A REMARK ON THE DESIGN OF TIME-VARYING STABILIZING FEEDBACK LAWS FOR CONTROLLABLE SYSTEMS WITHOUT DRIFT

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Abstract : This paper gives an approach to design time-varying feedback laws for controllable systems without drift. This approach is based on a time-varying Lyapunov function.

Keywords : Nonlinear systems, Asymptotic stabilization, Controllability, Time-varying feedback.

1 Introduction

Let $f = (f_1, \ldots, f_m)$ be m vector fields of class C^∞ on $\mathbb{R}^n$. We consider the control system without drift

$$\dot{x} = \sum_{k=1}^{m} u_k f_k \quad . \qquad (1)$$

Let Lie(f) be the Lie algebra of vector fields generated by f. Through all this paper, we assume that, for all x in $\mathbb{R}^n \backslash \{0\}$,

$$\text{Lie}(f)(x) = \{ h(x) ; h \in \text{Lie}(f) \} = \mathbb{R}^n \quad . \qquad (2)$$

It is well known that (2) implies (and is in fact equivalent if the vector fields $f_1, \ldots, f_m$ are analytic) that system (1) is completely controllable on $\mathbb{R}^n \backslash \{0\}$. However, it has been proven by R. Brockett in [1] that (2), even with $\mathbb{R}^n$ instead of $\mathbb{R}^n \backslash \{0\}$, does not imply that system (1) can be locally asymptotically stabilized by means of a continuous feedback law $u = u(x)$. In fact, a simple consequence of [1] is (see [7]) :

Proposition 1 *If $m < n$ and*

$$\text{Rank} \{ f_1(0), \ldots, f_m(0) \} = m \qquad (3)$$

then (1) cannot be locally asymptotically stabilized by means of a continuous feedback law $u = u(x)$, nor can it even be locally asymptotically stabilized by means of a continuous dynamic feedback law $u = u(\xi, x)$, $\dot{\xi} = g(\xi, x)$.

C. Samson has suggested in [8] (see also [9]) to stabilize some systems of the form (1) by means of a continuous time-varying periodic feedback law $u = u(t, x)$. He has proved in particular, with proposing an explicit expression for the feedback, that the system

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= x_1 u_2 \\ \dot{x}_3 &= u_2 \ , \end{aligned} \qquad (4)$$

which satisfies (2) but also (3), can be globally asymptotically stabilized by means of a time-varying continuous feedback law, periodic with respect to time. He studied some other systems from nonholonomic robotics, see [9]. This example was also extended by R. Sépulchre in [10].

Note that the interest of time-varying feedback laws for stabilization of systems (with a drift) had been previously shown by E. Sontag and H. Sussmann in [15].

It turns out that these time-varying stabilizing control laws exist in general for systems (1) meeting condition (2), and this is proved in [3]. On the other hand, [7] contains a method to actually design these laws under a more restrictive assumption than (2). The purpose of the present paper is to show how the results contained in [3] may be used to extend the design method proposed in [7] to the general situation (2), thus giving a way to design the control laws whose existence was established in [3].

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Next section 2 is an overview of the results presented in [3] and [7]. Section 3 presents the main result of this paper and the resulting design method. Section 4 is a brief conclusion.

2 Two different approaches

The following result, proved in [3], establishes existence of stabilizing time-varying laws under assumption (2) :

Theorem 2 ([3]) *If (2) is met, there exists, for any positive T, a map $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ such that*

$$\begin{aligned} u &\in C^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^m) \ , && (5) \\ u(t,0) &= 0 \quad \forall t \in \mathbb{R} \ , && (6) \\ u(t+T,x) &= u(t,x) \quad \forall t \in \mathbb{R},\ \forall x \in \mathbb{R}^n \ , && (7) \end{aligned}$$

and

$$\left.\begin{array}{l} 0 \in \mathbb{R}^n \text{ is a globally assymptotically stable} \\ \text{equilibrium point of } \ \dot{x} = \sum_{k=1}^{m} u_k(t,x) f_k(x) \ . \end{array}\right\} \quad (8)$$

The proof of this theorem relies on the analysis of the controllability of the linearized equation around some selected trajectories of (1). More precisely, for a given time-varying feedback law $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$, let $\phi(u) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be defined by

$$\begin{aligned} \frac{\partial \phi(u)}{\partial t}(t,x) &= \sum_{k=1}^{m} u_k(t,\phi(u)(t,x))\, f_k(\phi(u)(t,x)) && (9) \\ \phi(u)(0,x) &= x \ , && (10) \end{aligned}$$

i.e. $t \mapsto \phi(u)(t,x)$ is the solution starting from x at time 0 of the closed-loop system obtained by applying the time-varying feedback u to (1). The linearized control system along this trajectory is

$$\begin{aligned} \dot{y} \ = \ & \frac{\partial}{\partial x}\left[\sum_{i=1}^{m} u_i(t,\phi(u)(t,x)) f_i(\phi(u)(t,x))\right] y \\ & + \sum_{i=1}^{m} w_i f_i(\phi(u)(t,x)) \qquad (11) \end{aligned}$$

where $w = (w_1, \ldots, w_m)$ is the control and $x \in \mathbb{R}^n$ and the law u act as parameters. The main step in [3] is to prove :

Proposition 3 ([3, Sections 2,3,4]) *If system (1) satisfies (2), there exists $\overline{u}$ satisfying (5), (6), (7) such that :*

$$\begin{aligned} \left\| \sum_{k=1}^{m} \overline{u}_k(t,x) f_k(x) \right\| &\leq 1 \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^n && (12) \\ \phi(\overline{u})(x,T) &= x \quad \forall x \in \mathbb{R}^n && (13) \end{aligned}$$

and

$$\left.\begin{array}{l} \text{for } u = \overline{u} \text{ and for all } x \text{ in } \mathbb{R}^n\backslash\{0\}, \\ \text{the linear system (11) is controllable} \\ \text{with impulsive controls at time } t = \frac{3T}{4}. \end{array}\right\} \quad (14)$$

For a proper definition of "controllable with impulsive controls", see e.g. T. Kailath [6, p.614]. It follows easily from a theorem by L.M. Silverman and H.E. Meadows [11] (see also [6, p.614]) that (14) is equivalent to

$$\begin{aligned} & \forall x \in \mathbb{R}^n\backslash\{0\} \ , \\ & \text{Span}\left\{ \left[\mathcal{L}(\overline{u})^p \widetilde{f}_k\right](\tfrac{3T}{4}, x) \ ; \ \begin{array}{l} p \geq 0 \ , \\ k = 1, \ldots, m \end{array} \right\} \ ; \ \mathbb{R}^n \end{aligned} \qquad (15)$$

where $\widetilde{f}_k \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ is defined by :

$$\widetilde{f}_k(t,x) \ = \ f_k(x)$$

and where, for $u \in C^\infty(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^m)$, $\mathcal{L}(u) : C^\infty(\mathbb{R}\times \mathbb{R}^n, \mathbb{R}^n) \to C^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is defined by :

$$\mathcal{L}(u)X \ = \ \frac{\partial X}{\partial t} + [\sum_{k=1}^{m} u_k f_k \ , X \ ] \qquad (16)$$

where $[\,.\,,\,.\,]$ denotes the classical Lie bracket and, by convention, $\mathcal{L}(u)^0 X = X$.

In fact, (15) is proved directly in [3], see [3, Equation (1.15)]. Let us mention that a different and shorter proof of Proposition 3 has been obtained recently by E.D. Sontag in [13] when the f_k's are analytic and that, in this case, this proposition is related to [14] : the novelty in proposition 3 compared to [14] is condition (5).

From (13), the control $\overline{u}$ guarantees that all the solutions of the closed-loop system are periodic, and therefore this control does not solve our stabilization problem. However, controllability of (14) implies that for any η in $C^\infty(\mathbb{R}^n, [0,+\infty))$, there exists w satisfying (5), (6) and (7) such that :

$$\left(\frac{\partial \phi}{\partial u}(\overline{u}).w\right)(x,T) \ = \ -\eta(x)\,x \ ; \qquad (17)$$

hence one may hope that $u = \overline{u} + \varepsilon w$ is a solution of our problem if ε is small enough and η has been chosen positive on $\mathbb{R}^n\backslash\{0\}$. This is proven to be true in [3, Section 5].

In [7], following a different approach, theorem 2 is proved under the extra assumption that, for all x in $\mathbb{R}^n\backslash\{0\}$,

$$\text{Rank}\left\{ \text{ad}^j_{f_1} f_k(x) \ , \ j \geq 0 \ , \ 1 \leq k \leq m \right\} \ = \ n \qquad (18)$$

One of the main interests of [7] is that it provides a method to design explicitely the control laws, and leads to very simple expressions in many interesing cases (for example $f_1 = \frac{\partial}{\partial x_1}$). Moreover it has the advantage to provide also a Lyapunov function. This is convenient when the system to be stabilized is not (1), but (1) with pure integrators added, see J. Tsinias [16] or [3, Section 6]. This is also usefull when analysing the robustness of the obtained closed-loop stability.

The method used in [7] is the following. Let α in $C^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^m)$ be such that, for all t in $\mathbb{R}$ and x

in $\mathbb{R}^n$:

$$\begin{aligned} \alpha(t+2\pi, x) &= \alpha(t, x) & (19)\\ \alpha(-t, x) &= -\alpha(t, x) & (20)\\ \alpha(t, 0) &= 0 & (21)\\ |\alpha(t,x)|\,\|f_1(x)\| &\leq K(1+\|x\|) & (22) \end{aligned}$$

where K is some positive constant. Let $u^*(\alpha)$ be defined by

$$u^*(\alpha)(t,x) = (\alpha(t,x), 0, \ldots, 0) \ , \quad (23)$$

and the non-negative function V by :

$$V(t,x) = \tfrac{1}{2} \left\|\phi(u^*(\alpha))^{-1}(x,t)\right\|^2 \ . \quad (24)$$

where $\phi(u)^{-1}$ is defined by $\phi(u)\left(\phi(u)^{-1}(x,t),t\right) = x$, and let the time-varying stabilizing control be defined by

$$\begin{aligned} u_1(t,x) &= \alpha(t,x) - L_{f_1}V(t,x) & (25)\\ u_k(t,x) &= -L_{f_k}V(t,x) \quad \forall k \in [2,m] \ . & (26) \end{aligned}$$

Then the following theorem is proved in [7] :

Theorem 4 *If system (1) satisfies (18) and α is chosen such that (19), (20), (21) and (22) are met, as well as (27) :*

$$\left.\begin{array}{ll} \frac{\partial^j \alpha}{\partial t^j}(t,x) = 0 & j \geq 0 \\ L_{f_k}V(t,x) = 0 & 1 \leq k \leq m \end{array}\right\} \Rightarrow x = 0 \quad , (27)$$

then the map u given by (23)-(24) satisfies (5), (6), (7) and (8). Moreover, the function V is strictly decreasing along the nonzero solutions of $\dot{x} = \sum_{k=1}^m u_k(t,x) f_k(x)$.

There exists α meeting the required conditions. A possible choice is

$$\alpha(t,x) = \frac{\|x\|^2}{(1+\|x\|^2)(1+\|f_1(x)\|^2)} \sin t \ . \quad (28)$$

3 Main result

Although the approaches in [7] and [3] are somehow different, it is to be noticed that $u^* = (\alpha, 0, \ldots, 0)$ in (23) plays exactly the same role as $\overline{u}$. Actually, under the extra assumption (18), one may, in proposition 3, chose $\overline{u} = u^*$ with u^* given by (28); more precisely, such a $\overline{u}$ satisfies (12), (13) and (14). Most of the difficulties in [3] (contained in the proof of proposition 3) therefore disappear if (18) is satisfied. Without the extra assumption (18), the appraoch in [7] fails, at least if u^* is still restricted to be of the form (23), and it is no longer sufficient to choose $\overline{u}$ in proposition 3 with $m-1$ zero entries.

The idea of this paper is to take advantage of the generality [3] and the simplicity of [7]. This is done by applying the design method coming from [7], but with u^* given by [3], i.e. we allow all the entries of u^* to be nonzero, and take precisely $u^* = \overline{u}$ with $\overline{u}$ given by proposition 3. From now on, we no longer assume (18).

Let $\overline{u}$ be as in proposition 3, and let $V : \mathbb{R}\times\mathbb{R}^n \to [0,+\infty)$ be defined by :

$$V(t,x) = \tfrac{1}{2}\left\|\phi(\overline{u})^{-1}(x,t)\right\|^2 \quad (29)$$

where $\phi(\overline{u})^{-1}$ is defined by $\phi(\overline{u})(\phi(\overline{u})^{-1}(x,t),t) = x$, as in (24), and let the stabilizing control u be defined by

$$u_k(t,x) = \overline{u}_k(t,x) - L_{f_k}V(t,x) \quad \forall k \in [1,m] \ . \quad (30)$$

Our result is :

Theorem 5 *Under assumption (2), the above constructed map u satisfies (5), (6), (7) and (8). Moreover :*

$$\begin{aligned} V &\in C^\infty(\mathbb{R}\times\mathbb{R}^n, [0,+\infty)) & (31)\\ V(t+T,x) &= V(t,x) \quad \forall (t,x) \in \mathbb{R}\times\mathbb{R}^n & (32)\\ V(t,x) &= 0 \ \Leftrightarrow \ x = 0 & (33) \end{aligned}$$

$$\left.\begin{array}{c} \forall K > 0, \ \{x \mid \exists t, V(t,x) \leq K\} \\ \textit{is a bounded set} \end{array}\right\} \quad (34)$$

$$\frac{\partial V}{\partial t} + \sum_{k=1}^m \overline{u}_k L_{f_k} V = 0 \quad (35)$$

$$\left.\begin{array}{l} V \textit{ is non-increasing along the} \\ \textit{solutions of } \ \dot{x} = \sum_{k=1}^m u_k(t,x) f_k(x) \end{array}\right\} \quad (36)$$

$$\left.\begin{array}{r} V(T, \phi(u)(x,T)) < V(0, \phi(u)(x,0)), \\ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^n\backslash\{0\} \end{array}\right\} \quad (37)$$

Proof of theorem 5 : The proof is similar to the one given in [7]. Since $\overline{u}$ satisfies (5), (6), (7), (12) and (13), V given by (29) satisfies (31), (32) and (33), and u satisfies (5), (6) and (7). Property (34) is a consequence of relations (29), (32) and the fact that, thanks to (12), $x \mapsto \phi(\overline{u})(t,x)$ is a homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$. By (29), we have

$$V(t, \phi(\overline{u})(t,x)) = \tfrac{1}{2}\|x\|^2 \ . \quad (38)$$

Differentiating (38) with respect to t, we get (35). From (30) and (35), we have :

$$\frac{d}{dt} V(t,\phi(u)(x,t)) = -\sum_{k=1}^m \left(L_{f_k}V(t,\phi(u)(x,t))\right)^2 \quad (39)$$

which obviously implies (36). Note that (8) is a consequence of (7), (32), (33), (34), (36) and (37), which imply, for instance, asymptotic stability of the Poincaré map $x \mapsto \phi(u)(x,T)$.

The proof of (37) is an adaptation of V. Jurdjevic and J.-P. Quinn [4] –see also [7]–. Let us fix x in $\mathbb{R}^n$; for simplicity, we will write $\phi(t)$ for $\phi(u)(x,t)$ and $\overline{\phi}(t)$ for $\phi(\overline{u})(x,t)$. (39) clearly implies

$$V(T, \phi(u)(x,T)) \leq V(0, \phi(u)(x,0)) \quad (40)$$

where the inequality is an equality if and only if

$$L_{f_k} V(t, \phi(t)) = 0 \quad \forall k = 1, \dots, m, \ \forall t \in [0, T] . \quad (41)$$

We now assume that (40) is an equality. By (41) and (30),

$$\phi(t) = \overline{\phi}(t) \quad \forall t \in [0, T] . \quad (42)$$

Let now X be in $C^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$. For any such X, we have

$$\begin{aligned} & \frac{d}{dt} \left(L_X V(t, \overline{\phi}(t)) \right) \\ &= \frac{\partial L_X V}{\partial t}(t, \overline{\phi}(t)) + \left(L_{\sum_{k=1}^m \overline{u}_k f_k} L_X V \right)(t, \overline{\phi}(t)) \\ &= \left(L_{\mathcal{L}(\overline{u}) X} V \right)(t, \overline{\phi}(t)) \\ &\quad + L_X \left(\frac{\partial V}{\partial t} + \sum_{k=1}^m \overline{u}_k L_{f_k} V \right)(t, \overline{\phi}(t)) . \end{aligned}$$

Hence, from (35),

$$\frac{d}{dt} \left(L_X V(t, \overline{\phi}(t)) \right) = \left(L_{\mathcal{L}(\overline{u}) X} V \right)(t, \overline{\phi}(t)) \quad , \quad (43)$$

and, by induction on p,

$$\frac{d^p}{dt^p} \left(L_X V(t, \overline{\phi}(t)) \right) = \left(L_{\mathcal{L}(\overline{u})^p X} V \right)(t, \overline{\phi}(t)) \quad . \quad (44)$$

By (44), (42) and (41), we have

$$\left(L_{\mathcal{L}(\overline{u})^p f_k} V \right) \left(\frac{3T}{4}, \overline{\phi}\left(\frac{3T}{4}\right) \right) = 0 \qquad \begin{array}{l} k = 1, \dots, m \\ p \geq 0 \end{array} \quad (45)$$

Hence, by (15),

$$\frac{\partial V}{\partial x}\left(\frac{3T}{4}, \overline{\phi}\left(\frac{3T}{4}\right) \right) = 0 \quad . \quad (46)$$

Since $\phi(\overline{u})^{-1}(\frac{3T}{4}, .)$ is a diffeomorphism, (29) implies, together with (46), that $\overline{\phi}(\frac{3T}{4}) = 0$ and therefore that $x = 0$. This proves (37) and completes the proof of the theorem. ∎

4 Conclusion

Theorem 5 leads to a design method, made of three separate parts :

1. Find $\overline{u}$ meeting (12), (13) and (14). Existence of such a $\overline{u}$ is established by proposition 3. The proof of proposition 3 actually provides a guide to find a suitable $\overline{u}$.
2. Compute V from $\overline{u}$ according to (29).
3. The control u is then given by (30).

The method proposed in [7] for the case when assumption (18) was met was a particular case of this one. Finding a suitable $\overline{u}$ was however, due to the additionnal assumption, somehow simpler : it may be taken of the form $(\alpha, 0, \dots, 0)$ with α explicitly given by (28).

The method proposed in [3] for the same general case as here differs from the present one at steps "2" and "3" above. In [3], these two steps are replaced by a construction of u from $\overline{u}$ which relies on the controllability by impulsive controls of the time-varying linear system obtained with $\overline{u}$. This construction leads to much more complicated computations than those of step 2 and 3, even when the f_k's are simple (e.g. when assumption (18) is met).

We have therefore been able to derive here a method keeping most of the advantages of [7], but not restricted to the simpler case of assumption (18). The hard part in the design process remains the choice of a suitable $\overline{u}$. It turns out however that, in many practical cases, not only a suitable $\overline{u}$ may be exhibited, but there is a wide choice of possible solutions, and $\overline{u}$ may be considered as a "design" parameter. An interesting subject for future research is therefore to study the relationship between the choice of $\overline{u}$ and the "performances" of the controller.

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