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Mathematics of Control, Signals, and Systems

On the Stabilization of Controllable and Observable Systems by an Output Feedback Law*

Jean-Michel Coron†

Abstract. For control systems which can be locally stabilized in small time by means of a dynamic periodic time-varying *state* feedback law, we give a sufficient condition on Lie derivatives of the output for local stabilization in small time by means of a dynamic periodic time-varying *output* feedback law. If the system is analytic our sufficient condition is also necessary.

Key words. Nonlinear systems, Stabilization, Controllability, Observability, Output feedback.

1. Introduction

In many practical situations only part of the state—called the observation or the output—is measured and therefore state stabilizing feedback cannot be implemented; only output feedback is allowed. A natural question is: to what extent does stabilizability by means of state feedback plus some observability condition imply stabilizability by means of (dynamic) output feedback?

There are numerous results to this problem. In the linear case the situation is well understood, see, e.g., Section 6.2 of [S3]. For the nonlinear case, let us mention recent works by Sontag [S2], Isidori [I], Tornambé [T], Esfandari and Khalil [EK], Mazenc *et al.* [MPD], Teel and Praly [TP], Viel [V]—see also [GHO]—Mazenc and Praly [MP], and the references therein. In particular, in [TP] Teel and Praly prove the following nice result: if the state is a function of the output, the input, and their derivatives, then state local stabilizability implies (dynamic) output local stabilizability. Our goal is to weaken this observability condition, which, by a result due to Gauthier and Kupka [GK], is not generic if the dimension of the output is less or equal to the dimension of the input. Roughly speaking, Teel and Praly require that, using *any* output, two different states can be distinguished by just looking at the output. One may wonder if it is not sufficient to require only that:

(i) Given two different states a (time-varying) input exist which leads to different outputs.
(ii) If the input is identically 0 and the output is identically 0, then the state is also 0.

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† Ecole Normale Supérieure de Cachan, Centre de Mathématiques et de Leurs Applications, Unité associée au CNRS–URA-1611, 61 Avenue du Président Wilson, 94235 Cachan Cedex, France.

The reason for (ii) is that for output stabilizability it is required that the output feedback vanishes if the output vanishes. In this paper we give a Lie algebraic condition which, when the system is analytic, is equivalent to (i) and (ii); we also prove—and this is our main result—that this Lie algebraic condition is sufficient to ensure (dynamic) output feedback local stabilizability in small time if one has state feedback local stabilizability in small time, provided that one allows the output feedback to *depend on time*; moreover, we show that this Lie algebraic condition is in fact necessary for "fast" dynamic output feedback stabilizability if the system is analytic. To allow the output feedback to depend on time is necessary; indeed, for example, the control system $\dot{x} = u \in \mathbb{R}$, $y = x^2$, where the state is x, the control is u, and the observation y satisfies (i) and (ii), is asymptotically—and also in small time—stabilizable by means of (stationary) state feedback, but cannot be asymptotically stabilized by means of a dynamic output feedback which does not depend on time; see Section 2.

As is explained in detail in Section 2, the proof of our main result uses ideas which already appear in the paper by Lozano [L], in [C2], and in the paper by Mazenc and Praly [MP]. Roughly speaking, we alternatively learn the state and stabilize. In order to learn the state, we excite the system by a suitable time-varying output feedback: with this output feedback the state can be recovered by looking at the observation a sufficiently large number of times, which are chosen at random.

2. Statement of the Results

Let C be the control system

$$\dot{x} = f(x, u), \qquad y = h(x), \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control, and $y \in \mathbb{R}^p$ is the output. We assume that $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$, $h \in C^\infty(\mathbb{R}^n; \mathbb{R}^p)$, and

$$f(0, 0) = 0, \tag{2.2}$$

$$h(0) = 0. \tag{2.3}$$

In order to state our main result we first introduce some definitions.

Definition 2.1. System C is locally stabilizable in small time by means of a continuous static periodic time-varying *state* feedback law if, for any positive real number T, there are ε in $(0, +\infty)$ and u in $C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$ such that

$$u(0, t) = 0, \qquad \forall t \in \mathbb{R}, \tag{2.4}$$

$$u(x, t + T) = u(x, t), \qquad \forall x \in \mathbb{R}^n, \quad \forall t \in \mathbb{R}, \tag{2.5}$$

$$((\dot{x} = f(x, u(x, t)) \text{ and } x(s) = 0) \Rightarrow (x(t) = 0, \forall t \geq s)), \qquad \forall s \in \mathbb{R}, \tag{2.6}$$

$$((\dot{x} = f(x, u(x, t)) \text{ and } |x(s)| \leq \varepsilon) \Rightarrow (x(t) = 0, \forall t \geq s + T)), \qquad \forall s \in \mathbb{R}. \tag{2.7}$$

If, moreover, u can be chosen so that

$$u(x, t) = \bar{u}(h(x), t) \tag{2.8}$$

for some $\bar{u}$ in $C^0(\mathbb{R}^p \times \mathbb{R}; \mathbb{R}^n)$, then system C is said to be locally stabilizable in small time by means of a continuous static periodic time-varying *output* feedback law.

Throughout this paper, and in particular in (2.6) and (2.7), by $\dot{x} = f(x, u(x, t))$ we denote *any maximal* solution of this differential equation. We emphasize that since the feedback law is only continuous, the Cauchy problem $\dot{x} = f(x, u(x, t))$, $x(t_0) = x_0$, where t_0 and x_0 are given, may have many maximal solutions; in particular, (2.6) does not follow from (2.2) and (2.4); note also that without (2.6) 0 is not a stable point of $\dot{x} = f(x, u(x, t))$; but if u is as in the above definition, it can be easily checked that 0 is a—uniformly—asymptotically stable point of $\dot{x} = f(x, u(x, t))$ (see Lemma 2.15 of [C4]). We recall that 0 is a uniformly locally asymptotically stable point for $\dot{x} = X(x, t)$, with X in $C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n)$, if it is a uniformly locally stable point (i.e., $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall s \in \mathbb{R}$, $\dot{x} = X(x, t)$ and $|x(s)| < \delta$ imply $|x(\tau)| < \varepsilon$, $\forall \tau > s$) and is a uniformly locally attractive point with respect to time and state (i.e., $\exists \delta > 0$ s.t., $\forall \varepsilon > 0$, $\exists M > 0$ s.t., $\forall s \in \mathbb{R}$, $\dot{x} = X(x, t)$ and $|x(s)| < \delta$ imply $|x(\tau)| < \varepsilon$, $\forall \tau > s + M$).

We point out that it is proved in [C4] that many sufficient conditions for local controllability in small time and with small control—e.g., the Hermes condition [H], [S6] or even the Sussmann condition [S7, Theorem 7.3]—imply, if $n \geq 4$, that the system is locally stabilizable in small time by means of a continuous static periodic time-varying state feedback law.

Our next definition concerns dynamical stabilizability.

Definition 2.2. System C is locally stabilizable in small time by means of a continuous dynamic periodic time-varying state (resp. output) feedback law if, for some integer $k \geq 0$, the control system

$$\dot{x} = f(x, u), \qquad \dot{z} = v, \qquad \tilde{h}(x, z) = (h(x), z), \tag{2.9}$$

where the state is $(x, z) \in \mathbb{R}^n \times \mathbb{R}^k$, the control $(u, v) \in \mathbb{R}^m \times \mathbb{R}^k$, and the output $\tilde{h}(x, z) \in \mathbb{R}^p \times \mathbb{R}^k$, is locally stabilizable in small time by means of a continuous static periodic time-varying state (resp. output) feedback law.

In the above definition system (2.9) with $k = 0$ is, by convention, system C. We also point out that it is proved in Section 3 of [C2] that many sufficient conditions for local controllability in small time and with small control imply that the system is locally stabilizable in small time by means of a continuous dynamic periodic time-varying state feedback law; this also follow from [C4]—but the proof given in Section 3 of [C2], which gives a weaker result, is much simpler than the one given in [C4].

For our last definition we need to introduce some notation. For α in $\mathbb{N}^m$ and $\bar{u}$ in $\mathbb{R}^m$ let $f_{\bar{u}}^{\alpha}$ in $C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ be defined by

$$f_{\bar{u}}^{\alpha}(x) = \frac{\partial^{|\alpha|} f}{\partial u^\alpha}(x, \bar{u}), \qquad \forall x \in \mathbb{R}^n. \tag{2.10}$$

Let $\mathcal{O}(C)$ be the subspace of $C^\infty(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^p)$ spanned by the maps ω such that, for some integer $r \geq 0$—depending on ω—and for some sequence $\alpha_1, \ldots, \alpha_r$ of r multi-indices in $\mathbb{N}^m$,

$$\omega(x, u) = L_{f_u^{\alpha_1}} \cdots L_{f_u^{\alpha_r}} h(x), \qquad \forall x \in \mathbb{R}^n, \quad \forall u \in \mathbb{R}^m, \tag{2.11}$$

where $L_{f_u^{\alpha_i}}$ denotes Lie derivatives with respect to $f_u^{\alpha_i}$ and where, by convention, if $r = 0$, then the right-hand side of (2.11) is $h(x)$. With these notations our last definition is:

Definition 2.3. System C is locally Lie null-observable if a positive real number $\bar{\varepsilon}$ exists such that:

(i) For all a in $\mathbb{R}^n \backslash \{0\}$ such that $|a| < \bar{\varepsilon}$ there is q in $\mathbb{N}$ such that

$$L_{f_0}^q h(a) \neq 0 \tag{2.12}$$

with $f_0(x) = f(x, 0)$ and the usual convention $L_{f_0}^0 h = h$.

(ii) For all $(a_1, a_2) \in (\mathbb{R}^n \backslash \{0\})^2$ with $a_1 \neq a_2$, $|a_1| < \bar{\varepsilon}$, and $|a_2| < \bar{\varepsilon}$, and for all u in $\mathbb{R}^m$ with $|u| < \bar{\varepsilon}$, there is ω in $\mathcal{O}(C)$ such that

$$\omega(a_1, u) \neq \omega(a_2, u). \tag{2.13}$$

Note that (i) implies the following property:

(i)* For any $a \neq 0$ in $B_{\bar{\varepsilon}} := \{x \in \mathbb{R}^m, |x| < \bar{\varepsilon}\}$, a positive real number τ exists such that

$$x(\tau) \text{ exists and } \quad h(x(\tau)) \neq 0, \tag{2.14}$$

where $x(t)$ is defined by $\dot{x} = f(x, 0)$, $x(0) = a$.

Moreover, if f and g are analytic, (i)* implies (i). The reason of "null" in "null-observable" comes from condition (i) or (i)*: roughly speaking, we want to be able to distinguish from 0 any a in $B_{\bar{\varepsilon}} \backslash \{0\}$ by using the control law which vanishes identically.

When f is affine with respect to u, i.e., $f(x, u) = f_0(x) + \sum_{i=1}^m u_i f_i(x)$ with $f_0, \ldots, f_m$ in $C^\infty(\mathbb{R}^n; \mathbb{R}^n)$, then a slightly simpler version of (ii) can be given. Let $\tilde{\mathcal{O}}(C)$ be the observation space—see, e.g., [HK] or Remark 5.4.2 in [S3]—i.e., the set of maps $\tilde{\omega}$ in $C^\infty(\mathbb{R}^n; \mathbb{R}^p)$ such that, for some integer $r \geq 0$—depending on $\tilde{\omega}$—and for some sequence $i_1, \ldots, i_r$ of integers in $[0, m]$,

$$\tilde{\omega}(x) = L_{f_{i_1}} \cdots L_{f_{i_r}} h(x), \qquad \forall x \in \mathbb{R}^n, \tag{2.15}$$

with the convention that if $r = 0$, then the right-hand side of (2.15) is $h(x)$. Then (ii) is equivalent to

$$((a_1, a_2) \in B_{\bar{\varepsilon}}^2, \tilde{\omega}(a_1) = \tilde{\omega}(a_2), \qquad \forall \tilde{\omega} \in \tilde{\mathcal{O}}(C)) \quad \Rightarrow \quad (a_1 = a_2). \tag{2.16}$$

Finally, we remark that if f is a polynomial with respect to u or if f and g are analytic, then (ii) is equivalent to:

(ii)* For all $(a_1, a_2) \in \mathbb{R}^n \backslash \{0\}$ with $a_1 \neq a_2$, $|a_1| < \varepsilon$, and $|a_2| < \varepsilon$, there are u in $\mathbb{R}^m$ and ω in $\mathcal{O}(C)$ such that (2.13) holds.

Indeed in these cases the subspace of $\mathbb{R}^p$ spanned by $\omega(x, u)$, $\omega \in \mathcal{O}(C)$, does not depend on u: it is the observation space of C evaluated at x—as defined for example in [HK].

Our main result, which is proved in Section 3, is:

Theorem 2.4. *Assume that system C is locally stabilizable in small time by means of a continuous dynamic periodic time-varying state feedback law. Assume that system C is locally Lie null-observable. Then system C is locally stabilizable in small time by means of a continuous dynamic periodic time-varying output feedback law.*

Clearly, local stabilizability in small time by means of a continuous dynamic periodic time-varying state feedback law is a necessary condition for local stabilizability in small time by means of a continuous dynamic periodic time-varying output feedback law. Our next proposition, which is proved in Section 4, shows that, in the analytic case, Lie null-observability is also a necessary condition, i.e.:

Proposition 2.5. *Assume that f and h are analytic. Assume that system C is locally stabilizable in small time by means of a continuous dynamic periodic time-varying output feedback law. Then system C is locally Lie null-observable.*

We recall that in [TP] Teel and Praly have obtained a result closely related to Theorem 2.4; the main difference is that they use a different observability assumption: they assume the existence of two integers n_y, n_u and of a smooth function φ such that, for each solution of $\dot{x} = f(x, u_0)$, $\dot{u}_0 = u_1, \ldots, \dot{u}_{n_u} = v$, then, for all t such that $x(t)$ is defined,

$$x(t) = \varphi(y(t), \ldots, y^{(n_y)}(t), u_0(t), \ldots, u_{n_u}(t)). \tag{2.17}$$

Taking $u_1 \equiv \cdots \equiv u_{n_u} \equiv 0$ it is easily seen that this observability condition implies the Lie null-observability condition. However, in general, the converse is false. For example, let us take

$$n = m = 1, \qquad f(x, u) = uf_1 = u, \qquad y = h(x) = x^2; \tag{2.18}$$

then (2.17) does not hold (take $u_0 \equiv u_1 \equiv \cdots \equiv u_{n_u} \equiv 0$). Nevertheless, the Lie null-observability condition holds; indeed, $h(x) = 0$ if and only $x = 0$, so (i) holds and $L_{f_1}h(a_1) = L_{f_1}h(a_2)$ implies $a_1 = a_2$, so (2.16) and therefore (ii) hold. Let us emphasize that [TP] also deals with stationary—i.e., independent of time—feedback, which is not the case of Theorem 2.4; on the other hand, let us point out that the system defined by (2.18) shows that the stationary version of Theorem 2.4 in fact does not hold. Indeed, $\dot{x} = u \in \mathbb{R}$ can be locally stabilized in small time by means of a continuous stationary state feedback (choose, for example, $u(x) = -x^{1/3}$). However, $\dot{x} = u$, $y = x^2$ cannot be stabilized in small time or even asymptotically by means of a continuous stationary dynamic output feedback law. Indeed assume that, for some integer $k \geq 1$, there are $u \in C^0(\mathbb{R} \times \mathbb{R}^k; \mathbb{R})$ and $v \in C^0(\mathbb{R} \times \mathbb{R}^k; \mathbb{R}^k)$ such that

$$u(0, 0) = 0, \qquad v(0, 0) = 0, \tag{2.19}$$

$$(0, 0) \text{ is an asymptotically stable point of } \quad \dot{x} = u(x^2, z), \quad \dot{z} = v(x^2, z). \tag{2.20}$$

Then, by a theorem due to Krasnosel'skiĭ [KZ, Theorem 52.1], [Z]—see [C5] when one does not have uniqueness of the solutions to the Cauchy problem—the index of $F: (x, z) \to (u(x^2, z), v(x^2, z))$ at $(0, 0)$ is $(-1)^{k+1}$; but since $F(-x, z) = F(x, z)$ this index is either not defined or is zero. The case $k = 0$ can be treated similarly—then there is no v and $\mathbb{R} \times \mathbb{R}^k$ is replaced by $\mathbb{R}$. Theorem 2.4 applied to the system defined by (2.18) implies that this system is stabilizable in small time by means of a continuous dynamic periodic time-varying output feedback law. This shows the interest of a time-varying output feedback law compared with a stationary output feedback law, even if the system is stabilizable by means of a *stationary* state feedback law. For the interest of a time-varying state feedback law compared with a stationary state feedback law let us recall the pioneer papers by Sontag and Sussmann [SS] and by Samson [S1]; see also [C1], [C2], and [C4].

Concerning the proof of Theorem 2.4, let us emphasize that we use an idea due to Lozano [L] and Mazenc and Praly [MP]: as in [L] and [MP] we first recover the state from the output. A related idea is also used in Section 3 of [C2], where we first recover initial data from the state. Moreover, as in [MP] our proof relies on the existence of an output feedback which distinguishes every pair of distinct states. In [MP] it is established that distinguishability with a universal time-varying control, global stabilizability by state feedback, and observability of blow-up are sufficient conditions for the existence of a time-varying dynamic—of infinite dimension and in a sense more general than the one considered in Definition 2.2—output feedback guaranteeing boundedness and convergence of all the solutions defined at time $t = 0$; the methods developed in [MP] can be applied directly to our situation; in this case Theorem 2.4 gives two improvements: we get that 0 is asymptotically stable for the closed-loop system, instead of only an attractor for time 0, and our dynamical extension is of finite dimension, instead of infinite dimension. We remark that it follows from our proof of Theorem 2.4 that it suffices to consider the dynamic extension of dimension $n + (2n + 1)p$, i.e., under the assumption of Theorem 2.4, system (2.9) with $k = n + (2n + 1)p$ is locally stabilizable in small time by means of a continuous static periodic time-varying output feedback law. Note that this number k is independent of the size of the extension required in order to have local stabilizability in small time by means of a continuous static periodic time-varying state feedback law. We conjecture that, as in the linear case, this result still holds for $k = n - 1$.

We also point out that our method of proving Theorem 2.4 can also be used to obtain similar results for semiglobal stabilization and for stabilization in finite time. For semiglobal stabilization instead of local stabilization in Definition 2.1 "there are ε in $(0, +\infty)$ and u in ..." is replaced by "for any ε in $(0, +\infty)$ there is u in ..." and in Definition 2.3 it is required that (i) and (ii) are satisfied for $\varepsilon = +\infty$. Note that for global stabilization Theorem 2.4 does not hold due to some blow-up phenomena; see, e.g., [MPD] and the references therein. In the case of semiglobal stabilization we avoid blow-up, which might appear in Step 1—see Section 3—by taking $|v| \leq 1$ and T small enough. For local stabilization in finite time—instead of small time—in Definition 2.1 "for any positive real number T..." is replaced by "for some positive real number T ..."; for this case, see the end of Section 3.

We finally remark that $\dot{x} = f(x, u)$ is locally stabilizable in small time by means of a continuous dynamic periodic state feedback law if and only if 0 is locally continuously reachable in small time and with a small control in the sense of Definition 1.1 of [C4], i.e., for any positive real number T, there are u in $C^0(\mathbb{R}^n; L^1((0, T); \mathbb{R}^n))$ and a positive real number ε such that

$$\sup\{|u(a)(t)|; t \in [0, T]\} \to 0 \qquad \text{as} \quad a \to 0,$$
$$(\dot{x} = f(x, u(x(0), t)), |x(0)| < \varepsilon) \quad \Rightarrow \quad x(T) = 0.$$

The "if" part follows from Section 3 of [C3] and Lemma 2.3 of [C4]; the "only if" part is, as we will see, a consequence of Lemma 3.5 below—consider u defined by (3.67).

3. Proof of Theorem 2.4

3.1. Sketch of Proof

We assume that the assumptions of Theorem 2.4 are satisfied. Let T be a positive real number. Our proof of Theorem 2.4 is divided into three steps.

Step 1. Using the assumption that system C is locally Lie null-observable we prove, using [C3], that u^* in $C^\infty(\mathbb{R}^p \times [0, T]; \mathbb{R}^m)$ and a positive real number ε^* exist such that

$$u^*(y, T) = u^*(y, 0) = 0, \qquad \forall y \in \mathbb{R}^p, \tag{3.1}$$
$$u^*(0, t) = 0, \qquad \forall t \in [0, T], \tag{3.2}$$

and, for all (a_1, a_2) in $B^2_{\varepsilon^*}$, for all s in $(0, T)$,

$$(h^{(i)}_{a_1}(s) = h^{(i)}_{a_2}(s), \forall i \in \mathbb{N}) \quad \Rightarrow \quad (a_1 = a_2), \tag{3.3}$$

where $h_a(s) = h(x^*(a, s))$ with x^* defined by $\partial x^*/\partial t = f(x^*, u^*(h(x^*), t))$, $x^*(a, 0) = a$. Note that in [MP] a similar u^* was considered, but it was taken depending only on time and so (3.2), which is important to achieve stability, was, in general, not satisfied. In this step we do not use any stabilizability property of C.

Step 2. Let $q = 2n + 1$. In this step, using (3.3), we prove the existence of $(q + 1)$ real numbers $0 < t_0 < t_1 < \cdots < t_q < T$ such that the map K: $B_{\varepsilon^*} \to (\mathbb{R}^p)^q$ defined by

$$K(a) = \left(\int_{t_0}^{t_1} (s - t_0)(t_1 - s)h_a(s)\, ds, \ldots, \int_{t_0}^{t_q} (s - t_0)(t_q - s)h_a(s)\, ds\right) \tag{3.4}$$

is one-to-one and so, as we will see, a map θ: $(\mathbb{R}^p)^q \to \mathbb{R}^n$ exists such that

$$\theta \circ K(a) = x^*(a, T), \qquad \forall a \in B_{\varepsilon^*/2}. \tag{3.5}$$

Step 3. In this step we prove the existence of $\bar{u}$ in $C^0(\mathbb{R}^n \times [0, T]; \mathbb{R}^m)$ and $\bar{\varepsilon}$ in $(0, +\infty)$ such that

$$\bar{u} = 0 \qquad \text{on} \quad (\mathbb{R}^n \times \{0, T\}) \cup (\{0\} \times [0, T]), \tag{3.6}$$
$$(\dot{x} = f(x, \bar{u}(x(0), t)) \text{ and } |x(0)| < \bar{\varepsilon}) \quad \Rightarrow \quad (x(T) = 0). \tag{3.7}$$

Property (3.7) means that $\bar{u}$ is a "dead-beat" open-loop control. In this last step we use the stabilizability assumption on C, but do not use the Lie null-observability assumption.

Using these three steps let us end the proof of Theorem 2.4. The dynamic extension of system C that we consider is

$$\dot{x} = f(x, u), \qquad \dot{z} = v = (v_1, \ldots, v_q, v_{q+1}) \in \mathbb{R}^p \times \cdots \times \mathbb{R}^p \times \mathbb{R}^n \simeq \mathbb{R}^{pq+n}, \quad (3.8)$$

with $z_1 = (z_1, \ldots, z_q, z_{q+1}) \in \mathbb{R}^p \times \cdots \times \mathbb{R}^p \times \mathbb{R}^n \simeq \mathbb{R}^{pq+n}$. For this system the output is $\tilde{h}(x, z) = (h(x), z) \in \mathbb{R}^p \times \mathbb{R}^{pq+n}$. For $s \in \mathbb{R}$ let $s^+ = \max(s, 0)$ and let $\operatorname{sgn} s = 1$ if $s > 0$, 0 if $s = 0$, -1 if $s < 0$. Finally, for r in $\mathbb{N}\backslash\{0\}$ and $b = (b_1, \ldots, b_r)$ in $\mathbb{R}^r$, let

$$b^{1/3} = (|b_1|^{1/3} \operatorname{sgn} b_1, \ldots, |b_r|^{1/3} \operatorname{sgn} b_r). \quad (3.9)$$

We now define u: $\mathbb{R}^p \times \mathbb{R}^{pq+n} \times \mathbb{R} \to \mathbb{R}^m$ and v: $\mathbb{R}^p \times \mathbb{R}^{pq+n} \times \mathbb{R} \to \mathbb{R}^{pq+n}$ by requiring, for (y, z) in $\mathbb{R}^p \times \mathbb{R}^{pq+n}$ and for all i in $[1, q]$, that

$$u(y, z, t) = u^*(y, t), \qquad \forall t \in [0, T), \quad (3.10)$$

$$v_i(y, z, t) = -t(t_0 - t)^+ z_i^{1/3} + (t - t_0)^+(t_i - t)^+ y, \qquad \forall t \in [0, T), \quad (3.11)$$

$$v_{q+1}(y, z, t) = -t(t_q - t)^+ z_{q+1}^{1/3} + 6\frac{(T - t)^+(t - t_q)^+}{(T - t_q)^3}\theta(z_1, \ldots, z_q), \quad (3.12)$$

$$u(y, z, t) = \bar{u}(z_{q+1}, t - T), \qquad \forall t \in [T, 2T), \quad (3.13)$$

$$v(y, z, t) = 0, \qquad \forall t \in [T, 2T), \quad (3.14)$$

$$u(y, z, t) = u(y, z, t + 2T), \qquad \forall t \in \mathbb{R}, \quad (3.15)$$

$$v(y, z, t) = v(y, z, t + 2T), \qquad \forall t \in \mathbb{R}. \quad (3.16)$$

Roughly speaking, the strategy used is the following:

(i) During the interval of time $[0, T]$, system C is "excited" by means of $u^*(y, t)$ in order to deduce from the observation during this interval of time what is the state at time T: at time T we have $z_{q+1} = x$.

(ii) During the interval of time $[T, 2T]$, z_{q+1} does not move and the dead-beat open-loop $\bar{u}$, but tranformed into an output feedback by using z_q in its argument instead of the value of x at time T, is used—this method has been used previously in the proof of Theorem 1.7 of [C2].

In a context of adaptative control, a similar strategy has been used later by Kreisselmeier and Lozano [KL].

It is easily seen that u and v are continuous and vanish on $\{(0, 0)\} \times \mathbb{R}$. Let (x, z) be any maximal solution of the closed-loop system $\dot{x} = f(x, u(\tilde{h}(x, z), t))$, $\dot{z} =$

$v(\tilde{h}(x, z), t)$; then it is easily checked that, if $|x(0)| + |z(0)|$ is small enough,

$$z_i(t_0) = 0, \qquad \forall i \in [1, q], \tag{3.17}$$

$$(z_1(t), \ldots, z_q(t)) = K(x(0)), \qquad \forall t \in [t_q, T], \tag{3.18}$$

$$z_{q+1}(t_q) = 0, \tag{3.19}$$

$$z_{q+1}(T) = \theta \circ K(x(0)) = x(T), \tag{3.20}$$

$$x(t) = 0, \qquad \forall t \in [2T, 3T], \tag{3.21}$$

$$z(2T + t_q) = 0. \tag{3.22}$$

Equalities (3.17) (resp. (3.19)) are proved by computing explicitly, for $i \in [1, q]$, z_i on $[0, t_0]$ (resp. z_{q+1} on $[0, t_q]$), by seeing that this explicit solution reaches 0 before time t_0 (resp. t_q), and by pointing out that if, for some s in $[0, t_0]$ (resp. $[0, t_q]$), $z_i(s) = 0$ (resp. $z_{q+1}(s) = 0$), then $z_i = 0$ on $[s, t_0]$ (resp. $z_{q+1} = 0$ on $[s, t_q]$)—note that $z_i \dot{z}_i \leq 0$ on $[0, t_0]$ (resp. $z_{q+1}\dot{z}_{q+1} \leq 0$ on $[0, t_q]$).

Moreover, we also have, for all s in $\mathbb{R}$ and all $t \geq s$,

$$((x(s), z(s)) = (0, 0)) \quad \Rightarrow \quad ((x(t), z(t)) = (0, 0)). \tag{3.23}$$

Indeed, first note that without loss of generality we may assume $s \in [0, 2T]$ and $t \in [0, 2T]$. If $s \in [0, T]$, then, since u^* is of class C^∞, we get, using (3.2), that $x(t) = 0$, $\forall t \in [s, T]$, and then, using (2.3) and (3.11), get that, for all $i \in [1, q]$, $z_i \dot{z}_i \leq 0$ on $[s, T]$ and so z_i also vanishes on $[s, T]$; this, with (3.12) and $\theta(0) = 0$—see (3.4) and (3.5)—implies that $z_{q+1} = 0$ also on $[s, T]$. Hence we may assume that $s \in [T, 2T]$. However, in this case, using (3.14), we get that $z = 0$ on $[s, 2T]$ and, from (3.6) and (3.13), we get that $x = 0$ also on $[s, 2T]$.

From (3.21)–(3.23) we get—see Lemma 2.15 in [C4]—the existence of ε in $(0, +\infty)$ such that, for any s in $\mathbb{R}$ and any maximal solution (x, z) of $\dot{x} = f(x, u(\tilde{h}(x, z), t))$, $\dot{z} = v(\tilde{h}(x, z), t)$, we have

$$(|x(s)| + |y(s)| \leq \varepsilon) \quad \Rightarrow \quad ((x(t), z(t)) = (0, 0), \forall t \geq s + 5T). \tag{3.24}$$

Since T is arbitrary, Theorem 2.4 is proved. It remains to perform Steps 1–3.

3.2. *Step 1: Learning Excitation* (*Existence of* u^*)

We deduce the existence of u^* from Corollary 1.15 of [C3]. We mention that, if f is analytic, alternatively, [S5] or [WS] can be used. Let $F \in C^\infty(\mathbb{R}^n \times \mathbb{R}^{pm}; \mathbb{R}^n)$ be defined by

$$F(x, v) = f\left(x, \sum_{i=1}^{p} h_i(x) v^i\right), \qquad \forall x \in \mathbb{R}^n, \quad \forall v = (v^1, \ldots, v^p) \in (\mathbb{R}^m)^p \simeq \mathbb{R}^{pm}. \tag{3.25}$$

Following Section 4.1 of [WS], we define the control system $\mathscr{C}$:

$$\dot{z} = \mathscr{F}(z, v), \qquad y = \mathscr{H}(z), \tag{3.26}$$

where the state is $z = (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n$, the control $v = (v^1, \ldots, v^p) \in (\mathbb{R}^m)^p \simeq \mathbb{R}^{pm}$, the observation $y \in \mathbb{R}^p$, and where the maps $\mathscr{F} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{pm}; \mathbb{R}^n \times \mathbb{R}^n)$ and

$\mathscr{H} \in \mathscr{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^p)$ are defined by

$$\mathscr{F}(x_1, x_2, v) = (F(x_1, v), F(x_2, v)), \qquad \forall (x_1, x_2, v) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{pm}, \quad (3.27)$$

$$\mathscr{H}(x_1, x_2) = h(x_2) - h(x_1), \qquad \forall (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n. \quad (3.28)$$

We assume:

Lemma 3.1. *Let δ in $(0, \bar{\varepsilon})$ be such that, with $U = \{v \in \mathbb{R}^{pm}; |v| < \delta\}$,*

$$\left| \sum_{i=1}^{p} h_i(x) v^i \right| < \bar{\varepsilon}, \qquad \forall x \in B_\delta, \qquad \forall v \in U. \quad (3.29)$$

Then, for any $z = (x_1, x_2) \in B_\delta \times B_\delta$ and for any v in $\mathbb{R}^{pm}$ with $|v| < \delta$, we have

$$(\omega^*(z, v) = 0, \forall \omega^* \in \mathcal{O}(\mathscr{C})) \quad \Rightarrow \quad (x_1 = x_2). \quad (3.30)$$

Then, by Corollary 1.15 in [C3], for a generic v in $C^\infty((0, T); U)$—for the modified version of the Whitney C^∞ topology proposed in [C3]—we have, with $Z(z, t) = \mathscr{F}(z, v(t))$ and for all $z = (x_1, x_2) \in B_\delta^2$ with $x_1 \neq x_2$,

$$\left\{ \left(\left(\frac{\partial}{\partial t} + L_Z \right)^k \mathscr{H} \right)(z); k \geq 0 \right\} \neq \{0\}, \qquad \forall t \in (0, T). \quad (3.31)$$

Recall that, for the C^∞ topology constructed in [C3], the set of v in $C^\infty((0, T); U)$ such that

$$|v^{(k)}(t)| < t(T - t), \qquad \forall t \in (0, T), \quad \forall k \in \mathbb{N}, \quad (3.32)$$

is a—nonempty—open subset of $C^\infty((0, T); U)$. So, by the above genericity, a map v^* in $C^\infty((0, T); U)$ exists such that (3.31) and (3.32) hold. We extend v^* to $[0, T]$ by requiring

$$v^* = 0 \qquad \text{on } \{0, T\}. \quad (3.33)$$

We still denote this extension by v^*. By (3.32), $v^* \in C^\infty([0, T]; U)$. Let u^* in $C^\infty(\mathbb{R}^n \times [0, T]; \mathbb{R}^m)$ be defined by

$$u^*(y, t) = \sum_{i=1}^{p} y_i v^{*i}(t). \quad (3.34)$$

Then (3.2) holds and (3.1) follows from (3.33). We now choose ε^* small enough—but positive—so that

$$(|a| < \varepsilon^*) \quad \Rightarrow \quad (|x^*(a, t)| < \delta, \forall t \in [0, T]) \quad (3.35)$$

with x^* defined by (3.4). Note that

$$(a_1 \neq a_2) \quad \Rightarrow \quad (x^*(a_1, t) \neq x^*(a_2, t), \forall t \in [0, T]). \quad (3.36)$$

Moreover, it is easily checked that, for any (a_1, a_2, τ) in $\mathbb{R}^n \times \mathbb{R}^n \times [0, T]$,

$$h_{a_2}^{(k)}(\tau) - h_{a_1}^{(k)}(\tau) = \left(\left(\frac{\partial}{\partial t} + L_{Z^*} \right)^k \right) \mathscr{H}(x^*(a_1, \tau), x^*(a_2, \tau)), \qquad \forall k \in \mathbb{N}, \quad (3.37)$$

with $Z^*(z, t) = \mathscr{F}(z, v^*(t))$. Hence (3.3) follows from (3.31), (3.35), and (3.36). Lemma 3.1 is proved in the Appendix. This ends Step 1.

3.3. *Step 2: Learning Sampling* (*Existence of* $0 < t_0 < t_1 < \cdots < t_{q+1} < T$ *such that K is One-to-One*)

Throughout this step "manifold" always means the finite-dimensional Hausdorff second countable manifold of class C^∞. A subset S of a manifold is called C^∞-rectifiable if a manifold Y and a map $\varphi \in C^\infty(Y, M)$ exist such that $\varphi(Y) = S$. We say that the dimension of S is $\leq r$ if Y can be chosen so that the dimension of Y is $\leq r$. Note that the union of any countable family whose members are C^∞-rectifiable subsets of M of dimension $\leq r$ is a C^∞-rectifiable subset of M of dimension $\leq r$ and that a C^∞-rectifiable subset S of an open subset Ω of $\mathbb{R}^n$, whose dimension is $\leq n - 1$, is such that

$$\Omega \backslash S \text{ is residual,} \tag{3.38}$$

i.e., $\Omega \backslash S$ is the union of the countably many open and dense subsets of Ω, and

$$S \text{ has a zero Lebesgue measure.} \tag{3.39}$$

We assume:

Lemma 3.2. *Let N be a manifold of dimension r. Let $g \in C^\infty(N \times (0, T); \mathbb{R}^p)$. Assume that, for any x_0 in N and any t_0 in $(0, T)$, an integer i_0 (which may depend on x_0 and t_0) exists such that*

$$\frac{\partial^{i_0} g}{\partial t^{i_0}}(x_0, t_0) \neq 0. \tag{3.40}$$

Then a C^∞-rectifiable subset S of $(0, T)^{r+1}$ of dimension $\leq r$ exists such that, for all $t = (t_1, \ldots, t_{r+1})$ in $(0, T)^{r+1} \backslash S$,

$$\{x \in N; g(x, t_1) = \cdots = g(x, t_{r+1}) = 0\} = \varnothing. \tag{3.41}$$

Let us apply this lemma with $N = \{x = (x_1, x_2) \in B_{\varepsilon^*} \times B_{\varepsilon^*}; x_1 \neq x_2\}$ and, for some t_0 in $(0, T)$,

$$g(x, t) = \int_{t_0}^{t} (s - t_0)(t - s)(h_{x_2}(s) - h_{x_1}(s))\, ds. \tag{3.42}$$

Note that

$$\frac{\partial^2 g}{\partial t^2}(x, t) = (t - t_0)(h_{x_2}(t) - h_{x_1}(t)), \qquad \forall (x, t) \in N \times (0, T), \tag{3.43}$$

and that, for all integers $i \geq 3$,

$$\frac{\partial^i g}{\partial t^i}(x, t) = (t - t_0)(h_{x_2}^{(i-2)}(t) - h_{x_1}^{(i-2)}(t)) + (i - 2)(h_{x_2}^{(i-3)}(t) - h_{x_1}^{(i-3)}(t)). \tag{3.44}$$

Hence (3.40) follows from (3.3). So we get from Lemma 3.2 the existence of a C^∞-rectifiable subset S of $(0, T)^q$—with $q = 2n + 1$—of dimension $\leq 2n$ such that, for all $t = (t_1, \ldots, t_q)$ in $(0, T)^q \backslash S$, (3.41) holds, which implies that the map $K\colon B_{\varepsilon*} \to (\mathbb{R}^p)^q$ defined by (3.4) is one-to-one. Since S has a zero Lebesgue measure in $(0, T)^q$—see (3.39)—t in $(0, T)^q \backslash S$ exists such that $t_0 < t_1 < \cdots < t_q$.

We now deduce, from the fact that K is one-to-one, the existence of θ in $C^0((\mathbb{R}^p)^q; \mathbb{R}^n)$ such that (3.5) holds. Let $Q = K(\bar{B}_{\varepsilon*/2})$. Since K is one-to-one a map $\bar{\theta}\colon Q \to \bar{B}_{\varepsilon*/2}$ exists such that

$$\bar{\theta}(K(a)) = a, \qquad \forall a \in \bar{B}_{\varepsilon*/2}. \tag{3.45}$$

Since $\bar{B}_{\varepsilon*/2}$ is compact, the continuity of K implies the continuity of $\bar{\theta}$. Let $\tilde{\theta}\colon Q \to \mathbb{R}^n$ be defined by

$$\tilde{\theta}(b) = x^*(\bar{\theta}(b), T). \tag{3.46}$$

This map $\tilde{\theta}$ is continuous and since Q is closed, by the Tietze–Urysohn extension theorem, a map θ in $C^0((\mathbb{R}^p)^q; \mathbb{R}^n)$ equal to $\tilde{\theta}$ on Q exists, which, with (3.45) and (3.46), satisfies (3.5).

Finally, we prove Lemma 3.2. Clearly, we may assume $N = \mathbb{R}^r$ and it suffices to show that, for any $(\bar{x}, \bar{t})$ in $\mathbb{R}^r \times (0, T)^{r+1}$, there are $\varepsilon > 0$ and a C^∞-rectifiable subset S of $\mathbb{R}^{r+1}$ of dimension $\leq r$ such that

$$\tilde{S} := \{t \in B_\varepsilon(\bar{t}); \exists x \in B_\varepsilon(\bar{x}) \text{ s.t. } g(x, t_1) = \cdots = g(x, t_{r+1}) = 0\} \subset S, \tag{3.47}$$

where $B_\varepsilon(\bar{t}) = \{t \in (0, T)^{r+1}; |t - \bar{t}| < \varepsilon\}$ and $B_\varepsilon(\bar{x}) = \{x \in \mathbb{R}^r; |x - \bar{x}| < \varepsilon\}$. Clearly, we may assume $g(\bar{x}, \bar{t}_1) = \cdots = g(\bar{x}, \bar{t}_{r+1}) = 0$—otherwise we can take $S = \varnothing$. By (3.41), for any i in $[1, r + 1]$, a positive integer k_i exists such that

$$\frac{\partial^j g}{\partial t^j}(\bar{x}, \bar{t}_i) = 0, \quad \forall j \in [0, k_i - 1], \qquad \frac{\partial^{k_i} g}{\partial t^{k_i}}(\bar{x}, \bar{t}_i) \neq 0. \tag{3.48}$$

So, by the Malgrange preparation theorem, see, e.g., p. 95 of [GG], integers $(l_i; i \in [1, r + 1])$ in $[1, q]$, functions $(a_i^j; i \in [1, r + 1], j \in [0, k_i - 1])$ in $C^\infty(\mathbb{R}^r; \mathbb{R})$, functions $(q_i; i \in [1, r + 1])$ in $C^\infty(\mathbb{R}^r \times \mathbb{R}; \mathbb{R}\backslash\{0\})$, and ε in $(0, +\infty)$ exist such that, for any i in $[1, r + 1]$, any τ in $(\bar{t}_i - \varepsilon, \bar{t}_i + \varepsilon) \cap (0, T)$, and any x in $B_\varepsilon(\bar{x})$,

$$q_i(x, \tau) g_{l_i}(x, \tau) = (\tau - \bar{t}_i)^{k_i} + \sum_{j=0}^{k_i - 1} a_i^j(x)(\tau - \bar{t}_i)^j. \tag{3.49}$$

Therefore,

$$\tilde{S} \subset \left\{t \in \mathbb{R}^{r+1}; \exists x \in \mathbb{R}^r \text{ s.t. } (t_i - \bar{t}_i)^{k_i} + \sum_{j=0}^{k_i - 1} a_i^j(x)(t_i - \bar{t}_i)^j = 0, \forall i \in [1, r + 1]\right\}. \tag{3.50}$$

For $i \in [1, r + 1]$, let

$$M_i = \left\{(\tau, b_0, \ldots, b_{k_i - 1}) \in \mathbb{R}^{1 + k_i}; (\tau - \bar{t}_i)^{k_i} + \sum_{j=0}^{k_i - 1} b_j(\tau - \bar{t}_i)^j = 0\right\}, \tag{3.51}$$

and let $\pi_i\colon M_i \to \mathbb{R}^{k_i}$ be defined by $\pi_i(\tau, b) = b$, with $b = (b_0, \ldots, b_{k_i - 1})$. Since M_i is an algebraic subset of $\mathbb{R}^{1 + k_i}$ and π_i is a polynomial it is well known that π_i can be stratified, see, e.g., Section I.1.7 of [GM]—note that in the semialgebraic case the properness of the map is not needed; so a finite number of smooth submanifolds of

$\mathbb{R}^{k_i}$ $(M_{i,j}; j \in [1, s_i])$ and a finite number of maps $(\mu_{i,j}; j \in [1, s_i])$ exist such that

$$u_{i,j} \in C^\infty(M_{i,j}; \mathbb{R}), \qquad \forall i \in [1, r+1], \quad \forall j \in [1, s_i],$$
$$M_i = \bigcup_{j \in [1, s_i]} \{(\mu_{i,j}(b), b); b \in M_{i,j}\}, \qquad \forall i \in [1, r+1]. \tag{3.52}$$

Let $J = \prod_{i=1}^{r+1} [1, s_i]$. For $j = (j_1, \ldots, j_{r+1}) \in J$, let

$$\tilde{S}(j) = \{(\mu_{1,j_1}(a_1(x)), \ldots, \mu_{r+1,j_{r+1}}(a_{r+1}(x))); x \in \mathbb{R}^r \text{ s.t. } a_i(x) \in M_{i,j_i}, \forall i \in [1, r+1]\}, \tag{3.53}$$

where $a_i(x) = (a_i(x), \ldots, a_i^{k_i-1}(x))$. Then, by (3.50), (3.51), and (3.52), $\tilde{S} \subset \bigcup_{j \in J} \tilde{S}(j)$. So Lemma 3.2 follows from the following lemma.

Lemma 3.3. *Let M be a submanifold of $\mathbb{R}^k$, let $\mu \in C^\infty(M; \mathbb{R}^q)$, and let $a \in C^\infty(\mathbb{R}^r; \mathbb{R}^k)$. Then $\{\mu(a(x)); x \in \mathbb{R}^n, a(x) \in M\}$ is included in a C^∞-rectifiable subset of $\mathbb{R}^q$ of dimension $\leq r$.*

Proof. Clearly, a sequence $(M^i; i \in \mathbb{N})$ of subsets of M exists such that

$$\bigcup_{i \in \mathbb{N}} M^i = M \tag{3.54}$$

and, for any i in $\mathbb{N}$, there is $P_i \in C^\infty(\mathbb{R}^k; M)$ such that

$$P_i(x) = x, \qquad \forall x \in M^i. \tag{3.55}$$

From (3.54) and (3.55) we get

$$\{\mu(a(x)); x \in \mathbb{R}^r, a(x) \in M\} \subset \bigcup_{i \in \mathbb{N}} \{\mu \circ P_i \circ a_i(x); x \in \mathbb{R}^r\}, \tag{3.56}$$

which gives the conclusion of Lemma 3.3 and ends Step 2. ■

3.4. *Step 3: Dead-Beat Open-Loop Control (Existence of $\bar{u}$)*

In this step we prove the existence of $\bar{u}$ in $C^0(\mathbb{R}^n \times [0, T]; \mathbb{R}^m)$ and $\bar{\varepsilon}$ in $(0, +\infty)$ such that (3.6) and (3.7) hold.

It suffices to treat the case where system C can be locally stabilized in small time by means of a continuous static periodic time-varying state feedback law. Indeed, if, for the control system

$$\dot{x} = f(x, u), \qquad \dot{z} = v, \tag{3.57}$$

there is an open-loop control of $(u^\#, v^\#)$ in $C^0(\mathbb{R}^n \times \mathbb{R}^k \times [0, T]; \mathbb{R}^m \times \mathbb{R}^k)$ and $\varepsilon^\#$ in $(0, +\infty)$ such that

$$(u^\#, v^\#) = (0, 0) \quad \text{on} \quad (\mathbb{R}^n \times \mathbb{R}^k \times \{0, T\}) \cup (\{(0, 0)\} \times [0, T]), \tag{3.58}$$

$$(\dot{x} = f(x, u^\#(x(0), z(0), t)), \dot{z} = v^\#(x(0), z(0), t), |x(0)| + |z(0)| < \varepsilon^\#)$$
$$\Rightarrow \quad (x(T), z(T)) = 0,$$

then, if we define $\bar{u}$ in $C^0(\mathbb{R}^n \times [0, T]; \mathbb{R}^m)$ by

$$\bar{u}(a, t) = u^{\#}(a, 0, t), \qquad \forall a \in \mathbb{R}^n, \quad \forall t \in [0, T], \tag{3.59}$$

and, if we let $\bar{\varepsilon} = \varepsilon^{\#}$, we have (3.6) and (3.7).

We introduce the following definition—see also Section 1.1 in [S4]:

Definition 3.4. System C is locally stabilizable in small time by means of an almost smooth static periodic time-varying state feedback law if, for any positive real number T, there are ε in $(0, +\infty)$ and u in $C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m) \cap C^\infty((\mathbb{R}^n \backslash \{0\}) \times \mathbb{R}; \mathbb{R}^m)$ such that (2.4)–(2.7) hold.

Then we have the following lemma, which is in the spirit of the corollary in Section 7 of [S4]:

Lemma 3.5. *Assume system C is locally stabilizable in small time by means of a continuous static periodic time-varying state feedback law. Then it is locally stabilizable in small time by means of an almost smooth static periodic time-varying state feedback law. Moreover, we can impose the T-periodic feedback law to satisfy*

$$\partial^\alpha u = 0 \qquad on \quad (\mathbb{R}^n \backslash \{0\}) \times \{0, T\}, \quad \forall \alpha \in \mathbb{N}^{n+m}. \tag{3.60}$$

More precisely, for any positive real number T, there are u in $C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$ of class C^∞ on $(\mathbb{R}^n \backslash \{0\}) \times \mathbb{R}$ and ε in $(0, +\infty)$ such that (2.4)–(2.7) *and* (3.60) *hold.*

We assume that Lemma 3.5 holds. So $\tilde{u}$: $\mathbb{R}^n \times [0, T] \to \mathbb{R}^m$ and $\tilde{\varepsilon}$ in $(0, +\infty)$ exist such that

$$\tilde{u} \in C^0(\mathbb{R}^n \times [0, T]; \mathbb{R}^m) \cap C^\infty((\mathbb{R}^n \backslash \{0\}) \times [0, T]; \mathbb{R}^m), \tag{3.61}$$

$$\tilde{u} = 0 \qquad \text{on} \quad (\{0\} \times [0, T]) \cup (\mathbb{R}^n \times \{0, T\}), \tag{3.62}$$

$$((\dot{x} = f(x, \tilde{u}(x, t)), x(s) = 0) \Rightarrow (x(t) = 0, \forall t \in [s, T])), \qquad \forall s \in \left[\frac{T}{2}, T\right], \tag{3.63}$$

$$(\dot{x} = f(x, \tilde{u}(x, t)), |x(0)| < \tilde{\varepsilon}) \quad \Rightarrow \quad (\tilde{x}(T) = 0). \tag{3.64}$$

Note that, by (3.61), (3.63), and (3.64), for any a in $B_{\tilde{\varepsilon}}$ the Cauchy problem

$$\frac{\partial \tilde{x}}{\partial t} = f(\tilde{x}, \tilde{u}(\tilde{x}, t)), \qquad \tilde{x}(a, 0) = a, \tag{3.65}$$

has one and only one solution defined on $[0, T]$. Clearly,

$$\tilde{x} \in C^0(B_{\tilde{\varepsilon}} \times [0, T]; \mathbb{R}^n). \tag{3.66}$$

Let u: $B_{\tilde{\varepsilon}} \times [0, T] \to \mathbb{R}^m$ be defined by

$$u(a, t) = \tilde{u}(\tilde{x}(a, t), t), \qquad \forall (a, t) \in B_{\tilde{\varepsilon}} \times [0, T]. \tag{3.67}$$

By (3.61), (3.66), and (3.67), u is continuous. By (3.62), (3.63), (3.65), and (3.67),

$$u = 0 \qquad \text{on} \quad (\{0\} \times [0, T]) \cup (B_{\tilde{\varepsilon}} \times [0, T]) \tag{3.68}$$

and, by (3.64), (3.65), and (3.67),

$$(\dot{x} = f(x, u(x(0), t)), |x(0)| < \tilde{\varepsilon}) \quad \Rightarrow \quad (x(T) = 0). \tag{3.69}$$

Clearly, a map $\bar{u}$ in $C^0(\mathbb{R}^n \times [0, T]; \mathbb{R}^m)$ exists such that

$$\bar{u} = 0 \qquad \text{on} \quad (\{0\} \times [0, T]) \cup (\mathbb{R}^n \times \{0, T\}), \tag{3.70}$$

$$\bar{u} = u \qquad \text{on} \quad B_{\tilde{\varepsilon}/2} \times [0, T]. \tag{3.71}$$

By (3.69) and (3.71), (3.7) holds if $\bar{\varepsilon} = \tilde{\varepsilon}/2$.

It remains only to prove Lemma 3.5. Let T be a positive real number and let, for $i \in \mathbb{N}$, $T_i = T(1 - 2^{-i})$. By assumption, for any i in $\mathbb{N}$ there are ε_i in $(0, +\infty)$ and v_i in $C^0(\mathbb{R}^n \times [T_i, T_{i+1}]; \mathbb{R}^m)$ such that

$$v_i = 0 \qquad \text{on} \quad \{0\} \times [T_i, T_{i+1}], \tag{3.72}$$

$$((\dot{x} = f(x, v_i(x, t)), x(s) = 0) \Rightarrow (x(t) = 0, \forall t \in [s, T_{i+1}])), \qquad \forall s \in [T_i, T_{i+1}], \tag{3.73}$$

$$(\dot{x} = f(x, v_i(x, t)), |x(T_i)| < \varepsilon_i) \quad \Rightarrow \quad (x(T_{i+1}) = 0). \tag{3.74}$$

We fix $i \in \mathbb{N}$. Let δ be a positive real number and let $w_i^\delta \in C^\infty(\mathbb{R}^n \times [T_i, T_{i+1}]; \mathbb{R}^m)$ be such that

$$w_i^\delta = 0 \qquad \text{on} \quad \{0\} \times [T_i, T_{i+1}], \tag{3.75}$$

$$\sup\{|w_i^\delta - v_i|(x, t); (x, t) \in \mathbb{R}^m \times [T_i, T_{i+1}]\} < \delta. \tag{3.76}$$

Let $\beta \in C^\infty([0, +\infty); [0, 1])$ be such that

$$\beta = 0 \quad \text{on } [0, 1], \qquad \beta = 1 \quad \text{on } [1, +\infty), \tag{3.77}$$

and let v_i^δ: $\mathbb{R}^n \times [T_i, T_{i+1}] \to \mathbb{R}^m$ be defined by

$$v_i^\delta(x, t) = \beta\left(\frac{t - T_i}{\delta}\right)\beta\left(\frac{T_{i+1} - t}{\delta}\right)w_i^\delta(x, t). \tag{3.78}$$

Then $v_i^\delta \in C^\infty(\mathbb{R}^n \times [T_i, T_{i+1}]; \mathbb{R}^m)$ and satisfies

$$v_i^\delta = 0 \qquad \text{on} \quad \{0\} \times [T_i, T_{i+1}], \tag{3.79}$$

$$v_i^\delta = 0 \qquad \text{on a neighborhood of} \quad \mathbb{R}^n \times \{T_i, T_{i+1}\} \quad \text{in} \quad \mathbb{R}^n \times [T_i, T_{i+1}]. \tag{3.80}$$

Moreover, arguing by contradiction, we easily get from (3.73), (3.74), and (3.76)–(3.78) that if $\delta = \delta_i$ is small enough (depending on i), then

$$(\dot{x} = f(x, v_i^{\delta_i}(x, t)), |x(T_i)| < \varepsilon_i/2) \quad \Rightarrow \quad (|x(T_{i+1})| < \varepsilon_{i+1}/2). \tag{3.81}$$

Now let $\tilde{u}$: $\mathbb{R}^n \times [0, T) \to \mathbb{R}^m$ be defined by

$$\tilde{u}(x, t) = v_i^{\delta_i}(x, t), \qquad \forall (x, t) \in \mathbb{R}^n \times [T_i, T_{i+1}), \quad \forall i \in \mathbb{N}. \tag{3.82}$$

It follows from our construction, see in particular (3.79)–(3.82), that

$$\tilde{u} \in C^\infty(\mathbb{R}^n \times [0, T); \mathbb{R}^m), \tag{3.83}$$

$$\tilde{u} = 0 \qquad \text{on} \quad \{0\} \times [0, T), \tag{3.84}$$

$$\tilde{u} = 0 \qquad \text{on a neighborhood of} \quad \mathbb{R}^n \times \{0\} \quad \text{in} \quad \mathbb{R}^n \times [0, T], \tag{3.85}$$

$$(\dot{x} = f(x, \tilde{u}(x, t)), |x(0)| < \varepsilon_0/2) \quad \Rightarrow \quad (x(t) \to 0 \text{ as } t \to T). \tag{3.86}$$

In order to control $\tilde{u}$ near $\mathbb{R}^n \times \{T\}$, we remark that, decreasing ε_i if necessary and modifying, in an obvious way, v_i if necessary, we may assume

$$\sup\{|v_i(x, t)|; x \in \mathbb{R}^n, t \in [T_i, T_{i+1}]\} < 2^{-i}, \qquad \forall i \in \mathbb{N}. \tag{3.87}$$

On the other hand, clearly, we may choose δ_i so that

$$\delta_i < 2^{-i}, \qquad \forall i \in \mathbb{N}. \tag{3.88}$$

From (3.76), (3.78), (3.82), (3.87), and (3.83) we get

$$|\tilde{u}(x, t)| \leq 4(T - t), \qquad \forall (x, t) \in \mathbb{R}^n \times [0, T). \tag{3.89}$$

Now let $\tilde{x}: B_{\varepsilon_0/2} \times [0, T) \to \mathbb{R}^n$ be defined by

$$\frac{\partial \tilde{x}}{\partial t} = f(\tilde{x}, \tilde{u}(\tilde{x}, t)), \qquad \tilde{x}(a, 0) = a. \tag{3.90}$$

Let

$$K = \{(\tilde{x}(a, t), t); a \in \bar{B}_{\varepsilon_0/4}, t \in [0, T)\} \cup \{(0, T)\}. \tag{3.91}$$

From (3.86) and (3.89) we easily get

$$\left(\lim_{n \to +\infty} a_n = a \in \bar{B}_{\varepsilon_0/4}, \lim_{n \to +\infty} t_n = T\right) \Rightarrow \left(\lim_{n \to +\infty} \tilde{x}(a_n, t_n) = 0\right), \tag{3.92}$$

which implies that K is compact. Since $K \cap (\mathbb{R}^n \times \{T\}) = \{(0, T)\}$, by the compactness of K, a function $\varphi: \mathbb{R}^n \times [0, T] \to [0, 1]$ of class C^∞ on $(\mathbb{R}^n \times [0, T]) \backslash \{(0, T)\}$ exists such that

$$\varphi = 1 \qquad \text{on a neighborhood of} \quad K \backslash \{(0, T)\} \quad \text{in} \quad \mathbb{R}^n \times [0, T), \tag{3.93}$$

$$\varphi = 0 \quad \text{on a neighborhood of} \quad (\mathbb{R}^n \backslash \{0\}) \times \{T\} \quad \text{in} \quad (\mathbb{R}^n \backslash \{0\}) \times [0, T]. \tag{3.94}$$

So we define u on $\mathbb{R}^n \times [0, T)$ by

$$u(x, t) = \varphi(x, t)\tilde{u}(x, t), \qquad \forall x \in \mathbb{R}^n \times [0, T), \tag{3.95}$$

and extend u to all $\mathbb{R}^n \times \mathbb{R}$ by requiring

$$u(x, t + T) = u(x, t), \qquad \forall (x, t) \in \mathbb{R}^n \times \mathbb{R}. \tag{3.96}$$

Then, see in particular (3.82), (3.84)–(3.86), (3.89), and (3.93)–(3.95),

$$u \in C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m) \cap C^\infty((\mathbb{R}^n \times \mathbb{R}) \backslash \{(0, kT); k \in \mathbb{Z}\}; \mathbb{R}^m), \tag{3.97}$$

$$u \in C^\infty(\mathbb{R}^n \times [kT, (k+1)T); \mathbb{R}^m), \qquad \forall k \in \mathbb{Z}, \tag{3.98}$$

$$u(0, t) = 0, \qquad \forall t \in \mathbb{R}, \tag{3.99}$$

$$(\dot{x} = f(x, u(x, t)), |x(0)| < \varepsilon_0/4) \quad \Rightarrow \quad (x(T) = 0), \tag{3.100}$$

$$\partial^\alpha u = 0 \qquad \text{on} \quad (\mathbb{R}^n \backslash \{0\}) \times \{kT; k \in \mathbb{Z}\}, \quad \forall \alpha \in \mathbb{N}^{n+m}. \tag{3.101}$$

Note that from (3.98) we get

$$((\dot{x} = f(x, u(x, t)), x(s) = 0) \Rightarrow (x(t) = 0, \forall t \geq s)), \qquad \forall s \in \mathbb{R}. \tag{3.102}$$

From (3.100), (3.102), and Lemma 3.15 of [C4] we get that $\delta > 0$ exists such that

$$(\dot{x} = f(x, u(x, t)), |x(s)| < \delta) \quad \Rightarrow \quad (x(t) = 0, \forall t \geq s + 2T). \tag{3.103}$$

Since T is arbitrary this ends the proof of Lemma 3.5 and therefore of Theorem 2.4. ■

We end this section by a remark. Instead of defining (u, v) with (3.10)–(3.16) we can also proceed in the following way. Clearly, increasing if necessary the dimension of x, we may assume that $\dot{x} = f(x, u)$ is locally stabilizable in small time by means of a static periodic time-varying state feedback law. Then let $t_{q+1} \in (t_q, T)$ and define (u, v) by (3.10), (3.11), and, instead of (3.12)–(3.16),

$$v_{q+1}(y, z, t) = -t(t_q - t)z_{q+1}^{+1/3} + 6\frac{(t_{q+1} - t)(t - t_q)^+}{(t_{q+1} - t_q)^3} g(\theta(z_1, \dots, z_q)), \quad \forall t \in [0, t_{q+1}], \tag{3.104}$$

$$v_{q+1}(y, z, t) = \frac{(t - t_{q+1})}{(T - t_{q+1})} f(z_{q+1}, 0), \qquad \forall t \in [t_{q+1}, T], \tag{3.105}$$

$$u(y, z, t) = \tilde{u}(z_{q+1}, t), \qquad \forall t \in [T, 2T], \tag{3.106}$$

$$v_i(y, z, t) = -(t - T)^+(2T - t)z_i^{1/3}, \qquad \forall t \in [T, 2T], \tag{3.107}$$

$$v_{q+1}(y, z, t) = f(z_{q+1}, \tilde{u}(z_{q+1}, t)), \qquad \forall t \in [T, 2T]. \tag{3.108}$$

$$u(y, z, t + 2T) = u(y, z, t), \qquad \forall t \in \mathbb{R}, \tag{3.109}$$

$$v(y, z, t + 2T) = v(y, z, t), \qquad \forall t \in \mathbb{R}, \tag{3.110}$$

where $\tilde{u}$ satisfies (3.61)–(3.64) and $g \in C^0(\mathbb{R}^n; \mathbb{R}^n)$ satisfies, for some $\delta > 0$,

$$(\dot{x} = (t - t_q)f(x, 0)/(T - t_{q+1}), |x(T)| < \delta) \quad \Rightarrow \quad (x(t_{q+1}) = g(x(T))). \tag{3.111}$$

This new (u, v) also works. The advantage of the previous (u, v) is that, even if $\dot{x} = f(x, u)$ is not locally stabilizable in small time by means of a static periodic time-varying state feedback law,we do not need to increase the dimension of x. The advantage of this new (u, v) is that it allows us to prove the finite time—instead of small time—version of Theorem 2.4, i.e., it allows us to prove this theorem if, in Definition 2.1, "for any positive real number T..." is replaced by "a positive real number T exists ... ." Indeed we do not know if Lemma 3.5 holds if, in Definition 2.1 and in this lemma, "for any positive real number T..." is replaced by "for some positive real number T... ." However, with this new (u, v) this lemma is not needed. It is only necessary to know that ε in $(0, +\infty)$ and u in $C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$ exist such that u vanishes on $\mathbb{R}^n \times \{0, T\}$ and (2.4)–(2.7) hold; and it is not hard to see that, if C is locally stabilizable in finite time by means of a continuous static periodic time-varying state feedback law, then, for T large enough, such ε and u exist.

4. Proof of Proposition 2.5

In this section we prove a proposition which implies Proposition 2.5. Let us first introduce some definitions.

Definition 4.1. Let λ be a positive real number. System C is locally λ-exponentially asymptotically stabilizable from time 0 by means of a small continuous static time-varying state feedback law if, for any μ in $(0, +\infty)$, there are ε in $(0, +\infty)$ and u in $C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$ satisfying (2.4) such that

$$\forall \delta > 0, \quad \exists \eta > 0 \quad \text{s.t.} \quad (\dot{x} = f(x, u(x, t)) \text{ and } |x(0)| < \eta) \quad \Rightarrow \quad (|x(t)| < \delta, \forall t \in \mathbb{R}), \tag{4.1}$$

$$(\dot{x} = f(x, u(x, t)), |x(0)| < \varepsilon) \quad \Rightarrow \quad (\exists M \in (0, +\infty) \text{ s.t. } |x(t)| e^{\lambda t} \leq M, \forall t \in (0, +\infty)), \tag{4.2}$$

$$|u(x, t)| \leq \mu, \qquad \forall (x, t) \in \mathbb{R}^n \times \mathbb{R}. \tag{4.3}$$

If, moreover, u can be chosen so that (2.8) holds for some $\bar{u}$ in $C^0(\mathbb{R}^p \times \mathbb{R}; \mathbb{R}^m)$, then system C is said to be locally λ-exponentially stabilizable from time 0 by means of a continuous time-varying small output feedback law.

Similarly to Definition 2.2 we introduce:

Definition 4.2. Let λ be a positive real number. System C is locally λ-exponentially stabilizable from time 0 by means of a small continuous dynamic time-varying state (resp. output) feedback law if, for some integer k, the control system (2.9), where the state is $(x, z) \in \mathbb{R}^n \times \mathbb{R}^k$, the control $(u, v) \in \mathbb{R}^m \times \mathbb{R}^k$, and the output $\tilde{h}(x, z) \in \mathbb{R}^p \times \mathbb{R}^k$, is locally λ-exponentially stabilizable from time 0 by means of a small continuous static time-varying state (resp. output) feedback law.

We point out that, clearly, if system C is locally stabilizable in small—or even finite—time by means of a continuous dynamic periodic time-varying output feedback law, then it is locally λ-exponentially stabilizable from time 0 by means of a small continuous dynamic time-varying state output feedback law for all positive real numbers λ. So Proposition 2.5 is a consequence of:

Proposition 4.3. *Assume that f and h are analytic. Assume that, for some real number λ in $(\|\partial f/\partial x\|(0, 0), \infty)$, if system C is locally λ-exponentially stabilizable from time 0 by means of a small continuous dynamic time-varying output feedback law, then it is locally Lie null-observable.*

Proof. By assumption an integer k exists such that the control system (2.9), where the state is $(x, z) \in \mathbb{R}^n \times \mathbb{R}^k$, the control $(u, v) \in \mathbb{R}^m \times \mathbb{R}^k$, and the output $\tilde{h}(x, z) \in \mathbb{R}^p \times \mathbb{R}^k$, is locally λ-exponentially stabilizable from time 0 by means of a small continuous static time-varying output feedback law. So, given μ in $(0, +\infty)$, there are $u \in C^0(\mathbb{R}^p \times \mathbb{R}^k \times \mathbb{R}; \mathbb{R}^m)$, $v \in C^0(\mathbb{R}^p \times \mathbb{R}^k \times \mathbb{R}; \mathbb{R}^k)$, and ε in $(0, \mu)$ such that

$$u(0, 0, t) = 0, \qquad \forall t \in \mathbb{R}, \tag{4.4}$$

$$v(0, 0, t) = 0, \qquad \forall t \in \mathbb{R}, \tag{4.5}$$

$$|u(y, z, t)|^2 + |v(y, z, t)|^2 < \mu^2, \qquad \forall (y, z, t) \in \mathbb{R}^p \times \mathbb{R}^k \times \mathbb{R}, \tag{4.6}$$

and, for any maximal solution of $\dot{x} = f(x, u(h(x), z, t))$, $\dot{z} = v(h(x), z, t)$ which satisfies $|x(0)|^2 + |z(0)|^2 < \varepsilon^2$,

$$|x(t)|^2 + |z(t)|^2 < \mu^2, \qquad \forall t \in [0, +\infty), \tag{4.7}$$

and, for some M in $(0, +\infty)$, depending on x and z,

$$|x(t)|^2 + |z(t)|^2 \le M^2 e^{-2\lambda t}, \qquad \forall t \in [0, +\infty). \tag{4.8}$$

We choose μ so that $\bar{\lambda}$ in $(0, \lambda)$ exists such that

$$\left\| \frac{\partial f}{\partial x}(x, u) \right\| \le \bar{\lambda}, \qquad \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^m \quad \text{s.t.} \quad |x| < \mu, \quad |u| < \mu. \tag{4.9}$$

We first consider the case where condition (i) of Definition 2.3 does not hold with $\bar{\varepsilon} = \varepsilon$. Then a in $B_\varepsilon \backslash \{0\}$ exists such that

$$L_{f_0}^q(a) = 0, \qquad \forall q \in \mathbb{N}. \tag{4.10}$$

Let x be the maximal solution of

$$\dot{x} = f(x, 0) = f_0(x), \qquad x(0) = a. \tag{4.11}$$

This solution is defined on some open interval I of $\mathbb{R}$ which contains 0. From (4.10) and (4.11) we obtain

$$\left(\frac{d^i}{dt^i}(h \circ x) \right)(0) = 0, \qquad \forall i \in \mathbb{N}, \tag{4.12}$$

and, therefore, since f and h are analytic,

$$h(x(t)) = 0, \qquad \forall t \in I. \tag{4.13}$$

Let $z: I \to \mathbb{R}^k$ be defined by $z(t) = 0$, $\forall t \in I$. From (4.4), (4.5), (4.11), and (4.13) we get

$$\dot{x} = f(x, u(h(x), z, t)), \qquad \dot{z} = v(h(x), z, t). \tag{4.14}$$

Therefore, by (4.7), $[0, +\infty) \subset I$ and

$$|x(t)| < \mu, \qquad \forall t \in [0, +\infty). \tag{4.15}$$

Using (2.2), (4.9), (4.11), and (4.15) we get

$$|\dot{x}(t)| \le \bar{\lambda} |x(t)|, \qquad \forall \in [0, +\infty) \tag{4.16}$$

and so

$$|x(t)| \ge e^{-\bar{\lambda} t} |a|, \qquad \forall t \in [0, +\infty), \tag{4.17}$$

which is in contradiction with (4.8) since $\bar{\lambda} < \lambda$ and $a \ne 0$.

We now consider the case where condition (ii) of Definition 2.3 does not hold with $\bar{\varepsilon} = \varepsilon$. Since f is analytic this implies that (ii)* does not hold with $\bar{\varepsilon} = \varepsilon$. So two distinct points a_1 and a_2 in B_ε exist such that

$$\omega(a_1, u) = \omega(a_2, u), \qquad \forall \omega \in 0(C), \qquad \forall u \in \mathbb{R}^m. \tag{4.18}$$

Let (x_1, z_1) be a maximal solution of

$$\dot{x}_1 = f(x_1, u(h(x_1), z_1, t)), \qquad \dot{z}_1 = v(h(x_1), z_1, t), \qquad x_1(0) = a_1, \qquad z_1(0) = 0, \tag{4.19}$$

and let $u_1: [0, \infty) \to \mathbb{R}^m$, $v_1: [0, +\infty) \to \mathbb{R}^k$ be defined by

$$u_1(t) = u(h(x_1(t)), z_1(t), t), \qquad v_1(t) = v(h(x_1(t)), z_1(t), t), \qquad \forall t \in [0, +\infty). \tag{4.20}$$

Let (x_2, z_2) be the maximal solution of

$$\dot{x}_2 = f(x_2, u_1(t)), \qquad \dot{z}_2 = v_1(t), \qquad x_2(0) = a_2, \qquad z_2(0) = 0. \tag{4.21}$$

This solution is defined on some open interval I of $[0, +\infty)$ which contains 0. Clearly,

$$z_2 = z_1 \qquad \text{on } I. \tag{4.22}$$

We note that it follows from (4.18), (4.19), and (4.21) that, if u_1 is analytic on I,

$$\frac{d^i}{dt^i}(h \circ x_2)(0) = \frac{d^i}{dt^i}(h \circ x_1)(0), \qquad \forall i \in \mathbb{N}, \tag{4.23}$$

and so since f and h are analytic we get, if u_1 is analytic on I,

$$h \circ x_2 = h \circ x_1 \qquad \text{on } I. \tag{4.24}$$

In fact (4.24) also holds if u_1 is only continuous; indeed, if u_1 is continuous on I, then, for any compact interval K with $0 \in K$ and $K \subset I$, it can be approximated uniformly on K by analytic maps $(u_1^i; i \in \mathbb{N})$ with $|u_1^i| < \mu$ on K; then, if we denote by $(x_j^i, i \in \mathbb{N}, j \in \{1, 2\})$ the maximal solutions of

$$\dot{x}_j^i = f(x_j^i, u_1^i(t)), \qquad x_j^i(0) = a_j, \tag{4.25}$$

we get, as above, $h(x_1^i) = h(x_2^i)$ on the intersection of the domains of the definition of x_1^i and x_2^i; letting $i \to \infty$ we get (4.24) on K and so on I, since K is any compact interval with $0 \in K$ and $K \subset I$. From (4.22) and (4.24) we get

$$\dot{x}_2 = f(x_2, u(h(x_2), z_2, t)), \qquad \dot{z}_2 = v(h(x_2), z_2, t), \tag{4.26}$$

which implies, by (4.7), $I = [0, +\infty)$ and

$$|x_2(t)|^2 + |z_2(t)|^2 < \mu^2, \qquad \forall t \in [0, +\infty). \tag{4.27}$$

Similarly we have

$$|x_1(t)|^2 + |z_1(t)|^2 < \mu^2, \qquad \forall t \in [0, +\infty). \tag{4.28}$$

By (4.6), (4.9), (4.19)–(4.21), (4.27), and (4.28) we get

$$|\dot{x}_2 - \dot{x}_1| \leq \bar{\lambda}|x_2 - x_1| \qquad \text{on} \quad [0, +\infty) \tag{4.29}$$

and so

$$|x_2 - x_1|(t) \geq e^{-\bar{\lambda}t}|a_2 - a_1|, \qquad \forall t \in [0, +\infty). \tag{4.30}$$

Moreover, (4.8) gives the existence of M in $[0, +\infty)$ such that

$$|x_2(t)| \leq Me^{-\lambda t}, \qquad |x_1(t)| \leq Me^{-\lambda t}, \qquad \forall t \in [0, +\infty), \tag{4.31}$$

which is in contradiction with (4.30) since $\bar{\lambda} < \lambda$ and $a_1 \neq a_2$. This ends the proof of Proposition 4.3. ■

We finally note that when f is analytic the assumption that $\dot{x} = f(x, u)$ is locally stabilizable in small time by means of a continuous dynamic periodic time-varying state feedback law implies, by a theorem due to Sussmann and Jurdjevic [SJ], that $\dot{x} = f(x, u)$ satisfies the accessibility rank condition at 0, as defined, for example, on p. 730 of [HK]; let us also recall that, if this accessibility rank condition holds and if f and h are analytic, condition (ii), which is always implied by the observability rank condition at 0—as defined, for example, on p. 73 of [HK]—is equivalent to this observability rank condition by a theorem due to Hermann and Krener [HK, Theorem 3.12].

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Appendix

In this appendix we prove Lemma 3.1. The proof is divided into four steps:

Step 1. In this step we introduce some combinatorial notation, the goal of which is to describe $O(C^*)$, where C^* is the control

$$\dot{x} = F(x, v), \qquad y = h(x),$$

with F defined by (3.25) and where the state is $x \in \mathbb{R}^n$, the control $v = (v^1, \ldots, v^p) \in (\mathbb{R}^m)^p \simeq \mathbb{R}^{pm}$, and the output $y \in \mathbb{R}^p$.

Step 2. In this step we give an expression of functions in $O(C^*)$ in terms of the functions in $O(C)$. With this expression we end the proof by distinguishing two cases:

Step 3. $h(x_1) = h(x_2) \neq 0$,

Step 4. $h(x_1) = h(x_2) = 0$.

A.1. Step 1: Combinatorial Notation

Let, for a positive integer k, s_k be the set of sequences $\sigma = a_k a_{k-1} \cdots a_1$ of k elements $(a_i;\ i \in [1, k])$ of $\mathbb{N}^m$. Let, for $\sigma = a_k a_{k-1} \cdots a_1 \in s_k$, $d_0(\sigma) = k$, $d_1(\sigma) = \sum_{i=1}^k |a_i| := \sum_{i=1}^k \sum_{j=1}^m a_i^j$. For such a σ and for a in $\mathbb{N}^m$ we define $a * \sigma$ in s_{k+1} by

$a * \sigma = aa_k a_{k-1} \cdots a_1$. For convenience we denote by s_0 the set whose unique element is the empty sequence, denoted by $\varnothing$. We let $d_0(\varnothing) = d_1(\varnothing) = 0$. Let $s = \bigcup_{k=0}^{+\infty} s_k$. For σ in s we define a linear operator δ^σ: $C^\infty(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^p) \to C^\infty(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^p)$ by induction on $d_0(\sigma)$ and by requiring, for all g in $C^\infty(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^p)$, all a in $\mathbb{N}^m$, and all σ in s,

$$\delta^\varnothing g = g, \tag{A.1}$$

$$\delta^{a*\sigma} g = \sum_{i=1}^{n} \frac{\partial^{|a|} f^i}{\partial u^a} \frac{\partial}{\partial x_i}(\delta^\sigma g). \tag{A.2}$$

Considering $h \in C^\infty(\mathbb{R}^n; \mathbb{R}^p)$ as a map in $C^\infty(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^p)$, we let, for σ in s, $h(\sigma) = \delta^\sigma h$.

On the set of finite sequences $\beta = (\sigma_k, i_k) \cdots (\sigma_2, i_2)\sigma_1$, $k \geq 1$, made of $(k-1)$ elements $((\sigma_j, i_j); j \in [2, k])$ in $s \times [1, p]$ and of an element σ_1 in $s\backslash\{\varnothing\}$ we define the following relation of equivalence, denoted $\sim$, $\beta = (\sigma_k, i_k) \cdots (\sigma_2, i_2)\sigma_1 \sim \beta' = (\sigma'_{k'}, i'_{k'}) \cdots (\sigma'_2, i'_2)\sigma'_1$ if and only if $k = k'$, $\sigma_1 = \sigma'_1$, and a bijection π: $[2, k] \to [2, k]$ exists such that

$$\sigma'_j = \sigma_{\pi(j)}, \qquad i'_j = i_{\pi(j)}, \qquad \forall j \in [2, k]. \tag{A.3}$$

We denote by b the quotient space. Note that, taking $k = 1$, we have $s\backslash\{\varnothing\} \subset b$. For $\beta = (\sigma_k, i_k) \cdots (\sigma_1, i_2)\sigma_1$ in b we let

$$d_j(\beta) = \sum_{i=1}^{k} d_j(\sigma_i), \qquad \forall j \in \{0, 1\}, \tag{A.4}$$

$$d_2(\beta) = k, \tag{A.5}$$

$$d_3(\beta) = \max\{d_0(\sigma_j) + d_1(\sigma_j); j \in [1, k]\}, \tag{A.6}$$

and define $h(\beta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^p)$ by

$$h(\beta) = \left(\prod_{j=2}^{k} h(\sigma_j)_{i_j}\right) h(\sigma), \tag{A.7}$$

where $h(\sigma_j)_{i_j}$ denotes the i_jth component of $h(\sigma_j)$. Note that if $k = 1$, then $h(\beta) = h(\sigma_1)$.

We now introduce similar notation for F. Let, for a positive integer k, S_k be the set of sequences $A_k A_{k-1} \cdots A_1$ of elements $(A_i; i \in [1, k])$ in $\mathbb{N}^{pm}$. For $\Sigma = A_k A_{k-1} \cdots A_1 \in S_k$ we let

$$D_0(\Sigma) = kD_1(\Sigma) = \sum_{i=1}^{k} |A_i| = \sum_{i=1}^{k} \sum_{j=1}^{p} \sum_{r=1}^{m} A_i^{jr}, \tag{A.8}$$

$$A * \Sigma = AA_k A_{k-1} \cdots A_1 \in S_{k+1}, \qquad \forall A \in \mathbb{N}^{pm}. \tag{A.9}$$

Again, for convenience, we denote by S_0 the set whose unique element is the empty sequence, still denoted $\varnothing$. We let $D_0(\varnothing) = D_1(\varnothing) = 0$ and $S = \bigcup_{k \geq 0}^{+\infty} S_k$. For Σ in S we define a linear map Δ^Σ: $C^\infty(\mathbb{R}^n \times \mathbb{R}^{pm}; \mathbb{R}^p) \to C^\infty(\mathbb{R}^n \times \mathbb{R}^{pm}; \mathbb{R}^p)$ by induction on $D_0(\Sigma)$ and by requiring

$$\Delta^{\varnothing} G = G, \tag{A.10}$$

$$\Delta^{A*\Sigma} G = \sum_{i=1}^{n} \left(\frac{\partial^{|A|} f^i}{\partial a^A}\right)\left(\frac{\partial}{\partial x_i}(\Delta^{\Sigma} G)\right), \tag{A.11}$$

for all G in $C^\infty(\mathbb{R}^n \times \mathbb{R}^{pm}; \mathbb{R}^p)$, all A in $\mathbb{N}^{pm}$, and all Σ in S. We let, for Σ in S, $h(\Sigma) = \Delta^{\Sigma} h$, where we consider $h \in C^\infty(\mathbb{R}^n; \mathbb{R}^p)$ as a map in $C^\infty(\mathbb{R}^n \times \mathbb{R}^{pm}; \mathbb{R}^p)$. Then Lemma 3.1 can be rephrased in the following way: if x_1, x_2 are two distinct points in B_δ and if $\bar{v} \in \mathbb{R}^{pm}$ satisfies $|\bar{v}| < \delta$, then $\bar{\Sigma}$ in S exists such that

$$h(\bar{\Sigma})(x_2, \bar{v}) \neq h(\bar{\Sigma})(x_1, \bar{v}). \tag{A.12}$$

So we consider x_1 and x_2, two points in $\mathbb{R}^n$, and $\bar{v} = (\bar{v}^i; 1 \leq i \leq p, \bar{v}^i \in \mathbb{R}^m) = (\bar{v}^{ij}; 1 \leq i \leq p,\ 1 \leq j \leq m,\ \bar{v}^{ij} \in \mathbb{R}) \in \mathbb{R}^{pm}$. We assume $|\bar{v}| < \delta$, $|x_1| < \delta$, $|x_2| < \delta$, and $x_1 \neq x_2$. In order to prove the existence of $\bar{\Sigma}$ in S satisfying (A.12), an expression of $h(\Sigma)$ in terms of $h(\beta)$, $\beta \in b$, is given in the next section.

A.2. Step 2: Description of $O(C^*)$ by Means of $O(C)$

Let us define, for $v = (v^{ij}; i \leq p, 1 \leq j \leq m, v^{ij} \in \mathbb{R}) \in \mathbb{R}^{pm}$ and for $t = (t^{ij}; 1 \leq i \leq p, 1 \leq j \leq m, t^{ij} \in \mathbb{R}) \in \mathbb{N}^{pm}$, v^t in $\mathbb{R}$ by

$$v^t = \prod_{i=1}^{p} \prod_{j=1}^{m} (v^{ij})^{t_{ij}} \in \mathbb{R}, \tag{A.13}$$

$$d_4(t) = \sum_{i=1}^{p} \sum_{j=1}^{m} t_{ij} \in \mathbb{N}. \tag{A.14}$$

In (A.13) we use the convention $(v^{ij})^0 = 1$, even if $v^{ij} = 0$. Then, by induction on $D_0(\Sigma)$, it can be easily checked that a map $C: S \times \mathbb{N}^{pm} \times b \to \mathbb{N}$ independent of f, h, $x \in \mathbb{R}^n$, and $v \in \mathbb{R}^{pm}$ exists such that

$$\text{Cardinal}\{(t, \beta) \in \mathbb{N}^{pm} \times b;\ C(\Sigma, t, \beta) \neq 0\} < +\infty, \qquad \forall \Sigma \in S, \tag{A.15}$$

and, for all (x, v) in $\mathbb{R}^n \times \mathbb{R}^{pm}$ and all Σ in S,

$$h(\Sigma)(x, v) = \sum_{\beta \in b} \sum_{t \in \mathbb{N}^{pm}} C(\Sigma, t, \beta) v^t h(\beta)\left(x, \sum_{i=1}^{p} h_i(x) v^i\right). \tag{A.16}$$

Moreover, still by induction on $D_0(\Sigma)$, it is readily seen that if $C(\Sigma, t, \beta)$ is not zero, then

$$d_0(\beta) = D_0(\Sigma), \tag{A.17}$$

$$d_1(\beta) - d_4(t) = D_1(\sigma), \tag{A.18}$$

$$d_3(\beta) \leq D_0(\Sigma) + D_1(\Sigma), \tag{A.19}$$

$$d_2(\beta) = 1 + d_1(\beta). \tag{A.20}$$

We also point out that if $\Sigma = A_k \cdots A_1$ satisfies

$$A_i^{jr} = 0, \qquad \forall j \in [2, p], \quad \forall r \in [s, m], \quad \forall i \in [1, k], \tag{A.21}$$

then, if $\beta = (\sigma_k, i_k)\cdots(\sigma_2, i_2)\sigma_1$,

$$(C(\Sigma, q, \beta) \neq 0) \quad \Rightarrow \quad (i_k = \cdots = i_2 = 1). \tag{A.22}$$

For $\sigma = a_k \cdots a_1 \in \Delta \backslash \{\varnothing\}$ and for $\Sigma = A_r \cdots A_1 \in S \backslash \{\varnothing\}$ we say that $\sigma \subset \Sigma$ if $k \leq r$ and a sequence $1 \leq v_1 < v_2 < \cdots < v_k \leq r$ exists such that

$$a_i^j = \sum_{q=1}^{p} A_{v_i}^{qj}, \qquad \forall i \in [1, k], \quad \forall i \in [1, m]. \tag{A.23}$$

Finally, if $\beta = (\sigma_k, i_k)\cdots(\sigma_2, i_2)\sigma_1 \in b$ and $\Sigma \in S$, we say that $\beta \subset \Sigma$ if $\sigma_1 \subset \Sigma$. Then, still by induction on $D_0(\Sigma)$, it is easily verified that, for any Σ in $S \backslash \{\varnothing\}$ and any β in b,

$$(C(\Sigma, 0, \beta) \neq 0) \quad \Rightarrow \quad (\beta \subset \Sigma). \tag{A.24}$$

Note that if $h(x_1) \neq h(x_2)$, then (A.12) holds with $\bar{\Sigma} = \varnothing$. So from now on we assume

$$h(x_1) = h(x_2). \tag{A.25}$$

Let $\bar{u} = \sum_{i=1}^{p} h_i(x_1)$, $\bar{v}^i = \sum_{i=1}^{p} h_i(x_2)\bar{v}^i$.

A.3. Step 3: Case $h(x_1) = h(x_2) \neq 0$

In this step we assume

$$h(x_1) = h(x_2) \neq 0. \tag{A.26}$$

Without loss of generality we may assume that, for example,

$$h_1(x_1) \neq 0. \tag{A.27}$$

We equip s with a total order, denoted $\leq$, such that, for all $\tilde{\sigma}$ in s and all $\bar{\sigma}$ in s,

$$(d_0(\tilde{\sigma}) + d_1(\tilde{\sigma}) < d_0(\bar{\sigma}) + d_1(\bar{\sigma})) \quad \Rightarrow \quad (\tilde{\sigma} \leq \bar{\sigma}), \tag{A.28}$$

$$(d_0(\tilde{\sigma}) + d_1(\tilde{\sigma}) = d_0(\bar{\sigma}) + d_1(\bar{\sigma}) \text{ and } d_0(\tilde{\sigma}) < d_0(\bar{\sigma})) \quad \Rightarrow \quad (\tilde{\sigma} \leq \bar{\sigma}). \tag{A.29}$$

By (ii) of Definition 2.3, $\tilde{\sigma}$ in s exists such that $h(\tilde{\sigma})(x_1, \bar{u}) \neq h(\tilde{\sigma})(x_2, \bar{u})$. So $\bar{\sigma}$ in s exists such that

$$h(\bar{\sigma})(x_1, u) \neq h(\bar{\sigma})(x_2, \bar{u}), \tag{A.30}$$

$$(h(\sigma)(x_1, \bar{u}) \neq h(\sigma)(x_2, \bar{u})) \quad \Rightarrow \quad \sigma \geq \bar{\sigma}. \tag{A.31}$$

By (A.25) and (A.30), $\bar{\sigma} \neq \varnothing$. From $\bar{\sigma} = \bar{a}_r \cdots \bar{a} \in s \backslash \{\varnothing\}$ we construct $\Sigma = \bar{A}_r \cdots \bar{A}_1 \in S \backslash \{\varnothing\}$ by requiring

$$\bar{A}_q^{1j} = \bar{a}_r^j, \qquad \forall j \in [1, m], \qquad \forall q \in [1, r], \tag{A.32}$$

$$\bar{A}_q^{ij} = 0, \qquad \forall j \in [1, m], \quad \forall q \in [1, r], \quad \forall i \in [2, p]. \tag{A.33}$$

Let us verify that (A.12) holds. Let us first remark that

$$D_0(\bar{\Sigma}) = d_0(\bar{\sigma}), \qquad D_1(\bar{\sigma}) = d_1(\bar{\sigma}), \tag{A.34}$$

and let us define

$$\bar{\beta} = (\varnothing, 1)\cdots(\varnothing, 1)\bar{\sigma} \in S, \tag{A.35}$$

where $(\varnothing, 1)$ is repeated $d_1(\bar{\sigma})$ times. It is easily seen—for example, by induction on $d_0(\bar{\sigma})$—that

$$C(\bar{\Sigma}, 0, \bar{\beta}) = 1. \tag{A.36}$$

We assume that, for all $(t, \beta) \in \mathbb{N}^{pm} \times b$,

$$(C(\bar{\Sigma}, t, \beta) \neq 0 \text{ and } (t, \beta) \neq (0, \bar{\beta})) \quad \Rightarrow \quad (h(\beta)(x_1, \bar{u}) = h(\beta)(x_2, \bar{u})). \tag{A.37}$$

From (A.16), (A.36), and (A.37) we get

$$h(\bar{\Sigma})(x_2, \bar{v}) - h(\bar{\Sigma})(x_1, v) = (h_1(x_1))^{d_1(\bar{\sigma})}\{h(\bar{\sigma})(x_2, \bar{u}) - h(\bar{\sigma})(x_1, \bar{u})\}, \tag{A.38}$$

which, with (A.27) and (A.30), gives (A.12). We now prove (A.37). Let $\beta \in b$ be defined by $\beta = (\sigma_k, i_k)\cdots(\sigma_2, i_2)\sigma_1$ and let t in $\mathbb{N}^{pm}$ be such that

$$C(\bar{\Sigma}, t, \beta) \neq 0, \tag{A.39}$$

$$h(\beta)(x_1, \bar{u}) \neq h(\beta)(x_2, \bar{u}). \tag{A.40}$$

Using (A.19), (A.35), and (A.39) we get

$$d_0(\sigma_j) + d_1(\sigma_j) \leq d_0(\bar{\sigma}) + d_1(\bar{\sigma}), \qquad \forall j \in [1, k]. \tag{A.41}$$

Using (A.17), (A.34), and (A.39) we get

$$\sum_{j=1}^{k} d_0(\sigma_j) = d_0(\bar{\sigma}). \tag{A.42}$$

Since $\sigma_1 \in s \backslash \{\varnothing\}$,

$$d_0(\sigma_1) \geq 1. \tag{A.43}$$

So from (A.28), (A.29), (A.31), (A.40), and (A.42) we get

$$\sigma_j = \varnothing, \qquad \forall j \in [2, k]. \tag{A.44}$$

Indeed, if (A.44) does not hold we get, from (A.42) and (A.43),

$$d_0(\sigma_j) < d_0(\bar{\sigma}), \qquad \forall j \in [1, k]; \tag{A.45}$$

from (A.41) and (A.45) we get

$$\sigma_j \leq \bar{\sigma} \quad \text{and} \quad \sigma_j \neq \bar{\sigma}, \qquad \forall j \in [1, k]; \tag{A.46}$$

from (A.31) and (A.46) we get

$$h(\sigma_j)(x_1, \bar{u}) = h(\sigma_j)(x_2, \bar{u}), \qquad \forall j \in [1, k], \tag{A.47}$$

which is in contradiction with (A.40). So (A.44) holds. From (A.22), (A.33), and (A.39) we get

$$i_j = 1, \qquad \forall j \in [2, k]. \tag{A.48}$$

From (A.18), (A.35), (A.39), (A.42), and (A.44), we obtain

$$d_0(\sigma_1) = d_0(\bar{\sigma}), \qquad d_1(\sigma_1) = d_1(\bar{\sigma}), \tag{A.49}$$

$$d_4(t) = 0. \tag{A.50}$$

From (A.24), (A.39), and (A.50), we get

$$\sigma_1 \subset \bar{\Sigma}, \tag{A.51}$$

which, with (A.49), gives

$$\sigma_1 = \bar{\sigma}. \tag{A.52}$$

Moreover, from (A.20), (A.39), (A.44), and (A.49) we obtain

$$k = 1 + d_1(\bar{\sigma}). \tag{A.53}$$

Finally, using (A.44), (A.48), (A.52), and (A.53) we get $\beta = \bar{\beta}$ which ends the proof of (A.37) and therefore gives (A.12) if (A.26) holds.

A.4. Step 4: Case $h(x_1) = h(x_2) = 0$

In this step we turn to the case where (A.26) does not hold, i.e.,

$$h(x_1) = h(x_2) = 0, \tag{A.54}$$

and so $\bar{u} = 0$. With the convention $\mathrm{Min}(\varnothing) = +\infty$ we define, for $i \in \{1, 2\}$, $l_i \in \mathbb{N} \cup \{+\infty\}$ by

$$l_i = \mathrm{Min}\{r \geq 0;\ L^r_{f_0}(x_i) \neq 0\}. \tag{A.55}$$

Since x_1 and x_2 are distinct they cannot both be equal to 0; so, by (2.12),

$$l := \mathrm{Min}\{l_1, l_2\} < +\infty. \tag{A.56}$$

Let $\bar{\Sigma} = 0 \cdots 0 \in S$ where 0 is repeated k times. Then, using (A.54) and (A.55), it is seen that $h(\bar{\Sigma})(x_i, \bar{v}) = L^l_{f_0} h(x_i)$, $\forall i \in [1, 2]$. So, if $L^l_{f_0} h(x_2) \neq L^l_{f_0}(x_1)$, then (A.12) holds; therefore we may assume

$$L^l_{f_0} h(x_1) = L^l_{f_0} h(x_2) \neq 0 \tag{A.57}$$

and so, performing if necessary a permutation of the components of h,

$$L^l_{f_0} h_1(x_1) = L^l_{f_0} h_1(x_2) \neq 0. \tag{A.58}$$

We equip s with a total order, still denoted $\leq$, such that, for all $\tilde{\sigma}$ in s and all $\bar{\sigma}$ in s,

$$(d_0(\tilde{\sigma}) + l d_1(\tilde{\sigma}) < d_0(\bar{\sigma}) + l d_1(\bar{\sigma})) \quad \Rightarrow \quad (\tilde{\sigma} \leq \bar{\sigma}), \tag{A.59}$$

$$(d_0(\tilde{\sigma}) + l d_1(\tilde{\sigma}) = d_0(\bar{\sigma}) + l d_1(\bar{\sigma}) \text{ and } d_1(\tilde{\sigma}) > d_1(\bar{\sigma})) \quad \Rightarrow \quad (\tilde{\sigma} \leq \bar{\sigma}), \tag{A.60}$$

and, finally, if $\tilde{\sigma} = \tilde{a}_r \cdots \tilde{a}_1$, $\bar{\sigma} = \bar{a}_r \cdots \bar{a}_1$, for some $r \geq 1$, are such that $d_1(\tilde{\sigma}) = d_1(\bar{\sigma})$ and, for some integer k in $[1, r]$, $\tilde{a}_j = 0$, $\forall j \in [k, r]$, and $\bar{a}_k \neq 0$, then $\tilde{\sigma} \leq \bar{\sigma}$. By (ii) of Definition 2.3, $\tilde{\sigma}$ in s exists such that $h(\tilde{\sigma})(x_1, 0) \neq h(\tilde{\sigma})(x_2, 0)$. So $\bar{\sigma}$ in s exists such that

$$h(\bar{\sigma})(x_1, 0) \neq h(\bar{\sigma})(x_2, 0), \tag{A.61}$$

$$h(\sigma)(x_1, 0) = h(\sigma)(x_2, 0) \quad \Rightarrow \quad \sigma \geq \bar{\sigma}. \tag{A.62}$$

Again by (A.25) and (A.61), $\bar{\sigma} \neq \emptyset$. From $\bar{\sigma} = \bar{\sigma}_r \cdots \bar{\sigma}_1 \in s \backslash \{\emptyset\}$ we construct $\bar{\Sigma} \in S \backslash \{\emptyset\}$ by

$$\bar{\Sigma} = 0 \cdots 0 \bar{A}_r \cdots \bar{A}_1, \tag{A.63}$$

where 0 is repeated $ld_1(\sigma)$ times and $\bar{A}_1, \ldots, \bar{A}_r$ are defined again by (A.32) and (A.33). We have

$$D_0(\bar{\Sigma}) = d_0(\bar{\sigma}) + ld_1(\bar{\sigma}), \tag{A.64}$$

$$D_1(\bar{\Sigma}) = d_1(\bar{\sigma}). \tag{A.65}$$

We check that, with this $\bar{\Sigma}$, (A.12) holds. Let $\sigma^* = 0 \cdots 0 \in s$ with 0 repeated l times and let $\bar{\beta} = (\sigma^*, 1) \cdots (\sigma^*, 1)\bar{\sigma}$ with $(\sigma^*, 1)$ repeated $d_1(\bar{\sigma})$ times. By induction on $d_0(\bar{\sigma})$ it is easily checked that

$$C(\bar{\Sigma}, 0, \bar{\beta}) > 0. \tag{A.66}$$

We assume that, for all $(t, \beta) \in \mathbb{N}^{pm} \times b$,

$$(C(\bar{\Sigma}, y, \beta) \neq 0 \text{ and } (t, \beta) \neq (0, \bar{\beta})) \quad \Rightarrow \quad (h(\beta)(x_1, 0) = h(\beta)(x_2, 0)). \tag{A.67}$$

From (A.16) and (A.67) we get

$$h(\bar{\Sigma})(x_2, 0) - h(\bar{\Sigma})(x_1, 0) = C(\bar{\Sigma}, 0, \bar{\beta})(L^l_{f_0} h_1(x_1))^{d_1(\bar{\sigma})}(h(\bar{\sigma})(x_2, 0) - h(\bar{\sigma})(x_1, 0)) \tag{A.68}$$

and, therefore, (A.12) follows from (A.58), (A.61), and (A.66). It remains only to prove (A.67). Let $\beta = (\sigma_k, i_k) \cdots (\sigma_2, i_2)\sigma_1 \in b$ and let $t \in \mathbb{N}^{pm}$ be such that

$$C(\Sigma, t, \beta) \neq 0, \tag{A.69}$$

$$h(\beta)(x_1, 0) \neq h(\beta)(x_2, 0). \tag{A.70}$$

Note that, by the definition of l, for any σ in s,

$$(d_0(\sigma) \leq l - 1 \text{ and } d_1(\sigma) = 0) \quad \Rightarrow \quad (h(\sigma)(x_1, 0) = h(\sigma)(x_2, 0) = 0). \tag{A.71}$$

For σ in s, let $d(\sigma) = d_0(\sigma) + ld_1(\sigma)$. From (A.70) and (A.71) we get

$$\theta_i := d(\sigma_i) - l \geq 0, \qquad \forall i \in [1, k]. \tag{A.72}$$

Let i be in $[1, k]$; we have

$$d_0(\beta) + ld_1(\beta) = d(\sigma_i) + l(d_2(\beta) - 1) + \sum_{j \in [1,k] \backslash \{i\}} \theta_j. \tag{A.73}$$

Using (A.17), (A.20), (A.64), (A.69), and (A.73) we have

$$d(\sigma_i) = d(\bar{\sigma}) - \sum_{j \in [1,k] \backslash \{i\}} \theta_j. \tag{A.74}$$

Note that, for any σ in s, $d_0(\sigma) \geq 1$ if $d_1(\sigma) \geq 1$. So, from the definition of l, (A.57), and (A.61) we have

$$\theta_1 > 0. \tag{A.75}$$

We claim that

$$\theta_i = 0, \qquad \forall i \in [2, k]. \tag{A.76}$$

Indeed, if (A.76) does not hold, we get, from (A.72), (A.74), and (A.75),

$$d(\sigma_i) < d(\bar{\sigma}), \qquad \forall i \in [1, k], \tag{A.77}$$

which, by (A.59) and (A.62), implies

$$h(\sigma)(x_1, 0) = h(\sigma_i)(x_2, 0), \qquad \forall i \in [1, k]. \tag{A.78}$$

However, (A.78) is in contradiction with (A.70). So (A.76) holds and therefore

$$\sigma_1 = \sigma^*, \qquad \forall i \in [2, k]. \tag{A.79}$$

From (A.74), with $i = 1$, and (A.76) we get

$$d(\sigma_1) = d(\bar{\sigma}). \tag{A.80}$$

This implies

$$d_1(\sigma_1) \leq d_1(\bar{\sigma}). \tag{A.81}$$

Indeed, if (A.81) does not hold, using (A.60) and (A.80) we get $\sigma_1 \leq \bar{\sigma}$ and $\sigma_1 \neq \bar{\sigma}$ and so, by (A.62),

$$h(\sigma_1)(x_1, 0) = h(\sigma_1)(x_2, 0). \tag{A.82}$$

However, (A.57) and (A.79) imply

$$h(\sigma_i)(x_i, 0) = h(\sigma_i)(x_2, 0), \qquad \forall i \in [2, k], \tag{A.83}$$

which is in contradiction with (A.70) and (A.82). So (A.81) holds. Nevertheless, from (A.18), (A.64), (A.69), and (A.79) we obtain $d_1(\sigma_1) = d_1(\bar{\sigma}) + q(t)$, which, with (A.80) and (A.81), implies

$$t = 0, \tag{A.84}$$

$$d_0(\sigma_1) = d_0(\bar{\sigma}), \qquad d_1(\sigma_1) = d_1(\bar{\sigma}). \tag{A.85}$$

However, from (A.24), (A.69), and (A.84) we have $\sigma_1 \subset \bar{\Sigma}$, which, with (A.32), (A.33), (A.63), and (A.85), implies

$$\sigma_1 = \bar{\sigma}. \tag{A.86}$$

From (A.20), (A.69), and (A.85) we have

$$k = 1 + d_1(\bar{\sigma}). \tag{A.87}$$

Finally, from (A.22), (A.34), and (A.69) we get $i_j = 0$, $\forall j \in [2, k]$, which, with (A.79), (A.84), (A.86), and (A.87), implies $\beta = \bar{\beta}$ and so ends the proof of Lemma 3.1. ■

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