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SMOOTH STABILIZING TIME-VARYING CONTROL LAWS FOR A CLASS OF NONLINEAR SYSTEMS. APPLICATION TO MOBILE ROBOTS

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Abstract

Let $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a smooth map. We assume that for all x in $\mathbb{R}^n - \{0\}$ there exists α in $\mathbb{N}^m$ such that $x\,(\frac{\partial^{|\alpha|} f}{\partial u^\alpha})\,(x,0)$ is not zero. We prove that $\dot{x} = v\,f(x,u)$ can be globally asymptotically stabilized by means of a time-varying feedback law $v = v(x,t)$, $u = u(x,t)$. We give an application to the design of time-varying control laws for nonholonomic wheeled mobile robots.

Keywords :

Nonlinear systems, Asymptotic stabilization, Time-varying feedback law, Mobile robots.

1 Introduction

Let $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ and $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ be two maps of class C^∞. We assume :

$$V(x) \longrightarrow +\infty \text{ when } |x| \longrightarrow +\infty \qquad (1)$$

$$V(0) = 0 \qquad (2)$$

$$V(x) > 0 \text{ for all } x \in \mathbb{R}^n - \{0\} \qquad (3)$$

Let $q : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}$ be defined by :

$$q(x,u) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(x)\, f_i(x,u) \qquad (4)$$

We are interested in the asymptotic stabilization of the control system :

$$\dot{x} = v\, f(x,u) \qquad (5)$$

where the control is $(v,u) \in \mathbb{R} \times \mathbb{R}^m$. Our main result is : (with $\frac{\partial^{|\alpha|} f}{\partial u^\alpha} = \frac{\partial^{\alpha_1}}{\partial u_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_m} f}{\partial u_m^{\alpha_m}}$)

Theorem 1 *Assume that :*

$$\forall x \in \mathbb{R}^n - \{0\},\ \exists \alpha \in \mathbb{N}^m \text{ s.t. } \frac{\partial^{|\alpha|} q}{\partial u^\alpha}(x,0) \neq 0 \qquad (6)$$

Then for all T in $[0,+\infty)$ there exist $u : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^m$ and $v : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$ such that :

$$u \in C^\infty(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m) \qquad (7)$$

$$v \in C^\infty(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}) \qquad (8)$$

$$u(x,t+T) = u(x,t) \text{ for all } (x,t) \in \mathbb{R}^n \times \mathbb{R} \qquad (9)$$

$$v(x,t+T) = v(x,t) \text{ for all } (x,t) \in \mathbb{R}^n \times \mathbb{R} \qquad (10)$$

$$u(0,t) = 0 \text{ for all } t \in \mathbb{R} \qquad (11)$$

$$v(0,t) = 0 \text{ for all } t \in \mathbb{R} \qquad (12)$$

0 is a globally asymptotically stable point of :

$$\dot{x} = v(x,t)\, f(x, u(x,t)) \qquad (13)$$

In the abstract we have taken $V = \frac{1}{2} |x|^2$.
Let us remark that our assumptions do not imply that $\dot{x} = v\,f(x,u)$ can be asymptotically stabilized by means of at least continuous static state feedback laws $u = u(x)$, $v = v(x)$. For example let us choose $n = 2$, $m = 1$, $f(x_1, x_2, u) = (u, 1)$. Then system (5) becomes :

$$\begin{cases} \dot{x}_1 = u\,v \\ \dot{x}_2 = v \end{cases} \qquad (14)$$

We take $V(x) = \frac{x_1^2 + x_2^2}{2}$. Then (6) is satisfied but the image of the map $(x_1, x_2, u, v) \longrightarrow (uv, v)$ does not contain a neighborhood of $(0,0)$. Hence, by a Theorem due to Brockett [4], system (14) cannot be asymptotically stabilized by means of a continuous static state feedback law. Note that the advantage of time-varying feedback laws compared to static feedback laws has already been noticed by various authors, e.g. [13,7,10]; concerning the time-varying stabilization of mobile robots see also [11,12].
In **Section 2** we will give the proof of Theorem 1 when $m = 1$ and in **Section 3** when $m \geq 2$. In **Section 4** we illustrate the design of time-varying control laws by stabilizing nonholonomic wheeled mobile robots.

2 Proof of Theorem 1 when $m = 1$

Let $\omega : \mathbb{R}^n \longrightarrow \mathbb{R}$ be such that :

$$\omega \in C^\infty(\mathbb{R}^n, \mathbb{R}) \qquad (15)$$

$$\omega(0) = 0 \qquad (16)$$

$$\forall x \neq 0,\ \omega(x) = 0 \Longrightarrow q(x,0) \neq 0 \qquad (17)$$

$$\begin{cases} \forall x \text{ s.t. } \omega(x) \neq 0, \ \exists r \in \mathbf{N} \text{ s.t.} \\ \forall u \in [-|\omega(x)|, |\omega(x)|], \ \frac{\partial^r q}{\partial u^r}(x,u) \neq 0 \end{cases} \tag{18}$$

Note that by straightforward arguments relying on partition of unity one gets that there exists such a ω (note that we may also impose $\omega > 0$ on $\mathbf{R}^n - \{0\}$). Let now $g : \mathbf{R} \longrightarrow \mathbf{R}$ be such that :

$$g \in C^\infty(\mathbf{R}, [-1,1]) \tag{19}$$

$$g(t+T) = g(t) \ \forall t \in \mathbf{R} \tag{20}$$

$$\forall t \in \mathbf{R}, \ \exists l \in \mathbf{N} - \{0\} \text{ s.t. } g^{(l)}(t) \neq 0 \tag{21}$$

For example we can take $g(t) = sin(\frac{2\pi t}{T})$. Note also that generic g satisfying (19)-(20) satisfy (21). Let $u : \mathbf{R}^n \times \mathbf{R} \longrightarrow \mathbf{R}$, $v : \mathbf{R}^n \times \mathbf{R} \longrightarrow \mathbf{R}$ be defined by :

$$u(x,t) = \omega(x)\, g(t) \tag{22}$$

$$v(x,t) = -q(x, u(x,t)) \tag{23}$$

Then, using (15), (16), (19) and (20) we get (7)-(11). Moreover by (2) and (3) we have :

$$\nabla V(0) = 0 \tag{24}$$

Therefore using (4), (23) we have (12). It remains to prove (13). Let $\phi : \mathbf{R}^n \times [0, +\infty) \longrightarrow \mathbf{R}^n$ be defined by :

$$\frac{\partial \phi}{\partial t} = v(\phi, t)\, f(\phi, u(\phi, t)), \ \phi(x, 0) = x \tag{25}$$

We have :

$$\frac{\partial}{\partial t} V(\phi) = -v^2(\phi, t) \leq 0 \tag{26}$$

In particular, by (1), ϕ is defined on all $\mathbf{R}^n \times [0, \infty) \longrightarrow \mathbf{R}^n$. Let us assume for the moment that :

$$\forall y \in \mathbf{R}^n - \{0\}, \forall t, \ \exists k \in \mathbf{N} \text{ s.t. } \frac{\partial^k v}{\partial t^k}(y,t) \neq 0 \tag{27}$$

We are going to prove :

$$\begin{cases} V(\phi(x,t_2)) < V(\phi(x,t_1)), \ \forall x \in \mathbf{R}^n - \{0\}, \\ \forall (t_1, t_2) \in [0,\infty)^2 \text{ with } t_1 < t_2 \end{cases} \tag{28}$$

Note that by LaSalle's invariance principle in the periodic case (see [8, Thm. 55.1]), (28) with (1)-(3) implies (13). Let x in $\mathbf{R}^n$ and (t_1, t_2) in $[0,\infty)^2$ with

$$t_1 < t_2 \tag{29}$$

$$V(\phi(x,t_1)) = V(\phi(x,t_2)) \tag{30}$$

Using (26), (29) and (30) we get :

$$v(\phi(x,t), t) = 0, \ \forall t \in [t_1, t_2] \tag{31}$$

Let $y = \phi(x, t_1)$. By (25) and (31) we get :

$$\phi(x,t) = y, \ \forall t \in [t_1, t_2] \tag{32}$$

Differentiating (31) with respect to t and using (32) we get :

$$\frac{\partial^k v}{\partial t^k}(y,t) = 0, \ \forall k \in \mathbf{N}, \ \forall t \in [t_1, t_2] \tag{33}$$

Hence by (27) $y = 0$ and therefore

$$x = 0 \tag{34}$$

which proves (28). It remains to prove (27). Fix $(y_0, t_0) \in \mathbf{R}^n \times \mathbf{R}$. If $\omega(y_0) = 0$ and $y_0 \neq 0$ we have by (17) $v(y_0, t_0) \neq 0$. Hence we may assume :

$$\omega(y_0) \neq 0 \tag{35}$$

By (18) there exists an integer l such that :

$$\frac{\partial^l q}{\partial u^l}(y_0, \omega(y_0) g(t_0)) \neq 0 \tag{36}$$

and

$$\frac{\partial^r q}{\partial u^r}(y_0, \omega(y_0) g(t_0)) = 0, \ \forall r \in [0, l-1] \tag{37}$$

If $l = 0$ we have already $v(y_0, t_0) \neq 0$. We assume now that $l \geq 1$ and that :

$$\frac{\partial^k v}{\partial t^k}(y_0, t_0) = 0, \ \forall k \in \mathbf{N} \tag{38}$$

Using (37) we have :

$$\frac{\partial^l v}{\partial t^l}(y_0, t_0) = -\frac{\partial^l q}{\partial u^l}(y_0, \omega(y_0) g(t_0))\, (\omega(y_0) \dot{g}(t_0))^l \tag{39}$$

By (39), (38) and (36) we get :

$$\dot{g}(t_0) = 0 \tag{40}$$

Using (37) and (40) we have :

$$\frac{\partial^{2l} v}{\partial t^{2l}}(y_0, t_0) = -\frac{(2l)!}{l! 2^l} \frac{\partial^l q}{\partial u^l}(y_0, \omega(y_0) g(t_0))\, (\omega(y_0) \ddot{g}(t_0))^l \tag{41}$$

Hence by (36) and (38) we have :

$$\ddot{g}(t_0) = 0 \tag{42}$$

Keep going, we get by an easy induction argument :

$$g^{(r)}(t_0) = 0, \ \forall r \in \mathbf{N} - \{0\} \tag{43}$$

which is in contradiction with (21) and concludes the proof. □

Remark 1 *One could replace (21) by*

$$\dot{g} \not\equiv 0 \tag{44}$$

With (44) instead of (21), (28) does not necessarily hold but it follows from our study that (44) implies

$$V(\phi(x,T)) < V(\phi(x,0)), \ \forall x \in \mathbf{R}^n - \{0\} \tag{45}$$

and (45) clearly implies (13).

3 Proof of Theorem 1 when $m \geq 2$

In order to reduce the case $m \geq 2$ to the one-dimensional case, we first introduce, using (6), for any x in $\mathbf{R}^n - \{0\}$ a vector $s(x)$ in $\mathbf{R}^m - \{0\}$ such that :

$$| s(x) | = 1 \tag{46}$$

$$\exists r \in \mathbf{N} \text{ s.t. } h^{(r)}(0) \neq 0 \text{ where } h : \mathbf{R} \longrightarrow \mathbf{R} \quad h(\lambda) = q(x, \lambda s(x)) \tag{47}$$

By continuity we get that there exists a real number $\epsilon(x)$ such that :

$$0 < \epsilon(x) < | x | \tag{48}$$

$$\begin{cases} \forall y \in B(x; \epsilon(x)),\ \forall \lambda \in [-\epsilon(x), \epsilon(x)] \\ \text{there exists } r \in \mathbf{N} \text{ s.t. } \dfrac{\partial^r \tilde{q}}{\partial \lambda^r}(y, \lambda) \neq 0 \end{cases} \tag{49}$$

where $B(x; \epsilon(x))$ denotes the open ball of $\mathbf{R}^n$ centered at x with radius $\epsilon(x)$ and $\tilde{q}(y, \lambda) = q(y, \lambda s(x))$.
Since $\mathbf{R}^n - \{0\}$ is a countable union of compact sets, there exists a sequence of points $(x_i)_{i \in \mathbf{N}}$ in $\mathbf{R}^n - \{0\}$ such that :

$$\mathbf{R}^n - \{0\} = \cup_{i \in \mathbf{N}} B(x_i; \frac{\epsilon(x_i)}{2}) \tag{50}$$

For i in $\mathbf{N}^*$, let ω_i and g_i be such that :

$$\omega_i \in C^\infty(\mathbf{R}^n, \mathbf{R}),\ g_i \in C^\infty(\mathbf{R}, \mathbf{R}) \tag{51}$$

$$Supp\, \omega_i \subset B(x_i; \epsilon(x_i)) \tag{52}$$

$$\omega_i > 0 \text{ on } B(x_i; \frac{\epsilon(x_i)}{2}) \tag{53}$$

$$Max\, \{ | \frac{\partial^{|\alpha|} \omega_i}{\partial x^\alpha}(x) |,\ x \in \mathbf{R}^n,\ | \alpha | \leq i \} \leq 2^{-i} \tag{54}$$

$$Supp\, g_i \in [\frac{T}{2^{i+1}}, \frac{T}{2^i}] \tag{55}$$

$$Max\, \{ | g_i^{(r)}(t) |,\ t \in \mathbf{R},\ r \leq i \} \leq 2^{-i} \tag{56}$$

$$g_i \not\equiv 0 \tag{57}$$

We define u on $\mathbf{R}^n \times (0, T]$ by :

$$u(x,t) = \sum_{i \in \mathbf{N}^*} \omega_i(x)\, g_i(t)\, s(x_i) \tag{58}$$

Note that (x, t) being given, in the above sum there is at most one term which is not zero. One easily checks :

$$u \in C^\infty(\mathbf{R}^n \times (0, T]) \tag{59}$$

$$u(x,t) = 0\ \forall (x,t) \in \mathbf{R}^n \times [T/2, T] \tag{60}$$

and

$$u(0,t) = 0 \tag{61}$$

Moreover, using (46), (54), (56) and (58) we get that :

$$\forall \alpha \in \mathbf{N}^n,\ \forall k \in \mathbf{N}, \quad Max\, \{ \frac{\partial^{|\alpha|+k} u(x,t)}{\partial x^\alpha \partial t^k},\ x \in \mathbf{R}^n \} \longrightarrow 0 \text{ as } t \longrightarrow 0 \tag{62}$$

Hence, by (59) and (60), if we extend u to all $\mathbf{R}^n \times \mathbf{R}$ by requiring that u is T-periodic with respect to t, then :

$$u \in C^\infty(\mathbf{R}^n \times \mathbf{R}; \mathbf{R}^m) \tag{63}$$

We now define v by :

$$v(x,t) = -q(x, u(x,t)) \tag{64}$$

then (u, v) satisfies (7)-(12). It remains to prove (13). We have easily :

$$\begin{cases} V(\phi(x,t_2)) \leq V(\phi(x,t_1)),\ \forall x \in \mathbf{R}^n \\ \forall (t_1, t_2) \in [0, \infty)^2 \text{ with } t_1 \leq t_2 \end{cases} \tag{65}$$

We claim that :

$$V(\phi(x,T)) < V(\phi(x,0)),\ \forall x \in \mathbf{R}^n - \{0\} \tag{66}$$

Note that (1)-(3) and (66) imply (13). Let $x \in \mathbf{R}^n - \{0\}$, by (50) there exists an integer i such that :

$$x \in B(x_i; \frac{\epsilon(x_i)}{2}) \tag{67}$$

Using Section 2 (see in particular Remark 1), (46), (49), (53), (55)-(58) we get :

$$V(\phi(x, \frac{T}{2^i})) < V(\phi(x, \frac{T}{2^{i+1}})) \tag{68}$$

From (65) and (68) we obtain (66) which concludes the proof of Theorem 1. □

4 Application to mobile robots

Mobile robots constitute a typical example of non-holonomic systems (see e.g. [2,3,9,11,5]). We consider here a robot moving on an horizontal plane, constituted by a rigid trolley equipped with non deformable wheels. The contact between the wheels and the ground satisfies the conditions of **pure rolling and non slipping**.
Consider an inertial reference frame $\{0, I_1, I_2\}$ in the plane of motion. Define a reference point Q on the trolley, and a basis $\{e_1, e_2\}$ attached to the trolley (see Fig. 1). The position of the trolley in the plane is completely specified by the following 3 variables :

- x_1, x_2 : the coordinates of the reference point Q in the inertial frame,
- θ : the orientation of the basis $\{e_1, e_2\}$ with respect to the inertial basis.

We will consider the case of a 3-wheeled mobile robot. The 2 front wheels (index 2 and 3) have a fixed orientation while the orientation β of wheel 1 is varying. The rotation angles of the 3 wheels are denoted ϕ_i, $i = 1, \cdots, 3$. The reference point Q is the center of the segment B_2B_3. The basis vector x_1 is aligned with B_2B_3, see Fig. 2 and for more details [1].

The robot motion is then completely described by the following vector of 7 generalized coordinates :

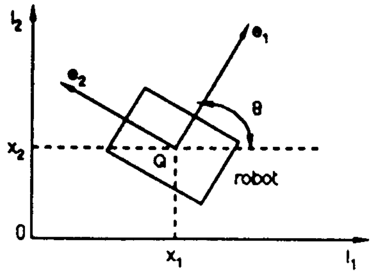

Figure 1: A mobile robot

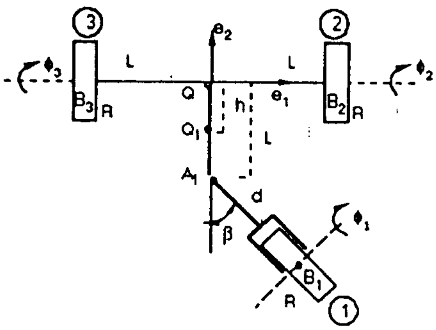

Figure 2: 3-wheels example

$$q(t) = (x_1 \;\; x_2 \;\; \theta \;\; \beta \;\; \phi_1 \;\; \phi_2 \;\; \phi_3)' \qquad (69)$$

A' denoting the transpose of A.
The system has 2 degrees of freedom (see [1]) and we consider two generalized forces applied to the system as follows :

- **Case 1**: The 2 motors provide the torques for the rotation of wheels 2 and 3, wheel 1 being self-aligning.
- **Case 2** : The 2 motors are implemented on wheel 1, the first one for its orientation, the second one for its rotation.

Using the Euler-Lagrange formalism to derive the dynamical model of the robot and the technique of Lagrange multipliers to deal with the pure rolling and the non slipping constraints, we obtain a general state-space representation of the system, which can be simplified by a first static state feedback law as explained in [1] :

$$\begin{cases} \dot{x}_1 = -\eta_1 sin\theta \\ \dot{x}_2 = \eta_1 cos\theta \\ \dot{\theta} = \eta_2 \\ \dot{\beta} = D_1(\beta)P\eta \\ \dot{\phi} = D_2(\beta)P\eta \\ \dot{\eta}_1 = v_1 \\ \dot{\eta}_2 = v_2 \end{cases} \qquad (70)$$

Due to the expression of $D_1(\beta)$ and $D_2(\beta)$, $\dot{\beta}$ and $\dot{\phi}$ will uniformly tend to zero provided η asymptotically converges to zero. Consequently, system (70) can be reduced to (see [1] for details) :

$$\begin{cases} \dot{x}_1 = -\eta_1 sin\theta \\ \dot{x}_2 = \eta_1 cos\theta \\ \dot{\theta} = \eta_2 \\ \dot{\eta}_1 = v_1 \\ \dot{\eta}_2 = v_2 \end{cases} \qquad (71)$$

By Brockett's Theorem [4], there is no smooth feedback law stabilizing system (71). Nevertheless static state feedback laws ensuring the stabilization of x_1, x_2, η with $\dot{\theta}$ converging to zero have been exhibited in [1,6], but nothing could be said about θ.
We will now apply Theorem 1 to compute globally stabilizing time-varying smooth control laws for system (71).
We first consider the reduced system :

$$\begin{cases} \dot{x}_1 = -\eta_1 sin\theta \\ \dot{x}_2 = \eta_1 cos\theta \end{cases} \qquad (72)$$

Theorem 1 applies with the controls $u = \theta$, $v = \eta_1$, $m = 1$, $n = 2$, $f(x,u) = (-sinu, cosu)'$ and $V(x) = \frac{x_1^2 + x_2^2}{2}$. Then :

$$q(x,u) = -x_1 sinu + x_2 cosu \qquad (73)$$

clearly satisfies assumption (6). Hence, by Theorem 1, system (72) can be globally asymptotically stabilized by means of smooth time-varying feedback laws. Moreover, section 2 gives a method to produce explicitely such a feedback law. For example one can take :

$$\begin{cases} \omega(x) = x_1 \\ g(t) = sint \end{cases} \qquad (74)$$

it is an easy computation to check that $\omega(x)$ satisfies (15)-(18) and that $g(t)$ satisfies (19)-(21). Consequently from (22) and (23) we deduce that the following time-varying smooth feedback law stabilizes system (72) :

$$\begin{cases} \theta = x_1\, sint \\ \eta_1 = -q(x,\theta) = x_1\, sin(x_1\, sint) - x_2\, cos(x_1\, sint) \end{cases} \qquad (75)$$

To obtain now a stabilizing control law for the complete system (71), we have to consider the problem of stabilizing cascaded systems. More precisely, a system (Σ) obtained by adding pure integrators to a system (Σ') which can be globally asymptotically stabilized by means of a smooth time-varying feedback law, can also be globally asymptotically stabilized by means of a smooth time-varying feedback law.
This is a straightforward time-varying formulation of a result due to Tsinias [14] (see also [7]). More precisely we have the following lemma :

Lemma 1 *Let us consider system* (Σ') *given by :*

$$\begin{aligned} \dot{x} = F(x, u_1, u_2) \,,\ x \in \mathbf{R}^n \,, \\ u = (u_1, u_2)' : \mathbf{R}^n \times \mathbf{R} \longrightarrow \mathbf{R}^{m_1} \times \mathbf{R}^{m_2} \end{aligned} \tag{76}$$

F belonging to $C^\infty(\mathbf{R}^n \times \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}; \mathbf{R}^n)$.
We make the following assumption on system (76) :
A1 : *there exists a* C^r *time-varying feedback law* $u(x,t)$ *such that :*

$$u(0,t) = 0 \ \text{for all}\, t \in \mathbf{R} \tag{77}$$

$$u(x, t+T) = u(x,t) \ \text{for all}\, (x,t) \in \mathbf{R}^n \times \mathbf{R} \tag{78}$$

making 0 *in* $\mathbf{R}^n$ *a globally asymptotically stable point of* $\dot{x} = F(x, u(x,t))$.
Under assumption A1, the following system (Σ)

$$\begin{cases} \dot{x} = F(x, v_1, y) \\ \dot{y} = v_2 \end{cases} \tag{79}$$

where $y \in \mathbf{R}^{m_2}$, *the control* $v = (v_1, v_2)' \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}$, *is such that* 0 *in* $\mathbf{R}^n \times \mathbf{R}^{m_2}$ *is globally asymptotically stabilizable using a* C^{r-1} *time-varying feedback law* $v(x,y,t)$ *such that :*

$$v(0,0,t) = 0 \ \text{for all}\, t \in \mathbf{R} \tag{80}$$

$$\begin{aligned} v(x,y,t+T) = v(x,y,t) \\ \text{for all}\, (x,y,t) \in \mathbf{R}^n \times \mathbf{R}^{m_2} \times \mathbf{R} \end{aligned} \tag{81}$$

Proof :
From assumption **A1**, using a classical converse of Lyapunov's second Theorem (see [8, Thm. 49.4]), we deduce that system (76) admits a T-periodic Lyapunov function $V(x,t) \in C^\infty(R^n \times R; [0, +\infty))$ satisfying :

$$V(0,t) = 0 \,,\ \forall t \in \mathbf{R} \tag{82}$$

$$V(x,t) > 0 \,,\ \forall (x,t) \in (\mathbf{R}^n - \{0\}) \times \mathbf{R} \tag{83}$$

$$V(x,t+T) = V(x,t) \,,\ \forall (x,t) \in \mathbf{R}^n \times \mathbf{R} \tag{84}$$

$$\lim_{|x| \to +\infty} Min(V(x,t) \,,\ t \in [0,T]) = +\infty \tag{85}$$

$$\begin{cases} V(\phi_{\Sigma'}(x,t_2),t_2) \le V(\phi_{\Sigma'}(x,t_1),t_1) \\ \forall x \in \mathbf{R}^n \,,\ \text{for all positive}\ t_1, t_2\ \text{s.t.}\ t_1 \le t_2 \end{cases} \tag{86}$$

$$V(\phi_{\Sigma'}(x,T),T) < V(\phi_{\Sigma'}(x,0),0) \,,\ \forall x \in \mathbf{R}^n - \{0\} \tag{87}$$

$\phi_{\Sigma'}$ denoting the flow of $\dot{x} = F(x, u(x,t))$.
We could in fact assume that (86) holds with strict inequalities, but in our case (see (26) and Remark 1) it is more useful to assume only (86)-(87).

Let us now introduce the following time-varying feedback law for system (Σ) :

$$\begin{cases} v_1(x,y,t) = u_1(x,t) \\ v_2(x,y,t) = \dfrac{\partial u_2}{\partial t}(x,t) - (y - u_2(x,t)) \\ + \displaystyle\sum_{i=1}^{n} \frac{\partial u_2}{\partial x_i}(x,t)\, F_i(x, v_1(x,t), y) - \\ H(x, u_1(x,t), u_2(x,t), y) \left(\dfrac{\partial V}{\partial x}(x,t)\right)' \end{cases} \tag{88}$$

where $H \in C^\infty(\mathbf{R}^n \times \mathbf{R}^{m_1} \times \mathbf{R}^{m_2} \times \mathbf{R}^{m_2}; \mathcal{L}(\mathbf{R}^{m_2}; \mathbf{R}^n))$ satisfies :

$$F(x,u_1,y) - F(x,u_1,u_2) = H'(x,u_1,u_2,y)(y - u_2) \tag{89}$$

$\mathcal{L}(\mathbf{R}^{m_2}; \mathbf{R}^n)$ denoting the set of linear maps from $\mathbf{R}^{m_2}$ into $\mathbf{R}^n$. Then we can check that v is a globally asymptotically stabilizing feedback law for system (Σ) by considering the following Lyapunov function $W : \mathbf{R}^n \times \mathbf{R}^{m_2} \times \mathbf{R} \longrightarrow [0, +\infty)$ defined by :

$$W(x,y,t) = \frac{1}{2} \mid y - u_2(x,t) \mid^2 + V(x,t) \tag{90}$$

then (v, W) satisfies eq. (77) to (87) modified in a natural way as follows :

$$W(0,0,t) = 0 \,,\ \forall t \in \mathbf{R} \tag{91}$$

$$\begin{aligned} W(x,y,t) > 0 \,, \\ \forall (x,y,t) \in (\mathbf{R}^n \times \mathbf{R}^{m_2}) - \{(0,0)\}) \times \mathbf{R} \end{aligned} \tag{92}$$

$$\begin{aligned} W(x,y,t+T) = W(x,y,t) \,, \\ \forall (x,y,t) \in \mathbf{R}^n \times \mathbf{R}^{m_2} \times \mathbf{R} \end{aligned} \tag{93}$$

$$\lim_{|x|+|y| \to +\infty} Min(W(x,y,t) \,,\ t \in [0,T]) = +\infty \tag{94}$$

$$\begin{cases} W(\phi_\Sigma(x,y,t_2),t_2) \le W(\phi_\Sigma(x,y,t_1),t_1) \\ \forall (x,y,t_1,t_2) \in \mathbf{R}^n \times \mathbf{R}^{m_2} \times \mathbf{R} \times \mathbf{R} \\ \text{with}\ t_1 \le t_2 \end{cases} \tag{95}$$

$$\begin{aligned} W(\phi_\Sigma(x,y,T),T) < W(\phi_\Sigma(x,y,0),0) \\ \forall (x,y) \in (\mathbf{R}^n \times \mathbf{R}^{m_2}) - \{(0,0)\} \end{aligned} \tag{96}$$

ϕ_Σ denoting the flow of

$$\begin{cases} \dot{x} = F(x, v_1(x,t), y) \\ \dot{y} = v_2(x,y,t) \end{cases} \tag{97}$$

This concludes the proof. □

Let us now apply Lemma 1 to stabilize our system (71). More precisely in our context this procedure leads to the following steps :

1st step
Considering the reduced system (72), let us recall that the time-varying control law (75) is smooth and globally stabilizing :

$$\begin{cases} \eta_1 = -q(x,\theta) = x_1\, sin(x_1\, sint) - x_2\, cos(x_1\, sint) \\ \theta = x_1\, sint \end{cases} \tag{98}$$

An associated Lyapunov function is for example $V(x_1, x_2)$ given by :

$$V(x_1,x_2) = \frac{1}{2}\,(x_1^2 + x_2^2) \tag{99}$$

2nd step
We consider now system (72) with two integrators as follows :

$$\begin{cases} \dot{x}_1 = -\eta_1 sin\theta \\ \dot{x}_2 = \eta_1 cos\theta \\ \dot{\eta}_1 = v_1 \\ \dot{\theta} = \eta_2 \end{cases} \tag{100}$$

The previous control variables η_1 and θ are both delayed once and applying Lemma 1, we have to compute the new stabilizing control $(v_1, \eta_2)' = \bar{u}(x_1, x_2, \eta_1, \theta, t)$ using equation (88) where :

- $y = (\eta_1, \theta)'$,
- $F(x_1, x_2, y) = (-\eta_1 sin\theta \ , \ \eta_1 cos\theta)'$,
- $V(x_1, x_2)$ is given by (99)
- and u_2 is given by (98), i.e. :

$$u_2 = \begin{pmatrix} u_{21} \\ u_{22} \end{pmatrix} = \begin{pmatrix} x_1 \, sin(x_1 \, sint) - x_2 \, cos(x_1 \, sint) \\ x_1 \, sint \end{pmatrix} \quad (101)$$

We obtain :

$$\begin{aligned}
&\bar{u}(x_1, x_2, \eta_1, \theta, t) = \begin{pmatrix} \bar{u_1} \\ \bar{u_2} \end{pmatrix} \text{ with} \\
&\bar{u_1} = x_1 \, cost \, [x_1 \, cos(x_1 sint) + x_2 \, sin(x_1 sint)] \\
&-\eta_1 + x_1 \, sin(x_1 sint) - x_2 \, sin(x_1 sint) \\
&-sin(x_1 sint)\,(1 + x_2 sint)\,\eta_1 \, sin\theta \\
&-x_1 \, sint \, cos(x_1 sint)\,\eta_1 \, sin\theta \\
&+cos(x_1 sint)\,\eta_1 \, cos\theta \\
&+x_1 sinu_{22} - x_2 cosu_{22} \text{ and} \\
&\bar{u_2} = x_1 \, cost - \theta + x_1 \, sint - \eta_1 \, sint \, sin\theta \\
&+x_1 \eta_1 S(\frac{u_{22} - \theta}{2}) cos(\frac{u_{22} + \theta}{2}) \\
&+x_2 \eta_1 S(\frac{\theta - u_{22}}{2}) sin(\frac{u_{22} + \theta}{2})
\end{aligned} \quad (102)$$

where $S(x) = \frac{sinx}{x}$ and u_{22} is given by (101).

Using (90) we deduce the associated Lyapunov function :

$$\begin{aligned}
&\bar{V}(x_1, x_2, \eta_1, \theta, t) = \frac{1}{2}(x_1^2 + x_2^2 + (\theta - x_1 sint)^2 \\
&+(\eta_1 + x_2 cos(x_1 sint) - x_1 sin(x_1 sint))^2)
\end{aligned} \quad (103)$$

3rd step

Finally, to stabilize system (71), we have to delay once more the previous control variable η_2 and applying Lemma 1, we obtain the final stabilizing control $(v_1, v_2)'$:

$$\begin{cases}
v_1 = \bar{u}_1 \\
v_2 = \frac{\partial \bar{u}_2}{\partial t} - (\eta_2 - \bar{u}_2) - \frac{\partial \bar{u}_2}{\partial x_1}\eta_1 sin\theta \\
+\frac{\partial \bar{u}_2}{\partial x_2}\eta_1 cos\theta + \frac{\partial \bar{u}_2}{\partial \eta_1}\bar{u}_1 \\
+\frac{\partial \bar{u}_2}{\partial \theta}\eta_2 - (\theta - x_1 sint)
\end{cases} \quad (104)$$

where $\bar{u} = (\bar{u}_1, \bar{u}_2)'$ is given by (102).

Using once more (90) we deduce the associated Lyapunov function for system (71) :

$$W(x_1, x_2, \eta_1, \theta, \eta_2, t) = \bar{V}(x_1, x_2, \eta_1, \theta, t) + \frac{1}{2}(\eta_2 - \bar{u}_2)^2 \quad (105)$$

where $\bar{V}$ is given by (103). □

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