Global Exact Controllability of the 2D Navier-Stokes Equations on a Manifold Without Boundary

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Global exact controllability in arbitrary small time is established for the Navier-Stokes equations of incompressible fluids on any connected orientable compact Riemannian surface without boundary when the control is defined on an arbitrary nonempty open subset of the surface.

Dedicated to Mark Iosifovich Vishik on the occasion of his 75th anniversary

§1. INTRODUCTION

We study global exact controllability of the Navier-Stokes equations. We consider these equations on a two-dimensional Riemannian manifold. This gives us the possibility to apply (in §5) the controllability result obtained here to a question that arises in climate theory, where the Navier-Stokes equations on a sphere are used.

Let (M, g) be a connected two-dimensional orientable compact Riemannian surface of class C^∞. For $x \in M$, we denote by T_xM the tangent space of M at x; let $TM = \cup T_xM$ be the tangent space to M. Let $f : M \times [0, +\infty) \to TM$ be a time-varying vector field of class C^∞, $y_0(x)$ be a state and let $\widehat{y}(x,t)$ be a solution of the Navier-Stokes equations on M with external force equal to f. More precisely, y_0 is a C^∞-map of M into TM and $\widehat{y}(x,t)$ is a C^∞ map of $M \times [0, +\infty)$ into TM such that

$$y_0(x) \in T_xM, \quad \widehat{y}(x,t) \in T_xM, \quad \forall (x,t) \in M \times [0, +\infty), \tag{1.1}$$

$$\operatorname{div} y_0(x) = 0, \quad \operatorname{div} \widehat{y}(x,t) = 0 \quad \text{on} \quad M \tag{1.2}$$

and, for some C^∞ function $\widehat{p} : M \times [0, +\infty) \to \mathbb{R}$,

$$\partial \widehat{y}/\partial t - \Delta \widehat{y} + \nabla_{\widehat{y}} \widehat{y} + \nabla \widehat{p} = f \quad \text{on} \quad M \times [0, +\infty). \tag{1.3}$$

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(In the next section we recall the definition of div, Δ, and ∇.) Let M_0 be a nonempty open subset of M. The controllability problem we consider is the following: given a positive real number τ, does there exist a C^∞-map u: $M \times [0, +\infty) \to TM$ (called the *control*) for which

$$u(x,t) \in T_xM, \quad \forall (x,t) \in M \times [0,+\infty); \qquad \text{Support}\, u \subset (0,\tau) \times M_0, \tag{1.4}$$

and such that if y is a C^∞-map for which

$$y(x,t) \in T_xM, \quad \forall (x,t) \in M \times [0,+\infty), \tag{1.5}$$

$$\operatorname{div} y = 0 \quad \text{in} \quad M \times [0,+\infty), \tag{1.6}$$

$$y(x,0) = y_0(x), \quad \forall x \in M, \tag{1.7}$$

and

$$\partial y/\partial t - \Delta y + \nabla_y y + \nabla p = f + u \quad \text{in} \quad M \times [0,+\infty) \tag{1.8}$$

for some C^∞-function $p : M \times [0,+\infty) \to \mathbb{R}$, then we have

$$y(x,t) = \widehat{y}(x,t) \quad \forall (x,t) \in M \times [\tau,+\infty). \tag{1.9}$$

Roughly speaking, using some suitable force acting on M_0 during the time interval $(0,\tau)$, we want to pass from the initial value y_0 to the solution $\widehat{y}$. If such controls u exist for any $f, y_0, \widehat{y}$, and τ as above, we say that the Navier-Stokes equations on M are *globally exactly controllable* in small time by a force acting on M_0. The goal of this paper is to prove the following assertion.

Theorem 1.1. *For any nonempty open subset M_0 of M, the Navier-Stokes equations on M are globally exactly controllable in small time by a force acting on M_0.*

The proof of Theorem 1.1 is divided in two steps.

Step 1. We prove that the Navier-Stokes equations on M are globally approximately controllable in small time. Roughly speaking, this means that using controls acting on M_0 during a small time interval $(0,\tau_1)$, we can pass from the state y_0 to a solution y which is arbitrarily close to $\widehat{y}(\cdot,\tau_1)$ at time τ_1.

Step 2. We prove that the Navier-Stokes equations on M are locally exactly controllable. Roughly speaking, this means that, given any positive time $\tau_2 > \tau_1$ and using control acting on M_0 and during the time interval (τ_1,τ_2), we can pass from $y(\cdot,\tau_1)$ to $\widehat{y}(\cdot,\tau_1+\tau_2)$ provided that $y(\cdot,\tau_1)$ is sufficiently close to $\widehat{y}(\cdot,\tau_1)$.

Theorem 1.1 readily follows from (the proof of) Step 1 and Step 2. The proof of Step 1, which is given in §3, is essentially an adaptation to our situation of the proof, given in [C1], of a result on global approximate controllability for the 2D Navier-Stokes equations on domains in $\mathbb{R}^2$ when the fluid slips on the boundary according to the Navier slip boundary conditions. Let us recall that the proof given in [C1] relies on the "return method," which was introduced in [C2] for a stabilization problem and used in [C3] and [C4] for the controllability of the 2D Euler equations of incompressible fluids. See also [So] for an application of this method to motion planning problems. The proof of Step 2, which is given in §4, is essentially an adaptation of [FI1, FI2, FI5], where local exact controllability is proved for the Boussinesq and the Navier-Stokes equations defined in a domain of $\mathbb{R}^n$, $n = 2,3$. The method of [FI1, FI2, FI5] is a development of the approach established in [FI3, F, FI4] for the investigation of local exact controllability for Navier-Stokes equations.

Remark 1.2. Here, as in [FI1, FI2, FI4], by *exact controllability* we mean passing from a state to a solution instead of passing from one state to another state. This notion of exact controllability is natural for PDE which are not reversible in time. The point is that for such PDE passing to a general state is impossible. This notion, in the framework of finite-dimensional control systems, was introduced by Willems [W].

§2. NOTATION

For $k \in \mathbb{N} \cup \{\infty\}$, let $C^k(TM)$ be the set of C^k vector fields on M; if $k < \infty$, by $|\cdot|_{C^k(TM)}$ we denote a norm defining the usual topology of $C^k(TM)$. Let ∇ be the Levi-Civita connection associated with the metric g. For any $y \in C^\infty(TM)$ and any $z \in C^\infty(TM)$, we have $\nabla_y z \in C^\infty(TM)$. In local coordinates, one has, adopting the usual summation convention,

$$\nabla_{y^i \partial/\partial x^i} z^j \frac{\partial}{\partial x^j} = y^i \frac{\partial z^j}{\partial x^i} \frac{\partial}{\partial x^j} + y^i z^j \Gamma^k_{ij} \frac{\partial}{\partial x^k}, \tag{2.1}$$

where

$$\Gamma^k_{ij} = \frac{g^{kl}}{2} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right) \tag{2.2}$$

are the Christoffel symbols of g; in (2.2) $(g^{kl};\ 1 \leqslant k \leqslant 2,\ 1 \leqslant l \leqslant 2)$ denotes the inverse of the matrix

$$(g_{ij};\ 1 \leqslant i \leqslant 2,\ 1 \leqslant j \leqslant 2) \quad \text{with} \quad g\left(y^i \frac{\partial}{\partial x^i}, z^j \frac{\partial}{\partial x^j} \right) = g_{ij} y^i z^j.$$

Let $C^\infty(M, \mathbb{R})$ be the set of real-valued C^∞ functions on M. For $y \in C^\infty(TM)$, we define $\operatorname{div} y \in C^\infty(M, \mathbb{R})$ by $\operatorname{div} y(x) = \operatorname{trace}(a \in T_x M \to \nabla_a y(x))$. In local coordinates,

$$\operatorname{div} \left(y^i \frac{\partial}{\partial x^i} \right) = \frac{\partial y^i}{\partial x^i} + y^k \Gamma^l_{kl}. \tag{2.3}$$

For $p \in C^\infty(M, \mathbb{R})$, we define $\nabla p \in C^\infty(TM)$ by $g(\nabla p, y) = dp(y)$, $\forall y \in C^\infty(TM)$, where dp is the differential of p. In local coordinates,

$$\nabla p = g^{ij} \frac{\partial p}{\partial x^i} \frac{\partial}{\partial x^j}. \tag{2.4}$$

Let us now recall the definition of $\Delta y \in C^\infty(TM)$ for $y \in C^\infty(TM)$. For $z \in C^\infty(TM)$, $\widetilde{z} \in C^\infty(TM)$ and $y \in C^\infty(TM)$, let $\nabla^2_{z,\widetilde{z}} y \in C^\infty(TM)$ be defined by $\nabla^2_{z,\widetilde{z}} y = \nabla_z(\nabla_{\widetilde{z}} y) - \nabla_{\nabla_z \widetilde{z}} y$. One easily verifies that $(z(x), \widetilde{z}(x)) \to (\nabla^2_{z,\widetilde{z}} y)(x)$ is a (well-defined) bilinear transformation on $T_x M$ with values in $T_x M$. The trace of this bilinear transformation is, by definition, $\Delta y(x)$. In local coordinates,

$$\Delta y = g^{ij} \left(\nabla_{\partial/\partial x^i} \left(\nabla_{\partial/\partial x^j} y \right) - \nabla_{\nabla_{\partial/\partial x^i} \partial/\partial x^j} y \right). \tag{2.5}$$

It is well known (see, e.g., [EL]) that Δ is a negative semi-definite symmetric operator with respect to the inner product on $C^\infty(TM)$ defined by

$$\langle y, z \rangle = \int_M g(y, z)\, dx, \tag{2.6}$$

where dx is the measure on M associated with g. (Recall that M is a compact orientable manifold without boundary.) In local coordinates, we have

$$dx = (g_{11}(x) g_{22}(x) - g_{12}(x)^2)^{1/2}\, dx^1\, dx^2, \quad g(y, z) = g_{ij}(x) y^i(x) z^j(x).$$

Let I be an interval of $\mathbb{R}$. For $k \in \mathbb{N} \cup \{\infty\}$, by $C^k(M \times I; TM)$ we denote the set of $y : M \times I \to TM$ such that $y(x,t) \in T_x M$ for any $x \in M$ and $t \in I$.

For $y \in C^\infty(M \times I; TM)$, we define $\operatorname{div} y \in C^\infty(M \times I, M)$ by

$$\operatorname{div} y(x,t) = (\operatorname{div} y(\cdot,t))(x), \quad \forall (x,t) \in M \times I. \tag{2.7}$$

In a similar way, for $y \in C^\infty(M \times I; TM)$, $z \in C^\infty(M \times I; TM)$ and $p \in C^\infty(M \times I, \mathbb{R})$, we define

$$\Delta y \in C^\infty(M \times I; TM), \ \ \nabla_z y \in C^\infty(M \times I; TM), \ \ \nabla p \in C^\infty(M \times I; TM).$$

With this notation, the Cauchy problem for the Navier-Stokes equations describing an incompressible fluid flow on M is the following one (see, e.g., [Sc, EM, G, P1, T1] or [P2]): given $\varphi \in C^\infty(TM)$ such that $\operatorname{div} \varphi = 0$ in M and $f \in C^\infty(M \times [0,+\infty); TM)$, find $y \in C^\infty(M \times [0,+\infty); TM)$ and $p \in C^\infty(M \times [0,+\infty); \mathbb{R})$ such that

$$\partial y/\partial t - \Delta y + \nabla_y y + \nabla p = f \quad \text{in} \quad M \times [0,+\infty), \tag{2.8}$$

$$\operatorname{div} y = 0 \quad \text{in} \quad M \times [0,+\infty), \tag{2.9}$$

$$y(x,0) = \varphi(x), \quad \forall x \in M. \tag{2.10}$$

It is well known that this problem has a unique solution up to an arbitrary function depending on time alone added to p; e.g., see [EM, T1, P2].

Remark 2.1. As pointed out by Priebe [P1], different Navier-Stokes equations are proposed in [AB, IF] and [Il1]. Theorem 1.1 still holds for these other Navier-Stokes equations. The proof is similar to the one given here.

§3. GLOBAL APPROXIMATE CONTROLLABILITY

The goal of this section is to prove the following proposition.

Proposition 3.1. *Let $y_i(x) \in C^\infty(TM)$, $\operatorname{div} y_i = 0$, $i = 0,1$, and let M_0 a nonempty open subset of M be given. Then for any $\varepsilon > 0$ there exist $\delta \in (0,\varepsilon]$, $y \in C^\infty(M \times [0,\delta]; TM)$, $p \in C^\infty(M \times [0,\delta], \mathbb{R})$ and $u \in C^\infty(M \times [0,\delta]; TM)$ such that*

$$\partial y/\partial t - \Delta y + \nabla_y y + \nabla p = f + u \quad \textit{in} \quad M \times [0,\delta], \tag{3.1}$$

$$\operatorname{support} u \subset M_0 \times (0,\delta), \tag{3.2}$$

$$\operatorname{div} y = 0 \quad \textit{in} \quad M \times [0,\delta], \tag{3.3}$$

$$y(x,0) = y_0(x), \quad \forall x \in M, \tag{3.4}$$

$$\int_M \left(|y(\cdot,\delta) - y_1(\cdot)|^2 + |\nabla y(\cdot,\delta) - \nabla y_1(\cdot)|^2 \right) dx < \varepsilon. \tag{3.5}$$

In (3.5) we have used the well-known notation

$$|\nabla z|^2 = g^{ij} g\left(\nabla_{\partial/\partial x_i} z, \nabla_{\partial/\partial x_j} z \right)$$

and $|z|^2 = g(z,z) = g_{ij} z^i z^j$ for $z \in C^\infty(TM)$.

As was mentioned in the introduction, our proof of Proposition 3.1 relies on a method, called the "return method", introduced in [C2] for a stabilization problem and used in [C3] and [C4] to prove the controllability of the 2D Euler equations of incompressible fluids. The strategy of the "return method" consists in using trajectories $\bar{y}$ of the control system going from an initial state equal to 0 to a final state equal (or close) to 0, and such that the linearized control system around $\bar{y}$ has "good" controllability. Then perturbing in a suitable way these $\bar{y}$, one may hope to be able to go from any state to any other state. The key point is that, in some cases, $\bar{y} \equiv 0$, which is usually a trajectory that is not a good choice. For example, for the Euler equation, the linearized control system around $\bar{y} = 0$ is not controllable. Our construction of $\bar{y}$ is similar to the one given in [C1]. It relies on the following lemma.

Proposition 3.2. *There exist $y^* \in C^\infty(M \times [0,1]; TM)$, $p^* \in C^\infty(M \times [0,1]; TM)$, $u^* \in C^\infty(M \times [0,1]; TM)$, and $C^* > 0$ such that*

$$\partial y^*/\partial t + \nabla_{y^*} y^* + \nabla p^* = u^* \quad \textit{in} \quad M \times [0,1], \tag{3.6}$$

$$\operatorname{support} u^* \subset M_0 \times (0,1), \tag{3.7}$$

$$\operatorname{div} y^* = 0 \quad \textit{in} \quad M \times [0,1], \tag{3.8}$$

$$y^*(x,0) = 0, \quad y^*(x,1) = 0, \tag{3.9}$$

and such that for any $z_0 \in C^\infty(TM)$ and $z_1 \in C^\infty(TM)$ satisfying

$$\operatorname{div} z_0 = 0 \quad \textit{and} \quad \operatorname{div} z_1 = 0 \quad \textit{in} \quad M \tag{3.10}$$

there exist $z^* = Z^*(z_0, z_1) \in C^\infty(M \times [0,1]; TM)$, $\pi^* = \Pi^*(z_0, z_1) \in C^\infty(M \times [0,1]; \mathbb{R})$, and $v^* = V^*(z_0, z_1) \in C^\infty(M \times [0,1]; TM)$ such that

$$\frac{\partial z^*}{\partial t} + \nabla_{y^*} z^* + \nabla \pi^* = v^* \quad \textit{in} \quad M \times [0,1], \tag{3.11}$$

$$\operatorname{support} v^* \subset M_0 \times (0,1), \tag{3.12}$$

$$\operatorname{div} z^* = 0 \quad \textit{in} \quad M \times [0,1], \tag{3.13}$$

$$z^*(x,0) = z_0(x), \quad z^*(x,1) = z_1(x) \quad \forall x \in M, \tag{3.14}$$

$$|z^*(\cdot,t)|_{C^2(TM)} \leqslant C^*\big(|z_0|_{C^3(TM)} + |z_1|_{C^3(TM)}\big) \quad \forall t \in [0,1]. \tag{3.15}$$

The proof of Proposition 3.2 follows directly by adapting the proof of [C1, Proposition 2.1] to the Riemannian situation. In fact, since M has no boundary, many of the arguments given in [C1] are no longer needed. So, for simplicity, we omit the proof of Proposition 3.2 and now explain how one uses Proposition 3.2 in order to prove Proposition 3.1. Again, we adapt [C1].

Let $\delta \in (0, \varepsilon]$. Let us define $\bar{y} \in C^\infty(M \times [0,\delta]; TM)$ by requiring, for all $(x,t) \in M \times [0,\delta]$,

$$\bar{y}(x,t) = \frac{1}{\delta} y^*\Big(x, \frac{t}{\delta}\Big), \tag{3.16}$$

$$\bar{p}(x,t) = \frac{1}{\delta^2} p^*\Big(x, \frac{t}{\delta}\Big), \tag{3.17}$$

$$\bar{u}(x,t) = \frac{1}{\delta^2} u^*\Big(x, \frac{t}{\delta}\Big). \tag{3.18}$$

By (3.6), (3.16), (3.17), and (3.18)

$$\partial \bar{y}/\partial t + \nabla_{\bar{y}} \bar{y} + \nabla \bar{p} = \bar{u} \quad \text{in} \quad M \times [0, f]. \tag{3.19}$$

By (3.7) and (3.18), we have support $\bar{u} \subset M_0 \times (0, \delta)$. By (3.8) and (3.16),

$$\operatorname{div} \bar{y} = 0 \quad \text{in} \quad M \times [0, \delta]. \tag{3.20}$$

By (3.9) and (3.16),

$$\bar{y}(x,0) = \bar{y}(x,\delta) = 0 \quad \forall x \in M. \tag{3.21}$$

Let $Y^* \in C^\infty(M \times [0,1]; TM)$ and $P^* \in C^\infty(M \times [0,1]; \mathbb{R})$ be such that

$$\partial Y^*/\partial t + \nabla_{Y^*} y^* + \nabla_{y^*} Y^* + \nabla P^* = \Delta y^* \quad \text{in} \quad M \times [0,1], \tag{3.22}$$

$$\operatorname{div} Y^* = 0 \quad \text{in} \quad M \times [0,1], \tag{3.23}$$

$$Y^*(x,0) = 0 \quad \forall x \in M. \tag{3.24}$$

Let us define $\overline{Y} \in C^\infty(M \times [0,\delta]; TM)$ and $\overline{P} \in C^\infty(M \times [0,\delta]; \mathbb{R})$ by requiring, for all $(x,t) \in M \times [0,\delta]$,

$$\overline{Y}(x,t) = Y^*\Big(x, \frac{t}{\delta}\Big), \quad \overline{P}(x,t) = \frac{1}{\delta} P^*\Big(x, \frac{t}{\delta}\Big). \tag{3.25}$$

By (3.16), (3.22), (3.23), (3.24), and (3.25),

$$\partial \overline{Y}/\partial t + \nabla_{\overline{Y}} \bar{y} + \nabla_{\bar{y}} \overline{Y} + \nabla \overline{P} = \Delta \bar{y} \quad \text{in} \quad M \times [0,1], \tag{3.26}$$

$$\operatorname{div} \overline{Y} = 0 \quad \text{in} \quad M \times [0,\delta], \tag{3.27}$$

$$\overline{Y}(x,0) = 0 \quad \forall x \in M. \tag{3.28}$$

Let us now define $z \in C^\infty(M \times [0,1]; TM)$, $\pi \in C^\infty(M \times [0,1]; \mathbb{R})$ and $v \in C^\infty(M \times [0,1]; TM)$ by requiring, for all $(x,t) \in M \times [0,\delta]$,

$$z(x,t) = Z^*(y_0(\cdot,0), y_1(\cdot,\delta) - \overline{Y}(\cdot,\delta))\Big(x, \frac{t}{\delta}\Big) + \overline{Y}(x,t), \tag{3.29}$$

$$\pi(x,t) = \frac{1}{\delta} \Pi^*(y_0(\cdot,0), y_1(\cdot,\delta) - \overline{Y}(\cdot,\delta))\Big(x, \frac{t}{\delta}\Big) + \overline{P}(x,t), \tag{3.30}$$

$$v(x,t) = \frac{1}{\delta} V^*(y_0(\cdot,0), y_1(\cdot,\delta) - \overline{Y}(\cdot,\delta))\Big(x, \frac{t}{\delta}\Big), \tag{3.31}$$

where $y_0(x,0) \equiv y_0(x)$, $y_1(x,\delta) \equiv y_1(x)$. By (3.11), (3.16), (3.26), (3.29), (3.30), and (3.31),

$$\partial z/\partial t - \Delta \bar{y} + \nabla_{\bar{y}} z + \nabla_z \bar{y} + \nabla(\pi + \bar{P}) = v \quad \text{in} \quad M \times [0,\delta]. \tag{3.32}$$

By (3.12) and (3.31), we have $\operatorname{support} v \subset M_0 \times (0,\delta)$. By (3.13), (3.27), and (3.29),

$$\operatorname{div} z = 0 \quad \text{in} \quad M \times [0,\delta]. \tag{3.33}$$

By (3.14), (3.28), and (3.29),

$$z(x,0) = y_0(x,0) \quad \forall x \in M. \tag{3.34}$$

By (3.14) and (3.29),

$$z(x,\delta) = y_1(x,\delta) \quad \forall x \in M. \tag{3.35}$$

Finally, let $y \in C^\infty(M \times [0,1]; TM)$ and $p \in C^\infty(M \times [0,1]; \mathbb{R})$ be such that for

$$u := \bar{u} + v, \tag{3.36}$$

we have

$$\partial y/\partial t - \Delta y + \nabla_y y + \nabla p = f + u \quad \text{in} \quad M \times [0,\delta], \tag{3.37}$$

$$\operatorname{div} y = 0 \quad \text{in} \quad M \times [0,\delta], \tag{3.38}$$

$$y(x,0) = y_0(x,0) \quad \forall x \in M. \tag{3.39}$$

Note that

$$\operatorname{support} u \subset M_0 \times (0,\delta). \tag{3.40}$$

Thus, in order to prove Proposition 3.1, it remains only to check that (3.5) holds if δ is sufficiently small. Let $r \in C^\infty(M \times [0,\delta]; TM)$ and $q \in C^\infty(M \times [0,\delta], \mathbb{R})$ be defined by

$$r = y - \bar{y} - z, \qquad q = p - \bar{p} - \overline{P} - \pi. \tag{3.41}$$

By (3.21), (3.35), and (3.41),

$$y(x,\delta) - y_1(x,\delta) = r(x,\delta). \tag{3.42}$$

Hence, it suffices to check that

$$\lim_{\delta\to 0} \int |r(\cdot,\delta)|^2 + |\nabla r(\cdot,\delta)|^2\,dx = 0. \tag{3.43}$$

By (3.19), (3.32), (3.37), and (3.41), we have

$$\partial r/\partial t - \Delta r + \nabla_r r + \nabla_{\bar{y}+z} r + \nabla_r(\bar{y}+z) - \Delta z + \nabla_z z + \nabla q = f \quad \text{in} \quad M\times[0,\delta]. \tag{3.44}$$

By (3.20), (3.33), (3.38), and (3.41),

$$\operatorname{div} r = 0 \quad \text{in} \quad M\times[0,\delta]. \tag{3.45}$$

For $(a,b,c)\in C^\infty(TM)$, let $B(a,b,c)$ be defined by

$$B(a,b,c) = \int_M \nabla_a b\cdot c\,dx = \int g(\nabla_a b, c)\,dx. \tag{3.46}$$

It is well known (e.g., see [P2, Lemma 12]) that the following assertion is true.

Lemma 3.3. *For any a and b in $C^\infty(TM)$ such that*

$$\operatorname{div} a = 0 \quad in \quad M, \tag{3.47}$$

one has

$$B(a,b,b) = 0. \tag{3.48}$$

For $a\in C^\infty(TM)$, let

$$|a|_{L^2(TM)} = \left(\int_M |a|^2\,dx\right)^{1/2}, \quad |\nabla a|_{L^2(TM)} = \left(\int_M |\nabla a|^2\,dx\right)^{1/2}, \tag{3.49}$$

$$|a|_{H^1(TM)} = \left(|a|^2_{L^2(TM)} + |\nabla a|^2_{L^2(TM)}\right)^{1/2}, \quad |a|_{H^2(TM)} = \left(|a|^2_{L^2(TM)} + |\Delta a|^2_{L^2(TM)}\right)^{1/2}. \tag{3.50}$$

One has the following estimates.

Lemma 3.4. *There exists a constant C_1 such that for all a, b, and c in $C^\infty(TM)$ one has*

$$|B(a,b,c)| \leqslant C_1|a|_{L^2(TM)}|b|_{C^1(TM)}|c|_{L^2(TM)}, \tag{3.51}$$

$$|B(a,b,c)| \leqslant C_1|a|^{1/2}_{L^2(TM)}|a|^{1/2}_{H^1(TM)}|b|^{1/2}_{H^1(TM)}|b|^{1/2}_{H^2(TM)}|c|_{L^2(TM)}, \tag{3.52}$$

$$|B(a,b,\Delta c)| \leqslant C_1|a|_{H^1(TM)}|b|_{C^2(TM)}|c|_{H^1(TM)}, \tag{3.53}$$

and if $\operatorname{div} a = 0$, *then*

$$|B(a,b,\Delta b)| \leqslant C_1|a|_{C^1(TM)}|b|^2_{H^1(TM)}. \tag{3.54}$$

Inequality (3.51) readily follows from the definition of B. Inequality (3.52) is classical if M is a smooth bounded domain of $\mathbb{R}^2$ equipped with the Euclidean metric (e.g., see [La, T1]); the general case can be derived from this special case by decomposing c with the help of a partition of unity. Inequalities (3.53) and (3.54) can be readily proved using "integrations by parts," i.e., the Green formula.

For example, in order to prove (3.54), it suffices to note that if by (e_i, $1 \leqslant i \leqslant 2$) we denote a local orthonormal frame near $x \in M$ with $\nabla_{e_i} e_j(x) = 0$ $\forall i \in \{1,2\}$, $\forall j \in \{1,2\}$, then at the point x we have

$$\nabla_a b \cdot \Delta b = \operatorname{div} X + \operatorname{div}\left(\frac{|\nabla b|^2}{2} a\right) - |\nabla b|^2 \operatorname{div} a + \nabla_{e_i} b \cdot R(e_i, a) b + \nabla_{e_i} b \cdot \nabla_{[a,e_i]} b,$$

where R is the curvature of the connection ∇ and where $X \in C^\infty(TM)$ is defined in local coordinates by $X = g^{ij} \nabla_{\partial/\partial x^i} b \cdot \nabla_a b \partial/\partial x^j$.

Let C_i, $i \geqslant 2$, denote various constants which may depend on $M, g, y_0, y_1, \varepsilon$, and f but are independent of δ in $(0, \varepsilon]$ and $t \in [0, \delta]$. For example (see, in particular, (3.15), (3.16), (3.25), and (3.29)), there exists a $C_2 > 0$ such that for all $\delta \in [0, \varepsilon]$ and all $t \in [0, \delta]$ we have

$$|\bar{y}(\cdot, t)|_{C^2(TM)} \leqslant C_2/\delta, \quad |z(\cdot, t)|_{C^2(TM)} \leqslant C_2. \tag{3.55}$$

Let us take the inner product of (3.44) with r and integrate the resulting equality over M. Using (3.20), (3.25), (3.45), and Lemma 3.3 and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}|r(t)|^2_{L^2(TM)} + \int_M |\nabla r(t)|^2\, dx + \int_M (\nabla_{r(t)}(\bar{y} + z)(\cdot, t)) \cdot r(t)\, dx + \int_M ((-\Delta z + \nabla_z z)(\cdot, t)) \cdot r(t)\, dx = \int_M f(\cdot, t) \cdot r(t)\, dx, \tag{3.56}$$

for all $t \in [0, \delta]$, where $r(t) = r(\cdot, t)$. From (3.21), (3.34), (3.39), and (3.41) we obtain

$$r(0) = 0 \quad \text{in} \quad M. \tag{3.57}$$

From (3.51), (3.55), (3.56), (3.57) and Gronwall's inequality, we obtain the existence of $C_3 > 0$ such that for all $\delta \in (0, \varepsilon]$ and all $t \in [0, \delta]$ one has

$$|r(t)|^2_{L^2(TM)} \leqslant C_3 \delta. \tag{3.58}$$

Let us now take the inner product of (3.44) with $-\Delta r$ and integrate the resulting equality over M. Using (3.45) and (3.51)-(3.55), we obtain the existence of a constant $C_4 > 0$ such that for all $\delta \in (0, \varepsilon]$ and all $t \in [0, \delta]$ one has

$$\frac{d}{dt}|\nabla r(t)|^2_{L^2(TM)} + 2|\Delta r(t)|^2_{L^2(TM)} \leqslant C_4\Big(1 + |r(t)|^{1/2}_{L^2(TM)} |r(t)|_{H^1(TM)} |r(t)|^{3/2}_{H^2(TM)} + \frac{1}{C_4}|\Delta r(t)|_{L^2(TM)} + \frac{1}{\delta}|r(t)|^2_{H^1(TM)}\Big). \tag{3.59}$$

From (3.58) and (3.59) we obtain the existence of $C_5 > 0$ such that for all $\delta \in (0, \varepsilon]$ and $t \in [0, \delta]$ one has

$$\frac{d}{dt}|\nabla r(t)|^2_{L^2(TM)} \leqslant C_5\Big(1 + \frac{1}{\delta}|\nabla r|^2_{L^2(TM)} + \delta |\nabla r|^4_{L^2(TM)}\Big). \tag{3.60}$$

By a standard argument this result, in conjunction with (3.57), implies the existence of $C_6 > \varepsilon^{-1}$ such that for all $\delta \in (0, C_6^{-1}]$ and all $t \in [0, \delta]$ we have

$$|\nabla r(t)|^2_{L^2(TM)} \leqslant C_6 t. \tag{3.61}$$

From (3.58) and (3.61) we obtain (3.43), which completes the proof of Proposition 3.1.

§4. LOCAL EXACT CONTROLLABILITY

Let $\{U^{(j)}, \Phi^{(j)}\}$ be a finite atlas of M, and let $\{\varphi_j\}$ be a partition of unity subordinate to the cover $\{U^{(j)}\}$. The Sobolev space $H^s(M;\mathbb{R})$, $s \in \mathbb{R}$, is defined as the completion of $C^\infty(M;\mathbb{R})$ with respect to the norm

$$\|y\|^2_{H^s(M)} = \sum_j \|(\varphi_j y) \circ (\Phi^{(j)})^{-1}\|^2_{H^s(\mathbb{R}^2)},$$

where $\|\cdot\|_{H^s(\mathbb{R}^2;\mathbb{R})}$ is the norm of the Sobolev space $H^s(\mathbb{R}^2)$ of functions defined on $\mathbb{R}^2$ (e.g., see [LM]). The Sobolev space $H^s(TM)$ of vector fields on M is defined similarly. Set

$$V^s(TM) = \{v(x) \in H^s(TM) \,:\, \operatorname{div} v = 0\}. \tag{4.1}$$

The following formula, derived in [I12], is true for an arbitrary vector field $u(x) \in C^\infty(TM)$:

$$\nabla_u u = \nabla|u|^2/2 - n \times u \operatorname{div}(n \times u), \tag{4.2}$$

where $n \times u(x)$ is the counterclockwise rotation of a vector $u(x)$ by an angle of $\pi/2$. In local coordinates, the operator $n \times u$ has the form:

$$(n \times u(x))^k \partial/\partial x^k = (G^{-1}C^{kj} g_{jl} u^l(x))\partial/\partial x^k, \tag{4.3}$$

where $G(x) = g_{11}(x)g_{22}(x) - g^2_{12}(x)$ and the matrix C with entries C^{kj} is as follows:

$$C = (C^{kj}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.4}$$

Taking into account (4.2), we can rewrite the Navier-Stokes equations (1.3) in the form:

$$\partial\widehat{y}(x,t)/\partial t - \Delta\widehat{y} - n \times \widehat{y} \operatorname{div}(n \times \widehat{y}) + \nabla\widehat{p}_1 = f(x,t), \quad \operatorname{div}\widehat{y} = 0, \quad (x,t) \in Q, \tag{4.5}$$

where we set $\nabla\widehat{p}_1 = \nabla(p + |\widehat{y}|^2/2)$; $Q = M \times (0,\tau)$, and $\tau > 0$, is given.

Now let us recall the statement of the local exact controllability problem. Suppose that $(\widehat{y}, \nabla\widehat{p}) \in (C^\infty(Q;TM))^2$ is a given solution of the Navier-Stokes equations (4.5) and $y_0(x) \in V^1(TM)$ is an initial condition close to $\widehat{y}(0,x)$:

$$\|y_0 - \widehat{y}(\cdot,0)\|_{V^1(TM)} < \varepsilon. \tag{4.6}$$

Recall that $M_0 \subset M$ is an arbitrary given subdomain of M. The local exact controllability problem is to find a control $u(x,t)$ concentrated in $M_0 \times (0,\tau)$,

$$\operatorname{supp} u \subset M_0 \times (0,\tau) \tag{4.7}$$

such that there exists a solution $(y(x,t), \nabla p(x,t))$ of the problem

$$\partial y/\partial t - \Delta y - n \times y \operatorname{div}(n \times y) + \nabla p = f(x,t) + u(x,t), \quad \operatorname{div} y = 0, \tag{4.8}$$

$$y\big|_{t=0} = y_0, \quad y(x,\tau) \equiv \widehat{y}(x,\tau). \tag{4.9}$$

The main result of this section is as follows.

Proposition 4.1. *Suppose that $f(x,t) \in C^\infty(\overline{Q};TM)$, and $y_0 \in V^1(TM) \cap C^\infty(TM)$ are given data, $\widehat{y}(x,t) \in C^\infty(\overline{Q};TM)$ is a given solution of (4.5), and inequality (4.6) is true with $\varepsilon < \varepsilon(\widehat{y},\tau)$, where $\varepsilon(\widehat{y},\tau)$ is sufficiently small. Then there exists a control $u(x,t) \in C^\infty(\overline{Q};TM)$ satisfying (4.7) such that the solution $(y(x,t), \nabla p(x,t)) \in (C^\infty(\overline{Q};TM))^2$ of problem (4.8), (4.9) exists. Moreover*

$$\|y(x,\cdot) - \widehat{y}(x,\cdot)\|_{V^1(TM)} \leqslant C \exp\{-k/(\tau - t)\} \quad \text{as} \quad t \to \tau, \tag{4.10}$$

where $C > 0$ and $k > 0$ are some constants that do not depend on y_0 (satisfying (4.6)).

As was mentioned in the introduction, the proof of this proposition is an adaptation to the case of manifolds of the proof given in [FI1, FI2].

We seek the solution of (4.8), (4.9) in the form

$$y(x,t) = \widehat{y}(x,t) + w(x,t), \quad \nabla p(x,t) = \nabla\widehat{p}(x,t) + \nabla q(x,t). \tag{4.11}$$

Then the triplet $(w, \nabla q, u)$ satisfies the problem

$$N(w,\nabla q,u) \equiv \partial w/\partial t - \Delta w - n \times \widehat{y}\,\mathrm{div}(n \times w) - n \times w\,\mathrm{div}(n \times (\widehat{y} + w)) + \nabla q - u = 0, \tag{4.12}$$

$$\mathrm{div}\, w = 0, \quad w(x,t)\big|_{t=0} = w_0(x) \equiv y_0(x) - \widehat{y}(x,0), \quad w(x,t)\big|_{t=\tau} \equiv 0, \tag{4.13}$$

where u satisfies (4.7). This problem can be reduced to the linear controllability problem with the help of the following version of the implicit function theorem (see [ATF]).

Theorem (on a right inverse operator). *Suppose that X and Z are Banach spaces $\mathcal{A} : X \to Z$ is a continuously differentiable map such that $\mathcal{A}(\chi_0) = z_0$, $\chi_0 \in X$, $z_0 \in Z$, and the derivative $\mathcal{A}'(\chi_0) : X \to Z$ is surjective. Then there exists an $\varepsilon > 0$ such that for any $z \in Z$ satisfying $\|z - z_0\|_Z < \varepsilon$ there exists a solution χ of the equation $\mathcal{A}(\chi) = z$.*

In our case X will be a space of triplets $\chi = (w, \nabla q, u)$,

$$\mathcal{A}(\chi) = (N(w,\nabla q,u), w|_{t=0}), \tag{4.14}$$

and the collections of components in (4.14) will form the space Z. Equation (4.13_3) will be satisfied by introducing a special weight into the norm of X. Taking $\chi_0 = (0,0,0)$, $z_0 = (0,0)$, we see that $\mathcal{A}(\chi_0) = z_0$. All other assumptions of the right inverse operator theorem, except the surjectivity of $\mathcal{A}'(\chi_0)$, can be verified easily for the spaces X and Z defined below. The surjectivity property is reduced to the solvability of the following linear exact controllability problem:

$$N'(0)(v,\nabla p,u) = \partial v/\partial t - \Delta v - n \times \widehat{y}\,\mathrm{div}(n \times v) - n \times v\,\mathrm{div}(n \times \widehat{y}) + \nabla p - u = f, \tag{4.15}$$

$$\mathrm{div}\, v = 0, \quad v\big|_{t=0} = v_0, \quad v\big|_{t=\tau} = 0. \tag{4.16}$$

Let us now define the function spaces X and Z corresponding to problem (4.15), (4.16). For a weight $\varkappa(x,t) > 0$, we set

$$L_2(Q;\mathbb{R},\varkappa) = \Big\{ z(x,t) : \|z\|^2_{L_2(Q;\mathbb{R},\varkappa)} = \int_Q |z(x,t)|^2 e^{2\varkappa(x,t)}\,dx\,dt \Big\}. \tag{4.17}$$

The space $L_2(Q;TM,\varkappa)$ of vector fields is defined similarly.

Let $M_1 \subset M_0 \subset M$, and let, by definition, the set M_1 be diffeomorphic to a closed disk in $\mathbb{R}^2$. Set

$$\eta(x,t) \equiv \eta^\lambda(x,t) = \Big(e^{(4\lambda/3)\|\beta\|_{C(M)}} - e^{\lambda\beta(x)} \Big)/(T - t), \tag{4.18}$$

where $\lambda > 0$ is a parameter and $\beta(x) \in C^\infty(M)$ is a function satisfying the following properties: $\beta(x) > 2$, $|\nabla\beta(x)| \neq 0$ for $x \in M \setminus M_1$, and $\min_{x\in M}\beta(x) > 3/4 \max_{x\in M}\beta(x)$. The existence of a function $\beta(x)$ satisfying all these properties can be proved just as in [Im, FI4].

The space of right-hand sides f in (4.15) is

$$F(Q,\eta) = \{f \in L_2(Q;TM) : \exists f_1 \in L_2(Q;TM,\eta), \\ \exists f_2 \in L_2(0,T;H^1(M;\mathbb{R}) \text{ such that } f = f_1 + \nabla f_2\}, \\ \|f\|^2_{F(Q,\eta)} = \inf_{f_1,\nabla f_2,\, f=f_1+\nabla f_2}\left(\|f_1\|^2_{L_2(Q;TM,\eta)} + \|\nabla f_2\|^2_{L_2(Q;TM)}\right). \tag{4.19}$$

Let us introduce the space

$$\Theta(Q,\eta) = \left\{v(x,t) \in L_2(Q;TM) : \|v\|^2_{\Theta(Q,\eta)} \equiv \left\|\frac{\partial v}{\partial t} - \Delta v\right\|^2_{L_2(Q;TM,\eta)}\right. \\ + \left\|(\tau-t)^{-3/2}v\right\|^2_{L_2(Q;TM,\eta)} + \sum_{i=1}^{2}\left\|(\tau-t)^{-1/2}\nabla_{\partial/\partial x_i}v\right\|^2_{L_2(Q;TM,\eta)} \\ \left. + \left\|(\tau-t)^{1/2}\frac{\partial v}{\partial t}\right\|^2_{L_2(Q;TM,\eta)} + \sum_{i,j=1}^{2}\left\|(\tau-t)^{1/2}\nabla_{\partial/\partial x^i}\nabla_{\partial/\partial x^j}v\right\|^2_{L_2(Q;TM,\eta)} < \infty\right\}. \tag{4.20}$$

The space $S(Q,\eta)$ of the components v in (4.15), (4.16) is defined by

$$S(Q,\eta) = \left\{v(x,t) \in L_2(Q;TQ) : \operatorname{div} v = 0,\ \|v\|^2_{S(Q,\eta)} = \left\|\frac{\partial v}{\partial t} - \Delta v\right\|^2_{F(Q,\eta)}\right. \\ + \|(\tau-t)^{-1}v\|^2_{L_2(Q;TM,\eta)} + \sum_{i=1}^{2}\|\nabla_{\partial/\partial x^i}v\|^2_{L_2(Q;TM,\eta)} + \left\|(\tau-t)\frac{\partial v}{\partial t}\right\|^2_{L_2(Q;TM,\eta)} \\ \left. + \sum_{i,j=1}^{2}\|(\tau-t)\nabla_{\partial/\partial x^i}\nabla_{\partial/\partial x^j}v\|^2_{L_2(Q;TM,\eta)} < \infty\right\}. \tag{4.21}$$

Now we can define the spaces X and Z for problems (4.12), (4.13) and (4.15), (4.16):

$$X \equiv X^\lambda(Q) = S(Q,\eta^\lambda) \times L_2(Q;TM) \times \widehat{L}_2(Q_0;TM_0,\eta^\lambda + \ln(\tau-t)), \tag{4.22}$$

where $Q_0 = M_0 \times (0,\tau)$, $\widehat{L}_2(Q_0;TM_0,\varkappa) = \{u \in L_2(Q;TM,\varkappa) : \operatorname{supp} u \subset \overline{Q}_0\}$, and

$$Z \equiv Z^\lambda(Q) = F(Q,\eta^\lambda) \times V^1(M). \tag{4.23}$$

Since the weight $\eta(x,t)$ exponentially increases as $t \to \tau$, it follows that any function $v \in S(Q,\eta)$ decays exponentially to 0 as $t \to \tau$, and therefore (4.16_3) holds.

Below we use the Carleman estimates for elliptic and inverse parabolic equations. Set

$$\varphi(x,t) \equiv \varphi^\lambda(x,t) = e^{\lambda\beta(x)}/(\tau-t), \tag{4.24}$$

where $\beta(x)$ is just the same function as in the definition (4.18) of $\eta(x,t)$, and consider the Cauchy problem for the Laplace equation:

$$\Delta z(x,t) = f(x,t), \quad x \in \Omega_1; \quad z\big|_{\partial\Omega_1} = \nabla z\big|_{\partial\Omega_1} \equiv 0, \tag{4.25}$$

where $z(x,t)$ is a scalar function on Ω, $\Delta z \equiv \operatorname{div}\nabla z$, $\Omega_1 = M \setminus M_1$, $\partial\Omega_1$ is the boundary of Ω_1, and $t \in (0,\tau)$ is a parameter.

Lemma 4.2. *Let z and f satisfy* (4.25), *let η^λ and φ^λ be the functions* (4.18), (4.24) *and let $\lambda > \widehat{\lambda}$, where $\widehat{\lambda}$ is sufficiently large. Then the following estimate holds:*

$$\int_{\Omega_1}\Big(\varphi^{2s-1}\sum_{i=1}^{2}|\nabla_{\partial/\partial x_i}\nabla z|^2+\lambda\varphi^{2s+1}|\nabla z|^2+\lambda^4\varphi^{2s+3}|z|^2\Big)e^{-2\eta^\lambda}\,dx \leqslant c\int_{\Omega_1}\varphi^{2s}|f(x,t)|^2e^{-2\eta^\lambda}\,dx, \qquad (4.26)$$

where $s > -3$ and the constant $c > 0$ is independent of f, z, λ, and t.

The proof of this estimate is an adaptation of the method in [H, §8.3] to the case of manifolds.

Now consider the inverse heat equation for a vector field $z(x,t)$:

$$\partial z(x,t)/\partial t+\Delta z=f(x,t),\quad (x,t)\in Q_1;\quad z\big|_{\Sigma_1}=\nabla z\big|_{\Sigma_1}=0, \qquad (4.27)$$

where $\Sigma_1=\partial\Omega_1\times(0,\tau)$, $Q_1=\Omega_1\times(0,\tau)$.

Lemma 4.3. *Suppose that z and f satisfy* (4.27), *η^λ and φ^λ are the functions* (4.18) *and* (4.24), *and $\lambda>\lambda_0$, where λ_0 is sufficiently large. Then*

$$\int_{Q_1}\varphi^{2s-1}\bigg[\Big(\lambda^{-1}\Big|\frac{\partial z}{\partial t}\Big|^2+\sum_{i,j=1}^{2}|\nabla_{\partial/\partial x^i}\nabla_{\partial/\partial x^j}z|^2\Big) +\lambda\varphi^2\sum_{i=1}^{2}\big|\nabla_{\partial/\partial x^i}z\big|^2+\lambda^4\varphi^4|z|^2\bigg]e^{-2\eta^\lambda}\,dx\,dt\leqslant c\int_{Q_1}\varphi^{2s}|f(x,t)|^2e^{-2\eta^\lambda}\,dx\,dt, \qquad (4.28)$$

where $s > -3$ and C is independent of z and f.

The proof of Lemma 4.3 is an adaptation. of the proof given in [FI4] to the case of manifolds.

Our aim is to establish that the operator

$$\mathcal{A}'(0)\equiv(N'(0)(v,\nabla p,u),v|_{t=0}):X^\lambda(Q)\to Z^\lambda(Q) \qquad (4.29)$$

is surjective, where $N'(0)$ is defined in (4.15) and $X^\lambda(Q)$ and $Z^\lambda(Q)$ are spaces (4.22) and (4.23). To this end, we shall prove that $\Im\mathcal{A}'(0)$ is closed and dense in the space $Z^\lambda(Q)$. We begin from the proof of density. First, we consider $\mathcal{A}'(0)$ acting in spaces which are different from (4.29):

$$\mathcal{A}'(0):\mathcal{U}^\lambda(Q)\to\Phi^\lambda(Q), \qquad (4.30)$$

where

$$\mathcal{U}^\lambda(Q)=\Theta(Q,\eta^\lambda)\times L_2(Q;TM,\eta^\lambda)\times\widehat{L}_2(Q_0;TM_0,\eta^\lambda), \qquad (4.31)$$

$$\Phi^\lambda(Q)=L_2(Q;TM,\eta^\lambda)\times V^1(M). \qquad (4.32)$$

One can prove that the operator (4.30) is continuous.

Lemma 4.4. *The image of the operator* (4.30) *is dense in* $\Phi^\lambda(Q)$ *for any* $\lambda > 0$.

Proof. (Compare with [FI1].) Suppose that $\Im\mathcal{A}'(0)$ is not dense in $\Phi^\lambda(Q)$. Then there exists an element $(m, z_0) \in \Phi^\lambda(Q)$ such that for every $(w, \nabla p, u) \in \mathcal{U}^\lambda(Q)$ we have

$$\int_Q g\left(\frac{\partial w}{\partial t} - \Delta w - n \times \widehat{y}\operatorname{div}(n \times w) - n \times w \operatorname{div}(n \times \widehat{y})\right.$$
$$\left. + \nabla p - u, m\right) e^{2\eta}\,dx\,dt + (w(\cdot, 0), z_0)_{V^1(M)} = 0 \quad (4.33)$$

$(\eta = \eta^\lambda)$. In (4.33) we set

$$z(x,t) = m(x,t)e^{2\eta} \quad (4.34)$$

and take $\nabla p(x,t) \equiv 0$, $u \equiv 0$, and $w \in \Theta(Q,\eta) \cap C_0^\infty(Q;T)$. Then integrating by parts yields

$$\partial z/\partial t + \Delta z = n^* \times z \operatorname{div}(n \times \widehat{y}) - n^* \times (\nabla g(n \times \widehat{y}, z)) + \nabla \widetilde{p} \quad \text{on} \quad Q, \quad (4.35)$$

where $(n^* \times v)(x)$ is the clockwise rotation of the vector $v(x)$ by an angle of $\pi/2$ and $\widetilde{p}(x,t) \in L_2(0,T;H^1(M;\mathbb{R}))$ is some function. If we substitute

$$u \in \widehat{L}_2(Q_0; TM_0, \eta), \quad \nabla p \equiv 0, \quad w \equiv 0$$

into (4.33), then we obtain

$$z(x,t) \equiv 0, \quad (x,t) \in M_0 \times (0,\tau). \quad (4.36)$$

Setting $\nabla p \in L_2(Q;TM,\eta)$, $w = 0$, and $u = 0$ in (4.33) yields

$$\operatorname{div} z \equiv 0 \quad \text{on} \quad Q. \quad (4.37)$$

Equations (4.35) and (4.36) imply

$$\nabla \widetilde{p}(x,t) \equiv 0, \quad (x,t) \in M_0 \times (0,\tau). \quad (4.38)$$

Note that for any $f \in C^1(M;\mathbb{R})$ we have

$$\operatorname{div}(n^* \times \nabla f) \equiv 0. \quad (4.39)$$

Since this equation is local, we verify it in local coordinates. We choose local coordinates (x^1, x^2) which are orthonormal at a point $\widehat{x}$ and such that, at $\widehat{x}$, $\nabla_{\partial/\partial x^k} g^{ij} = 0$. Then at $\widehat{x}$ (4.39) is rewritten in the form

$$\operatorname{div}\left(\frac{\partial f}{\partial x^2}\frac{\partial}{\partial x^1} - \frac{\partial f}{\partial x^1}\frac{\partial}{\partial x^2}\right) \doteq \frac{\partial^2 f}{\partial x^1 \partial x^2} - \frac{\partial^2 f}{\partial x^1 \partial x^2} \equiv 0.$$

This proves (4.39). Similarly we obtain, by straightforward calculations,

$$\operatorname{div}\frac{\partial z}{\partial t} = \frac{\partial}{\partial t}\operatorname{div} z, \quad \operatorname{div}\Delta z = \Delta \operatorname{div} z + \mathcal{A}_1(z), \quad (4.40)$$

where $\mathcal{A}_1(z)$ is some first-order differential operator, which in the local coordinates has the form $\mathcal{A}_1(z) = a_j^k \nabla_{\partial/\partial x^k}(z^j) + b_j z^j$. Here a_j^k and b_j are C^∞ tensor fields defined on Q.

Application of the operator div to both parts in (4.35), and relations (4.37), (4.39), and (4.40), yields

$$\Delta\widetilde{p} = -\operatorname{div}(n^* \times z \operatorname{div}(n \times \widehat{y})) + \mathcal{A}_1(z). \tag{4.41}$$

Since $\widetilde{p}$ is defined in (4.37) and (4.41) up to an arbitrary constant depending on t, it follows from (4.38) that we can choose this constant so that

$$\widetilde{p}(x,t) \equiv 0, \quad (x,t) \in M_0 \times (0,\tau). \tag{4.42}$$

Applying the Carleman estimate (4.26) with $\lambda = \mu > \widehat{\lambda}$ and $s = -1/2$ to (4.41) and (4.42) and then integrating with respect to t, we get

$$\int_Q |\nabla\widetilde{p}|^2 e^{-2\eta^\mu}\,dx\,dt \leqslant c\int_{Q_1} \mu^{-1}\varphi^{-1}e^{-2\eta^\mu}\Big(|z|^2 + \sum_{j=1}^{2}|\nabla_{\partial/\partial x^j} z|^2\Big)\,dx\,dt. \tag{4.43}$$

Application of the Carleman estimate (4.28) with $\lambda = \mu > \lambda_0$, $s = 0$ to (4.35), (4.36) and substitution of (4.43) into the right-hand side of the obtained inequality yields

$$\int_{Q_1} e^{-2\eta^\mu}\Big(\mu\varphi\sum_{j=1}^{2}\big|\nabla_{\partial/\partial x^j} z\big|^2 + \mu^4\varphi^3|z|^2\Big)\,dx\,dt$$
$$\leqslant c_1\int_{Q_1} e^{-2\eta^\mu}(1 + c\mu^{-1}\varphi^{-1})\Big(|z|^2 + \sum_{j=1}^{2}\big|\nabla_{\partial/\partial x^j} z\big|^2\Big)\,dx\,dt. \tag{4.44}$$

By virtue of the definition (4.24) of φ^μ, for sufficiently large μ the inequalities

$$\mu\varphi > c_1(1 + c\mu^{-1}\varphi^{-1}), \quad \mu^4\varphi^3 > c_1(1 + c\mu^{-1}\varphi^{-1}) \tag{4.45}$$

hold uniformly with respect to $(x,t) \in Q_1$, where c_1 is the same constant as in (4.44). Therefore (4.44), (4.45), and (4.36) imply $z(x,t) \equiv 0$, $(x,t) \in Q$. Taking $z(x,t) \equiv 0$ and $w \in \Theta(Q,r)$ in (4.33), we obtain $z_0(x) \equiv 0$. Hence, $(m, z_0) = 0$.

Proposition 4.5. *The image of the operator* (4.29) *is dense in* $Z^\lambda(Q)$ *for any* $\lambda > 0$.

Proof. Let $(f, v_0) \in Z(Q)$. Then by (4.23) and (4.19) we have $f = f_1 + \nabla f_2$, where $f_1 \in L_2(Q;TM,\eta)$ and $\nabla f_2 \in L_2(Q;TM)$. Hence, by (4.32) $f_1, v_0 \in \Phi^\lambda(Q)$. By virtue of Lemma 4.4, for any $\varepsilon > 0$ there exists an $(f_1^\varepsilon, v_0^\varepsilon) \in \Phi^\lambda(Q)$ such that $\mathcal{A}'(0)(w^\varepsilon, \nabla p^\varepsilon, u^\varepsilon) = (f_1^\varepsilon, v_0^\varepsilon)$ for some $(w^\varepsilon, \nabla p^\varepsilon, u^\varepsilon) \in \mathcal{U}^\lambda(Q)$ and the estimate

$$\|f_1 - f_1^\varepsilon\|^2_{L_2(Q;TM,\eta)} + \|v_0 - v_0^\varepsilon\|^2_{V^1(TM)} < \varepsilon \tag{4.46}$$

holds. By virtue of (4.15) we have $\mathcal{A}'(0)(w^\varepsilon, \nabla(p^\varepsilon + f_2), u^\varepsilon) = (f_1^\varepsilon + \nabla f_2, v_0^\varepsilon)$. Obviously, $(w^\varepsilon, \nabla(p^\varepsilon + f_2), u^\varepsilon) \in X(Q)$ and, by (4.46),

$$\|f - (f_1^\varepsilon + \nabla f_2)\|^2_{F(Q,\eta)} + \|v_0 - v_0^\varepsilon\|^2_{V^1(M)} < \varepsilon.$$

This proves that the image of the operator (4.29) is dense.

We now investigate a decomposition of the Weyl type:

$$y(x,t) = v(x,t) + \nabla q(x,t), \quad (x,t) \in Q_1, \tag{4.47}$$

where $\operatorname{div} v = 0$, $Q_i = \Omega_i \times (0,\tau)$, and $\Omega_i = M \setminus M_i$. Recall that $M_1 \subset M_0 \subset M$ and M_1 is diffeomorphic to a closed plane disk. We look for $v \in S(Q_0,\eta)$ if $y \in \Theta(Q_1,\eta)$ instead of imposing any boundary conditions on v or ∇q. Besides, we require the following assumption to be satisfied:

$$\text{if} \quad \operatorname{div} y(x,0) \equiv 0, \quad \text{then} \quad y(x,0) \equiv v(x,0). \tag{4.48}$$

Of course, these assumptions do not guarantee the uniqueness of the decomposition (4.47).

To find the decomposition (4.47), we consider the extremal problem

$$J(q) = \int_{Q_1} \frac{|q(x,t)|^2}{(\tau-t)^4} e^{2\eta^\lambda}\, dx\, dt \to \inf, \tag{4.49}$$

$$\Delta q(x,t) = \operatorname{div} y(x,t), \quad (x,t) \in Q_1, \tag{4.50}$$

where $y(x,t) \in \Theta(Q_1,\eta^\lambda)$ is a given function. If there exists a solution $m(x,t)$ of (4.49), (4.50), then we set $v = y - \nabla m$. Then, by (4.50), we have $\operatorname{div} v \equiv 0$, whence the decomposition (4.47) follows.

Lemma 4.6. *There exists a λ_1 such that for $y \in \Theta(Q_1,\eta^\lambda)$, $\lambda \geqslant \lambda_1$, problem (4.49), (4.50) has a unique solution $m(x,t) \in L_2(Q_1;\mathbb{R},\eta^\alpha - 2\ln(\tau-t))$. This solution satisfies the estimates*

$$\int_{Q_1} \frac{|m(x,t)|^2}{(\tau-t)^4} e^{2\eta^\lambda}\, dx\, dt \leqslant c_1 \int_{Q_1} \frac{|\operatorname{div} y|^2}{(T-t)} e^{2\eta^\lambda}\, dx\, dt, \tag{4.51}$$

$$\int_{Q_1} \left|\frac{\partial m(x,t)}{\partial t}\right|^2 e^{2\eta^\lambda}\, dx\, dt \leqslant c_2 \|y\|^2_{\Theta(Q_1;TM_1,\eta^\lambda)}, \tag{4.52}$$

where c_1 and c_2 are constants independent of y.

The idea of the proof is as follows. We replace Q_1 in (4.51), (4.52) by $Q_{1,\varepsilon} \equiv \Omega_1 \times (0,\tau-\varepsilon)$, prove the solvability of the obtained extremal problem, and derive its optimality system

$$\Delta p(x,t) + \frac{m(x,t)}{(\tau-t)^4} e^{2\eta} = 0 \quad \text{in} \quad \Omega_1; \quad p\big|_{\partial\Omega_1} = \nabla p\big|_{\partial\Omega_1} = 0, \tag{4.53}$$

where m is a solution of the extremal problem and p is the conjugate function. We establish that m, p do not depend on ε and are respectively the solution and the conjugate function of problem (4.49), (4.50) defined on the cylinder Q_1.

Substitute $q = m$ into (4.50), multiply the obtained equality by p and integrate over $Q_{1,\varepsilon}$. After that, integrating by parts, and applying (4.53), we obtain

$$0 = \int_{Q_{1,\varepsilon}} (m\Delta p - p \operatorname{div} y)\, dx\, dt = -\int_{Q_{1,\varepsilon}} \Big(\frac{m^2}{(\tau-t)^4} + p \operatorname{div} y\Big)\, dx\, dt.$$

Let us apply the Carleman estimate (4.26) with $s = -2$ to (4.53) and integrate with respect to t. Then we obtain

$$\int_{Q_{1,\varepsilon}} (\tau-t) p^2 e^{-2\eta}\, dx\, dt \leqslant c_3 \int_{Q_{1,\varepsilon}} \frac{m^2}{(\tau-t)^4} e^{2\eta}\, dx\, dt,$$

where c_3 is independent of ε. The last two relations yield

$$\int_{Q_{1,\varepsilon}} (\tau-t) p^2 e^{-2\eta}\, dx\, dt \leqslant c \int_{Q_{1,\varepsilon}} \frac{|\operatorname{div} y|^2}{(\tau-t)} e^{2\eta}\, dx\, dt + \frac{1}{2} \int_{Q_{1,\varepsilon}} \frac{m^2}{(\tau-t)^4} e^{2\eta}\, dx\, dt.$$

This inequality implies (4.51). Differentiating (4.50), (4.53) with respect to t and making transformations approximately as above, we obtain (4.52) (see [FI1] for details).

Let $\rho(x) \in C^\infty(\overline{\Omega}_1)$, $\rho|_{\partial\Omega_1} = 0$, and $\rho(x) > 0$ $\forall x \in \Omega_1$. We define the space

$$M(Q_1, \eta) = \Big\{ f : \|f\|^2_{M(Q_1,\eta)} \equiv \|(\tau - t)^{-1} f\|^2_{L_2(Q_1;TM_1,\eta)} + \sum_{i=1}^{2} \left\| \nabla_{\partial/\partial x^i} f \right\|^2_{L_2(Q_1;TM_1,\eta)}$$

$$+ \left\| (\tau - t)\frac{\partial f}{\partial t} \right\|^2_{L_2(Q_1;TM_1,\eta)} + \sum_{i,j=1}^{2} \left\| (\tau - t) \nabla_{\partial/\partial x^i} \nabla_{\partial/\partial x^j} f \right\|^2_{L_2(Q_1;TM_1,\eta)} < \infty \Big\}. \quad (4.54)$$

Lemma 4.7. *Let $m(x,t)$ be the solution of problem* (4.52), (4.53). *Then*

$$\sum_{i=1}^{2} \left\| \rho^3 \nabla_{\partial/\partial x^i} m \right\|^2_{M(Q_1,\eta)} \leqslant c \|y\|^2_{\Theta(Q_1,\eta)}. \quad (4.55)$$

This lemma can be proved in the same way as Lemma 4.2 in [FI1].

Lemmas 4.6 and 4.7 imply the following assertion.

Proposition 4.8. *Let λ satisfy the assumptions of Lemma 4.6. Then a vector field $y \in \Theta(Q_1, \eta^\lambda)$ admits the decomposition* (4.47), *where* $\operatorname{div} v(x,t) \equiv 0$, $\rho^3 \nabla q \in M(Q_1, \eta^\lambda)$, *and if* $\operatorname{div} y(0,x) \equiv 0$, *then* $y(0,x) \equiv v(0,x)$.

Proof (cf. [FI1]). For the solution $m(x,t)$ of (4.49), (4.50), set $\zeta(x,t) = \varphi(t) m(x,t)$, $t \in (0,\tau)$, $x \in \partial\Omega_1$ and consider the Dirichlet problem

$$\Delta q(x,t) = \operatorname{div} y(x,t), \quad (x,t) \in Q_1; \quad q\big|_{\partial\Omega_1} = \zeta,$$

where $t \in (0,\tau)$ is a parameter. The solution $q(x,t)$ of this problem and the vector fields $v(x,t) = y(x,t) - \nabla q(x,t)$ give the desired decomposition (4.47).

To prove that the image of the operator (4.29) is closed, we first consider the controllability problem for the parabolic system

$$\frac{\partial y(x,t)}{\partial t} - \Delta y - n \times \hat{v} \operatorname{div}(n \times y) = f_1(x,t), \quad (x,t) \in Q_1, \quad (4.56)$$

$$y(x,t)\big|_{t=0} = y_0(x), \quad y(x,t)\big|_{t=\tau} \equiv 0, \quad x \in \Omega_1. \quad (4.57)$$

Proposition 4.9. *Let $\hat{y}(x,t) \in C^\infty(Q_1; TM_1)$ be given. Then there exists a $\lambda_2 > 0$ such that for $\lambda \geqslant \lambda_2$ and for an arbitrary $y_0 \in (H^1(\Omega_1))^2$, $f_1 \in L_2(Q_1; TM_1, \eta^\lambda)$ there exists a solution $y \in \Theta(Q_1, \eta^\lambda)$ of problem* (4.56), (4.57).

The proof of this proposition can be obtained by methods of the papers [FI3, F, Im].

Proposition 4.10. *Let $\hat{y} \in C^\infty(Q, TM)$. Then the image of the operator* (4.29) *is closed in the space $Z^\lambda(Q)$ if λ is sufficiently large.*

Proof. We decompose the operator (4.29) into the sum $B + K$ ($\mathcal{A}'(0) = B + K$), where B is the operator generated by the problem

$$\partial v/\partial t - \Delta v - n \times \hat{y} \operatorname{div}(n \times v) + \nabla p + \mathcal{E}_1(q) - u = f, \quad \operatorname{div} v \equiv 0, \quad v\big|_{t=0} = v_0, \quad (4.58)$$

$$v(x,T) \equiv 0, \quad (4.59)$$

and $\mathcal{E}_1(q)$ is the first-order differential operator defined in (4.63) (see below).

The operator K is defined by the formula

$$K(v, \nabla p, u, q) = (-n \times v \operatorname{div}(n \times \widehat{y}) - \mathcal{E}_1(q), 0). \tag{4.60}$$

Let $\Omega_i \subset \Omega$ and $Q_i = \Omega_i \times (0, \tau)$. We introduce the space

$$N^\lambda(Q_i) = \left\{ q \in L_2(Q_i; \mathbb{R}, \eta^\lambda) : \|q\|^2_{N^\lambda(Q_i)} = \|(\tau - t)^{-2} q\|^2_{L_2(Q_i;\mathbb{R},\eta^\lambda)} + \|\nabla q\|^2_{M(Q_i,\eta^\lambda)} < \infty \right\},$$

where the space M was defined in (4.54).

The boundedness of the operator

$$B : X^\lambda(Q) \times N^\lambda(Q) \to Z^\lambda(Q) \tag{4.61}$$

is obvious. To prove that $\Im B = Z^\lambda(Q)$, we first use Eq. (4.56) instead of the more complicated equations (4.58).

We choose λ satisfying the conditions of Propositions 4.8 and 4.9. By virtue of Proposition 4.9, there exists a solution $y \in \Theta(Q_1, \eta^\lambda)$ of problem (4.56), (4.57). By Proposition 4.8, the vector field y admits the decomposition (4.47) on Q_1, where

$$\operatorname{div} v \equiv 0, \ \ \rho^3 \nabla q \in M(Q_1; \mathbb{R}, \eta^\lambda), \ \ (\tau - t)^{-2} q \in L_2(Q_1, \eta^\lambda), \ \ y(x, 0) = v_0(x) = y_0(x).$$

We substitute this into (4.56). Then we claim that the following equation for v is true:

$$\frac{\partial v}{\partial t} - \Delta v - (n \times \widehat{y}) \operatorname{div}(n \times v) + \nabla m + \mathcal{E}_1(q) = f_1, \quad \operatorname{div} v = 0, \quad v(x, 0) = y_0(x), \tag{4.62}$$

where $\nabla m(x, t) = \nabla(\partial_t q - \Delta q)$. To prove this assertion, we must verify that

$$\operatorname{div}(n \times \nabla q) \equiv 0, \quad \Delta \nabla q = \nabla \Delta q + \mathcal{E}_1(q), \tag{4.63}$$

where $\mathcal{E}_1(q)$ is a first-order differential operator, which in local coordinates has the following form: $\mathcal{E}(q) = (d^i_j g^{jk} \partial q / \partial x^k + \mathcal{D}^i q) \partial / \partial x^i$, where d^i_j and $\mathcal{D}^i$ are C^∞ tensor fields. Equation (4.63_1) can be verified just as relation (4.39). To obtain (4.63_2), we pass in (4.40_2) to the adjoint operators (with respect to inner product (2.6)). In fact, straightforward computations show that $\mathcal{E}_1(q) = \mathcal{K} \nabla q$, where $\mathcal{K}$ is the Gaussian curvature of (M, g).

To eliminate the factor ρ^3 in the relation $\rho^3 \nabla q \in M(Q_1, \eta^\lambda)$, we restrict (4.58) and (4.59) to Q_0. Then $q \in N^\lambda(Q_0)$, and hence, $\nabla m \in L_2(Q_0; TM_0, \eta^\lambda + \ln(\tau - t))$. One can extend $v \in S(Q_0; TM_0, \eta^\lambda)$ to a vector field v_e (see [La, T2, FI1]) such that $v_e(x, 0) \equiv v_0(x)$, $x \in M$, and extend ∇m to a vector field $\nabla m_e \in L_2(Q; TM, \eta^\lambda + \ln(\tau - t))$ and q to $q_e \in N^\lambda(Q)$.

Now we can prove that the operator (4.61) is surjective. Indeed, let $(f, v_0) \in Z^\lambda(Q)$, where $f = f_1 + \nabla f_2$. We set

$$\nabla p = \nabla(m_e + f_2), \quad u = \frac{\partial v_e}{\partial t} - \Delta v_e - n \times \widehat{y} \operatorname{div}(n \times v_e) + \nabla m_e - f_1 + \mathcal{E}_1(q_e). \tag{4.64}$$

Equations (4.64) imply that the collection $(v_e, \nabla p, u, q_e)$ satisfies (4.58). Since $v_e \in S(Q; TM, \eta)$, Eq. (4.59) is true. Equations (4.62) and (4.64) yield $\operatorname{supp} u \subset M_0 \times (0, T)$.

By straightforward verification we establish the compactness of operator $K : X^\lambda(Q) \times N^\lambda(Q) \to Z^\lambda(Q)$ defined in (4.60). Since $\Im B = Z^\lambda(Q)$, $K : X^\lambda(Q) \times N^\lambda(Q) \to Z^\lambda(Q)$ is compact and $X^\lambda(Q), N^\lambda(Q), Z^\lambda(Q)$ are separable Hilbert spaces, it follows that $\Im(B + K)$

is closed in $Z^{\lambda}(Q)$ (see [FI1]). By definition of B and K, $B+K$ coincides with the operator (4.32) (and is defined on $X^{\lambda}(Q)$).

Proof of Proposition 4.1. Propositions 4.10 and 4.5 imply that the operator (4.31) is surjective. Hence, by the right inverse operator theorem, if (4.6) is satisfied with sufficiently small ε, then there exists a solution $(w, \nabla q, u) \in X^{\lambda}(Q)$ of problem (4.12), (4.13). We define y by (4.11). Since $w \in S(Q, \eta^{\lambda})$, it follows that inequality (4.10) holds with some $C > 0$ and $k > 0$, and therefore, w satisfies (4.13_3). Relations (4.12), (4.13) for $(w, \nabla q, u)$ and definition (4.11) of $(y, \nabla p)$ imply that $(y, \nabla p, u)$ satisfy (4.8), (4.9). Moreover, the proof given above allows us to construct a control u satisfying all the previous conditions and equal to zero in a neighborhood of $\Sigma_0 = \partial\Omega_0 \times (0, \tau)$. Then by the smoothness theorem for solutions of the Navier-Stokes equations, the assumption $(f(x,t), y_0(x)) \in C^{\infty}(\overline{Q}; TM) \times C^{\infty}(M, TM)$ implies the relation $(y, \nabla p) \in (C^{\infty}(\overline{Q}_0, TM_0))^2$. We can extend $(y, \nabla p)$ from Q_0 to the vector fields $(y_e, \nabla p_e) \in (C^{\infty}(\overline{Q}, TM))^2$ such that $\operatorname{div} y_e = 0$ and (4.11) with $y = y_e$ is true. If we set

$$u_e = \frac{\partial y_e}{\partial t} - \Delta y_e - n \times y_e \operatorname{div}(n \times y_e) + \nabla p_e - f,$$

then the triplet $(y_e, \nabla p_e, u_e)$ will satisfy all assertions of Proposition 4.1.

§5. AN APPLICATION OF THEOREM 1.1

The controllability problem for the Navier-Stokes equations describing incompressible fluid flow has been raised by J.-L. Lions in [Li1, Li2]. To motivate the study of this problem, J.-L. Lions in [Li1] discussed a certain reversibility problem for processes simulated by the so-called Planet Earth System. (Roughly speaking, the climate process simulation is discussed.) Being very complicated, the Planet Earth System consists of basic blocks of nonlinear PDEs: equations of fluid dynamics (atmosphere and ocean), equations of solid dynamics (for the ice caps), and many others; all these blocks are connected by coupling terms.

On this system, there are some permanent actions (mainly of the sun). Besides, the possibility of major catastrophes is admitted, which can essentially change the process. The crucial question is: are some of these changes irreversible, or are they all reversible?

It is possible to answer this question when instead of the Planet Earth System the Navier-Stokes equations

$$\frac{\partial y(x,t)}{\partial t} - \Delta y + \nabla_y y + \nabla p(x,t) = f(x,t), \quad \operatorname{div} y(x,t) = 0 \tag{5.1}$$

defined on the sphere $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ are considered. We assume that the right-hand side $f(x,t) \in C^{\infty}(Q; TS^2)$, where $Q = S^2 \times \mathbb{R}$, simulating the permanent (solar) action is periodic with respect to t: $f(x, t+2\pi) = f(x,t)$ (the case of quasiperiodic f can also be considered). Since climate is described by long-term processes, it is natural to define climate as an element of the trajectory attractor A of system (5.1). The definition of trajectory attractor $\mathcal{A}$ of the nonautonomous Navier-Stokes system, a demonstration of the existence of $\mathcal{A}$, and a description of its structure can be found in [CV1, CV2]. Here only one fact is essential to us: each solution $y \in C^{\infty}(Q; TS^2)$ of (5.1) satisfying the inequality

$$\sup_{t \in \mathbb{R}} \|y(\cdot, t)\|_{H^1(S^2, TS^2)} < \infty \tag{5.2}$$

belongs to the attractor $\mathcal{A}$, and the set of such solutions is not empty. This assertion can also be extracted from [CV1, CV2]. The class of such functions will be denoted by $\mathcal{A}_1$. Thus, it is natural to define climate in the framework of model (5.1) as a solution $y(x,t) \in \mathcal{A}_1$ of (5.1). Then global change of climate means jumping from $y_0(x,t) \in \mathcal{A}_1$ to $y_1(x,t) \in \mathcal{A}_1$ and the answer to the question about reversibility is given by the following assertion.

Proposition 5.1. *Let $y_\alpha \in \mathcal{A}_1$, $\alpha = 0, 1$, a time interval (t_1, t_2), and a subset ω of the sphere S^2 be given. Then there exists a control function $u(x,t) \in C^\infty(Q; TS^2)$ with support belonging to $\omega \times (t_1, t_2)$ that realizes the jump from y_0 to y_1. This means that there exists a solution $z(x,t)$ of the problem*

$$\frac{\partial z(x,t)}{\partial t} - \Delta z + \nabla_z z + \nabla q = f(x,t) + u(x,t), \quad \operatorname{div} z(x,t) \equiv 0$$

such that $z \in C^\infty(Q; TS^2)$, $z(x,t) \equiv y_0(x,t)$ for $t \in (-\infty, t_1)$, and $z(x,t) = y_1(x,t)$ for $t \in (t_2, \infty)$.

This proposition is an obvious corollary of Theorem 1.1.

Remark 5.1. We realize (as did J.-L. Lions in [Li1]) that the model (5.1) is a strong oversimplification of the above-mentioned Planet Earth System. Nevertheless, note that in mathematical climate theory [DF], the barotropic model, which is close to problem (5.1), is considered. The barotropic model gives an average of the atmosphere circulation (see [DF]).

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