LINKS BETWEEN LOCAL CONTROLLABILITY AND LOCAL CONTINUOUS STABILIZATION

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Abstract: We prove that a control system which satisfies well known sufficient conditions for small time local controllability — for example the Hermes Condition — can be dynamically locally asymptotically stabilized by means of a continuous time-varying feedback law. For special systems (including systems without drift) we get local stabilization in finite time by means of a continuous time-varying feedback law.

Keywords: Nonlinear control systems, local controllability, local stabilization, time-varying feedback law.

1. Introduction

For f in $C^\infty(\mathbf{R}^n \times \mathbf{R}^m; \mathbf{R}^n)$ we consider the control system

$$\dot{x} = f(x, u) \tag{1.1}$$

where u in $\mathbf{R}^m$ is the control. H. Sussmann and V. Jurdjevic have proved in [SJ] that the set of reachable points from x_0 in small time and with small controls has x_0 in the closure of its interior if (and only if f is analytic)

$$f(x_0, 0) = 0\,, \tag{1.2}$$

$$\left\{h(x_0); h \in \text{Lie}\left\{\frac{\partial^{|\alpha|} f}{\partial u^\alpha}(\cdot, 0); \alpha \in \mathbf{N}^m\right\}\right\} = \mathbf{R}^n \tag{1.3}$$

where, for a family $\mathcal{F}$ of vector fields on $\mathbf{R}^n$, Lie $\mathcal{F}$ denotes the Lie algebra generated by the vector fields in $\mathcal{F}$. Condition (1.3) for all x_0 in $\mathbf{R}^n$ implies, for special f, the complete controllability of $\dot{x} = f(x, u)$. This is in particular the case if

$$f(x, u) = \sum_{i=1}^{m} u_i f_i(x)\,. \tag{1.4}$$

Let us recall that H. Sussmann has proved in [Su1] that, if f is analytic, the complete controllability implies that $\dot{x} = f(x, u)$ can be steered to the origin by means of a piecewise analytic feedback law: $u = u(x)$. Let us also mention that in [DMK] and [Kaw2] it is proved that if $n = 2$, $m = 1$ and f is affine then local controllability implies that $\dot{x} = f(x, u)$ is asymptotically stabilizable by means of a continuous feedback law. Unfortunately, as it has been shown by R. Brockett in [B], the complete controllability does not imply in more general situations - even if (1.4) holds - that $\dot{x} = f(x, u)$ can be asymptotically stabilized by means of a *continuous* feedback law. For example

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = x_1 u_2 - x_2 u_1 \tag{1.5}$$

is a control system which satisfies (1.3) for all x_0 in $\mathbf{R}^n$ and (1.4). Therefore it is completely controllable, but it is proved in [B] that it cannot be asymptotically stabilized by means of a continuous feedback law. In [Sa] C. Samson has proved that (1.5) can be globally asymptotically stabilized by means of a smooth time-varying feedback law $u = u(x, t)$. It turns out to be true in general under conditions (1.4) and (1.3) for all x_0 in $\mathbf{R}^n \backslash \{0\}$. More precisely it is proved in [C1] (see [P] and [Se] for special cases but with explicit feedback laws; see also [CPo] and [So3]).

Theorem 1.1. Assume (1.3) for all x_0 in $\mathbf{R}^n \backslash \{0\}$ and (1.4). Then, for any positive T, there exists u in $C^\infty(\mathbf{R}^n \times \mathbf{R}; \mathbf{R}^m)$ such that

$$u(0, t) = 0 \quad \text{for all } t \text{ in } \mathbf{R}\,, \tag{1.6}$$

$$u(x, t+T) = u(x, t) \quad \forall (x, t) \in \mathbf{R}^n \times \mathbf{R}\,, \tag{1.7}$$

$$\left.\begin{array}{l}\text{the origin of } \mathbf{R}^n \text{ is a globally asymptotically} \\ \text{stable point of } \dot{x} = f(x\,,\, u(x, t))\end{array}\right\} \tag{1.8}$$

Theorem 1.1 can be slightly generalized in the following

Proposition 1.2. Assume that (1.3) holds for all x_0 in $\mathbf{R}^n \backslash \{0\}$ and that there exists φ in $C^\infty(\mathbf{R}^m; \mathbf{R}^m)$ such that

$$\varphi(0) = 0 \tag{1.9}$$

$$f(x, \varphi(u)) = -f(x, u) \quad \forall (x, u) \in \mathbf{R}^n \times \mathbf{R}^m\,. \tag{1.10}$$

Then the conclusion of theorem 1.1 holds.

Examples 1.3. a) Choosing $\varphi(u) = -u$ we get Theorem 1.1. b) If $f(x, u) = u_1 g(x, u_2), u_1 \in \mathbf{R}, u_2 \in \mathbf{R}^{m-1}$ we can take $\varphi(u_1, u_2) = (-u_1, u_2)$.

We will sketch in Section 2 the modifications of the proof given in [C1] in order to get Proposition 1.2. Let us notice that (1.9) and (1.10) implies $f(x, 0) = 0$ and therefore do not allow a drift term. In presence of a drift term many studies have been carried out on sufficient conditions for the Small Time Local Controllability (STLC — this means that the attainable set from x_0 at time $t > 0$ contains x_0 in its interior for all $t > 0$); see e.g. the recent nice survey [Kaw1] by M. Kawski on this question. An important tool to study STLC is the local approximation cones of the attainable set and the associated families of admissible control variations, see [Kaw1]. For stabilization it seems natural to modify the definition of p-th order tangent vector to the attainable set at zero in the following way (we assume $f(0,0)=0$)

Definition 1.4. For a positive integer p, let D^p be the set of vectors ξ in $\mathbf{R}^n$ such that there exists u in $C^0\left([0,1];L^1((0,1);\mathbf{R}^m)\right)$ such that

$$|u(s)(t)| \leq s \quad \text{for all } (s,t) \text{ in } [0,1]\times[0,1]\,, \tag{1.11}$$

$$u(s)(t) = 0 \quad \text{if } t \geq s \tag{1.12}$$

and

$$\psi(u(s)) = s^p\xi + o(s^p) \quad \text{as } s \longrightarrow 0 \tag{1.13}$$

where $\psi(u(s))$ denotes the value at time 0 of the solution of $\dot{x} = f(x, u(s)(t))$, $x(1) = 0$.
In order to state our main results on the stabilization of systems with a drift term let us introduce some other definitions

Definition 1.5. The system $\dot{x} = f(x,u)$ is locally asymptotically stabilizable by means of a T-periodic feedback law if there exists $u : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^m$ such that

$$u \in C^\infty((\mathbf{R}^n\backslash\{0\})\times\mathbf{R};\mathbf{R}^m)\cap C^0(\mathbf{R}^n\times\mathbf{R};\mathbf{R}^m)\,, \tag{1.14}$$

$$u(0,t) = 0 \quad \text{for all } t \text{ in } \mathbf{R}\,, \tag{1.15}$$

$$u(x,t+T) = u(x,t) \quad \forall (x,t)\in\mathbf{R}^n\times\mathbf{R}\,, \tag{1.16}$$

$$\left.\begin{array}{l}\exists\delta>0 \text{ such that for } |x_0|<\delta,\ t_0\leq t_1 \text{ there}\\ \text{exists one and only one solution on } [t_0,t_1]\\ \text{of } \dot{x}=f(x,u(x,t)),\ x(t_0)=x_0\end{array}\right\}, \tag{1.17}$$

$$\left.\begin{array}{l}0\in\mathbf{R}^n \text{ is a locally asymptotically stable}\\ \text{point of } \dot{x}=f(x\,,\,u(x,t))\end{array}\right\} \tag{1.18}$$

If such a u exists we will say that $\dot{x} = f(x,u)$ is T-LAS.
If, moreover, for all small enough x_0, we have

$$\left(\dot{x} = f(x,u(x,t)) \text{ and } x(0) = x_0\right) \Longrightarrow x(T) = 0 \tag{1.19}$$

we will say that $\dot{x} = f(x,u)$ is T-Locally Stabilizable (T-LS).

Definition 1.6. Let k be an integer; $\dot{x} = f(x,u)$ is k-dynamically T-LAS (resp. T-LS) if the system $\dot{x} = f(x,u)$, $\dot{y} = v$ where the control is $(u,v)\in\mathbf{R}^m\times\mathbf{R}^k$ is T-LAS (resp. T-LS).
Note that

$$T\text{-LS} \Longrightarrow T\text{-LAS}, \tag{1.20}$$

$$T\text{-LAS} \Longrightarrow k\text{-dynamically } T\text{-LAS}, \tag{1.21}$$

$$\left.\begin{array}{l}(k\text{-dynamically } T\text{-LAS and } k\leq k')\\ \Longrightarrow\\ k'\text{-dynamically } T\text{-LAS}\end{array}\right\}. \tag{1.22}$$

Until the end of this paper we will assume

$$f(0,0) = 0\,, \tag{1.23}$$

$$\left\{h(0); h\in \text{Lie}\left\{\frac{\partial^{|\alpha|}}{\partial u^\alpha}f(\cdot,0);\alpha\in\mathbf{N}^m\right\}\right\} = \mathbf{R}^n\,. \tag{1.24}$$

Let us remark that (1.23) and (1.24) for the system $\dot{x} = f(x,u)$ are equivalent to (1.23) and (1.24) for the system $\dot{x} = f(x,y)$, $\dot{y} = u \in \mathbf{R}^m$. Let $D = \cup_{p\geq 1} D^p$ and let $\text{int}(D)$ its interior. We will prove in Section 3

Theorem 1.7. Assume

$$0 \in \text{int}\, D\,. \tag{1.25}$$

Then, for all positive real number T, $\dot{x} = f(x,u)$ is n-dynamically T-LAS.
In Section 5 we will make some remarks concerning (1.25). In particular we will see that the Hermes Condition ([Su2; Section 7.3]) implies (1.25). Let us recall that this condition for $m = 1$ and $f(x,u) = f_0(x) + uf_1(x)$ (see Section 5 for the general case) means that any iterated Lie bracket of f_0 and f_1 with an even number of f_1 and an odd number of f_0 can be expressed at 0 as the sum of iterated Lie brackets containing fewer f_1. Let us notice that the interest of time-varying feedback law (resp. dynamical stabilization) for systems with drift has already been pointed out in [SS] (resp. [CPr]).

It would be interesting to know if (1.25) implies that $\dot{x} = f(x,u)$ is T-LS; our next theorem is a partial result in this direction (see [C3] for other cases; see also [SCW] for different cases and T-LAS, instead of T-LS, but with explicit feedback laws).

Theorem 1.8. Let $u = (u_1,u_2)\in\mathbf{R}^{m-1}\times\mathbf{R}$ and $x = (x_1,x_2)\in\mathbf{R}^{n-1}\times\mathbf{R}$. Assume $n\geq 4$, (1.25),

$$f(x,u) = (f_1(x,u),u_2)\in\mathbf{R}^{n-1}\times\mathbf{R}\simeq\mathbf{R}^n\,, \tag{1.26}$$

$$f_1((0,x_2),(0,u_2)) = 0\ \forall\,(x_2,u_2)\in\mathbf{R}\times\mathbf{R}\,. \tag{1.27}$$

Then $\dot{x} = f(x,u)$ is T-LS for all positive T.

For $n\geq 3$ our next proposition is a consequence of Theorem 1.8.

Proposition 1.9. Assume (1.25) holds. Then for all positive T, $\dot{x} = f(x,u)$ is 1-dynamically T-LS.

Proof: For $n\geq 3$ apply Theorem 1.8 to the system $\dot{x} = f(x,u)$, $\dot{y} = v\in\mathbf{R}$. For $n\leq 2$ see Remark 4.1 b) or [C3].

Our next proposition is a corollary of Theorem 1.8 if $n\geq 4$ and is proved in [C3] if $n\leq 3$

Proposition 1.10. Assume $f(x,u) = \sum_{i=1}^m u_i f_i(x)$. Then $\dot{x} = f(x,u)$ is T-LS for all positive T.

Proof: Since $f(x,u) = \sum_{i=1}^m u_i f_i(x)$, $\dot{x} = f(x,u)$ satisfies the Hermes Condition and therefore (1.25). Moreover without loss of generality we may assume $f_m = e_n$ and $f_i.\ e_n = 0$ for all i in $[1,m-1]$. The conclusion follows from Theorem 1.8 if $n\geq 4$; if $n\leq 3$ see Remark 4.2 b) or [C3].

Some of the proofs are only sketched. The details of these proofs are given in [C2] and [C3]. Section 3 deals with Theorem 1.7, Section 4 deals with Theorem 1.8. One of the tools we use is the study of the controllability of the linearized equations around trajectories. Roughly speaking we will see in Section 2 that, under some Lie algebra rank condition, the linearized equations around a generic family of trajectories near $x_0 = 0$ and with u "small" are locally controllable with impulsive controls. This is a generalization of a result contained in [C1].

Finally let us mention that many results on continuous feedback stabilization have been found recently. For a recent nice survey on this subject see [So2].

2. Study of the linearized equation

Throughout this paper "manifold" always means finite dimensional, Hausdorff second countable manifold of class C^∞. Unless otherwise specified the manifolds have no boundary. For two manifolds V and W, and for p in $\mathbf{N} \cup \{\infty\}$, $C^p(V;W)$ is provided with the Whitney topology (see e.g. [GG;p.42]). On $C^\infty(V;W)$ we define a topology, called the C^∞-topology, in the following way. For an integer k, let $J^k(V;W)$ be the set of k-jets of C^∞-mappings from V into W. Let $(K_i; i \in \mathbf{N})$ be a sequence of compact subsets of V such that $K_i \subset \overset{\circ}{K}_{i+1}$ for all integer i, $\cup_{i\in\mathbf{N}} K_i = V$, and $K_0 = \phi$. For a sequence $k = (k_i; i \in \mathbf{N})$ of integers and for a sequence $U = (U_i; i \in \mathbf{N})$ where U_i is an open subset of $J^{k_i}(V;W)$ for all integer i, let $O(k,U)$ be the set of u in $C^\infty(V;W)$ such that $j^{k_i}(V \setminus \overset{\circ}{K}_i) \subset U_i$ for all integer i. Our topology is the topology whose a basis is the family of set $O(k,U)$ where k and U are as above. This topology is independent of the choice of $(K_i; i \in \mathbf{N})$ and is finer than the Whitney C^∞-topology if V is not compact. Note also that $C^\infty(V;W)$ with our topology, as $C^\infty(V;W)$ with the Whitney C^∞-topology, is a Baire space (adapt the proof of [GG;Proposition II.3.3]). For a C^∞-smooth fibration $p: W \to V$, $C^\infty(V)$ denotes the set of the C^∞-smooth sections of this fibration. Let N and Λ be two manifolds and let U be an open set of $\mathbf{R}^m$. We denote by $\pi: TN \to N$ the tangent bundle of N and by $C^\infty(T \times U \times \Lambda)$ the C^∞-smooth sections of the fibration $\tilde\pi : TN \times U \times \Lambda \to N \times U \times \Lambda$, $\tilde\pi(x,u,\lambda) = (\pi(x),u,\lambda)$. Let E be a vector subbundle of the tangent bundle of N. Let g be in $C^\infty(TN \times U \times \Lambda)$. Throughout this section we assume that, for any $(x_0,u_0,\lambda_0,\alpha)$ in $N \times U \times \Lambda \times \mathbf{N}^m$

$$\frac{\partial^{|\alpha|} g}{\partial u^\alpha}(x_0,u_0,\lambda_0) \in E(x_0) . \qquad (2.1)$$

We will say that g satisfies hypothesis $H(k)$ at (x_0,λ_0) if

$$\left.\begin{array}{l} \text{Span}\left(\{\frac{\partial^{|\alpha|} g}{\partial u^\alpha}(x_0,0,\lambda_0); 1 \le |\alpha| \le k\} \right. \\ \left. \bigcup \text{Br}_2^k\{\frac{\partial^{|\alpha|} g}{\partial u^\alpha}(\cdot,0,\lambda_0); |\alpha| \le k\}(x_0) \right) = E(x_0) \end{array}\right\} .$$
(2.2)

where $\text{Br}_2^k\mathcal{F}$ denotes the set of iterated Lie brackets of vectors in $\mathcal{F}$ of total length between 2 and k and where $\text{Br}_2^k\mathcal{F}(x_0) = \{h(x_0); h \in \text{Br}_2^k\mathcal{F}\}$. Let, for λ_0 chosen in Λ and for u in $C^\infty([0,T];U)$, $\gamma: [0,T] \to N$ be a solution of

$$\dot\gamma = g(\gamma, u(t), \lambda_0) . \qquad (2.3)$$

The linearized control system around γ is :

$$\dot y(t) = A(t)\,y(t) + \sum_{i=1}^m v_i\, b_i(t) \qquad (2.4)$$

where $v \in \mathbf{R}^m$ is the control and

$$A(t) = \frac{\partial g}{\partial x}(\gamma(t),u(t),\lambda_0),\ b_i(t) = \frac{\partial g}{\partial u_i}(\gamma(t),u(t),\lambda_0) . \qquad (2.5)$$

We will say that γ is (E,r)-controllable (and r-controllable if $E = TM$) at time t if

$$\text{Span}\left\{ \left(\frac{d}{dt} - A(\tau)\right)^j b_i\Big|_{\tau=t} : 1 \le i \le m, 0 \le j \le r \right\} = E(\gamma(t)) . \quad (2.6)$$

For the reason of this definition, see e.g. [SM] or [Kal: p.614]. Finally we will say that γ is $(E,\mathbf{N})$-controllable on $(0,T)$ (and $\mathbf{N}$-controllable if $E = TM$) if, for any t in $(0,T)$, there exists an integer r such that γ is (E,r)-controllable at time t.

Let θ in $C^\infty(\Lambda;M)$ be such that the Cauchy problem

$$\frac{\partial \bar x}{\partial t} = g(\bar x, 0, \lambda) \text{ and } \bar x(0,\lambda) = \theta(\lambda) \qquad (2.7)$$

has a solution on $[0,T] \times \Lambda$. Let d be a metric on $N \times \Lambda$. Then we have

Theorem 2.1. For any neighborhood Ω of 0 in $C^\infty((0,T) \times \Lambda; \mathbf{R}^m)$ there exists $\bar u$ in $\Omega \cap C^\infty([0,T] \times \Lambda; \mathbf{R}^m)$ such that

(i) the solution of the Cauchy problem

$$\frac{\partial \bar x}{\partial t} = g(\bar x, \bar u(t,\lambda), \lambda) \text{ and } \bar x(0,\lambda) = \theta(\lambda) \quad (2.8)$$

is defined on $[0,T] \times \Lambda$;

(ii) for all (t,λ) in $(0,T) \times \Lambda$ and all integer k, if g satisfies $H(k)$ at (x,λ') for all (x,λ') in $N \times \Lambda$ such that $d\Big((x,\lambda'),(\bar x(t,\lambda),\lambda)\Big) \le 1$ then

$$\bar x(\cdot,\lambda) \text{ is } (E,\mathbf{N})-\text{controllable at time } t . \quad (2.9)$$

Let us sketch the proof of Theorem 2.1 when Λ is reduced to a point λ_0 (the same proof works when Λ is compact) and Ω is a neighborhood of 0 in $C^p([0,T] \times \{\lambda_0\}; \mathbf{R}^m)$ (for the general case see [C2; Corollary 1.8]). From now on we will omit λ_0. One first notices that Theorem 2.1 for $\dot x = g(x,y)$, $\dot y = v$ and Ω a neighborhood of 0 in the $C^{p-1}([0,T];\mathbf{R}^m)$- topology implies Theorem 2.1 for $\dot x = g(x,u)$ and Ω a neighborhood of 0 in the $C^p([0,T];\mathbf{R}^m)$-topology. Hence we may assume that $g(x,u) = g_0(x) + \sum_{i=1}^m u_i g_i(x)$. The idea is to choose $\bar u(t) = b(\mu t)/\mu^{p+1}$ where $b = (b_1,\ldots,b_m) \in C^\infty(\mathbf{R};\mathbf{R}^m)$ is T-periodic and μ is a large integer. The real number $1/\mu$ plays the role of a in [C1; (4.1)]: it allows to "neglect" iterated Lie brackets of too large length if it is small. Proceeding in a similar way as in [C1] one can prove that, for generic b, if μ is large enough $\bar u$ is suitable; the only main modification is that we have, for ℓ given and with the notations of [C1], to increase q in such a way that the condition

$\mathrm{rank}\{C_p(I)(t); 1 \le p \le q, |I| \le \ell\}$

$$< q^*(\ell) = (m+1)\frac{(m+1)^\ell - 1}{m} \quad (2.11)$$

is now of codimension 2 — instead of codimension 1 in [C1] — in the space of jets $\{b_i^{(j)}(t); j \le q-1, 1 \le i \le m\}$; note that now I may contain the index 0 and $b_0 = 1$.

Remark 2.2. a) One of the reasons for having not assumed $E = TM$ is to allow time varying system — this is useful, for example, for step 1 in section 4: in order to study systems like $\dot{x} = g(x,t,u,\lambda)$ one has just to apply theorem 2.1 to $\dot{x} = g(x,s,u,\lambda)$, $\dot{s} = 1$ which is a system on $M \times \mathbf{R}$; clearly for this system, if E is equal to $T(M \times \mathbf{R})$, $H(k)$ is never satisfied indeed, the left hand side of (2.2) is included in $TM \times \{0\}$.
b) It follows from Theorem 2.1 (see [C2, Corollary 1.8]) that there exists an open neighborhood Ω of 0 in $C^\infty((0,T) \times \Lambda; \mathbf{R}^m)$ included in $C^\infty([0,T] \times \Lambda; \mathbf{R}^m)$ such that any u in Ω satisfies (2.8) and such that the set of u in Ω, satisfying (i) and (ii), is generic in Ω (for the $C^\infty((0,T) \times \Lambda; \mathbf{R}^m)$-topology).
c) Theorem 2.1 still holds even if (2.1) does not hold and if one replaces in (2.2) and (2.6) = by $\supset$; see again [C2; Corollary 1.8].
d) Theorem 2.1 is related to a previous paper due to E.D. Sontag [So1] ; the main novelties of our result is the $(E,\mathbf{N})$-controllability and the smoothness with respect to λ.
e) After our paper has been completed E.D. Sontag has obtained in [So3] an interesting result related to Theorem 2.1 ; using his method one can also get Theorem 2.1 if N is an open subset of $\mathbf{R}^n$, g is analytic with respect to u and x, and $E = TN$. Note that using our method we can get [So3;Thm 2] with even controllability with impulsive controls, N any manifold and g only C^∞ (in this case one replaces the stong accessibility condition by $E = TN$) ; see [C2] for more details.
As an application of Theorem 2.1 we explain how to modify the proof of Theorem 1.1 given in [C1] in order to get Proposition 1.2. We apply Theorem 2.1 with $N = \Lambda = \mathbf{R}^n\backslash\{0\}$, $g(x,u,\lambda) = f(x,u)$, $\theta(\lambda) = \lambda$, $d((x,\lambda),(x',\lambda')) = |x-x'| + |\lambda - \lambda'| + \big|\, |x|^{-1} - |x'|^{-1} \big| + \big|\, |\lambda|^{-1} - |\lambda'|^{-1} \big|$ and for T we take $T/2$. We obtain the existence of $\bar{u}$ in $C^\infty(\mathbf{R}^n \times [0,\frac{T}{2}]; \mathbf{R}^m)$ such that, on $(\{0\} \times \left[0,\frac{T}{2}\right]) \cup (\mathbf{R}^n \times \left\{0,\frac{T}{2}\right\})$,

$$\frac{\partial^{\beta+|\alpha|}}{\partial t^\beta \partial x^\alpha}\bar{u} = 0 \quad \forall(\alpha,\beta) \in \mathbf{N}^n \times \mathbf{N} \qquad (2.11)$$

and, for any x_0 in $\mathbf{R}^n\backslash\{0\}$, the solution of $\dot{\bar{x}} = f(\bar{x}, u(x_0,t))$, $\bar{x}(0) = x_0$ is $\mathbf{N}$-controllable on $(0,T)$. Moreover we can impose that $\bar{u}$ is small enough (for the $C^\infty((\mathbf{R}^n\backslash\{0\}) \times (0,\frac{T}{2}); \mathbf{R}^m)$-topology) in such a way that, for all t in $[0,\frac{T}{2}]$, $x_0 \to \bar{x}(x_0,t)$ is a diffeomorphism of $\mathbf{R}^n$. We extend $\bar{u}$ to $\mathbf{R}^n \times \mathbf{R}$ by requiring (as in [C1]) :

$$\bar{u}(x,T-t) = \varphi(\bar{u}(x,t)) \; \forall(x,t) \in \mathbf{R}^n \times \left[\frac{T}{2},T\right] \quad (2.12)$$

and

$$\bar{u}(x,t+T) = \bar{u}(x,t) \text{ for all } (x,t) \text{ in } \mathbf{R}^n \times \mathbf{R} \,. \quad (2.13)$$

By (1.9), (2.11), (2.12) and (2.13) $\bar{u} \in C^\infty(\mathbf{R}^n \times \mathbf{R}; \mathbf{R}^m)$. Moreover, by (1.10) and (2.12), if $\dot{\bar{x}} = f(\bar{x}, \bar{u}(x_0,t))$ and $\bar{x}(0) = x_0$, then $\bar{x}(T) = x_0$. The remaining part of the proof of Proposition 1.2 is similar to the one given in [C1; Section 5].

3. Proof of Theorem 1.7

Let, for $\varepsilon > 0$, B_ε be the open ball of $\mathbf{R}^n$ of radius ε and let T be a positive real number. We assume (1.25). Our first step comes from [Kaw1; Appendix]-see also [H1]-

Lemma 3.1. There exist $\varepsilon > 0$ and u in $C^0\Big(B_\varepsilon; L^1((0,T); \mathbf{R}^m)\Big)$ such that :

$$\mathrm{Sup}\{|u(a)(t)| : t \in [0,T]\} \longrightarrow 0 \text{ as } |a| \longrightarrow 0 \quad (3.1)$$

$$x(T,a;u) = 0 \text{ for all } a \text{ in } B_\varepsilon \qquad (3.2)$$

where x is defined by $x(0,a;u) = a$ and

$$\frac{\partial x}{\partial t}(t,a;u) = f\big(x(t,a;u), u(a)(t)\big) \qquad (3.3)$$

Proof: In [Kaw1, Appendix], M. Kawski has proved the existence of $u : B_\varepsilon \to L^1((0,T); \mathbf{R}^m)$ satisfying (3.1) and (3.2). His u is not continuous — even if (1.25) holds — but a very slight modification of his proof gives a continuous u. Let $(e_i)_{1 \le i \le n}$ be the usual basis of $\mathbf{R}^n$ and let $e_i = -e_{i-n}$ for i in $[n+1, 2n]$. By (1.25) and noting that $D^\ell \subset D^{\ell+1}$ we may assume, after possibly a change of scale, that for some $p \ge 1$ $e_i \in D^p$ for all i in $[1,2n]$. Let u_i be as in Definition 1.11 with $\xi = e_i$. For a in $\mathbf{R}^n$ we write $a = \sum_{i=1}^{2n} a_i e_i$ with $a_i \ge 0$ for all i in $[1,2n]$ and $a_i a_{i+n} = 0$ for all i in $[1,n]$. Let $\mu(a) = \sum_{i=1}^{2n} a_i^{1/p} e_i$. Let for a in B_1, $u(a) : [0, \mu(a)) \to \mathbf{R}^m$ be defined by

$$u(a)(t) = u_j(a_j^{1/p})(t - t_j) \qquad (3.4)$$

if $t_j = \sum_{i=1}^{j-1} a_i^{1/p} \le t < \sum_{i=1}^{j} a_i^{1/p}$. By Gronwall's lemma, (1.11), (1.12) and (1.13) we get

$$\mu(x_1) = o(\mu(a)) \text{ as } a \longrightarrow 0 \qquad (3.5)$$

where $x_1 = x(\mu(a), a; u(a))$. Hence, for a small enough,

$$\mu(x_1) \le \mu(a)/2 \,. \qquad (3.6)$$

We now define $u(a)$ on $[\mu(a), \mu(a) + \mu(x_1))$ by

$$u(a)(t) = u(x_1)(t - \mu(a)) \qquad (3.7)$$

We have, if a is small enough, $\mu(x_2) \le \mu(x_1)/2$ with $x_2 = x(\mu(a) + \mu(x_1), a; u)$. We keep going and define in this way $u(a)$ on $[0, \mu(a) + \sum_{i=1}^\infty \mu(x_i))$. Note that, if a is small enough, $\mu(a) + \sum_{i=1}^\infty \mu(x_i) \le 2\mu(a) \le T$. We extend $u(a)$ on $[0,T]$ by $u(a)(t) = 0$ if $t \in [\mu(a) + \sum_{i=1}^\infty \mu(x_i), T]$; $u(a)$ satisfies all the required properties.
For ε in $(0,+\infty]$ let $B'_\varepsilon = B_\varepsilon\backslash\{0\} = \{x \in \mathbf{R}^n; 0 <$

$|x| < \varepsilon\}$ and let $C_0^\infty(B'_\varepsilon \times [0,T]; \mathbf{R}^m)$ be the set of functions u in $C^\infty(B'_\varepsilon \times [0,T]; \mathbf{R}^m) \cap C^0(B_\varepsilon \times [0,T]; \mathbf{R}^m)$ such that for all α in $\mathbf{N}^{n+1}$

$$\partial^\alpha u = 0 \text{ on } B'_\varepsilon \times \{0,T\}, \qquad (3.8)$$

$$u(0,t) = 0 \text{ for all } t \text{ in } [0,T]. \qquad (3.9)$$

Our next statement is a corollary of Lemma 3.1 and Theorem 2.1. We do not need it for the proof of Theorem 1.7, but it will be useful in Section 4 (see also the end of this section).

Corollary 3.2. In Lemma 3.1 u can be chosen in $C_0^\infty(B'_\varepsilon \times [0,T]; \mathbf{R}^m)$.

Proof: By Theorem 2.1 there exist $\delta > 0$ and $\bar{u}$ in $C_0^\infty(B'_\delta \times [0,T/3]; \mathbf{R}^m)$ such that the solutions of $\dot{\bar{x}} = f(\bar{x}, \bar{u}(a,t))$, $\bar{x}(0,a;\bar{u}) = a \in B'_\delta$ are $\mathbf{N}$-controllable on $(0,T/3)$. Let u be as in Lemma 3.1 but with $[T/3, 2T/3]$ instead of $[0,T]$; we extend $\bar{u}$ on $(T/3,T]$ by $\bar{u}(a,t) = u(\bar{x}(T/2,a;\bar{u}),t)$ for $t \in (T/3, 2T/3]$ and by 0 for $t \in (2T/3, T]$. This makes sense if $|a| < \eta$ with $0 < \eta$ small enough. Then $\bar{u}$ satisfies the conclusion of Lemma 3.1; this map has not the regularity required by Corollary 3.2 but the $\mathbf{N}$-controllability allows to smooth it in a map in $C_0^\infty(B'_\eta \times [0,T]; \mathbf{R}^m)$ satisfying (3.2) (proceed as in [C1]; see [C3] for more details).

Now the proof of Theorem 1.7 goes as follows. Let $\bar{u}$ be as in Lemma 3.1 but with $T/3$ instead of T. Let $(u,v) : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^m \times \mathbf{R}^n$ be T-periodic in t and such that for $t \in [0,T)$ and $|x| + |y|$ small enough

$$u = 0 \text{ if } t \in [0,T/3] \cup [2T/3, T) \qquad (3.10)$$

$$u(x,y,t) = \bar{u}(y, t-(T/3)) \text{ if } t \in (T/3, 2T/3) \qquad (3.11)$$

$$v = 0 \text{ if } t \in (T/3, 2T/3) \qquad (3.12)$$

$$\left.\begin{array}{r} v(x,y,t) = -|y-\nu(x,t)|^{-1/3}(y-\nu(x,t)) \\ \text{if } t \in [0,T/3] \cup [2T/3,T) \end{array}\right\} \qquad (3.13)$$

with $\nu(x,t) = z(T/3)$ where $\dot{z} = f(z,0)$, $z(t) = x$. For $|x_0| + |y_0|$ small enough one has $x\left(\frac{T}{3}\right) = y\left(\frac{T}{3}\right)$ and $x\left(\frac{2T}{3}\right) = x(T) = y(T) = 0$ where x and y are solutions of the *closed loop* system $\dot{x} = f(x, u(x,y,t))$, $\dot{y} = v(x,y,t)$, $x(0) = x_0$, and $y(0) = y_0$. It remains only to smooth (u,v) in a suitable way in order to get Theorem 1.7 — with even a decay of $|x(t)| + |y(t)|$ "as fast as we want", for example exponential. Note also that if one requires for the feedback law to be continuous and only smooth outside a locally finite union of smooth manifolds instead of being in $C^\infty\left((\mathbf{R}^n \times \mathbf{R}^n \backslash \{0,0\}) \times \mathbf{R}; \mathbf{R}^m \times \mathbf{R}^n\right)$ then $\dot{x} = f(x,u)$ is n-dynamically T-LS: in the above proof one has only to use corollary 3.2 instead of Lemma 3.1 and to multiply v by $\rho \in C^\infty(\mathbf{R}; [0,1])$, T-periodic in t, with $\rho = 1$ on $[T/12, T/4] \cup [3T/4, 11T/12]$, $\rho = 0$, on a neighborhood of $\{0, T/3, 2T/3\}$.

4. Sketch of the proof of Theorem 1.8.

We only give the steps of the proof (see [C3] for a detailed proof).

Step 1. Using Corollary 3.2 and Proposition 2.1 one can prove that there exist $\varepsilon > 0$ and u_1 in $C_0^\infty(B'_\varepsilon \times [0,T]; \mathbf{R}^m)$ such that

$$x(T,a;u_1) = 0 \text{ for all } a \text{ in } B_\varepsilon \qquad (4.1)$$

$$\left.\begin{array}{l} \text{the trajectories } t \longrightarrow x(t,a;u_1) \text{ are } \mathbf{N}\text{-} \\ \text{controllable on } (0,T) \text{ for all } a \text{ in } B'_\varepsilon. \end{array}\right\} \qquad (4.2)$$

Let u_2 be the restriction of u to $(0,\varepsilon e_n) \times [0,T]$ and let $C_0^\infty((0,\varepsilon e_n) \times [0,T]; \mathbf{R}^m)$ be the set of maps u in $C^\infty((0,\varepsilon e_n) \times [0,T]; \mathbf{R}^m) \cap C^0([0,\varepsilon e_n) \times [0,T]; \mathbf{R}^m)$ such that

$$\partial^\alpha u = 0 \;\; on \;\; (0,\varepsilon e_n) \times \{0,T\} \text{ for all } \alpha \text{ in } \mathbf{N}^2, \qquad (4.3)$$

$$u(0,t) = 0 \text{ for all } t \text{ in } [0,T]. \qquad (4.4)$$

Note that $u_2 \in C_0^\infty((0,\varepsilon e_n) \times [0,T]; \mathbf{R}^n)$. We now use $n \geq 4$; the next step is wrong for $n \leq 3$.

Step 2. Perturbing u_1 slightly and in a suitable way, if necessary, we obtain a new u satisfying again the properties of Step 1 such that the corresponding u_2, called u_3, is such that, for all t in $(0,T)$ the map: $(0,\varepsilon) \to \mathbf{R}^n$, $a_n \to x(t, a_n e_n; u_3)$ is an embedding. Note that by (4.2), for all a_n in $(0,\varepsilon)$, the trajectories $t \to x(t, a_n e_n; u_3)$ are $\mathbf{N}$-controllable on $(0,T)$. The proof of this relies essentially on (4.2) and on the classical proof of Whitney's embedding theorem; see e.g. [GG;II.5]. One could alternatively slightly perturb only u_2 (instead of u_1) and use ideas due to M. Gromov [G; (E') p. 121] as well as [GG; II.5].

Step 3. Using the $\mathbf{N}$-controllability of the trajectories $t \to x(t, a_n e_n; u_3)$ on $(0,T)$ for all a_n in $(0,\varepsilon)$ and the above embedding property one can prove that there exists an open neighborhood $\mathcal{N}_1$ of $(0, \varepsilon e_n/2)$ and u_4 in $C_0^\infty(B'_\varepsilon \times [0,T]; \mathbf{R}^m)$ such that, for all t in $[0,T)$, the map $a \in \mathcal{N}_1 \to x(t,a;u_4) \in \mathbf{R}^m$ is an embedding and $x(T,a;u_4) = 0$ for all a in $\mathcal{N}_1$. From this we get that there exists a neighborhood $\mathcal{N}_2 (\subset \mathcal{N}_1)$ of $(0, \varepsilon e_n/4)$ and u_5 in $C_0^\infty((\mathbf{R}^n \backslash \{0\}) \times [0,T]; \mathbf{R}^m)$ such that (1.17) holds with $u = u_5$, $0 \leq t_0 \leq t_1 \leq T$ and that (1.19) holds with $u = u_5$ and $x_0 \in \mathcal{N}_2 \cup \{0\}$.

Step 4. Finally we replace in the above steps $[0,T]$ by $[T/2, T]$ and define u on $\mathbf{R}^n \times (T/2, T]$ by $u = u_5$ on this set. On $\mathbf{R}^n \times [0, T/2]$ we choose u in $C_0^\infty((\mathbf{R}^n \backslash \{0\}) \times [0,T/2]; \mathbf{R}^m)$ such that (1.17) holds for $0 \leq t_0 \leq t_1 \leq T/2$ and there exists $\delta_1 > 0$ such that if $|x(0)| < \delta_1$ and $\dot{x} = f(x, u(x,t))$ then $x(T/2) \in \mathcal{N}_2 \cup \{0\}$.

The existence of such a u follows from (1.26) and (1.27). The map u, extended by T-periodicity (in time) on all $\mathbf{R}^n \times \mathbf{R}$, satisfies (1.14) to (1.19).

Remark 4.1. a.We use $u \geq 4$ only at Step 2. The existence of u_3 as in Step 2 can be proved (see [C3]) in a different manner if $f_1(x, (-u_1, -u_2)) = -f_1(x, (u_1, u_2))$ or $f_1((x_1, x_2), (u_1, u_2)) = \bar{f}_1(x_1, u_1)$; in these cases we do not need $n \geq 4$.b.Assumptions (1.26) and (1.27) can be omitted,see [C3].

5. Links between the Sussmann condition and $0 \in \text{int}\, D$

Let us assume, for the time being, that

$$f(x,u) = f_0(x) + \sum_{i+1}^{m} u_i f_i(x) .$$

Let $\text{Br}(f)$ be the set of iterated Lie brackets of $\{f_0, f_1, ..., f_m\}$. For h in $\text{Br}(f)$ let $\delta_i(h)$ the number of times that f_i appears in h. Recall (see [Su2; Section 7]) that, for $\theta \in [0, +\infty]$, $\dot{x} = f_0(x) + \sum_{i+1}^{m} u_i f_i(x)$ satisfies the Sussmann condition $S(\theta)$ if whenever $h \in \text{Br}(f)$ with $\delta_0(h)$ odd and $\delta_i(h)$ even for all i in $[1,m]$ then $h(0)$ is the Span of the $g(0)$'s where the g's are in $\text{Br}(f)$ and satisfy

$$\theta\delta_0(g) + \sum_{i=1}^{m} \delta_i(g) < \theta\delta_0(h) + \sum_{i=1}^{m} \delta_i(h)\} . \qquad (5.1)$$

with the convention that when $\theta = +\infty$, (5.1) is replaced by $\delta_0(g) < \delta_0(h)$. H. Sussmann has proved :

Theorem 5.1. [Su2; Thm. 7.3]. If, for some θ in $[0,1]$, $\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i f_i(x)$ satisfies $S(\theta)$ then it is STLC. Moreover it follows directly from the proof of [Su2; Thm. 7.3] that we have

Proposition 5.2. Under the hypothesis of Theorem 5.1, $0 \in \text{int}(D)$.

Let us notice that one can check

Proposition 5.3. Let θ be in $[0,1)$. Then $\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i f_i(x)$ satisfies $S(\theta)$ if and only if $\dot{x} = f_0(x) + \sum_{i=1}^{m} y_i f_i(x)$, $\dot{y} = u$ satisfies $S(\theta/(1-\theta))$.

This proposition allows us to extend $S(\theta)$ to $\dot{x} = f(x,u)$ in the following way

Definition 5.4. Let $\theta \in [0,1]$; we will say that $\dot{x} = f(x,u)$ satisfies $S(\theta)$ if $\dot{x} = f(x,y)$, $\dot{y} = u$ satisfies $S(\theta/(1-\theta))$.

What we have called in this paper the Hermes condition is $S(0)$; the true Hermes condition is in fact more restrictive (see [H2] or [Su2; Section 7.3]). Moreover it follows from [Su2] that

Proposition 5.5. If, for some θ in $[0,1]$, $\dot{x} = f(x,u)$ satisfies $S(\theta)$ then it is STLC.

Proof: Apply [Su2] to $\dot{x} = f(x,y)$, $\dot{y} = u$ with the constraint $\int_0^t |u(s)|\, ds \le 1$ (instead of $|u| \le 1$).

Remark 5.6. Similar comments can be done for the sufficient condition for STLC due to R. Bianchini and G. Stefani [BS]. In particular the hypothesis of [BS; Corollary p. 970] implies also $0 \in \text{int}(D)$.

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References

[BS] R.M. Bianchini, G. Stefani, Sufficient conditions of local controllability; Proceedings of 25th Conference on Decision and Control, Athens, Greece. December 1986, p. 967 - 970.

[B] R.W. Brockett, Asymptotic stability and feedback stabilization; in: R.W. Brockett, R.S. Millman and H.J. Sussmann eds., *Differential Geometric Control Theory*; Birkhäuser, Basel - Boston, 1983.

[C1] J.-M. Coron, Global asymptotic stabilization for controllable systems without drift; To appear in MCSS, 1992.

[C2] J.-M. Coron, Linearized controlled systems and applications to smooth stabilization. Preprint Université Paris-Sud, 1992.

[C3] J.-M. Coron;Stabilisation in finite time of locally controllable systems by means of continuous time-varying feedback laws, Preprint CMLA Nov. 1992.

[CPo] J.-M. Coron, J.-B. Pomet, A remark on the design of time-varying stabilizing feedback laws for controllable systems without drift; These proceedings.

[CPr] J.-M. Coron, L. Praly, Adding an integrator for the stabilization problem; Systems and Control Letters 17 (1991) 89 - 104.

[DMK] W.P. Dayawansa, C.F. Martin and G. Knowles, Asymptotic stabilization of a class of smooth two dimensional systems; SIAM J. Control Optim. 28 (1990) 1321 - 1349.

[G] M. Gromov, *Partial Differential Relations*, Ergebnisse Mathematik 3, Folge 9, Springer, 1986.

[GG] M. Golubitsky, V. Guillemin, *Stable mappings and their singularities*; Grad. Texts in Math. 14, Springer 1973.

[H1] H. Hermes,On the synthesis of a stabilizing feedback control via Lie algebraic methods;SIAM J. Control and Optimization 18 (1980) 352-361.

[H2] H. Hermes, Control systems which generate decomposable Lie algebras J. Diff. Equ. 44 (1982) 166-187.

[Kai] T. Kailath, *Linear Systems*; Prentice-Hall, 1980.

[Kaw1] M. Kawski, High-order small-time local controllability, in: H.J. Sussmann ed. *Nonlinear Controllability and Optimal Control*; Monographs and Textbooks in Pure and Applied Mathematics, 133, Marcel Dekker, 1990, 431 - 467.

[Kaw2] M. Kawski, Stabilization of nonlinear systems in the plane, Systems and Control Letters 12 (1989) 169 - 175.

[P] J.-B. Pomet, Explicit design of time-varying stabilizing control laws for a class of controllable system without drift; Systems and Control Letters, to appear.

[Sa] C. Samson, Velocity and torque feedback law control of a wheeled mobile robot, stability analysis; preprint, INRIA Sophia Antipolis, 1990.

[Se] R. Sepulchre; private communication, 1991.

[So1] E.D. Sontag, Finite dimensional open loop control generators for nonlinear control systems, Int. J. Cont. 47 (1988) 537-556.

[So2] E.D. Sontag, Feedback stabilization of nonlinear systems; in: M.A. Kaashoek, J.H. van Schuppen, and A.C.M. Ran, eds., *Robust Control of Linear Systems and Nonlinear Control*, Birkhäuser, Cambridge, MA, 2, 1990, 61 - 81.

[So3] E.D. Sontag, Universal nonsingular controls, Preprint Rutgers University, Dec. 1991.

[SCW] R. Sepulchre, G. Campion, V. Wertz, some remarks about periodic feedback stabilization; These Proceedings.

[SM] L.M. Silverman and H.E. Meadows, Controllability and observability in time variable linear systems; SIAM J. Control, 5 (1967) 64 - 73.

[SS] E.D. Sontag, H.J. Sussmann, Remarks on continuous feedback, IEEE, CDC, Albuquerque, 2 (1980) 916 - 921.

[Su1] H.J. Sussmann, Subanalytic sets and feedback control; J. Differential equations, 31 (1979) 31 - 52.

[Su2] H.J. Sussmann, A general theorem on local controllability; SIAM J. Control and Optimization, 25 (1987) 158 - 194.

[SJ] H.J. Sussmann, V. Jurdjevic, Controllability of nonlinear systems; J. Differential Equations, 12 (1972) 95 - 116.

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