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Explicit Feedbacks Stabilizing the Attitude of a Rigid Spacecraft with Two Control Torques*

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Key Words—Attitude control; stabilization.

Abstract—We construct explicit time-varying feedbacks that locally asymptotically stabilize the attitude of a rigid spacecraft with two controls. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

Rigid-body models for rigid spacecraft with only one or two controls—control torques provided by thruster jets—have received much attention.

The controllability of such systems is well understood. Bonnard (1982) and Crouch (1984) proved that, in general, such systems are globally controllable. In Keraï (1995), following Hashémi (1992), it was proved that, with two controls, the system is small time locally controllable in general. Moreover in Keraï (1995) it has been shown that, with one control, the system is never small-time locally controllable.

Stabilizability has been studied only recently. The first result that was obtained on this subject is the stabilizability of only part of the system, namely the angular velocity. Aeyels and Szafranski (1988) proved that, in general and even with one control, the angular velocity can be locally asymptotically stabilized by means of smooth state feedback—see also Brockett (1983), Crouch and Irving (1983), Byrnes and Isidori (1988), Aeyels (1985), Sontag and Sussmann (1988) and Outbib and Sallet (1992). For stabilization of other parts of the system see Hermes (1980) and Byrnes and Isidori (1991). Unfortunately, as shown by Byrnes and Isidori (1991), this is no longer true for the full system, even with two controls; indeed the system, with one or two controls, never satisfies the Brockett condition (Brockett, 1983) and so cannot be locally asymptotically stabilized by means of a smooth state feedback law or even by a discontinuous feedback law if one considers, as proposed by Hermes (1967), Filippov solutions for the closed-loop system (Coron and Rosier, 1994).

In pioneering works, Sontag and Sussmann (1980) and Samson (1991) pointed out that there are systems that cannot be locally asymptotically stabilized by means of state feedback but can be locally asymptotically stabilized by means of time-varying feedback. So one may wonder if our system can be locally asymptotically stabilized by means of time-varying feedback.

In Samson and Morin (see Morin, 1992) proposed, for a special case (see below) an explicit time-varying feedback that displays asymptotic stabilization on simulation; but no proof of the asymptotic stability has been established. Following Hashémi (1992), it was proved by Keraï (1995) that, after a suitable change of variables, the rigid spacecraft with two controls satisfies, in general, a sufficient condition for small-time local controllability due to Sussmann (1987, Theorem 7.3), and so, by Coron (1993, 1995), can be locally asymptotically stabilized by means of time-varying feedbacks. However, it remained to construct such feedbacks explicitly. This construction was performed independently by Walsh *et al.* (1994) (for more details, see also Walsh *et al.*, 1995) and by Morin *et al.* (1995)—both in the special case (considered also in Morin, 1992; Hashémi, 1992) where the torque actions are exerted about the principal axis of the inertia matrix of the spacecraft. See also Krishnan *et al.* (1992) for explicit time-varying feedbacks such that, for the same special case, 0 is a local attractor for time 0 for the closed-loop system—but note that, with these feedbacks, 0 is not locally asymptotically stable for the closed-loop system. The goal of this paper is to perform the construction of a

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locally asymptotically stabilizing time-varying feedback in the general case.

2. SKETCH OF THE CONSTRUCTION

In this section we sketch our construction of a locally asymptotically stabilizing time-varying feedback law in the generic case where there are two controls. After a suitable change of variables (see Keraï (1995) and the Appendix of this paper), the system can be written as

$$\Sigma: \quad \dot{x} = f(x, u) = X(x) + R(x) + uY(x), \quad (1)$$

where $x = (x_1, \ldots, x_6) \in \mathbb{R}^6$ is the state, $u = (u_1, u_2) \in \mathbb{R}^2$ is the control,

$$\begin{aligned} uY &= u_1 Y_1 + u_2 Y_2 = (0, 0, 0, 0, u_1, u_2), \\ X(x) &= (x_5 x_6, x_1 + c x_3 x_6, x_5, x_6, 0, 0). \end{aligned} \quad (2)$$

with c a constant in $(0, +\infty)$, and R is a *perturbation* term. More precisely, note that X is homogeneous of degree 0 with respect to the dilation

$$\begin{aligned} \delta_\varepsilon(x) &= (\varepsilon^{r_1} x_1, \ldots, \varepsilon^{r_6} x_6) \\ &:= (\varepsilon^2 x_1, \varepsilon^2 x_2, \varepsilon x_3, \varepsilon x_4, \varepsilon x_5, \varepsilon x_6), \end{aligned} \quad (3)$$

which means (see e.g. Kawski, 1988; Hermes, 1990) that

$$X_i(\delta_\varepsilon(x)) = \varepsilon^{r_i} X_i(x) \quad \forall \varepsilon \in (0, +\infty), \ \forall x \in \mathbb{R}^6, \ \forall i \in [1, 6]. \quad (4)$$

Then, for a suitable constant C_0 the vector field R satisfies, for all ε in $(0, 1)$ and all x in $\mathbb{R}^6$ with $|x|$ small enough,

$$|R_i(\delta_\varepsilon x)| \leq C_0 \varepsilon^{1+r_i} |R_i(x)| \quad \forall i \in [1, 6]; \quad (5)$$

roughly speaking, R_i is small compared with X for the dilation δ_ε. Let us point out that the case studied in Morin (1992), Hashémi (1992) and Morin *et al.* (1994) is that where $c = 1$; note that in this case Rouchon (1992) has proved that the system Σ is flat, a notion introduced by Fliess *et al.* (1992).

Keeping in mind this homogeneity with respect to δ_ε and following Kawski (1988) and Hermes (1990), it is natural to consider time-varying feedback laws u that are homogeneous of degree 1 with respect to δ_ε, i.e. that satisfy

$$u(\delta_\varepsilon x, t) = \varepsilon u(x, t) \quad \forall x \in \mathbb{R}^6, \ \forall t \in \mathbb{R}. \quad (6)$$

Indeed, assume that u is homogeneous of degree 1 with respect to δ_ε and that it is a periodic time-varying feedback law that globally asymptotically stabilizes (see Definition 3.1) the control system

$$\Sigma^*: \quad \dot{x} = X(x) + uY. \quad (7)$$

Then u locally asymptotically stabilizes the control system Σ; this follows from (5), the fact that the vector field $X(x) + u(x, t)Y$ is homogeneous of degree 0 with respect to δ_ε, and from (the time-varying version of) a result due to Rosier (1992, 1993)—see also Massera (1956), Kawski (1988) and Hermes (1990) when u is smooth enough.

Now the strategy is to construct a periodic time-varying feedback with good homogeneity that globally asymptotically stabilizes the control system

$$\bar{\Sigma}: \quad \dot{x}_1 = x_5 x_6, \quad \dot{x}_2 = x_1 + c x_3 x_6, \quad \dot{x}_3 = x_5, \quad (8)$$

where the state is $(x_1, x_2, x_3) \in \mathbb{R}^3$ and the control $(x_5, x_6) \in \mathbb{R}^2$. By 'good homogeneity', we mean that, for all t in $\mathbb{R}$, all (x_1, x_2, x_3) in $\mathbb{R}^3$, all ε in $(0, +\infty)$ and all i in $\{5, 6\}$,

$$x_i(\varepsilon^2 x_1, \varepsilon^2 x_2, \varepsilon x_3, t) = \varepsilon x_i(x_1, x_2, x_3, t). \quad (9)$$

Using (as in Coron and Praly, 1991) the method of desingularization (Praly *et al.*, 1991) and Sontag's (1988) proof of Artstein's (1983) theorem, we obtain from such a feedback a feedback $\tilde{u}: \mathbb{R}^5 \times \mathbb{R} \to \mathbb{R}^2$, $(x_1, x_2, x_3, x_5, x_6, t) \to \tilde{u}(x_1, x_2, x_3, x_5, x_6, t)$ that is periodic in time, has good homogeneity, and globally asymptotically stabilizes the control system obtained from $\bar{\Sigma}$ by adding an integrator on x_5 and on x_6, i.e. the control system

$$\begin{aligned} \tilde{\Sigma}: \quad & \dot{x}_1 = x_5 x_6, \quad \dot{x}_2 = x_1 + c x_3 x_6, \quad \dot{x}_3 = x_5, \\ & \dot{x}_5 = u_1, \quad \dot{x}_6 = u_2, \end{aligned} \quad (10)$$

where the state is $(x_1, x_2, x_3, x_5, x_6) \in \mathbb{R}^5$ and the control $u = (u_1, u_2) \in \mathbb{R}^2$. Now by 'good homogeneity' we mean that for all t in $\mathbb{R}$, all $(x_1, x_2, x_3, x_5, x_6) \in \mathbb{R}^5$, all ε in $(0, +\infty)$ and all i in $\{1, 2\}$,

$$\begin{aligned} u_i(\varepsilon^2 x_1, \varepsilon^2 x_2, \varepsilon x_3, \varepsilon x_5, \varepsilon x_6, t) \\ = \varepsilon u_i(x_1, x_2, x_3, x_5, x_6, t). \end{aligned} \quad (11)$$

It remains to take care of x_4; in order to do this, we note that if $x: \mathbb{R} \to \mathbb{R}^6$ and $u_2: \mathbb{R} \to \mathbb{R}$ are such that $\dot{x}(t) = X(x(t)) + u_2(t) Y_2$, and if for some time $\bar{t} \in \mathbb{R}$, $x_1(\bar{t}) = x_2(\bar{t}) = x_3(\bar{t}) = x_5(\bar{t}) = 0$, then, for all $t \geq \bar{t}$ (and in fact for all t), $x_1(t) = x_2(t) = x_3(t) = x_5(t) = 0$. So a natural idea is to define $u: \mathbb{R}^6 \times \mathbb{R} \to \mathbb{R}^2$ in the following way: let τ and τ' be in $(0, +\infty)$ and let, for $x = (x_1, \ldots, x_6) \in \mathbb{R}^6$,

$$u(x, t) = \tilde{u}(x_1, x_2, x_3, x_5, x_6, t) \quad \forall t \in [0, \tau), \quad (12)$$

$$u(x, t) = (0, \zeta_4 x_4 + \zeta_6 x_6) \quad \forall t \in [\tau, \tau + \tau'), \quad (13)$$

where ζ_4 and ζ_6 are constants such that $0 \in \mathbb{R}^2$ is globally asymptotically stable for $\dot{x}_4 = x_6$ and $\dot{x}_6 = \zeta_4 x_4 + \zeta_6 x_6$; finally, we extend u to all

$\mathbb{R}^6 \times \mathbb{R}$ by $(\tau + \tau')$-periodicity with respect to time. We see that, for a suitable choice of τ, τ', ζ_4 and ζ_6, the periodic time-varying feedback u locally asymptotically stabilizes the control system Σ^*, and therefore also the control system Σ, since u is homogeneous of degree 1 with respect to δ_ε.

Our paper is organized as follows. We design periodic time-varying feedbacks, with the required homogeneities, that locally asymptotically stabilize $\bar{\Sigma}$ in Section 3, $\tilde{\Sigma}$ in Section 4 and Σ^* (and therefore Σ) in Section 5.

3. STABILIZATION OF $\bar{\Sigma}$

Let us first introduce a definition.

Definition 3.1. A function $h: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ is *almost continuous* if there exist a sequence of time $(t_i; i \in \mathbb{Z})$ and a sequence of functions $(h_i \in C^0(\mathbb{R}^n \times [t_i, t_{i+1}]; \mathbb{R}^m); i \in \mathbb{Z})$ such that

$$t_i < t_{i+1} \quad \forall i \in \mathbb{Z}, \tag{14}$$

$$\lim_{i \to -\infty} t_i = -\infty, \quad \lim_{i \to +\infty} t_i = +\infty, \tag{15}$$

$$h(x, t) = h_i(x, t) \quad \forall i \in \mathbb{Z}, \quad \forall t \in (t_i, t_{i+1}), \ \forall x \in \mathbb{R}^n. \tag{16}$$

Let us point out that if h is almost continuous then there exists a unique function $\tilde{h}$ that is equal to h almost every where and satisfies, for all (x, t) in $\mathbb{R}^n \times \mathbb{R}$, $\tilde{h}(x', t') \to \tilde{h}(x, t)$ if $x' \to x$ and $t' \to t$ with $t' \geq t$; we use this representative of h systematically and also call it h for simplicity.

The feedbacks that we consider are always almost-continuous. Let $F: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be an almost-continuous function; it is well known (see e.g. Filippov, 1988, Theorem 1 in Chapter 1) that the Cauchy problem, where $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}$ are given,

$$\dot{x} = F(x, t), \quad x(t_0) = x_0, \tag{17}$$

always has a solution, which means that there exists an open interval I containing t_0 and an absolutely continuous function $x: I \to \mathbb{R}^n$ such that $x(t_0) = x_0$ and $\dot{x}(t) = F(x, t)$ for almost every $t \in I$. Note that (17) may have more than one solution.

Let us now define locally asymptotically stable—we should in fact say uniformly locally asymptotically stable.

Definition 3.2. Let $F: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be an almost-continuous function. Then 0 is *locally asymptotically stable for* $\dot{x} = F(x, t)$ if

(i) for all $\varepsilon > 0$, there exists $\rho > 0$ such that, for any t_0 in $\mathbb{R}$ and any (maximal) solution of $\dot{x} = F(x, t)$, $|x(t_0)| \leq \rho$ implies that, for any $t \geq t_0$, $|x(t)| \leq \varepsilon$;

(ii) there exists $r > 0$ such that, for any $\varepsilon > 0$, there exists $\tau > 0$ such that, for any t_0 in $\mathbb{R}$ and any (maximal) solution of $\dot{x} = F(x, t)$, $|x(t_0)| \leq r$ implies that, for any $t \geq t_0 + \tau$, $|x(t)| \leq \varepsilon$.

If, in (ii), one can replace 'there exists $r > 0$ such that, for any $\varepsilon > 0$' by 'for any $r > 0$ and any $\varepsilon > 0$', we say that 0 is *globally asymptotically stable for* $\dot{x} = F(x, t)$.

Of course, for a control system $\dot{x} = g(x, u)$, where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control, and $g \in C^0(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$, we say that the almost-continuous time-varying feedback locally (respectively globally) asymptotically stabilizes this control system if 0 is locally (respectively globally) asymptotically stable for the closed-loop system $\dot{x} = g(x, u(x, t))$.

After these definitions, we now turn to the stabilization of $\bar{\Sigma}$. Let

$$K \in (\tfrac{1}{4}, +\infty), \tag{18}$$

let $T \in (0, +\infty)$, and let $a: \mathbb{R} \to (0, +\infty)$ be any function of class C^1, or, more generally, any Lipschitz function whose derivative in the distributional sense is equal—almost everywhere—to an almost-continuous function. We assume that a is t-peridoic with respect to time and satisfies the following conditions:

$$\exists \bar{v}_1 > 0 \text{ such that } 1 + 2\dot{a}(t) > \bar{v}_1 \quad \text{for almost every } t \text{ in } \mathbb{R}, \tag{19}$$

$$\exists \bar{v}_2 > 0 \text{ such that } 1 + 2K\dot{a}(t) > \bar{v}_2 \quad \text{for almost every } t \text{ in } \mathbb{R}, \tag{20}$$

$$a \text{ is not a constant function.} \tag{21}$$

Let $\delta: \mathbb{R} \to \mathbb{R}$ be defined by

$$\delta(t) = \left\{\frac{1}{c(4K-1)}[2 + \dot{a}(t)]\right\}^{3/4}. \tag{22}$$

For $M > 0$, let us define the continuous time-varying feedback $(\bar{x}_5, \bar{x}_6): \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ by

$$\bar{x}_5(x_1, x_2, x_3, t) = -M\left\{\delta(t)^{1/3}\left[2Kx_1 + \frac{1}{a(t)}x_2\right]^{\{1/2\}} + x_3\right\}, \tag{23}$$

$$\bar{x}_6(x_1, x_2, x_3, t) = \delta(t)\left|2Kx_1 + \frac{1}{a(t)}x_2\right|^{1/2}. \tag{24}$$

In (23), as throughout this paper, we use the notation $\beta^{\{\alpha\}} = |\beta|^{\alpha-1}\beta$ if $\beta \neq 0$, $0^{\{\alpha\}} = 0$

$\forall \alpha \in \mathbb{R}\backslash\{0\}$. Note that $(\bar{x}_5, \bar{x}_6)$ is T-periodic with respect to time and has good homogeneity:

$$\bar{x}_5(\epsilon^2 x_1, \epsilon^2 x_2, \epsilon x_3, t) = \varepsilon \bar{x}_5(x_1, x_2, x_3, t) \quad \forall (x_1, x_2, x_3, t) \in \mathbb{R}^4,$$

$$\bar{x}_6(\epsilon^2 x_1, \epsilon^2 x_2, \epsilon x_3, t) = \varepsilon \bar{x}_6(x_1, x_2, x_3, t) \quad \forall (x_1, x_2, x_3, t) \in \mathbb{R}^4.$$

The result of this section is the following.

Proposition 3.1. If M is large enough then the feedback $(\bar{x}_5, \bar{x}_6)$ defined by (23) and (24) globally asymptotically stabilizes the control system $\bar{\Sigma}$.

Proof. Let $b: \mathbb{R} \to \mathbb{R}$ be defined by

$$b(t) = \frac{1}{c}\left\{-\frac{4K-1}{2}a(t) - \left[\int_0^t \frac{4K-1}{4}\dot{a}(s)^2\,\mathrm{d}s\right] + mt\right\}, \tag{25}$$

where (see (21))

$$m = \frac{4K-1}{4T}\int_0^T \dot{a}(s)^2\,\mathrm{d}s > 0. \tag{26}$$

Note that b is T-periodic with respect to time. For $\mu \in [0, +\infty)$, let $V: \mathbb{R}^3 \times \mathbb{R} \to [0, +\infty)$ be defined by

$$V(x_1, x_2, x_3, t) = Kx_1^2 + \frac{1}{a(t)}x_1 x_2 + \frac{1}{a(t)}x_2^2 + \frac{1}{4}x_3^4 - \mu[a(t)x_3^4 + b(t)x_1^2], \tag{27}$$

for all (x_1, x_2, x_3, t) in $\mathbb{R}^4$. Note that V is T-periodic with respect to time, and has the following homogeneity

$$V(\varepsilon^2 x_1, \varepsilon^2 x_2, \varepsilon x_3, t) = \varepsilon^4 V(x_1, x_2, x_3, t) \quad \forall (\varepsilon, x_1, x_2, x_3, t) \in (0, +\infty) \times \mathbb{R}^4. \tag{28}$$

We choose μ small enough that

$$K - \mu b(t) > \tfrac{1}{4}, \quad \tfrac{1}{4} - \mu a(t) > 0 \quad \forall t \in \mathbb{R}. \tag{29}$$

From (27) and (29), we find that $V > 0$ on $(\mathbb{R}^3\backslash\{0\}) \times \mathbb{R}$. Let $W: \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ be defined by

$$W(=\dot{V}) := \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1}\bar{x}_5\bar{x}_6 + \frac{\partial V}{\partial x_2}(x_1 + cx_3\bar{x}_6) + \frac{\partial V}{\partial x_3}\bar{x}_5 \tag{30}$$

and let

$$X = \delta^{1/3}(2Kx_1 + x_2/a)^{\{1/2\}} + x_3,$$
$$Y = (x_1 + 2x_2/a)^{\{1/2\}},$$
$$Z = (2Kx_1 + x_2/a)^{\{1/2\}}.$$

By (18), we have

$$(X = Y = Z = 0) \Rightarrow (x_1 = x_2 = x_3 = 0). \tag{31}$$

Straightforward computations give

$$W = -W_1 - \mu W_2, \tag{32}$$

with

$$W_1 = MX^2(X^2 - 3\delta^{1/3}XZ + 3\delta^{2/3}Z^2) - \frac{c}{a}\delta XY^{\{2\}}|Z| + \frac{1+2K\dot{a}}{(4K-1)a}Y^4, \tag{33}$$

$$W_2 = (\dot{a} - 4aM)X^4 + (-4\dot{a} + 12aM)\delta^{1/3}X^3Z + (6\dot{a} - 12aM)\delta^{2/3}X^2Z^2 + \frac{2M}{4K-1}\delta bXY^{\{2\}}|Z| + \left(-4\dot{a} - \frac{4Mb}{4K-1} + 4aM\right)\delta XZ^3 + \frac{\dot{b}}{(4K-1)^2}Y^4 - \frac{4\dot{b}}{(4k-1)^2}Y^{\{2\}}Z^{\{2\}} + \frac{4m}{c(4K-1)^2}Z^4. \tag{34}$$

Let us denote by A_i $(1 \le i \le 6)$ various positive constants that may depend on a, c but are independent of X, Y, Z, M and μ. These constants can be easily estimated from above: for simplicity we shall not do this. From (18)–(20), (22), (28) and (33), one gets that there are constants A_1, A_2 and A_3 such that

$$W_1 \ge \frac{M}{5}X^4 + \left(\frac{M}{A_1} - A_2\right)X^2Z^2 + \frac{1}{A_3}Y^4; \tag{35}$$

from (20), (26) and (34) we find positive constants A_4, A_5 and A_6 such that

$$W_2 \ge \frac{1}{A_4}Z^4 - A_5(1+M)(X^4 + X^2Z^2) - A_6Y^4. \tag{36}$$

From (32), (35), and (36), we find positive constants M_0, μ_0 and ν_0 such that, for all M in $[M_0, +\infty)$ and all μ in $[0, \mu_0]$,

$$W \le -\nu_0(MX^4 + Y^4 + \mu Z^4), \tag{37}$$

which, with (31) and Lyapunov's second theorem, ends the proof of Proposition 3.1. □

Remark 3.1. We could in fact take $\mu = 0$ and conclude by using LaSalle's theorem instead of Lyapunov's second theorem. The interest in taking μ in $(0, \mu_0]$ is that we get an estimate of the decay of V with time; this is useful if one wants to estimate some constants that appear in our time-varying feedback that stabilizes Σ (see Section 5).

4. STABILIZATION OF $\tilde{\Sigma}$

Recall that $\tilde{\Sigma}$ is obtained from $\bar{\Sigma}$ by adding an integrator on x_5 and on x_6. Our stabilizing feedback $(\bar{x}_5, \bar{x}_6)$ for $\bar{\Sigma}$ is not smooth enough to use directly the procedure due to Byrnes and Isidori (1989) and Tsinias (1989); we proceed as in Praly *et al.* (1991) and Coron and Praly (1991): we 'homogeneously desingularize' $\bar{x}_5$ and $\bar{x}_6$. Let $\phi_1: \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $\phi_2: \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be defined by

$$\phi_1(x_1, x_2, x_3, x_5, t) = (x_5 + Mx_3)^3 + M^3\delta(t)\left[2Kx_1 + \frac{1}{a(t)}x_2\right]^{\{3/2\}}, \quad (38)$$

$$\phi_2(x_1, x_2, x_3, x_6, t) = x_6^3 - \delta(t)\left|2Kx_1 + \frac{1}{a(t)}x_2\right|^{3/2}. \quad (39)$$

Note that ϕ_1 and ϕ_2 are of class C^1 with respect to the state variables—and of class C^1 with respect to all the variables if a is also of class C^1. Note also that

$$(\phi_1(x_1, x_2, x_3, x_5, t) = 0) \Leftrightarrow (x_5 = \bar{x}_5(x_1, x_2, x_3, t)), \quad (40)$$

$$(\phi_2(x_1, x_2, x_3, x_6, t) = 0) \Leftrightarrow (x_6 = \bar{x}_6(x_1, x_2, x_3, t)). \quad (41)$$

As in Praly *et al.* (1991), we define $\psi_1: \mathbb{R}^4\times\mathbb{R}\to\mathbb{R}$, $\psi_2: \mathbb{R}^4\times\mathbb{R}\to\mathbb{R}$, and $\tilde{V}: \mathbb{R}^5\times\mathbb{R}\to\mathbb{R}$

$$\psi_1(x_1, x_2, x_3, x_5, t) = \int_{\bar{x}_5(x_1,x_2,x_3,t)}^{x_5} \phi_1(x_1, x_2, x_3, s, t)\,ds, \quad (42)$$

$$\psi_2(x_1, x_2, x_3, x_6, t) = \int_{\bar{x}_6(x_1,x_2,x_3,t)}^{x_6} \phi_2(x_1, x_2, x_3, s, t)\,ds, \quad (43)$$

$$\tilde{V}(x_1, x_2, x_3, x_5, x_6, t) = \gamma V(x_1, x_2, x_3, t) + \psi_1(x_1, x_2, x_3, x_5, t) + \psi_2(x_1, x_2, x_3, x_6, t), \quad (44)$$

where γ is a positive constant. Note that $\tilde{V}$ is T-periodic with respect to time, vanishes on $\{0\}\times\mathbb{R}$, is positive outside $\{0\}\times\mathbb{R}$, and has good homogeneity, i.e., for any $\varepsilon>0$ and any $(x_1, x_2, x_3, x_5, x_6, t)\in\mathbb{R}^6$, we have $\tilde{V}(\varepsilon^2x_1, \varepsilon^2x_2, \varepsilon x_3, \varepsilon x_5, \varepsilon x_6, t) = \varepsilon^4\tilde{V}(x_1, x_2, x_5, x_6, t)$. Straightforward computations give

$$\tilde{V} = \gamma V + \tfrac{1}{4}(x_5 + MX - M\delta^{1/3}Z)^4 - \tfrac{1}{4}M^4\delta^{4/3}Z^4 + M^3\delta Z^3(x_5 + mX) + \tfrac{1}{4}x_6^4 - \tfrac{1}{4}\delta(t)^4Z^4 - \delta(t)\,|Z|^3[x_6 - \delta(t)\,|Z|]. \quad (45)$$

We use $\tilde{V}$ as a (time-varying) control Lyapunov function (see Artstein, 1983; Sontag, 1988; Praly *et al.*, 1991). Let $\tilde{x} = (x_1, x_2, x_3, x_5, x_6)\in\mathbb{R}^5$ and let $\tilde{f}: \mathbb{R}^5\times\mathbb{R}^2\to\mathbb{R}^5$ be defined by $\tilde{f}(\tilde{x}, u) = (x_5x_6, x_1 + cx_3x_6, x_5, u_1, u_2)$; hence $\tilde{\Sigma}$ is the control system $\dot{\tilde{x}} = \tilde{f}(\tilde{x}, u)$. Let $H: \mathbb{R}^5\times\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}$ be defined by

$$H(=\dot{\tilde{V}}) := \frac{\partial\tilde{V}}{\partial t} + \frac{\partial\tilde{V}}{\partial x_1}x_5x_6 + \frac{\partial\tilde{V}}{\partial x_2}(x_1 + cx_3x_6) + \frac{\partial\tilde{V}}{\partial x_3}x_5 + \frac{\partial\tilde{V}}{\partial x_5}u_1 + \frac{\partial\tilde{V}}{\partial x_6}u_2. \quad (46)$$

We can write $H = A(\tilde{x}, t) + B_1(\tilde{x}, t)u_1 + B_2(\tilde{x}, t)u_2$, where the functions A, B_1 and B_2 have the following homogeneity, with $\tilde{\delta}_\varepsilon\tilde{x} = (\varepsilon^2x_1, \varepsilon^2x_2, \varepsilon x_3, \varepsilon x_5, \varepsilon x_6)$:

$$A(\tilde{\delta}_\varepsilon\tilde{x}, t) = \varepsilon^4A(\tilde{x}, t) \quad \forall(\varepsilon, \tilde{x}, t)\in(0, +\infty)\times\mathbb{R}^5\times\mathbb{R}, \quad (47)$$

$$B_1(\tilde{\delta}_\varepsilon\tilde{x}, t) = \varepsilon^3B_1(\tilde{x}, t) \quad \forall(\varepsilon, \tilde{x}, t)\in(0, +\infty)\times\mathbb{R}^5\times\mathbb{R}, \quad (48)$$

$$B_2(\tilde{\delta}_\varepsilon\tilde{x}, t) = \varepsilon^3B_2(\tilde{x}, t) \quad \forall(\varepsilon, \tilde{x}, t)\in(0, +\infty)\times\mathbb{R}^5\times\mathbb{R}. \quad (49)$$

Moreover, if $B_1(\tilde{x}, t) = B_2(\tilde{x}, t) = 0$ then

$$x_5 = \bar{x}_5 = -MX \quad x_6 = \bar{x}_6 = \delta Z, \quad A(\tilde{x}, t) = \gamma W(x_1, x_2, x_3, t). \quad (50)$$

We choose μ in $(0, \mu_0]$ and M in $[M_0, +\infty)$ (see Section 3). In particular, we have (37); using this, with (31) and (50), we have

$$(B_1(\tilde{x}, t) = B_2(\tilde{x}, t) = 0 \text{ and } \tilde{x}\neq 0) \Rightarrow (A(\tilde{x}, t) < 0). \quad (51)$$

Let $\tilde{M}$ be any real number in $(0, +\infty)$. Inspired by Sontag (1988), we define $\tilde{u} = (\tilde{u}_1, \tilde{u}_2): \mathbb{R}^5\times\mathbb{R}\to\mathbb{R}^2$ by

$$\tilde{u}(\tilde{x}, t) = 0 \quad \text{if } B(\tilde{x}, t) = 0, \quad (52)$$

$$\tilde{u}(\tilde{x}, t) = -|B(\tilde{x}, t)|^{-2}[A(\tilde{x}, t) + |A(\tilde{x}, t)| + \tilde{M}\,|B(\tilde{x}, t)|^{4/3}]\,B(\tilde{x}, t) \quad \text{if } B(\tilde{x}, t)\neq 0, \quad (53)$$

where $B = (B_1, B_2)$. Then $\tilde{u}$ is T-periodic with respect to time, $\tilde{\delta}_\varepsilon$-homogeneous of degree 1, i.e.

$$\tilde{u}(\tilde{\delta}_\varepsilon(\tilde{x}, t)) = \varepsilon\tilde{u}(\tilde{x}, t) \quad \forall(\varepsilon, \tilde{x}, t)\in(0, +\infty)\times\mathbb{R}^5\times\mathbb{R}, \quad (54)$$

and (see in particular (51)–(54)) is almost continuous. Let $\tilde{W}: \mathbb{R}^5 \times \mathbb{R} \to \mathbb{R}$ be defined by $\tilde{W} := \dot{\tilde{V}} = H(\tilde{x}, t, \tilde{u}(\tilde{x}, t))$. It follows from (51)–(53) that

$$\tilde{W} = -|A(\tilde{x}, t)| - \tilde{M}\,|B(\tilde{x}, t)|^{4/3}, \qquad (55)$$

which, with (51), implies that $\tilde{W} < 0$ on $(\mathbb{R}^5\backslash\{0\}) \times \mathbb{R}$; hence, from Lyapunov's second theorem, we find that $\tilde{u}$ globally asymptotically stabilizes the control system $\tilde{\Sigma}$.

Remark 4.1. Again one could take $\mu = 0$ and conclude by LaSalle's theorem instead of Lyapunov's second theorem.

5. STABILIZATION OF Σ^* AND Σ

We proceed as described in Section 2. Let ζ_4 and ζ_6 be two real numbers such that $0 \in \mathbb{R}^2$ is locally asymptotically stable for $\dot{x}_4 = x_6$ and $\dot{x}_6 = \zeta_4 x_4 + \zeta_6 x_6$. let τ and τ' be two positive real numbers; we define an almost-continuous function $u: \mathbb{R}^6 \times \mathbb{R} \to \mathbb{R}^2$ by

$$u(x, t) = \tilde{u}(x_1, x_2, x_3, x_5, x_6, t) \quad \forall (x, t) \in \mathbb{R}^6 \times [0, \tau), \qquad (56)$$

$$u(x, t) = (0, \zeta_4 x_4 + \zeta_6 x_6) \quad \forall (x, t) \in \mathbb{R}^6 \times [\tau, \tau + \tau'), \qquad (57)$$

$$u(x, t + \tau + \tau') = u(x, t) \quad \forall (x, t) \in \mathbb{R}^6 \times \mathbb{R}. \qquad (58)$$

Let $r: \mathbb{R}^6 \to [0, +\infty)$ be a continuous function that vanishes only on $\{0\}$ and satisfies, $\forall (x, \varepsilon) \in \mathbb{R}^6 \times (0, +\infty)$, $r(\delta_\varepsilon x) = \varepsilon^4 r(x)$. One can take, for example, $r(x) = x_1^2 + x_2^2 + x_3^4 + x_4^4 + x_5^4 + x_6^4$. Note that u has the required homogeneity: it satisfies (6). This homogeneity property implies that any (maximal) solution of $\dot{x} = X(x) + u(x, t)Y(x)$ is defined on all $\mathbb{R}$ and satisfies, for some constant C_1 that may depend on $a, c, r, M, \tilde{M}, \mu, \zeta_4, \zeta_6, \tau$ and τ' but is independent of the maximal solution,

$$r(x(t_2)) \le r(x(t_1)) \exp[C_1(t_2 - t_1)] \quad \forall t_1 \le t_2. \qquad (59)$$

Let us denote by A_i $(i \ge 7)$ various positive constants that may depend on $a, c, r, M, \tilde{M}, \mu, \zeta_4, \zeta_6$ and $\tau_0 > 0$ but are independent of τ in $[\tau_0 + \infty)$, of τ' in $[\tau_0, +\infty)$ and of x, the (maximal) solution of $\dot{x} = X(x) + u(x, t)Y(x)$. These positive constants can be easily estimated from above; but, for simplicity, we do not give such estimates. From (51), (55) and homogeneity, we get

$$r(x_1(t), x_2(t), x_3(t), 0, x_5(t), x_6(t)) \le A_7 r(x(0)) \exp\left(-\frac{t}{A_8}\right) \quad \forall t \in [0, \tau]. \qquad (60)$$

From $\dot{x}_4 = x_6$ and (60), we get

$$|x_4(\tau)|^4 \le A_9 r(x(0)). \qquad (61)$$

Since $0 \in \mathbb{R}^2$ is locally asymptotically stable for $\dot{x}_4 = x_6$ and $\dot{x}_6 = \zeta_4 x_4 + \zeta_6 x_6$, we have

$$|x_4(t)| + |x_6(t)| \le A_{10}(|x_4(\tau)| + |x_6(\tau)|) \exp\left(-\frac{t - \tau}{A_{11}}\right) \quad \forall t \in [\tau, \tau + \tau'], \qquad (62)$$

which, with (60) and (61), implies

$$|x_4(t)| + |x_6(t)| \le A_{12} r(x(0))^{1/4} \exp\left(-\frac{t - \tau}{A_{11}}\right) \quad \forall t \in [\tau, \tau + \tau']. \qquad (63)$$

On $[\tau, \tau + \tau']$ we have $\dot{x}_5 = 0$, so, from (60), we have

$$|x_5(t)| \le A_{13} r(x(0))^{1/4} \exp\left(-\frac{\tau}{4A_8}\right) \quad \forall t \in [\tau, \tau + \tau'], \qquad (64)$$

which, with $\dot{x}_3 = x_5$ and (60), implies

$$|x_3(t)| \le A_{14} r(x(0))^{1/4} \tau' \exp\left(-\frac{\tau}{4A_8}\right) \quad \forall t \in [\tau, \tau + \tau']. \qquad (65)$$

Using $\dot{x}_1 = x_5 x_6$, (60), (63) and (64), we get

$$|x_1(t)| \le A_{15} r(x(0))^{1/2} \exp\left(-\frac{\tau'}{A_{16}}\right) \quad \forall t \in [\tau, \tau + \tau']. \qquad (66)$$

which, with $\dot{x}_2 = x_1 + c x_3 x_6$, (63), and (65), implies

$$|x_2(t)| \le A_{17} r(x(0))^{1/2} \tau' \exp\left(-\frac{\tau}{A_{18}}\right) \quad \forall t \in [\tau, \tau + \tau']. \qquad (67)$$

Finally, from (63)–(67), we can get

$$r(x(t)) \le A_{19} r(x_0)\left[\tau' \exp\left(-\frac{\tau}{A_{20}}\right) + \exp\left(-\frac{\tau'}{A_{21}}\right)\right] \quad \forall t \in [\tau, \tau + \tau']. \qquad (68)$$

We choose τ' large enough that $\exp(-\tau'/A_{21}) \le (4A_{19})^{-1}$ and then choose τ large enough that $\tau' \exp(-\tau/A_{20}) \le (4A_{19})^{-1}$. So, from (68), we get

$$r(x(\tau + \tau')) \le \tfrac{1}{2} r(x(0)), \qquad (69)$$

which proves that

$$0 \text{ is globally asymptotically stable for } \dot{x} = X(x) + u(x, t)Y(x). \quad (70)$$

Indeed, from (59) and (69), we get

$$r(x(t_2)) \leq 2^{-(t_2-t_1)/(\tau+\tau')} 4r(x(t_1)) \exp[2C_1(\tau+\tau')] \quad \forall t_1 \leq t_2, \quad (71)$$

which gives (70). By homogeneity and the Massera (1956)–Kawski (1988)–Hermes (1990)–Rosier (1992, 1993) theorem, (70) implies that 0 is also locally asymptotically stable for $\dot{x} = f(x, u(x, t))$. Let us remark that Massera (1956), Kawski (1988), Hermes (1990) and Rosier (1992) deal with vector fields that do not depend on time, but their results can be adapted to time-varying vector fields—note in particular that, by a result due to Rosier (1993), the converse of Lyapunov's second theorem holds for time-varying almost-continuous vector field. Moreover, in our case the fact that 0 is also locally asymptotically stable for $\dot{x} = f(x, u(x, t))$ follows directly from (69). Indeed r is a δ_ε-homogeneous Lyapunov function for the Poincaré map $x(0) \to x(\tau + \tau')$ for $\dot{x} = X(x) + uY(x)$ and so also for $\dot{x} = f(x, u(x, t))$, near 0, as follows from standard estimates.

Let us end this section with a comment. One has 'r-exponential asymptotic stability': it follows from our construction of u that there exists $r_0 > 0$, $C_0 > 0$, $\lambda > 0$ such that, for any maximal solution of $\dot{x} = f(x, u(x, t))$, any $\bar{t}$ in $\mathbb{R}$ and any $t \geq \bar{t}$, $|x(\bar{t})| \leq r_0$ implies that $r(x(t)) \leq C_0 r(x(\bar{t})) \exp[-\lambda(t - \bar{t})]$. One can increase the value of λ by the following procedure—which of course, and unfortunately, decreases r_0 and increases C_0. Let κ, ξ_0, ξ_7 and ξ_8 be four real numbers and let us define six new real numbers $\xi_6 = \xi_8 - \xi_0$, $\xi_5 = \xi_7 - \xi_0$, $\xi_4 = \xi_8 - 2\xi_0$, $\xi_3 = \xi_7 - 2\xi_0$, $\xi_1 = \xi_7 + \xi_8 - 3\xi_0$ and $\xi_2 = \xi_7 + \xi_8 - 4\xi_0$. Let $u^{\kappa,\xi} = (u_1^{\kappa,\xi}, u_2^{\kappa,\xi})\colon \mathbb{R}^6 \times \mathbb{R} \to \mathbb{R}^2$ be the almost-continuous function defined by $u_1^{\kappa,\xi}(x, t) = \kappa^{\xi_7} u(\kappa^{-\xi_1} x_1, \ldots, \kappa^{-\xi_6} x_6, \kappa^{\xi_0} t)$ and $u_2^{\kappa,\xi}(x, t) = \kappa^{\xi_8} u(\kappa^{-\xi_1} x_1, \ldots, \kappa^{-\xi_6} x_6, \kappa^{\xi_0} t)$. Then $u^{\kappa,\xi}$ is δ_ε-homogeneous of degree 1, i.e. it satisfies (6). Moreover, if $x\colon \mathbb{R} \to \mathbb{R}^6$ is a solution of $\dot{x} = X(x) + u(x, t)Y$ then $x^{\kappa,\xi}\colon \mathbb{R} \to \mathbb{R}^6$ defined by $x_i^{\kappa,\xi}(t) = \kappa^{\xi_i} x_i(\kappa^{\xi_0} t)$ $\forall i \in [1, 6]$ is a solution of $\dot{x}^{\kappa,\xi} = X(x^{\kappa,\xi}) + u^{\kappa,\xi}(x^{\kappa,\xi}, t)Y$. Easy estimations show the existence of $r_0^{\kappa,\xi} > 0$ and $C_0^{\kappa,\xi} > 0$ such that, for any maximal solution of $\dot{x} = f(x, u^{\kappa,\xi}(x, t))$, any $\bar{t}$ in $\mathbb{R}$ and any $t \geq \bar{t}$, $r(x(\bar{t})) \leq r_0^{\kappa,\xi}$ implies that $r(x(t)) \leq C_0^{\kappa,\xi} r(x(\bar{t})) \exp[-\lambda\kappa^{\xi_0}(t - \bar{t})]$; hence $u^{\kappa,\xi}$ also locally asymptotically stabilizes Σ and λ is now replaced by $\lambda\kappa^{\xi_0}$. So, in order to get large λ, it suffices to take $\xi_0 = 1$ and κ large.

Remark 5.1. Our stabilizing feedback u, which is only almost-continuous, can be easily transformed into a continuous stabilizing feedback. Indeed, one can always take a to be of class C^1, and then, for $\varepsilon > 0$, define $u_\varepsilon\colon \mathbb{R}^6 \times \mathbb{R} \to \mathbb{R}^2$ by

$$u_\varepsilon(x, t) = \min\left(\frac{t}{\varepsilon}, \frac{|t - \tau|}{\varepsilon}, \frac{\tau + \tau' - t}{\varepsilon}\right) u(x, t) \quad \forall (x, t) \in \mathbb{R}^6 \times [0, \tau + \tau'], \quad (72)$$

$$u_\varepsilon(x, t + \tau + \tau') = u_\varepsilon(x, t) \quad \forall (x, t) \in \mathbb{R}^6 \times \mathbb{R}. \quad (73)$$

Then u_ε is continuous and, for some $\varepsilon_0 > 0$ that can be estimated from below, u_ε locally asymptotically stabilizes Σ for any ε in $(0, \varepsilon_0)$.

6. CONCLUSIONS

We have constructed time-varying feedbacks that locally asymptotically stabilize the attitude of a rigid spacecraft with only two control torques. Such feedbacks, the existence of which was proved in Coron (1993, 1995) and Keraï 1995), had been constructed previously by Morin *et al.* (1994) in a special case.

Our time-varying feedbacks depend on various constants and on two functions (a and r). This offers some flexibility. It would be interesting to study the influence of these parameters on the 'quality' of the stabilization (the size of the basin of attraction, the rate of convergence, robustness, etc.).

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APPENDIX

In this Appendix we derive (1). Let $\eta = (\phi, \theta, \psi)$ be the Euler angles of a frame attached to the spacecraft, representing rotations about a reference frame, let $\omega = (\omega_1, \omega_2, \omega_3)$ be the angular velocity of the frame attached to the spacecraft with respect to the reference frame, expressed in the frame attached to the spacecraft; let J be the inertia matrix of the satellite. The evolution of the satellite is governed by the equations.

$$J\dot{\omega} = S(\omega)J\omega + v_1 b_1 + v_2 b_2, \quad \dot{\eta} = A(\eta)\omega, \tag{A.1}$$

where $v_1 \in \mathbb{R}$ and $v_2 \in \mathbb{R}$ are the controls, $v_1 b_1 \in \mathbb{R}^3$ and $v_2 b_2 \in \mathbb{R}^3$ are the torques applied to the satellite, $S(\omega)$ is the matrix representation of the wedge product, i.e.

$$S(\omega) = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}, \tag{A.2}$$

and

$$A(\eta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ \sin\theta\tan\phi & 1 & -\cos\theta\tan\phi \\ -\sin\theta/\cos\phi & 0 & \cos\theta/\cos\phi \end{pmatrix}. \tag{A.3}$$

Without loss of generality, we may assume that $\{v_1 b_1 + v_2 b_2 : (v_1, v_2) \in \mathbb{R}^2\} = \{0\} \times \mathbb{R}^2$. So, after a change of control variables, (A.1) can be replaced by

$$\dot{\omega}_1 = Q(\omega) + \omega_1 L_1(\omega), \quad \dot{\omega}_2 = V_1, \dot{\omega}_3 = V_2, \quad \dot{\eta} = A(\eta)\omega, \tag{A.4}$$

with $L_1\omega = D_1\omega_1 + E_1\omega_2 + F_1\omega_3$ and $q(\omega) = A\omega_2^2 + B\omega_2\omega_3 + C\omega_3^2$. For the system (A.4), the controls are V_1 and V_2. It is proved in Keraï (1995) that Q changes sign if and only if the control system (A.1) satisfies the Lie-algebra rank condition—which is a necessary condition for controllability (Sussmann and Jurdjevic, 1972). From now on, we assume that Q changes sign—this is a generic situation. Hence, after a suitable change of coordinates of the form

$$\omega = P\tilde{\omega} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_p & b_p \\ 0 & c_p & d_p \end{pmatrix} \tilde{\omega}, \tag{A.5}$$

the system (A.4) can be written as

$$\dot{\tilde{\omega}}_1 = \tilde{\omega}_2\tilde{\omega}_3 + \tilde{\omega}_1 L_2(\tilde{\omega}), \quad \dot{\tilde{\omega}}_2 = u_1, \quad \dot{\tilde{\omega}}_3 = u_2, \quad \dot{\eta} = A(\eta)P\tilde{\omega}, \tag{A.6}$$

with $L_2\tilde{\omega} = D_2\tilde{\omega}_1 + E_2\tilde{\omega}_2 + F_2\tilde{\omega}_3$. let $c = \det P$; we can always choose P so that $c > 0$. Let

$$x_1 = \tilde{\omega}_1, \quad x_5 = \tilde{\omega}_2, \quad x_6 = \tilde{\omega}_3, \tag{A.7}$$

$$x_3 = \frac{1}{c}(d_p\theta - b_p\psi), \quad x_4 = \frac{1}{c}(-c_p\theta + a_p\psi), \qquad x_2 = \phi - \tfrac{1}{2}b_pd_px_4^2 - \tfrac{1}{2}a_pc_px_3^2 - b_pc_px_3x_4. \tag{A.8}$$

In these coordinates, our system can be written as

$$\dot{x}_1 = x_5x_6 + R_1(x), \quad \dot{x}_2 = x_1 + cx_3x_6 + R_2(x), \qquad \dot{x}_3 = x_5 + R_3(x), \quad \dot{x}_4 = x_6 + R_4(x), \quad \dot{x}_5 = u_1, \quad \dot{x}_6 = u_2, \tag{A.9}$$

where R_1, R_2, R_3 and R_4 are analytic functions on a neighborhood of 0 such that, for a suitable positive constant C, one has, for all x in $\mathbb{R}^6$ with $|x|$ small enough,

$$|R_1(x)| + |R_3(x)| + |R_4(x)| \le C(|x_1| + |x_2| + |x_3|^2 + |x_4|^2 + |x_5|^2 + |x_6|^2)^{3/2}, \tag{A.10}$$

$$|R_2(x)| \le C(|x_1| + |x_2| + |x_3|^2 + |x_4|^2 + |x_4|^2 + |x_5|^2 + |x_6|^2)^2, \tag{A.11}$$

which gives (5).