Stabilization of a Rotating Body Beam Without Damping

Jean-Michel Coron and Brigitte d'Andréa-Novel

Abstract—**This paper deals with the stabilization of a rotating body-beam system with torque control. The system we consider is the one studied by Baillieul and Levi in [1]. In [12] it has been proved by Xu and Baillieul that, for any constant angular velocity smaller than a critical one, this system can be stabilized by means of a feedback torque control law if there is damping. We prove that this result also holds if there is no damping.**

Index Terms—**Distributed control system, elastic beam, hybrid system, nonlinear control, stabilization.**

I. Introduction

The goal of this paper is to study the stabilization of a system, already considered in [1], consisting of a disk with a beam attached to its center and perpendicular to the disk's plane. The beam is confined to another plane which is perpendicular to the disk and rotates with the disk; see Fig. 1.

The dynamics of motion is (see [1] and [2])

$$\rho u_{tt}(x,t)+EIu_{xxxx}(x,t)+\rho Bu_t(x,t)=\rho\omega^2(t)u(x,t) \quad (1)$$

$$\begin{gathered} u(0,t)=u_x(0,t)=u_{xx}(L,t)=u_{xxx}(L,t)=0\\ \frac{d}{dt}\left\{\omega(t)\left(I_d+\rho\int_0^L u^2(x,t)dx\right)\right\}=\Gamma(t) \end{gathered} \quad (2)$$

where L is the length of the beam, ρ is the mass per unit length of the beam, EI is the flexural rigidity per unit length of the beam, $\omega(t)=\dot{\theta}(t)$ is the angular velocity of the disk at time t, I_d is the disk's moment of inertia, $u(x,t)$ is the beam's displacement in the rotating plane at time t with respect to the spatial variable x, Bu_t is the damping term, and $\Gamma(t)$ is the torque control variable applied to the disk at time t (see Fig. 1).

If there is no damping $B=0$, and therefore (1) reads

$$\rho u_{tt}(x,t)+EIu_{xxxx}(x,t)=\rho\omega^2(t)u(x,t). \quad (3)$$

Two types of damping are considered in [12].

1) Viscous damping: $Bu_t=ku_t$ with $k>0$.
2) Structural damping: $Bu_t=ku_{xxxxt}$ with $k>0$.

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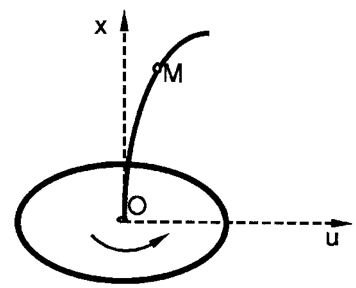

Fig. 1. The body-beam structure.

The asymptotic behavior of the solutions of (1) and (2) when there is no control (i.e., $\Gamma=0$) has been studied by Baillieul and Levi in [1] and by Bloch and Titi in [4].

For both types of damping, Xu and Baillieul have constructed in [12] a feedback torque control law which globally asymptotically stabilizes the equilibrium point $(u,\omega)=(0,\bar{\omega})$ with

$$\bar{\omega}\in(-\omega_c,\omega_c) \quad (4)$$

where ω_c is an explicit critical angular velocity. This critical angular velocity is given by

$$\omega_c=\sqrt{\frac{EI\mu_1}{\rho}} \quad (5)$$

where μ_1 is the first eigenvalue of the unbounded linear operator (d^4/dx^4) in $L^2(0,1)$ with domain

$$\begin{aligned}\mathrm{Dom}(d^4/dx^4)=\{f;f\in H^4(0,1),\ f(0)=f_x(0)\\ =f_{xx}(1)=f_{xxx}(1)=0\}.\end{aligned} \quad (6)$$

They also prove that ω_c is optimal: if $|\bar{\omega}|>\omega_c$, they prove that there is no feedback law which asymptotically stabilizes $(0,\bar{\omega})$. The asymptotically stabilizing feedback law constructed in [12] is linear, and the stabilization is strong and exponential. In [8] Laousy *et al.* have constructed a globally asymptotically stabilizing feedback in the case where there is no damping but when there is a control also on the free boundary of the beam $(x=L)$.

The goal of this paper is to investigate the stabilization problem when there is no damping and no control on the free boundary. We construct in this case a (nonlinear) feedback torque control law which globally asymptotically stabilizes the equilibrium point $(0,\bar{\omega})$, provided that (4) holds.

Our paper is organized as follows: in Section II we introduce some notations and construct stabilizing feedback laws. In

Sections III and IV we prove the asymptotic stability for $0 < |\bar{\omega}| < \omega_c$ and $\bar{\omega} = 0$ respectively.

II. Notations and Stabilizing Feedback Laws

Of course, by suitable scaling arguments, we may assume that $L = EI = \rho = 1$. Let $H^2(0,1)$ be the usual Sobolev space

$$H^2(0,1) = \{u \in L^2(0,1); u_{xx} \in L^2(0,1)\}.$$

Let

$$H = \{w = (u,v) \in H^2(0,1) \times L^2(0,1);\ u(0) = u_x(0) = 0\}.$$

The space H with inner product

$$\langle (u_1, v_1), (u_2, v_2)\rangle_H = \int_0^1 (u_{1xx}u_{2xx} + v_1 v_2)\, dx$$

is a Hilbert space. For $w \in H$, let

$$\mathcal{E}(w) = |w|_H^2.$$

We consider the unbounded linear operator A in H

$$A(u,v) = (-v, u_{xxxx})$$

with domain

$$\mathrm{Dom}(A) = \{(u,v) \in H^4(0,1) \times H^2(0,1); u(0) = u_x(0) = u_{xx}(1) = u_{xxx}(1) = v(0) = v_x(0) = 0\}.$$

It is well known that A is an unbounded skew-adjoint operator and therefore generates a unitary group e^{tA} of bounded linear operator on H. With this notation, our control system (2) and (3) reads

$$\frac{dw}{dt} + Aw = \omega^2(0,u) \tag{7}$$
$$\frac{d\omega}{dt} = \gamma \tag{8}$$

with

$$\gamma = \left(\Gamma - 2\omega \int_0^1 uv\, dx\right) \Big/ \left(I_d + \int_0^1 u^2\, dx\right). \tag{9}$$

By (9), we may consider γ as the control.

Let us now give our stabilizing feedback law when $0 < |\bar{\omega}| < \omega_c$. Without loss of generality we may assume $\bar{\omega} > 0$. In order to explain how we have constructed our stabilizing feedback law, let us first consider (7) as a control system where w is the state and $\omega - \bar{\omega}$ is the control. Then a natural candidate for a control Lyapunov function for this system is

$$\tilde{\mathcal{E}}(w) = \mathcal{E}(w) - \bar{\omega}^2 \int_0^1 u^2\, dx, \qquad \forall w = (u,v) \in H. \tag{10}$$

By the definition of μ_1

$$\int_0^1 u_{xx}^2\, dx \ge \mu_1 \int_0^1 u^2\, dx, \qquad \forall w \in H. \tag{11}$$

From (4), (5), (10), and (11), we get the existence of a constant $C_1 > 0$ such that

$$\frac{1}{C_1}\mathcal{E}(w) \le \tilde{\mathcal{E}}(w) \le \mathcal{E}(w), \qquad \forall w \in H. \tag{12}$$

Moreover, the time derivative of $\tilde{\mathcal{E}}$ along the trajectories of (7) is given by

$$\dot{\tilde{\mathcal{E}}}(w) = 2(\omega^2 - \bar{\omega}^2) \int_0^1 uv\, dx.$$

Therefore, in order to stabilize (7) where w is the state and $(\omega - \bar{\omega})$ is the control, it is natural to propose the feedback law

$$\omega - \bar{\omega} = -\sigma\left(\int_0^1 uv\, dx\right)$$

where $\sigma : R \to R$ is a function of class C^1 such that

$$(2\bar{\omega} - \sigma(s))s\sigma(s) > 0, \qquad \forall s \in R\backslash\{0\}. \tag{13}$$

Note that control system (7) and (8) is obtained by adding an integrator to (7). Therefore, following Byrnes and Isidori [5] or Tsinias [11], a natural candidate for a control Lyapunov function for control system (7) and (8) is

$$\tilde{V}(w,\omega) = \frac{1}{2}\left[\tilde{\mathcal{E}}(w) + \left(\omega - \bar{\omega} + \sigma\left(\int_0^1 uv\, dx\right)\right)^2\right]. \tag{14}$$

Then the time derivative of $\tilde{V}$ along the trajectories of (7) and (8) is

$$\begin{aligned}\dot{\tilde{V}}(w,\omega) &= (\omega^2 - \bar{\omega}^2)\int_0^1 uv\, dx + \left(\omega - \bar{\omega} + \sigma\left(\int_0^1 uv\, dx\right)\right) \\ &\quad \times \left(\gamma + \sigma'\left(\int_0^1 uv\, dx\right)\int_0^1 \left(v^2 - u_{xx}^2 + \omega^2 u^2\right) dx\right)\end{aligned} \tag{15}$$

and a natural candidate for a stabilizing feedback law is, with $C_2 > 0$

$$\begin{aligned}\gamma = &-\left(\omega + \bar{\omega} - \sigma\left(\int_0^1 uv\, dx\right)\right)\int_0^1 uv\, dx \\ &- C_2\left(\omega - \bar{\omega} + \sigma\left(\int_0^1 uv\, dx\right)\right) \\ &- \sigma'\left(\int_0^1 uv\, dx\right)\int_0^1 \left(v^2 - u_{xx}^2 + \omega^2 u^2\right) dx.\end{aligned} \tag{16}$$

With this feedback law one has, using (15)

$$\begin{aligned}\dot{\tilde{V}} = &-\left(2\bar{\omega} - \sigma\left(\int_0^1 uv\, dx\right)\right)\left(\int_0^1 uv\, dx\right)\sigma\left(\int_0^1 uv\, dx\right) \\ &- C_2\left(\omega - \bar{\omega} + \sigma\left(\int_0^1 uv\, dx\right)\right)^2.\end{aligned} \tag{17}$$

Hence, by (13)

$$\dot{\tilde{V}} \le 0. \tag{18}$$

We require that σ is of class C^2, so that $\gamma : H \times R \to R$ is Lipschitz on any bounded set, and therefore the Cauchy problem associated to (7) and (8) has, for this feedback law and for each initial datal in $H \times R$, one and only one (maximal) weak solution defined on an open interval containing zero; see, e.g., [9]. For technical reasons, we require also that

$$\exists C_3 > 0 \quad \text{s.t.} \quad \lim_{s \to 0,\, s \neq 0} \frac{\sigma(s)}{s} = C_3. \tag{19}$$

In Section III we will prove the following theorem.

Theorem 1: The feedback law γ defined by (16) globally strongly asymptotically stabilizes the equilibrium point $(0, \bar{\omega})$ of the control system (7) and (8).

Let us recall that by "globally strongly asymptotically stabilizes the equilibrium point $(0, \bar{\omega})$," one means that:

1) for every solution of (7)–(8) and (16)

$$\lim_{t \to +\infty} |w(t)|_H + |\omega(t) - \bar{\omega}| = 0; \tag{20}$$

2) for every $\epsilon > 0$, there exists $\eta > 0$ such that, for every solution of (7)–(8) and (16),

$$\begin{aligned} &(|w(0)|_H + |\omega(0) - \bar{\omega}| < \eta) \\ &\quad \Rightarrow (|w(t)|_H + |\omega(t) - \bar{\omega}| < \epsilon, \quad \forall t \geq 0). \end{aligned}$$

Let us now turn to the case where $\bar{\omega} = 0$. In order to explain how we have constructed our stabilizing feedback law, let us first consider again (7) as a control system where w is the state and ω is the control. Then natural candidates for a control Lyapunov function and a stabilizing feedback law are, respectively, $\mathcal{E}$ and $\omega = \sigma^\star(\int_0^1 uv\,dx)$, where $\sigma^\star \in C^0(R, R)$ satisfies $\sigma^\star > 0$ on $(-\infty, 0)$ and $\sigma^\star = 0$ on $[0, +\infty)$. One can prove that (see Appendix A) such feedbacks always give *weak* asymptotic stabilization, i.e., one gets instead of (20)

$$w(t) \rightharpoonup 0 \text{ weakly in } H \text{ as } t \to +\infty. \tag{21}$$

But it is not clear that such feedbacks give *strong* asymptotic stabilization. It is possible to prove that one gets such stabilization for the particular case where the feedback is

$$\omega = \left(\max\left\{0, -\int_0^1 uv\,dx\right\}\right)^{\frac{1}{2}}. \tag{22}$$

Let us recall that control system (7) and (8) is obtained by adding an integrator to control system (7). Unfortunately, ω defined by (22) is not of class C^1, and so one cannot use the techniques given in [5] and [11]. The smoothness of this ω is also not sufficient to apply the desingularization techniques introduced in [10]. For these reasons, we use a different control Lyapunov function and a different feedback law to asymptotically stabilize control system (7). For the control Lyapunov function, we take

$$J(w) = \frac{1}{2}\left\{\mathcal{E}(w) - F(\mathcal{E}(w)) \int_0^1 u^2\,dx\right\}, \quad \forall w = (u, v) \in H$$

where $F \in C^3([0, +\infty); [0, +\infty))$ satisfies

$$\mathrm{Sup}_{s \geq 0} F(s) < \frac{\mu_1}{2} \tag{23}$$

so that, by (11)

$$\frac{1}{4}\mathcal{E}(w) \leq J(w) \leq \frac{1}{2}\mathcal{E}(w), \qquad \forall w \in H. \tag{24}$$

Computing the time derivative $\dot{J}$ of J along the trajectories of (7) one gets

$$\dot{J} = (K\omega^2 - F(\mathcal{E}))\left(\int_0^1 uv\,dx\right) \tag{25}$$

where, for simplicity, we write $\mathcal{E}$ for $\mathcal{E}(w)$ and where

$$K(=K(w)) := 1 - F'(\mathcal{E}) \int_0^1 u^2\,dx. \tag{26}$$

Let us impose that

$$0 \leq F'(s)s < \mu_1 - F(s), \qquad \forall s \in [0, +\infty) \tag{27}$$

$$\exists C_4 > 0 \quad \text{s.t.} \quad \lim_{s \to 0,\, s > 0} \frac{F(s)}{s} = C_4. \tag{28}$$

It is then natural to consider the feedback law for (7) vanishing at zero and such that on $H \backslash \{0\}$

$$\omega = K^{-1/2}\left(F(\mathcal{E}) - \bar{\sigma}\left(\int_0^1 uv\,dx\right)\right)^{\frac{1}{2}} \tag{29}$$

where $\bar{\sigma} \in C^2(R; R)$ is such that

$$s\bar{\sigma}(s) > 0, \qquad \forall s \in R \backslash \{0\} \tag{30}$$

$$\exists C_5 > 0 \quad \text{s.t.} \quad \lim_{s \to 0,\, s \neq 0} \frac{\bar{\sigma}(s)}{s} = C_5 \tag{31}$$

$$\bar{\sigma}(s) < F(2\sqrt{\mu_1}s), \qquad \forall s > 0. \tag{32}$$

Note that, using (11), one gets that for every $w = (u, v) \in H$

$$\begin{aligned} \int_0^1 uv\,dx &\leq \frac{1}{2}\left(\sqrt{\mu_1}\int_0^1 u^2\,dx + \frac{1}{\sqrt{\mu_1}}\int_0^1 v^2\,dx\right) \\ &\leq \frac{1}{2\sqrt{\mu_1}}\left(\int_0^1 u_{xx}^2 dx + \int_0^1 v^2\,dx\right) \\ &= \frac{1}{2\sqrt{\mu_1}}\mathcal{E}(w) \end{aligned} \tag{33}$$

which, with (27), (28), and (32), implies that

$$F(\mathcal{E}(w)) - \bar{\sigma}\left(\int_0^1 uv\,dx\right) > 0, \qquad \forall w \in H \backslash \{0\}. \tag{34}$$

From (11), (26), and (27), one gets

$$\begin{aligned} K(w) &\geq 1 - \frac{F'(\mathcal{E}(w))}{\mu_1}\int_0^1 u_{xx}^2\,dx \\ &\geq 1 - \frac{F'(\mathcal{E}(w))}{\mu_1}\mathcal{E}(w) > 0, \qquad \forall w \in H. \end{aligned} \tag{35}$$

From (34) and (35), we get that ω is well defined by (29) and is of class C^2 on $H \backslash \{0\}$. This regularity is sufficient to apply the desingularization technique of [10]: we note that (29) is equivalent to

$$\omega^3 = \psi(w) := K^{-3/2}\left(F(\mathcal{E}(w)) - \bar{\sigma}\left(\int_0^1 uv\,dx\right)\right)^{\frac{3}{2}} \tag{36}$$

and therefore, following [10], one considers the following control Lyapunov function for control system (7) and (8):

$$V(w,\omega) = J + \int_{\psi^{\frac{1}{3}}}^{\omega} (s^3 - \psi)\, ds = J + \frac{1}{4}\omega^4 - \psi\omega + \frac{3}{4}\psi^{\frac{4}{3}}$$

where, for simplicity, we write J for $J(w)$ and ψ for $\psi(w)$. Then, by (24)

$$\begin{aligned} &V(w,\omega) \to +\infty \quad \text{as } |w|_H + |\omega| \to +\infty \\ &V(w,\omega) > 0, \qquad \forall (w,\omega) \in H \times R \backslash \{(0,0)\} \\ &V(0,0) = 0. \end{aligned}$$

Moreover, if one computes the time derivative $\dot{V}$ of V along the trajectories of (7) and (8), one gets, using in particular (25)

$$\begin{aligned} \dot{V} = &-\left(\int_0^1 uv\,dx\right)\bar{\sigma}\left(\int_0^1 uv\,dx\right) \\ &+ \left(\omega - \psi^{\frac{1}{3}}\right)\left[\gamma\left(\omega^2 + \psi^{\frac{1}{3}}\omega + \psi^{\frac{2}{3}}\right) + D\right] \end{aligned} \tag{37}$$

where

$$D = -\dot{\psi} + K\left(\omega + \psi^{\frac{1}{3}}\right)\int_0^1 uv\,dx \tag{38}$$

with (39), as shown at the bottom of the page. Hence it is natural to define feedback law γ by

$$\gamma(0,0) = 0 \tag{40}$$

and, for every $(w,\omega) \in (H \times R)\backslash\{(0,0)\}$

$$\gamma = -\left(\omega - \psi^{\frac{1}{3}}\right) - \frac{D}{\omega^2 + \psi^{\frac{1}{3}}\omega + \psi^{\frac{2}{3}}}. \tag{41}$$

Note that by (34)–(36)

$$\psi(w) > 0, \qquad \forall w \in H\backslash\{0\}. \tag{42}$$

Moreover, by (26), (28), and (31)–(33), there exists $\delta > 0$ such that

$$\psi(w) > \delta\mathcal{E}(w)^{3/2}, \qquad \forall w \in H \quad \text{s.t.} \quad \mathcal{E}(w) < \delta. \tag{43}$$

Using (26), (35), (36), (38), (39), and (41)–(43), one easily checks that γ is Lipschitz on any bounded set of $H \times R$. Therefore, the Cauchy problem associated to (7) and (8) has, for feedback law γ, one and only one (maximal) solution defined on an open interval containing zero. By (32), (37), (38), (40), and (41), one has

$$\begin{aligned} \dot{V} = &-\left(\int_0^1 uv\,dx\right)\bar{\sigma}\left(\int_0^1 uv\,dx\right) \\ &- \left(\omega - \psi^{\frac{1}{3}}\right)^2\left(\omega^2 + \psi^{\frac{1}{3}}\omega + \psi^{\frac{2}{3}}\right) \le 0. \end{aligned} \tag{44}$$

In Section IV, we prove the following.

Theorem 2: The feedback law γ defined by (40) and (41) globally strongly asymptotically stabilizes the equilibrium point $(0,0)$ for the control system (7) and (8).

III. Proof of Theorem 1

Throughout this section, γ is defined by (16). The Proof of Theorem 1 is divided in two parts.

1) First we prove that the trajectories of (7) and (8) are precompact in H for $t \ge 0$.
2) Then we conclude by LaSalle's theorem.

The main difficult point is to prove 1). More precisely, one needs to prove that the energy associated to the high-frequency modes is uniformly small [see (64)]. In order to prove this uniform smallness, a key point is that all these modes satisfy the same equation as w [see (62)]. Finally, in Lemma 1 we get some estimates on $\int_0^1 uv\,dx$ for any solution of (62) which will allow us to prove the uniform smallness.

A. Precompactness of the Trajectories

Let $\tilde{\omega} = \omega - \bar{\omega}$, $\tilde{\gamma}(w,\tilde{\omega}) = \gamma(w,\omega)$, $\tilde{A}w = Aw - (0,\bar{\omega}^2 u)$. Then system (7) and (8) is equivalent to

$$\frac{dw}{dt} + \tilde{A}w = (0, (2\bar{\omega}\tilde{\omega} + \tilde{\omega}^2)u) \tag{45}$$

$$\frac{d\tilde{\omega}}{dt} = \tilde{\gamma}. \tag{46}$$

Let $(w,\tilde{\omega})$ be a trajectory of (45) and (46). By (12), (14), and (18)

$$\{(w(t),\tilde{\omega}(t));\ t \ge 0\} \text{ is bounded in } H \times R. \tag{47}$$

In this subsection we prove that

$$\{(w(t),\tilde{\omega}(t));\ t \ge 0\} \text{ is precompact in } H \times R. \tag{48}$$

Let us point out that one cannot apply the classical method [6] due to Dafermos and Slemrod, since the operator from $H \times R$ into $H \times R$

$$(w,\tilde{\omega}) \to (-\tilde{A}w + (2\bar{\omega}\tilde{\omega} + \tilde{\omega}^2)u, \tilde{\gamma}(w,\tilde{\omega}))$$

is not monotone.

From (13) and (17) we get

$$\begin{aligned} &\int_0^{+\infty}\left(2\bar{\omega} - \sigma\left(\int_0^1 uv\,dx\right)\right) \\ &\quad\times\left(\int_0^1 uv\,dx\right)\sigma\left(\int_0^1 uv\,dx\right)dt < +\infty \end{aligned} \tag{49}$$

$$\int_0^{+\infty}\left(\tilde{\omega} + \sigma\left(\int_0^1 uv\,dx\right)\right)^2 dt < +\infty. \tag{50}$$

$$\begin{aligned} \dot{\psi} = &\frac{3\psi^{\frac{1}{3}}}{2K}\left[2F'(\mathcal{E})\omega^2\int_0^1 uv\,dx - \sigma'\left(\int_0^1 uv\,dx\right)\left(\int_0^1 \left(v^2 - u_{xx}^2 + \omega^2 u^2\right)dx\right)\right] + 3\frac{\psi}{K}\int_0^1 uv\,dx \\ &\times\left(\omega^2 F''(\mathcal{E})\int_0^1 u^2\,dx + F'(\mathcal{E})\right) \end{aligned} \tag{39}$$

By (47)

$$\left\{\int_0^1 uv\,dx;\, t \geq 0\right\} \text{ is bounded}$$

which, with (13) and (19), implies the existence of $\delta > 0$ such that for all $t \geq 0$

$$\left(2\bar{\omega} - \sigma\left(\int_0^1 uv\,dx\right)\right)\left(\int_0^1 uv\,dx\right)\sigma\left(\int_0^1 uv\,dx\right) \geq \delta\left[\left(\int_0^1 uv\,dx\right)^2 + \left(\sigma\left(\int_0^1 uv\,dx\right)\right)^2\right]. \tag{51}$$

From (49) and (51) we get that

$$\int_0^{+\infty}\left(\int_0^1 uv\,dx\right)^2 dt < +\infty \tag{52}$$

$$\int_0^{+\infty}\left(\sigma\left(\int_0^1 uv\,dx\right)\right)^2 dt < +\infty. \tag{53}$$

From (50) and (53) we get

$$\int_0^{+\infty} \tilde{\omega}^2\,dt < +\infty. \tag{54}$$

Note that (47) and (54) give

$$\int_0^{+\infty}(2\bar{\omega}\tilde{\omega} + \tilde{\omega}^2)^2\,dt < +\infty. \tag{55}$$

The unbounded linear operator (d^4/dx^4) in $L^2(0,1)$ with domain given by (6) is self-adjoint with compact resolvent. Thus, it has a discrete spectrum $\{\mu_j\}_{j\in N\setminus\{0\}}$, with $0 < \mu_j < \mu_{j+1}$, $\forall i \in N\setminus\{0\}$, and its eigenvectors $\{\phi_i\}_{i\in N\setminus\{0\}}$ may be taken to form an orthonormal basis in $L^2(0,1)$; let us recall that its eigenvalues are simple.

For $\bar{w} = (\bar{u}, \bar{v}) \in H$, let us define

$$\tilde{Q}(\bar{w}) = \sum_{j=1}^{+\infty}\mu_j\left(\bar{a}_{2j}^2 + \bar{a}_{2j-1}^2\right)^2 \tag{56}$$

where the $\bar{a}_j$, $j \geq 1$, are defined by

$$\bar{u} = \sum_{j=1}^{+\infty}\bar{a}_{2j}\phi_j(x) \tag{57}$$

$$\bar{v} = \sum_{j=1}^{+\infty}(\mu_j - \bar{\omega}^2)^{1/2}\bar{a}_{2j-1}\phi_j(x). \tag{58}$$

Note that the convergence in (57) is in $H^2(0,1)$ and the convergence in (58) is in $L^2(0,1)$. Moreover, $\sum_{j=1}^{+\infty}\mu_j\bar{a}_{2j}^2 < +\infty$ and $\sum_{j=1}^{+\infty}\mu_j\bar{a}_{2j-1}^2 < +\infty$ and therefore $\tilde{Q}(\bar{w}) < +\infty$.

Let us assume for the moment that the following lemma holds.

Lemma 1: There exist $c^\star \geq 1$ and $T^\star > 0$ such that for every $h^\star \in C^0([0,T^\star]; L^2(0,1))$ and for every $w^\star = (u^\star, v^\star) \in C^0([0,T^\star]; H)$ such that

$$\frac{dw^\star}{dt} + \tilde{A}w^\star = (0, h^\star) \tag{59}$$

one has

$$\tilde{Q}(w^\star(.,0)) \leq C^\star\left(\int_0^{T^\star}\left(\int_0^1 u^\star v^\star\,dx\right)^2 dt + \left(\int_0^{T^\star}\int_0^1 h^{\star 2}dx\,dt\right)^2\right) \tag{60}$$

$$\int_0^{T^\star}\left(\int_0^1 u^\star v^\star\,dx\right)^2 dt \leq C^\star\left(\tilde{Q}(w^\star(.,0)) + \left(\int_0^{T^\star}\int_0^1 h^{\star 2}dx\,dt\right)^2\right). \tag{61}$$

For $k \geq 1$ and $t \geq 0$, let $u_k(.,t)$ [respectively, $v_k(.,t)$] be the L^2-orthogonal projection of $u(.,t)$ [respectively, $v(.,t)$] on the L^2-closed linear subspace spanned by the ϕ_j, $j \geq k$. Let $w_k = (u_k, v_k)$. Then w_k belongs to $C([0,+\infty); H)$ and satisfies

$$\frac{dw_k}{dt} + \tilde{A}w_k = m(t)(0, u_k) \tag{62}$$

with

$$m(t) = 2\bar{\omega}\tilde{\omega}(t) + \tilde{\omega}^2(t). \tag{63}$$

By (47), in order to prove (48), it suffices to prove that

$$\forall \epsilon > 0, \qquad \exists k \geq 0 \quad \text{s.t.} \quad (\tilde{\mathcal{E}}(w_k(.,t)) \leq \epsilon, \qquad \forall t \geq 0). \tag{64}$$

Indeed, let $\epsilon > 0$ be given and let us denote by $B(\psi, r) \subset H$ the open ball of H centered at ψ and of radius r. By (64) there exists a positive integer k such that

$$\{w(t);\, t \geq 0\} \subset \{(w - w_k)(t);\, t \geq 0\} \cup B(0, \epsilon/2). \tag{65}$$

But, by (47), the set $\{(w - w_k)(t);\, t \geq 0\}$ is a bounded subset of H and, since it is also a subset of the *finite* dimensional space spanned by the ϕ_j, $1 \leq j < k$, this set is precompact. Therefore, there exists a finite number of functions $\psi_1, \cdots, \psi_p$ of H such that

$$\{(w - w_k)(t);\, t \geq 0\} \subset \bigcup_{i=1}^{p} B(\psi_i, \epsilon/2). \tag{66}$$

From (65) and (66) we get

$$\{w(t);\, t \geq 0\} \subset \bigcup_{i=1}^{p} B(\psi_i, \epsilon).$$

Hence, $\{w(t);\, t \geq 0\}$ can be covered by finitely many balls of radius ϵ for every $\epsilon > 0$ and so is precompact. Since by (47) $\{\omega(t);\, t \geq 0\}$ is also precompact, we get (48).

From (62), we get

$$\dot{u}_k = v_k. \tag{67}$$

Taking the scalar product of (62) with $(0, v_k)$ in the Hilbert space H and using (67) we get, with $\tilde{\mathcal{E}}_k(t) = \tilde{\mathcal{E}}(w_k(.,t))$

$$\frac{d}{dt}\tilde{\mathcal{E}}_k = 2m(t)\int_0^1 u_k v_k\,dx \le m(t)^2 + \left(\int_0^1 u_k v_k\,dx\right)^2. \tag{68}$$

Clearly, for every $t \in [0, +\infty)$

$$\tilde{Q}(w_k(.,t)) \le \tilde{Q}(w(.,t))$$
$$|u_k(.,t)|_{L^2(0,1)} \le |u(.,t)|_{L^2(0,1)}.$$

Hence, using (11), using (61) with $w_k^\star(.,t) = w_k(.,t+T)$, $h_k^\star(.,t) = m(t)u_k(.,t+T)$, and using (60) with $w^\star(.,t) = w(.,t+T)$, $h^\star(.,t) = m(t)u(.,t+T)$ we get that for every $T \ge 0$ and all $k \ge 1$

$$\begin{aligned}
&\int_T^{T+T^\star}\left(\int_0^1 u_k v_k\,dx\right)^2 dt\\
&\quad\le C^\star(1+C^\star)\left(\int_T^{T+T^\star}\left(\int_0^1 uv\,dx\right)^2 dt\right.\\
&\qquad\qquad\left.+\left(\int_T^{T+T^\star} m(t)^2\int_0^1 u^2\,dx\right)^2\right)\\
&\quad\le C^\star(1+C^\star)\left(1+\mu_1^{-1}|\mathcal{E}|_{L^\infty(0,\infty)}\right)\\
&\qquad\times\left(1+\mu_1^{-1}T^\star|\mathcal{E}|_{L^\infty(0,\infty)}|m|^2_{L^\infty(0,\infty)}\right)\\
&\qquad\times\left(\int_T^{T+T^\star} m(t)^2 + \left(\int_0^1 uv\,dx\right)^2 dt\right)
\end{aligned} \tag{69}$$

with

$$|\mathcal{E}|_{L^\infty(0,\infty)} := \mathrm{Sup}\{\mathcal{E}(w(t));\, t\in[0,+\infty)\}$$
$$|m|_{L^\infty(0,\infty)} := \mathrm{Sup}\{|m(t)|;\, t\in[0,+\infty)\}.$$

From (68) we get

$$\begin{aligned}
&\tilde{\mathcal{E}}(w_k(.,t+nT^\star))\\
&\quad\le \tilde{\mathcal{E}}(w_k(.,t)) + \int_t^{t+nT^\star} m(t)^2 + \left(\int_0^1 u_k v_k\,dx\right)^2 dt\\
&\qquad\qquad \forall k\ge 1,\qquad \forall n\in N,\qquad \forall t\ge 0
\end{aligned}$$

which, with (47), (52), (55), (63), and (69) gives that for every $\epsilon > 0$, there exists T_1 such that for every $k \ge 1$

$$\tilde{\mathcal{E}}(w_k(.,t+nT^\star)) \le \frac{\epsilon}{2} + \tilde{\mathcal{E}}(w_k(.,t)),\qquad \forall n\in N,\qquad \forall t\ge T_1. \tag{70}$$

Therefore, for every $k \ge 1$ and every $t \ge 0$

$$\tilde{\mathcal{E}}(w_k(.,t)) \le \frac{\epsilon}{2} + \max_{s\in[0,T_1+T^\star]}\tilde{\mathcal{E}}(w_k(.,s)). \tag{71}$$

Indeed, (71) is true if $t \in [0, T_1 + T^\star]$ and, if $t > T_1 + T^\star$, it suffices to apply (70) with t replaced by t_1 such that

$$T_1 \le t_1 < T_1 + T^\star \quad\text{and}\quad (t - t_1) \in \{nT^\star,\ n \in N\}.$$

Moreover, since $[0, T_1 + T^\star]$ is compact, there exists $k \ge 1$ such that

$$\tilde{\mathcal{E}}(w_k(.,s)) \le \frac{\epsilon}{2},\qquad \forall s\in[0,T_1+T^\star] \tag{72}$$

which, with (71), proves (64).

Finally, let us prove Lemma 1. Let us first point out that we may assume that

$$h^\star = 0. \tag{73}$$

Indeed, let us write

$$w^\star = w_1^\star + w_2^\star \tag{74}$$

where $w_1^\star \in C^0([0,T^\star];H)$ and $w_2^\star \in C^0([0,T^\star];H)$ are defined by

$$\frac{dw_1^\star}{dt} + \tilde{A}w_1^\star = 0$$
$$w_1^\star(.,0) = w^\star(.,0)$$
$$\frac{dw_2^\star}{dt} + \tilde{A}w_2^\star = (0, h^\star) \tag{75}$$
$$w_2^\star(.,0) = 0. \tag{76}$$

Taking the scalar product of (75) with $w_2^\star$ in H, we get, with $w_2^\star = (u_2^\star, v_2^\star)$

$$\frac{d}{dt}\tilde{\mathcal{E}}(w_2^\star) = 2\int_0^1 h^\star v_2^\star\,dx \le \int_0^1 h^{\star 2}dx + \tilde{\mathcal{E}}(w_2^\star). \tag{77}$$

Let us denote by C_j, $6 \le j \le 11$, various positive constants which are independent of $w^\star$ and $h^\star$, but may depend on $T^\star$. From (76) and (77) and Gronwall's lemma, one gets the existence of $C_6 > 0$ such that

$$\tilde{\mathcal{E}}(w_2^\star) \le C_6\int_0^{T^\star}\int_0^1 h^{\star 2}\,dx\,dt,\qquad \forall t\in[0,T^\star] \tag{78}$$

which implies that, for some $C_7 > 0$

$$\int_0^{T^\star}\left(\int_0^1 u_2^\star v_2^\star\,dx\right)^2 \le C_7\left(\int_0^{T^\star}\int_0^1 h^{\star 2}\,dx\,dt\right)^2. \tag{79}$$

Let $(u_1^\star, v_1^\star) = w^\star$ and let $\eta > 0$. We require that the C_j, $6 \le j \le 11$ are also independent of η. From (78) and the Cauchy–Schwarz inequality, we get the existence of C_8 such that for every $\eta > 0$

$$\begin{aligned}
\left(\int_0^1 u_1^\star v_2^\star\,dx\right)^2 &\le \eta\left(\int_0^1 u_1^{\star 2}\,dx\right)^2\\
&\quad + \frac{C_8}{\eta}\left(\int_0^{T^\star}\int_0^1 h^{\star 2}\,dx\,dt\right)^2,\\
&\qquad \forall t\in[0,T^\star].
\end{aligned} \tag{80}$$

Let us define $\{a_j(t)\}_{j\ge 1}$ and $\{b_j(t)\}_{j\ge 1}$, $t \in [0, T^\star]$ by requiring

$$u_1^\star(.,t) = \sum_{j=1}^{+\infty} a_{2j}(t)\phi_j \tag{81}$$

$$u_2^\star(.,t) = \sum_{j=1}^{+\infty} b_{2j}(t)\phi_j \tag{82}$$

$$v_1^\star(.,t) = \sum_{j=1}^{+\infty}(\mu_j - \bar{\omega}^2)^{\frac{1}{2}} a_{2j-1}(t)\phi_j \tag{83}$$

$$v_2^\star(.,t) = \sum_{j=1}^{+\infty}(\mu_j - \bar{\omega}^2)^{\frac{1}{2}} b_{2j-1}(t)\phi_j. \tag{84}$$

The convergences in (81) and (82) [respectively, (83) and (84)] are in $H^2(0,1)$ [respectively, $L^2(0,1)$]. Then

$$\begin{aligned}\left(\int_0^1 u_1^{\star 2}\,dx\right)^2(t) &= \left(\sum_{j=1}^{+\infty} a_{2j}^2\right)^2(t)\\ &\le \left(\sum_{j=1}^{+\infty}\frac{1}{\mu_j}\right)\left(\sum_{j=1}^{+\infty}\mu_j a_{2j}^4\right)(t)\\ &\le \left(\sum_{j=1}^{+\infty}\frac{1}{\mu_j}\right)\tilde{Q}(w_1^\star(.,t)),\\ &\qquad \forall t\in[0,T^\star].\end{aligned} \tag{85}$$

Let us first study (61). Let us recall that (see, e.g., [7, pp. 74–75])

$$\mu_j \in ((j-3/4)^4\pi^4, (j-1/4)^4\pi^4), \qquad \forall j \ge 1. \tag{86}$$

Hence by (80) and (85), there exists $C_9 > 0$ such that for every $\eta > 0$

$$\begin{aligned}&\left(\int_0^1 u_1^\star v_2^\star\,dx\right)^2(t)\\ &\quad\le \eta\tilde{Q}(w_1^\star(.,t)) + \frac{C_9}{\eta}\left(\int_0^{T^\star}\int_0^1 h^{\star 2}\,dx\,dt\right)^2,\\ &\qquad \forall t\in[0,T^\star].\end{aligned} \tag{87}$$

Similarly, using (78) and (86), one gets the existence of $C_{10} > 0$ such that for every $t \in [0,T^\star]$

$$\begin{aligned}&\left(\int_0^1 u_2^\star v_2^\star\,dx\right)^2(t)\\ &= \left(\sum_{j=1}^{+\infty}(\mu_j-\bar{\omega}^2)^{\frac{1}{2}} a_{2j-1}(t) b_{2j}(t)\right)^2\\ &\le \left(\sum_{j=1}^{+\infty}\mu_j a_{2j-1}^4(t)\right)^{\frac{1}{2}}\left(\sum_{j=1}^{+\infty}\frac{(\mu_j-\bar{\omega}^2)^{\frac{2}{3}}}{\mu_j^{\frac{1}{3}}}|b_{2j}(t)|^{\frac{4}{3}}\right)^{\frac{3}{2}}\\ &\le \eta\tilde{Q}(w_1^\star(.,t)) + \frac{1}{4\eta}\left(\sum_{j=1}^{+\infty}(\mu_j-\bar{\omega}^2) b_{2j}^2(t)\right)^2\\ &\le \eta\tilde{Q}(w_1^\star(.,t)) + \frac{C_{10}}{\eta}\tilde{\mathcal{E}}(w_2^\star)^2\\ &\le \eta\tilde{Q}(w_1^\star(.,t)) + \frac{C_6^2 C_{10}}{\eta}\left(\int_0^{T^\star}\int_0^1 h^{\star 2}\,dx\,dt\right)^2.\end{aligned} \tag{88}$$

Moreover, by (74) and (76)

$$\tilde{Q}(w^\star(.,0)) = \tilde{Q}(w_1^\star(.,0)) \tag{89}$$

and by (95), which is proved below

$$\tilde{Q}(w_1^\star(.,t)) = \tilde{Q}(w_1^\star(.,0)), \qquad \forall t\in[0,T^\star]. \tag{90}$$

Taking $\eta = 1$, we get from (79) and (87)–(90)

$$\begin{aligned}&\int_0^{T^\star}\left(\int_0^1 u^\star v^\star\,dx\right)^2 dt\\ &\quad\le 4\int_0^{T^\star}\left(\int_0^1 u_1^\star v_1^\star\,dx\right)^2 dt + 8T^\star\tilde{Q}(w^\star(.,0))\\ &\qquad + 4(C_7 + C_9 + C_6^2 C_{10})\left(\int_0^{T^\star}\int_0^1 h^{\star 2}\,dx\,dt\right)^2\end{aligned} \tag{91}$$

which, with (89), shows that in order to prove (61), we may assume, without loss of generality, (73).

Let us now turn to (60). It follows from (79) and (87)–(90) that

$$\begin{aligned}&\int_0^{T^\star}\left(\int_0^1 u_1^\star v_1^\star\,dx\right)^2 dt\\ &\le 4\Bigg[\int_0^{T^\star}\left(\int_0^1 u^\star v^\star\,dx\right)^2 dt + \int_0^{T^\star}\left(\int_0^1 u_1^\star v_2^\star\,dx\right)^2 dt\\ &\qquad + \int_0^{T^\star}\left(\int_0^1 u_2^\star v_1^\star\,dx\right)^2 dt + 9\int_0^{T^\star}\left(\int_0^1 u_2^\star v_2^\star\,dx\right)^2 dt\Bigg]\\ &\le 4\Bigg[\int_0^{T^\star}\left(\int_0^1 u^\star v^\star\,dx\right)^2 dt + 8\eta\tilde{Q}(w_1(.,0))\\ &\qquad + \left(\frac{C_9}{\eta} + \frac{C_6^2 C_{10}}{\eta} + 9C_7\right)\left(\int_0^{T^\star}\int_0^1 h^{\star 2}\,dx\,dt\right)^2\Bigg].\end{aligned} \tag{92}$$

Using (89) and (92) with $\eta > 0$ small enough, one gets that in order to prove (60) we may assume without loss of generality that (73) holds.

From now on, we assume that (73) holds. Then, by (59) we can write, with $\tilde{\mu}_j = \mu_j - \bar{\omega}^2$

$$\begin{aligned}u^\star(x,t) = \sum_{j=1}^{+\infty}\big(&\alpha_{2j-1}\sin\big(\sqrt{\tilde{\mu}_j}t\big)\\ &+ \alpha_{2j}\cos\big(\sqrt{\tilde{\mu}_j}t\big)\big)\phi_j(x)\end{aligned} \tag{93}$$

$$\begin{aligned}v^\star(x,t) = \sum_{j=1}^{+\infty}\sqrt{\tilde{\mu}_j}\big(&\alpha_{2j-1}\cos\big(\sqrt{\tilde{\mu}_j}t\big)\\ &- \alpha_{2j}\sin\big(\sqrt{\tilde{\mu}_j}t\big)\big)\phi_j(x).\end{aligned} \tag{94}$$

The convergences in (93) and (94) are in $C^0([0,T^\star]; H^2(0,1))$ and $C^0([0,T^\star]; L^2(0,1))$, respectively. We have

$$\tilde{Q}(w^\star(.,t)) = \sum_{j=1}^{+\infty}\mu_j\big(\alpha_{2j-1}^2 + \alpha_{2j}^2\big)^2, \qquad \forall t\in[0,T^\star] \tag{95}$$

which, in particular, implies (90) for $w_1^\star = w^\star$. Moreover, we have

$$\int_0^1 u^\star v^\star \, dx = \sum_{j=1}^{+\infty} \frac{\sqrt{\tilde{\mu}_j}}{2} \left(\alpha_{2j-1}^2 - \alpha_{2j}^2\right) \sin\left(2\sqrt{\tilde{\mu}_j} t\right) + \sqrt{\tilde{\mu}_j} 2\alpha_{2j-1}\alpha_{2j} \cos\left(2\sqrt{\tilde{\mu}_j} t\right).$$

By (86) we have, for $T^\star$ large enough and if we let $\tilde{\mu}_0 = 0$

$$4\left(\sqrt{\tilde{\mu}_j} - \sqrt{\tilde{\mu}_{j-1}}\right) > \frac{3\pi}{T^\star}, \qquad \forall j \in N \backslash \{0\}.$$

Therefore, by a classical result on lacunary Fourier series (see, e.g., [7, Lemmas 1.4.4 and 1.4.5]), for $T^\star$ large enough there exists $C_{11} > 0$ such that

$$\int_0^{T^\star} \left(\int_0^1 u^\star v^\star \, dx\right)^2 dt \leq C_{11} \sum_{j=1}^{+\infty} \tilde{\mu}_j \left(\left(\alpha_{2j-1}^2 - \alpha_{2j}^2\right)^2 + 4\alpha_{2j-1}^2 \alpha_{2j}^2\right)$$

$$\int_0^{T^\star} \left(\int_0^1 u^\star v^\star \, dx\right)^2 dt \geq C_{11}^{-1} \sum_{j=1}^{+\infty} \tilde{\mu}_j \left(\left(\alpha_{2j-1}^2 - \alpha_{2j}^2\right)^2 + 4\alpha_{2j-1}^2 \alpha_{2j}^2\right) \tag{96}$$

which, with (95), gives (60) and (61) for $C^\star$ large enough.

B. LaSalle's Theorem

By (13), (48), and LaSalle's theorem, in order to finish the proof of Theorem 1, it suffices to check that if $w \in C^0([0, +\infty); H)$ is such that

$$\frac{dw}{dt} + \tilde{A}w = 0 \tag{97}$$

$$\int_0^1 uv \, dx = 0 \text{ in } [0, +\infty) \tag{98}$$

then

$$w = 0. \tag{99}$$

But (99) follows directly from (60), (97), and (98).

Remark 1: Let us point out that the main difficulty in the proof of Theorem 1 is the strong convergence of $w(t)$ to zero in H (as $t \to +\infty$). Indeed the convergence of $\omega(t)$ to zero can be directly deduced from the fact that $\tilde{\omega}$ is in $L^2(0, +\infty)$ and its derivative is bounded (see (54), (8), (16), (47), and [8, Lemma 1]). Moreover, the weak convergence of $w(t)$ to zero in H can be proved by proceeding as in Appendix A.

IV. Proof of Theorem 2

Throughout this section, γ is defined by (41) and $\bar{\omega} = 0$. We proceed as in the previous section, i.e.,

1) first we prove that the trajectories of (7) and (8) are precompact in $H \times R$ for $t \geq 0$;
2) then conclude by LaSalle's theorem.

A. Precompactness of the Trajectories

Let (w, ω) be a solution of (7) and (8). By (44)

$$(w, \omega) \text{ is bounded in } H \times R \text{ on } [0, +\infty). \tag{100}$$

In order to prove that

$$\{(w(t), \omega(t));\, t \geq 0\} \text{ is precompact in } H \times R \tag{101}$$

we are going to check that, again

$$\forall \epsilon > 0, \quad \exists k \geq 0 \quad \text{s.t.} \quad (\mathcal{E}(w_k(.,t)) \leq \epsilon, \ \forall t \geq 0). \tag{102}$$

From (7) we get

$$\frac{dw_k}{dt} + Aw_k = \omega^2(0, u_k), \qquad \forall k \geq 1 \tag{103}$$

which implies that

$$\frac{d}{dt}\mathcal{E}(w_k) = 2\omega^2 \int_0^1 u_k v_k \, dx, \qquad \forall k \geq 1. \tag{104}$$

Let $f : [0, +\infty) \to R$ and $g : [0, +\infty) \to R$ be defined by

$$f(t) = F(\mathcal{E}(w(.,t)))/K(w(.,t)) \tag{105}$$

$$g(t) = \omega^2(t) - \psi^{\frac{2}{3}}(w(.,t)) - \left[\bar{\sigma}\left(\int_0^1 u(.,t)v(.,t)\, dx\right) \Big/ K(w(.,t))\right]. \tag{106}$$

By (36), (105), and (106), we have

$$\omega^2 = g + f. \tag{107}$$

For $\bar{w} = (\bar{u}, \bar{v}) \in H$, let

$$Q(\bar{w}) = \sum_{j=1}^{+\infty} \mu_j \left(a_{2j}^2 + a_{2j-1}^2\right)^2 \tag{108}$$

where the a_j, $j \geq 1$, are defined by (57) and (58) with $\bar{\omega} = 0$. It is clear that for any $\theta \in [0, \mu_1)$ there exists $C_\theta > 0$ such that for any $\bar{\omega} \in [0, \theta^{\frac{1}{2}})$

$$Q(\bar{w}) \leq \tilde{Q}(\bar{w}) \leq C_\theta Q(\bar{w}), \qquad \forall \bar{w} \in H. \tag{109}$$

It follows easily from (109) and from the Proof of Lemma 1 that the following generalization of this lemma holds.

Lemma 2: For any $\theta \in [0, \mu_1)$, there exist $C^\star \geq 1$ and $T^\star \geq 1$ such that, for every $\lambda \in [0, \theta]$, for every $h^\star \in C^0([0, T^\star]; L^2(0,1))$, and for every $w^\star = (u^\star, v^\star) \in C^0([0, T^\star]; H)$ such that

$$\frac{dw^\star}{dt} + Aw^\star - \lambda(0, u^\star) = (0, h^\star)$$

one has

$$Q(w^\star(.,0)) \leq C^\star \left(\int_0^{T^\star} \left(\int_0^1 u^\star v^\star \, dx\right)^2 dt + \left(\int_0^{T^\star}\int_0^1 h^{\star 2} \, dx \, dt\right)^2\right) \tag{110}$$

$$\int_0^{T^\star}\left(\int_0^1 u^\star v^\star\,dx\right)^2 dt \leq C^\star\left(Q(w^\star(.,0)) + \left(\int_0^{T^\star}\int_0^1 h^{\star 2}\,dx\,dt\right)^2\right). \quad (111)$$

Let

$$\theta = \mathrm{Sup}\{f(t);\, t\in[0,+\infty)\}. \quad (112)$$

By (35) and (105), one has on $[0,+\infty)$

$$0 \leq f \leq \frac{F(\mathcal{E})}{1-\mu_1^{-1}\mathcal{E}F'(\mathcal{E})}. \quad (113)$$

By (27), (100), (112), and (113)

$$\theta \in [0,\mu_1). \quad (114)$$

Let $T^\star$ be as in Lemma 1 and let, for $T\geq 0$

$$\lambda(T) = \frac{1}{T^\star}\int_T^{T+T^\star} f(s)\,ds. \quad (115)$$

From (112), (113), and (115), we get that

$$\lambda(T)\in[0,\theta], \qquad \forall T\geq 0. \quad (116)$$

Moreover, from (103) and (107), we obtain, for every $T\geq 0$ and for every $k\geq 1$

$$\frac{dw_k}{dt} + Aw_k - \lambda(T)(0,u_k) = (f(t)+g(t)-\lambda(T))(0,u_k) \quad (117)$$

which implies that

$$\frac{d}{dt}\left(\mathcal{E}(w_k) - \lambda(T)\int_0^1 u_k^2 dx\right) = 2(f(t)+g(t)-\lambda(T))\int_0^1 u_k v_k\,dx. \quad (118)$$

For $T\geq 0$ and $\bar{w}=(\bar{u},\bar{v})\in H$, let

$$\mathcal{E}_T(\bar{w}) = \mathcal{E}(\bar{w}) - \lambda(T)\int_0^1 \bar{u}^2\,dx. \quad (119)$$

Let us denote by C_i, $i\geq 12$, various constants which are independent of $T\geq 0$ and $k\geq 1$. Using (100), (117), (118), and (119), using (110) for $w^\star(.,t) = w(.,t+T)$, $h^\star(.,t) = (f(t+T)+g(t+T)-\lambda(T))(0,u(.,t+T))$, $\lambda=\lambda(T)$, and using (111) for $w^\star(.,t)=w_k(.,t+T)$, $h^\star(.,t) = (f(t+T)+g(t+T)-\lambda(T))(0,u_k(.,t+T))$, $\lambda=\lambda(T)$, we get the existence of $C_{12}>0$ such that

$$|\mathcal{E}_T(w_k(.,T+T^\star)) - \mathcal{E}_T(w_k(.,T))| \leq C_{12}\int_T^{T+T^\star}\left((f(t)-\lambda(T))^2 + g(t)^2 + \left(\int_0^1 uv\,dx\right)^2\right)dt. \quad (120)$$

Note that (30) and (44) give

$$\int_0^{+\infty}\left(\int_0^1 uv\,dx\right)\bar{\sigma}\left(\int_0^1 uv\,dx\right)dt < +\infty \quad (121)$$

$$\int_0^{+\infty}\left(\omega - \psi^{\frac{1}{3}}\right)^2\left(\omega^2 + \psi^{\frac{1}{3}}\omega + \psi^{\frac{2}{3}}\right)dt < +\infty. \quad (122)$$

From (30), (31), (100), and (121) we get

$$\int_0^{+\infty}\left(\int_0^1 uv\,dx\right)^2 dt < +\infty \quad (123)$$

$$\int_0^{+\infty}\left(\bar{\sigma}\left(\int_0^1 uv\,dx\right)\right)^2 dt < +\infty. \quad (124)$$

From (35), (100), (106), (122), and (124), we get

$$\int_0^{+\infty} g^2\,dt < +\infty. \quad (125)$$

By (115) and the Poincaré inequality, there exists $C_{13}>0$ such that

$$\int_T^{T+T^\star}|f-\lambda(T)|^2\,dt \leq C_{13}\int_T^{T+T^\star}\dot{f}^2\,dt. \quad (126)$$

By (7), (115), and (119), there exists $C_{14}>0$ such that if

$$\Delta = |\mathcal{E}_{T+T^\star}(w_k(.,T+T^\star)) - \mathcal{E}_T(w_k(.,T+T^\star))| \quad (127)$$

then

$$\begin{aligned}\Delta &= \left|\int_T^{T+T^\star}\frac{d\lambda}{dt}(t)\,dt\right|\int_0^1 u_k^2(x,T+T^\star)\,dx \\ &\leq \frac{1}{2}\left(T^\star\int_T^{T+T^\star}\left(\frac{d\lambda}{dt}(t)\right)^2 dt + \left(\int_0^1 u^2(x,T+T^\star)\,dx\right)^2\right) \\ &\leq \frac{1}{2}\left(\frac{1}{T^\star}\int_T^{T+T^\star}(f(t+T^\star)-f(t))^2\,dt + \left(\int_0^1 u^2(x,T)\,dx + 2\int_T^{T+T^\star}\int_0^1 uv\,dx\,dt\right)^2\right) \\ &\leq C_{14}\left(\int_T^{T+2T^\star}\dot{f}^2\,dt + \left(\int_0^1 u^2(x,T)\,dx\right)^2 + \int_T^{T+T^\star}\left(\int_0^1 uv\,dx\right)^2 dt\right).\end{aligned} \quad (128)$$

By the Cauchy–Schwarz inequality and (108)

$$\left(\int_0^1 u^2(x,T)\,dx\right)^2 \leq \left(\sum_{j=1}^{+\infty}\frac{1}{\mu_j}\right)Q(w(.,T))$$

which, with (86), (110) for $w^\star = w(.,t+T)$, $h^\star(.,t) = (f(t+T)-\lambda(T)+g(t+T))(0,u(.,t+T))$ and $\lambda=\lambda(T)$, (120), (126), (127), and (128), gives the existence of $C_{15} > 0$ such that for every $T \geq 0$ and every $k \geq 1$

$$\begin{aligned} &|\mathcal{E}_{T+T^\star}(w_k(.,T+T^\star)) - \mathcal{E}_T(w_k(.,T))| \\ &\quad \leq C_{15}\Bigg(\int_T^{T+2T^\star} \dot{f}^2\,dt \\ &\qquad + \int_T^{T+T^\star}\left(g^2 + \left(\int_0^1 uv\,dx\right)^2\right)dt\Bigg). \end{aligned} \tag{129}$$

By (116) and (119)

$$\begin{aligned} (1-(\theta/\mu_1))\mathcal{E}(\bar{w}) \leq \mathcal{E}_T(\bar{w}) \leq \mathcal{E}(\bar{w}), \\ \forall \bar{w} \in H, \qquad \forall T \geq 0. \end{aligned} \tag{130}$$

Straightforward computations show that

$$\begin{aligned} \dot{f} = 2\Bigg(K^{-1}\omega^2 F'(\mathcal{E}) + K^{-2}F(\mathcal{E})\Bigg(\omega^2 F''(\mathcal{E})\int_0^1 u^2\,dx \\ + F'(\mathcal{E})\Bigg)\Bigg)\int_0^1 uv\,dx \end{aligned} \tag{131}$$

which, with (35), (100), and (123), gives

$$\int_0^{+\infty} \dot{f}^2\,dt < +\infty. \tag{132}$$

From (114), (123), (125), (129), (130), and (132), we get that for every $\epsilon > 0$, there exist $T_2 \geq 0$ such that for every $k \in N^\star$, for every $n \in N^\star$, and for every $t \geq T_2$

$$\mathcal{E}(w_k(.,t+nT^\star)) \leq \frac{\epsilon}{2} + \frac{1}{1-(\theta/\mu_1)}\mathcal{E}(w_k(.,t))$$

which, as in Section III-A, implies (102).

B. LaSalle's Theorem

By LaSalle's theorem, (30), (36), (44), and (101), in order to finish the proof of Theorem 2, it suffices to check that if $w \in C^0([0,+\infty); H)$ is such that

$$\frac{dw}{dt} + Aw = (F(\mathcal{E}(w))/K(w))(0,u) \tag{133}$$

$$\int_0^1 uv\,dx = 0 \text{ on } [0,+\infty) \tag{134}$$

then $w = 0$. But, from (26), (133), and (134), we get that $K(w(.,t))$ and $\mathcal{E}(w(.,t))$ do not depend on time and so there exists $\theta \in R$ such that

$$F(\mathcal{E}(w(.,t)))/K(w(.,t)) = \theta, \qquad \forall t \in [0,+\infty). \tag{135}$$

By (27), (35), and (135)

$$\theta \in [0, \mu_1). \tag{136}$$

From (133)–(136) and (110) of Lemma 2 applied with $\lambda = \theta$, $w^\star = w$, and $h^\star = 0$, we get $w = 0$.

Appendix A

In this Appendix, we prove (21). By LaSalle's theorem (for H with the weak topology) it suffices to check that if $w = (u,v) \in C^0([0,+\infty); H)$ satisfies

$$\frac{dw}{dt} + Aw = 0 \tag{137}$$

$$\int_0^1 uv\,dx \geq 0 \tag{138}$$

then

$$w = 0. \tag{139}$$

By (137) and (138)

$$\frac{d}{dt}\int_0^1 u^2\,dx = 2\int_0^1 uv\,dx \geq 0. \tag{140}$$

Therefore, since $w \in L^\infty([0,+\infty); H)$, there exists $l \in [0,+\infty)$ such that

$$\lim_{t\to+\infty}\int_0^1 u^2(x,t)\,dx = l. \tag{141}$$

Clearly, in order to prove (139), it suffices to check that

$$l = 0. \tag{142}$$

Since $w \in L^\infty([0,+\infty); H)$ there exists a sequence (t_n), $n \in N$ of positive real numbers tending to $+\infty$ as n tends to $+\infty$ such that for some $\bar{w}_0 \in H$,

$$w(.,t_n) \rightharpoonup \bar{w}_0 \text{ weakly in } H. \tag{143}$$

Since $e^{tA} : H \to H$ is weakly continuous, we have, using (143)

$$\begin{aligned} w(.,t+t_n) \rightharpoonup \bar{w}(.,t) \quad \text{weakly in } H \text{ as } n \to +\infty, \\ \forall t \in R \end{aligned} \tag{144}$$

with $\bar{w}(.,t) = e^{tA}\bar{w}_0$. From (141) and (144) we get, if $\bar{w} = (\bar{u},\bar{v})$

$$\int_0^1 \bar{u}^2(x,t)\,dx = l, \qquad \forall t \in R. \tag{145}$$

Taking the time derivative of (145) with respect to time, we get

$$\int_0^1 \bar{u}\bar{v}\,dx = 0. \tag{146}$$

Using (96) with $(u^\star, v^\star) = (\bar{u},\bar{v})$ we get, with (146), $\bar{w} = 0$, which with (145) implies (142).

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