_-. By (3.48) and (A.2), there are two constants C_+ and C_- such that

$$\psi = C_+ \text{ on } \Gamma_+,\ \psi = C_- \text{ on } \Gamma_-. \tag{A.4}$$

Relabeling, if necessary, Γ_+ and Γ_-, we may assume that

$$C_- \leqslant C_+. \tag{A.5}$$

By (3.48) and (A.2),

$$\frac{\partial\psi}{\partial\tau} < 0 \text{ on } \gamma_-, \ \frac{\partial\psi}{\partial\tau} > 0 \text{ on } \gamma_+,$$

which, with (A.4) and (A.5), implies that

$$C_- < C_+, \tag{A.6}$$

$$\psi(x) \in [C_-, C_+] \ \forall x \in \partial\Omega. \tag{A.7}$$

Using (A.4), (A.6), and (A.7), together with the strong maximum principle applied to the harmonic function ψ, one gets that

$$\frac{\partial\psi}{\partial n} < 0 \text{ on } \Gamma_-, \ \frac{\partial\psi}{\partial n} > 0 \text{ on } \Gamma_+, \tag{A.8}$$

which, with (A.3), implies that

$$\frac{\partial\theta}{\partial\tau} > 0 \text{ on } \Gamma_-, \ \frac{\partial\theta}{\partial\tau} < 0 \text{ on } \Gamma_+. \tag{A.9}$$

From (2.3), (3.48), and (A.9), one gets that

$$\nabla\theta(x) \neq 0 \ \forall x \in \partial\Omega, \tag{A.10}$$

$$\text{degree } (\nabla\theta, \Omega, 0) = 0. \tag{A.11}$$

Let $f : \overline{\Omega} \subset \mathbb{R}^2 \cong \mathbb{C} \to \mathbb{C} \cong \mathbb{R}^2$ be defined by

$$f(x_1 + ix_2) = \frac{\partial\theta}{\partial x_1}(x_1, x_2) - i\frac{\partial\theta}{\partial x_2}(x_1, x_2). \tag{A.12}$$

Then f is holomorphic and, by (A.10), does not vanish on $\partial\Omega$; therefore the degree deg $(f, \Omega, 0)$ is well defined and is equal to the number of zeros of f, counted according to their multiplicity. But, by (A.11) and (A.12), degree $(f, \Omega, 0) = -$ degree $(\nabla\theta, \Omega, 0)$ $= 0$. Therefore f does not vanish on $\overline{\Omega}$, which proves Lemma 3.7.

Appendix B. Proof of Lemma 3.3. Let $\tau_- \in C^\infty(\Gamma_+ \cup \Gamma_-; \mathbb{R}^2)$ be defined by requiring

$$\tau_-(x) \in \{\tau(x), -\tau(x)\} \, \forall x \in \Gamma_+ \cup \Gamma_-, \tag{B.1}$$

$$\tau_-(x) \text{ is pointing outside } \Gamma_+ \cup \Gamma_- \, \forall x \in \partial\gamma_-. \tag{B.2}$$

Note that $\partial\gamma_-$ has two elements and is included in $\Gamma_+ \cup \Gamma_-$. It follows from (A.9) that

$$(\nabla\theta(x)) \cdot \tau_-(x) < 0 \ \forall x \in \partial\gamma_-,$$

which, with standard elliptic estimates, implies the existence of $M_0 > 0$ such that, for any $z \in C^0(\overline{\Omega}; \mathbb{R}^2)$ and for any $x \in \Gamma_+ \cup \Gamma_-$ such that dist $(x, \partial\gamma_-) \leqslant 1/M_0$,

$$\left(\begin{cases} z \cdot n = g \text{ on } \partial\Omega \\ \text{div } z = 0 \text{ in } \partial\Omega \\ M_0 | \text{ curl } z|_{L^\infty(\Omega)} \leqslant 1 \end{cases}\right) \Rightarrow \left(z(x) \cdot \tau_-(x) \leqslant -\frac{1}{M_0}\right). \tag{B.3}$$

Finally, let us remark that M_0 has the property required by Lemma 3.3. Indeed, let $y \in L^\infty\left((0,T);C^0\left(\overline{B_R};\mathbb{R}^2\right)\right)$ satisfying (3.5), (3.6) for some $K > 0$, (3.38)–(3.39) for some $\alpha \in L^\infty((0,T);(0,+\infty))$, and (3.40). We argue by contradiction and therefore assume the existence of $(\tilde{t},\tilde{x}) \in (0,T] \times \overline{\gamma_-}$, and $\nu \in (0,\tilde{t})$ such that

$$\Phi^y\left(t,\tilde{t},\tilde{x}\right) \in \overline{\Omega}\, \forall t \in (\tilde{t}-\nu,\tilde{t}). \tag{B.4}$$

For $t \in [\tilde{t}-\nu,\tilde{t}]$, let $k(t) = \Phi^y\left(t,\tilde{t},\tilde{x}\right)$. We claim that there exists $\nu_1 \in (0,\nu]$ such that

$$k(t) \in \partial\Omega\, \forall t \in [\tilde{t}-\nu_1,\tilde{t}]. \tag{B.5}$$

Indeed, (B.5) follows easily from (3.38) and (B.4) if y is smooth enough, e.g., locally Lipschitz with respect to x. As in the proof of (3.41), the case where y is not smooth follows from the main ingredient, due to Wolibner [25], to prove the uniqueness of Φ^y in Theorem 3.1; since the proof is very similar to the proof of (3.41) given above, we omit it. From (3.38) and (B.5), we get that

$$k(t) \in \Gamma_+ \cup \Gamma_-\, \forall t \in [\tilde{t}-\nu_1,\tilde{t}]. \tag{B.6}$$

By (3.38), (3.39), (3.40), and (B.3), there exists $\nu_2 \in (0,\nu_1]$ such that

$$\dot{k}(t)\cdot\tau_-(k(t)) \leqslant -\frac{1}{M_0}\alpha(t)\, \forall t \in [\tilde{t}-\nu_2,\tilde{t}]. \tag{B.7}$$

Since $k(\tilde{t}) \in \overline{\gamma_-}$, and $\alpha(t) > 0$ for a.e. $t \in (0,T)$, (B.6) and (B.7) are in contradiction—recall (B.2).

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