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Harmonic Maps with Defects

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Abstract. Two problems concerning maps φ with point singularities from a domain $\Omega \subset \mathbb{R}^3$ to S^2 are solved. The first is to determine the minimum energy of φ when the location and topological degree of the singularities are prescribed. In the second problem Ω is the unit ball and $\varphi = g$ is given on $\partial\Omega$; we show that the only cases in which $g(x/|x|)$ minimizes the energy is $g = \text{const}$ or $g(x) = \pm Rx$ with R a rotation. Extensions of these problems are also solved, e.g. points are replaced by "holes," $\mathbb{R}^3, S^2$ is replaced by $\mathbb{R}^N, S^{N-1}$ or by $\mathbb{R}^N, \mathbb{R}P^{N-1}$, the latter being appropriate for the theory of liquid crystals.

I. Introduction

Suppose $U \subset \mathbb{R}^3$ is open and $a \in U$. Consider maps $\varphi : U \to S^2$ which are continuous except (possibly) at a. If S is a sphere in U centered at a, φ restricted to S defines a map from S^2 to S^2 and so has a topological degree in $\mathbb{Z}$ (also known as winding or covering number). By continuity this number is independent of S and we shall denote it by d. If φ is also continuous at a, then $d=0$.

Suppose now that $\varphi \in C^1(U \setminus \{a\}; S^2)$ and consider its energy

$$E(\varphi) = \int_U |\nabla \varphi|^2 \tag{1.1}$$

possibly finite or infinite. The fact that $E(\varphi) < \infty$ does not imply that φ is continuous at a or even that $d=0$. An example with $d=1$, U bounded and $a=0$ is $\varphi(x) = x/|x|$. However if $U = \mathbb{R}^3$ and $E(\varphi) < \infty$, then d must be zero (since φ goes to a constant at infinity).

A natural problem is to minimize $E(\varphi)$ given the degree, d, of φ at a (assuming $U \neq \mathbb{R}^3$). We shall prove that the minimum energy is

$$E = 8\pi L, \tag{1.2}$$

where L is $|d|$ times the distance of a to ∂U.

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Another simple case is to consider two points $a_1, a_2 \in \mathbb{R}^3$ and maps

$$\varphi \in C(\mathbb{R}^3 \backslash \{a_1, a_2\}; S^2).$$

As above, one can define $\deg(\varphi, a_i)$, $i=1,2$, by restricting φ to small spheres around a_i, $i=1,2$. Assuming that $\varphi \in C^1(\mathbb{R}^3 \backslash \{a_1, a_2\}; S^2)$ and $E(\varphi) < \infty$, then we must have $d \equiv \deg(\varphi, a_1) = -\deg(\varphi, a_2)$. A natural problem is to minimize $E(\varphi)$ given d. We shall prove that the minimum energy is given by (1.2) with $L = |a_1 - a_2||d|$. The infimum is not achieved; however if φ^n is a minimizing sequence, we shall prove that φ^n tends to a constant a.e. and $|\nabla \varphi^n|^2$ tends to a uniform measure on the segment $[a_1, a_2]$ (after passing to a subsequence if necessary).

There are various generalizations of the two-point problem just mentioned, and they all give rise to the same formula (1.2) provided L is interpreted appropriately. We shall discuss four examples of increasing generality. Let U be an open set in $\mathbb{R}^3$. Let $H_1, \ldots, H_k$ be k disjoint compact subsets of U, which will be called the *holes*. Let $\Omega = U \backslash \left(\bigcup_{i=1}^{k} H_i \right)$. If $\varphi \in C(\Omega; S^2)$, then it is possible to define $\deg(\varphi, H_i)$, the degree of φ around H_i. If H_i is a point, $\deg(\varphi, H_i)$ is the usual topological degree, as above. For general H_i the degree can also be defined, but a bit of analysis is required; this is carefully discussed in Appendix B. Essentially, $\deg(\varphi, H_i)$ is the degree of φ restricted to a surface surrounding H_i.

Given integers $d_1, \ldots, d_k \in \mathbb{Z}$ (possibly including zero), consider the class

$$\mathscr{E} = \left\{ \varphi \in C(\Omega; S^2) | \deg(\varphi, H_i) = d_i \text{ and } \int_\Omega |\nabla \varphi|^2 < \infty \right\}. \tag{1.3}$$

Set

$$E = \inf_{\varphi \in \mathscr{E}} \int_\Omega |\nabla \varphi|^2. \tag{1.4}$$

[Note that E is unchanged if $C(\Omega; S^2)$ is replaced by $C^1(\Omega; S^2)$; this is explained in Appendix A.]

Example 1. $U = \mathbb{R}^3$ and the H_i are points a_i in $\mathbb{R}^3$.

Example 2. $U = \mathbb{R}^3$ and the H_i are not necessarily points.

Example 3. $U \neq \mathbb{R}^3$ and the H_i are not necessarily points.

Example 4. This is the same as Example 3, except that we consider the smaller class

$$\mathscr{E}' = \{\varphi \in C(\bar{U} \backslash (\cup H_i); S^2) | \varphi \in \mathscr{E} \text{ and } \varphi = \text{const on } \partial U\}.$$

and let

$$E = \inf_{\varphi \in \mathscr{E}'} \int_\Omega |\nabla \varphi|^2. \tag{1.5}$$

In Examples 1, 2 (respectively 4), $\mathscr{E}$ (respectively $\mathscr{E}'$) is empty unless $\sum d_i = 0$.

Our main result concerning this problem is

Theorem 1.1. *In all four examples,*

$$E = 8\pi L, \tag{1.6}$$

where L is defined in Sect. II.

L is a quantity which has the dimension of a length and depends on U, on the relative distances between the holes and on the d_i's. It is easiest to visualize L in Example 1 and when $d_i = \pm 1$ for all i. We shall say that a_i is a positive (respectively negative) point if $d_i = +1$ (respectively -1). Since $\sum d_i = 0$ we can pair the positive points with the negative points. This pairing, or *connection* as we call it in Sect. II, has a length which is the sum of the distances between the paired points. L is defined to be the minimum possible length. If the d_i's are not ± 1, then simply repeat the point a_i $|d_i|$ times.

In Example 2 the rule is the same as for Example 1, except that one has to use the following reduced distance between holes. Given two holes H_i, H_j we let dist (H_i, H_j) be the usual Euclidean distance between the holes. Then we define the reduced distance to be

$$D(H_i, H_j) = \min \sum_{m=1}^{p} \operatorname{dist}(H_{i_{m-1}}, H_{i_m}),$$

where $i_0, \ldots, i_p$ is a finite sequence with $i_0 = i$, $i_p = j$ and the above minimum is over all such sequences.

In Example 3 just pretend that $H_0 \equiv \mathbb{R}^3 \setminus U$ is a hole of degree $d_0 = -\sum d_i$ and use the above rule to compute L.

The rule in Example 4 is the same as in Example 2 except that $\operatorname{dist}(H_i, H_j)$ is replaced by the geodesic distance in U.

The proof of Theorem 1.1 has two steps. In Sect. III we show that $E \leqq 8\pi L$ by an explicit construction of an almost minimizer, which is obtained by gluing together "*dipoles*," i.e. almost minimizers for the two-point problem which are concentrated near the lines joining paired points. The lower bound $E \geqq 8\pi L$ is more delicate. For this purpose, we introduce in Sect. IV a useful vector field D associated to $\varphi \in \mathscr{E}$, with components

$$D = (\varphi \cdot \varphi_y \wedge \varphi_z, \varphi \cdot \varphi_z \wedge \varphi_x, \varphi \cdot \varphi_x \wedge \varphi_y). \tag{1.7}$$

In all examples $\operatorname{div} D = 0$ in Ω and $2|D| \leqq |\nabla \varphi|^2$. We sketch the essence of the argument for Example 1. In that case,

$$\operatorname{div} D = 4\pi \sum_{i=1}^{k} d_i \delta_{a_i} \equiv 4\pi \varrho \text{ in } \mathscr{D}'(\mathbb{R}^3), \tag{1.8}$$

so that

$$E \geqq 8\pi \inf \left\{ \int_{\mathbb{R}^3} |D| \,|\, \operatorname{div} D = \varrho \right\}. \tag{1.9}$$

By duality, as explained in Appendix C,

$$\inf \left\{ \int_{\mathbb{R}^3} |D| \,|\, \operatorname{div} D = \varrho \right\} = \max \left\{ \int_{\mathbb{R}^3} \zeta d\varrho \,|\, \zeta \in K \right\},$$

where

$$K = \{\zeta : \mathbb{R}^3 \to \mathbb{R} \,|\, \|\zeta\|_{\mathrm{Lip}} \leqq 1\} \quad \text{and} \quad \|\zeta\|_{\mathrm{Lip}} = \sup |\zeta(x) - \zeta(y)|/|x-y|.$$

We conclude by showing that

$$\max \left\{ \int_{\mathbb{R}^3} \zeta d\varrho \,|\, \zeta \in K \right\} = L \tag{1.10}$$

with the help of a theorem of Kantorovich [20] and Birkhoff's theorem [2, 26] on doubly stochastic matrices.

In general, there is no minimizer for the φ problem (1.4) [or (1.5)] and thus we are led in Sect. VI to investigate the behavior of minimizing sequences. However, the D problem defined by (1.9) and its analogue for the other examples does have a minimum as a vector-valued *measure*. Some properties of these D minimizers are described in Sect. V; for example we prove that $\operatorname{supp} D \subset G$, the union of the minimal connections. Our main result, in the context of Example 1, is that a minimizing sequence φ^n tends (modulo a subsequence) to a constant a.e. and $|\nabla \varphi^n|^2$ tends to a uniform measure distributed on a minimal connection. This is a striking fact since, if there is more than one minimal connection, a D minimizer can be supported by the union of two (or more) connections. This quantization phenomenon is based on the analysis in Appendix E.

A very different problem, one with a more classical flavor, is the subject of Sect. VII. Instead of specifying singularities we investigate the problem of minimizing $E(\varphi)$ on a domain $U \subset \mathbb{R}^3$ when $\varphi = g$ is specified on ∂U and we allow as admissible functions all H^1 maps from U into S^2. Clearly,

$$E(g) = \min\left\{\int_U |\nabla \varphi|^2 \,|\, \varphi \in H^1(U; S^2),\ \varphi = g \text{ on } \partial U\right\}$$

is achieved and it is known from the work of Schoen and Uhlenbeck [31, 32] that any minimizing φ has only point singularities and there are only finitely many of these. Our main result is

Theorem 1.2. *These singularities always have degree* ± 1 *and more precisely, near a singularity* x_0,

$$\varphi(x) \simeq \pm R(x - x_0)/|x - x_0|,$$

where R *is a rotation.*

This is a consequence of another result proved in Sect. VII, that if U is a ball, then $g(x/|x|)$ is a minimizer if and only if $\pm g$ is a rotation.

It is obvious that in the foregoing results one can replace the domain of φ by three dimensional manifolds other than $\mathbb{R}^3$, but we have not investigated these extensions. However other extensions are considered in Sect. VIII, for example we have replaced $\mathbb{R}^3, S^2$ by $\mathbb{R}^N, S^{N-1}$ and by $\mathbb{R}^N, \mathbb{R}P^{N-1}$. This replacement does not change the conclusions in any significant way. The $\mathbb{R}P^2$ extension is important for liquid crystals as explained below. We also touch upon a minimization problem where the minimum energy is proportional to an area (and not a length). A simple example of this kind of problem is to consider a closed Jordan curve $\Gamma \subset \mathbb{R}^3$ and $\varphi \in C(\mathbb{R}^3 \backslash \Gamma; S^1)$ having unit circulation around Γ. The energy to be minimized is $E(\varphi) = \int |\nabla \varphi|$. We conjecture that the minimum E is $2\pi A$, where A is the area of a minimal area surface spanning Γ.

In order not to interrupt the main thread of the paper, we have placed many of the technical facts in appendices. Some of these are of independent interest. For example, Appendix D contains a proof of the uniqueness of a divergence free vector-field supported on a curve. In Appendix E we present some noteworthy properties of certain nonlinear expressions involving weakly convergent sequences.

The mathematical analysis in this paper, summarized above, may be relevant to certain problems in physics.

A. Liquid Crystals

A nematic liquid crystal can be described by a vector field φ on a domain U in $\mathbb{R}^3$ (the container). The direction (optic axis) of the rod-like molecules at x is $\varphi(x)$ (called the director), so $|\varphi(x)|=1$, and therefore we can view $\varphi(x)$ as a point in S^2. Normally, the ends of the molecules cannot be distinguished, so $\varphi(x)$ should really take values in $\mathbb{R}P^2$, i.e. the quotient of S^2 by the equivalence relation $\varphi \simeq -\varphi$.

Except for defects, which are points or curves in Ω, $\varphi(x)$ varies continuously. Frequently the liquid crystal energy is taken to be [7, 9, 13, 14, 17, 18, 21]:

$$\tilde{E}(\varphi)=K_1 \int_U (\operatorname{div}\varphi)^2+K_2 \int_U (\varphi\cdot\operatorname{curl}\varphi)^2+K_3\int_U |\varphi\wedge\operatorname{curl}\varphi|^2. \tag{1.11}$$

A special case that has been frequently studied is the one-constant approximation $K_1=K_2=K_3\equiv K$. Then the integrand on the right side of (1.11) is

$$K\{(\operatorname{div}\varphi)^2+|\operatorname{curl}\varphi|^2\}=K\{|\nabla\varphi|^2+2D\cdot\varphi\}=K\{|\nabla\varphi|^2+\operatorname{div}W\} \tag{1.12}$$

with D given by (1.7) and

$$W=\varphi\operatorname{div}\varphi-(\varphi\cdot\nabla)\varphi=\varphi\operatorname{div}\varphi+\varphi\wedge\operatorname{curl}\varphi. \tag{1.13}$$

Both (1.12) and (1.13) hold in the sense of distributions for all φ with $\nabla\varphi\in L^2$. Taking $K=1$, and integrating (1.12) we find

$$\tilde{E}(\varphi)-E(\varphi)=\int_U \operatorname{div}W=\int_{\partial U} W\cdot n. \tag{1.14}$$

It is easy to check that $W\cdot n$ depends only on φ and its tangential derivatives on ∂U. Therefore, in all problems in which φ is prescribed on the boundary (such as Example 4 or the problems in Sect. VII) the boundary integral, $\int W\cdot n$, plays no role; the minimization of $\tilde{E}$ and E are the same problem. However, in Example 3, φ is not prescribed on the boundary and the two minimization problems are different. We shall discuss only the $E(\varphi)$ problem in this paper. It would be interesting to analyze the $\tilde{E}$ problem.

It is to be noted that $\varphi\to|\nabla\varphi|^2$ is $SO(3)$ invariant, namely if $R\in SO(3)$ and $\varphi'(x)\equiv R\varphi(x)$, then $|\nabla\varphi'|^2=|\nabla\varphi|^2$. Also, D is $SO(3)$ invariant, i.e. $D(x)=D'(x)$, where D' is the D field of φ'. On the other hand, $H(\varphi)\equiv(\operatorname{div}\varphi)^2+|\operatorname{curl}\varphi|^2$ is not $SO(3)$ invariant; it is only invariant under the simultaneous action of $SO(3)$ on φ and on x, i.e. $\varphi(x)\to R\varphi(Rx)$. From these observations one can conclude that $\tilde{E}\leqq E$ in Example 3. Indeed, let $d\mu$ be Haar measure on $SO(3)$ so that $\int d\mu(R)D\cdot R\varphi=0$. Thus, for all φ

$$\int d\mu(R)\int_U H(R\varphi)=\int_U |\nabla\varphi|^2, \tag{1.15}$$

so $\int H(R\varphi)\leqq\int|\nabla\varphi|^2$ for some R.

Long lived point singularities are observed in nature [6] and have degree one, consistent with our Theorem 1.2.

B. The Classical $O(3)$ Nonlinear Sigma Model

The Euler-Lagrange equation corresponding to (1.1) is

$$-\Delta\varphi=\varphi|\nabla\varphi|^2, \tag{1.16}$$

which is the equation of harmonic maps. It is also the equation of the classical nonlinear sigma model, but in the physics literature this is usually studied in $\mathbb{R}^2$, namely $\varphi: \mathbb{R}^2 \to S^2$. Our analysis suggests that the $O(3)$ nonlinear sigma model from $\mathbb{R}^3 \to S^2$ may be interesting, when singularities are included, although it is known that the quantized version of such a field theory is non-renormalizable. In any event, the expression for the energy needed to create two singularities separated by a distance L, namely $8\pi L$, is amusing. This is precisely the energy expression used in the semiclassical theory of quark confinement. Also, the fact that $\operatorname{supp}|\nabla\varphi^n|^2$ converges to a "string" is consistent with some pictures of quark-quark interactions.

Previously, Parisi [28] described a classical, relativistic field theory having some features in common with our φ field. In the static limit it reduces to monopoles embedded in a superconductor. However, to obtain strict linearity for the effective monopole-monopole interaction potential it seems to be necessary to take the limit of infinite critical field for the superconductor. For our Example 1, on the other hand, no limits are needed.

II. Minimal Connections

This section is concerned with defining some geometric quantities associated with a configuration of points or holes (disjoint compact subsets of $\mathbb{R}^N$) in certain domains in $\mathbb{R}^N$. From this construction we derive a number (with the dimension of a length) which, it will turn out, is proportional to the minimum energy.

A common feature of all the cases of interest to us is that we are given k disjoint holes in $\mathbb{R}^N$, $H_1, \ldots, H_k$. According to the case, a certain distance function $D(H_i, H_j)$ will be defined between pairs of holes. D will satisfy the usual properties of a metric ($D(H_i, H_j) + D(H_j, H_k) \geqq D(H_i, H_k)$ and $D(H_i, H_j) > 0$ for $i \neq j$ and $=0$ for $i=j$). The different choices of D will be defined subsequently.

Associated with each H_i is a degree $d_i \in \mathbb{Z}$. We assume that

$$\sum_{i=1}^{k} d_i = 0. \tag{2.1}$$

The holes with $d_i > 0$ (respectively $d_i < 0$) are called *positive* (respectively *negative*) *holes*. Let

$$Q = \sum_{d_i > 0} d_i = -\sum_{d_i < 0} d_i \tag{2.2}$$

be the total positive degree.

Definition of a Connection and Its Length. List the positive holes with each H_i repeated d_i times in the list. Write this list as $P_1, \ldots, P_Q$, with each P_j being some H_i. Likewise, list the negative holes, with each one repeated $|d_i|$ times. Write this as $N_1, \ldots, N_Q$. Note that the holes of degree zero are omitted from these two lists. A *connection*, C, is a pairing of the two lists $(P_1, N_{\sigma 1}), (P_2, N_{\sigma 2}) \ldots (P_Q, N_{\sigma Q})$, where σ is a permutation in S_Q.

The *length* of this connection is defined to be

$$L(C) = \sum_{i=1}^{Q} D(P_i, N_{\sigma i}). \tag{2.3}$$

The *minimal length* is

$$L = \min_C L(C), \tag{2.4}$$

and a *minimal connection* is a connection (which may not be unique) such that

$$L(C) = L.$$

Example 1. The holes are k distinct points, $a_1, \ldots, a_k$ in $\mathbb{R}^N$. $D(a_i, a_j) \equiv |a_i - a_j|$ = Euclidean distance. Note that in this case, holes of degree zero play no role whatsoever. We denote the minimal length by $L(\mathbb{R}^N, \{a\}, \{d\})$.

Example 2. $H_1, \ldots, H_k$ are k disjoint compact subsets of $\mathbb{R}^N$. (H_i could be a point or an object of any "dimension" from 1 to N.) $D(H_i, H_j)$ is defined as follows. First, let $\operatorname{dist}(H_i, H_j)$ be the usual Euclidean distance (i.e. $\min\{|x-y| \,|\, x \in H_i, y \in H_j\}$). Consider a chain $K = (i_0, i_1, \ldots, i_p)$ with each $1 \leqq i_m \leqq k$ and $i_0 = i$, $i_p = j$ and let $\Delta(K) = \sum_{m=1}^{p} \operatorname{dist}(H_{i_{m-1}} H_{i_m})$. Then

$$D(H_i, H_j) \equiv \min_K \Delta(K). \tag{2.5}$$

Note that holes of degree zero that are not points may now play a role in the definition of D since their presence may reduce D (see Fig. 1). Also, one only has to consider chains K without repetition, so the minimum in (2.5) is over a finite set of chains. We denote the minimal length by $L(\mathbb{R}^N, \{H\}, \{d\})$. If all the H_i are points this notation is consistent with Example 1.

Example 3. Let $U \neq \mathbb{R}^N$ be an open set in $\mathbb{R}^N$. Let $H_1, \ldots, H_k$ be disjoint compact subsets of U with degrees $d_1, \ldots, d_k$ but we *do not assume* (2.1). Introduce one more hole, $H_0 \equiv \mathbb{R}^N \backslash U$ (which is closed but not necessarily bounded), and let $d_0 \equiv -\sum_{i=1}^{k} d_i$. We repeat the construction of D and L in Example 2 (on $H_0, H_1, \ldots, H_k$). Note that even though H_0 may not be compact, $D(H_0, H_i) > 0$ for $i \neq 0$. Also note that even if $d_0 = 0$, the presence of H_0 influences D and therefore L. We call the minimal length $L(U, \{H\}, \{d\})$.

Example 4. Let $U \neq \mathbb{R}^N$ be a connected open set in $\mathbb{R}^N$. Let $H_1, \ldots, H_k$ be disjoint compact subsets of U with degrees satisfying (2.1). For $x, y \in U$ let $\operatorname{dist}_G(x, y)$ be the geodesic distance within U, which will be defined in a moment. $\operatorname{Dist}_G(H_i, H_j)$ is defined as in Example 2, but with the Euclidean distance $|x-y|$ being replaced by $\operatorname{dist}_G(x, y)$. Then $D(H_i, H_j)$ is given by (2.5), using dist_G in $\Delta(K)$. The minimal

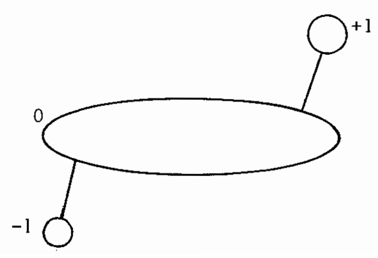

Fig. 1

length in this case will be denoted by $L_G(U, \{H\}, \{d\})$. The geodesic distance $\mathrm{dist}_G(x, y)$ is defined as follows. Let $\kappa = \{x_1, \ldots, x_m\}$, with $x_i \in U$ and $x_0 = x$, $x_m = y$ be a chain with the property that every line segment $[x_i, x_{i+1}] \subset U$. Note that such chains always exist since U is connected and hence arcwise connected. Let $\Lambda(\kappa) = \sum_{i=1}^{m} |x_{i-1} - x_i|$ and

$$\mathrm{dist}_G(x, y) \equiv \inf_{\kappa} \Lambda(\kappa). \tag{2.6}$$

Note that there exists a function $X : [0, 1] \to \bar{U}$ with the properties that $X(0) = x$, $X(1) = y$, and $|X(t) - X(s)| \leqq |t - s| \, \mathrm{dist}_G(x, y)$ for all $t, s \in [0, 1]$. This follows easily from Ascoli's theorem. Furthermore the length of the curve $X([0, 1])$ equals $\mathrm{dist}_G(x, y)$ $\left(\text{the length is } \int_0^1 |\dot{X}(t)| dt\right)$.

If U is convex then $\mathrm{dist}_G(x, y) = |x - y|$ and therefore $L_G(U, \{H\}, \{d\}) = L(\mathbb{R}^N, \{H\}, \{d\})$.

Properties of Minimal Connections. In each example we introduce a distance $D(H_i, H_j)$. It is to be noted that this distance can be realized as the length of a finite union of continuous paths (which may or may not be unique). In Example 1 the path is the line segment $[a_i, a_j]$. In Examples 2 and 3, there is always a certain minimizing chain K in (2.5) and the paths are just line segments which realize $\mathrm{dist}(H_{i_{m-1}}, H_{i_m})$. In Example 4 the line segments are replaced by curves in $\bar{U}$ of length $\mathrm{dist}_G(H_{i_{m-1}}, H_{i_m})$.

Definition. A *string* is a continuous curve $X(t) : [0, 1] \to \bar{U}$ with the following properties:

i) $X(0)$ belongs to some hole H, $X(1)$ belongs to some hole H'.

ii) The length of the curve is $D(H, H')$.

iii) For $t \in (0, 1)$, $X(t)$ belongs to none of the holes. The string carries an *orientation* from H to H'.

In Examples 1–3 a string is just a directed line segment running from H to H'. Given an arbitrary pair (H, H') there need not be a string from H to H', but $D(H, H')$ can always be realized as a finite chain of strings with the obvious consistent sequence of orientations.

Let C be a minimal connection: it has a pairing of the positive and negative holes and a length L given by (2.3). In an obvious way we can associate a finite union of strings with C, namely, first realize $D(P_i, N_{\sigma i})$ as a union of strings as above, and then take the union of all those strings including multiplicity. The sum of the lengths of all the strings is just L.

For descriptive purposes we can think of putting an arrow on each string in the direction of the orientation of the string. Some properties of the strings are the following:

a) For each hole H_i the number of arrows pointing out minus the number of arrows pointing in is just d_i.

b) If more than one string runs between H and H', all these strings are oriented in the same direction.

c) Given two strings in a minimal connection in Examples 1, 2 or 3, either
1) they are identical, or
2) they do not intersect, or
3) they intersect precisely at one point x with x in some hole.

The reason for this is the triangle inequality. Suppose S_1 (respectively S_2) is a string running from H_1 to H_1' (respectively H_2 to H_2'). Possibly some of these four holes are identical. Suppose $z \in S_1 \cap S_2$ and z does not belong to a hole. We claim that $S_1 = S_2$. Let x_1, y_1 (respectively x_2, y_2) be the end points of S_1 (respectively S_2) on H_1, H_1' (respectively H_2, H_2'). Consider the two paths $T_1 = [x_1, z] \cup [z, y_2]$ and $T_2 = [x_2, z] \cup [z, y_1]$. S_1 (respectively S_2) is part of a path joining some P_1 (respectively P_2) to some N_1 (respectively N_2). If we replace S_1 (respectively S_2) by T_1 (respectively T_2) we obtain a new connection in which P_1 (respectively P_2) is paired with N_2 (respectively N_1). The length is the same since $|T_1| + |T_2| = |S_1| + |S_2|$. But T_1 and T_2 are not line segments unless $S_1 = S_2$.

In Example 4 the situation is more complicated. Two different strings can have a non-empty intersection.

III. Upper Bound to the Energy

For simplicity we restrict our attention to $\mathbb{R}^3$. In each of the four examples we have:

Theorem 3.1. *$E \leqq 8\pi L$ with L given by* (2.4).

The proof requires a construction, which we call the basic dipole. Take two distinct points a_+, a_- in $\mathbb{R}^3$ and some positive integer d. Given any $\varepsilon > 0$ we construct a function $\varphi \in C(\mathbb{R}^3 \backslash \{a_+, a_-\}; S^2)$ such that:

a) $$E(\varphi) \leqq 8\pi d |a_+ - a_-| + \varepsilon. \tag{3.1}$$

b) φ is constant outside some set $N_\varepsilon(a_+, a_-)$, which we will henceforth call the support of φ, and which will be defined later.

c) $$\deg(\varphi, \{a_\pm\}) = \pm d. \tag{3.2}$$

Without loss of generality take $a_\pm = (0, 0, \pm l)$. Given $\varepsilon > 0$ we fix a smooth map $\omega : \mathbb{R}^2 \to S^2$ such that:

$$\int_{\mathbb{R}^2} |\nabla \omega|^2 \leqq 8\pi d + \varepsilon/2, \tag{3.3}$$

$$\omega \equiv \text{const} = e \text{ outside the unit disc}, \tag{3.4}$$

$$\deg \omega = -d. \tag{3.5}$$

Here, $\deg \omega$ is defined to be the degree of ω considered as a map from $S^2 \simeq \mathbb{R}^2 \cup \{\infty\}$ (by stereographic projection) to S^2. The existence of such a map is standard (see e.g. [4, proof of Theorem 2, Part C] used with $\underline{u} \equiv \text{const}$). The idea for constructing ω is the following:

(i) Let $v(x, y) = (\mathrm{Re}(x + iy)^{-d}, \ \mathrm{Im}(x + iy)^{-d})$. (ii) Let $\omega(x, y) = (\Pi \circ v)(x, y)$, where Π is stereographic projection from $\mathbb{R}^2$ to S^2. One finds that (3.3) and (3.5) are satisfied with $\varepsilon = 0$. (iii) Replace v by $\chi v = \tilde{v}$, where $0 \leqq \chi \leqq 1$ and χ has compact support and $\chi = 1$ on a large disc D. Equations (3.3) and (3.5) are satisfied if D is chosen large enough. (iv) Now replace $\tilde{v}(x, y)$ by $\tilde{v}(\lambda x, \lambda y) \equiv \hat{v}$ with λ large enough so that $\operatorname{supp} \hat{v} \subset$ unit disc. The left side of (3.3) is independent of λ.

Next, define $\varphi : \mathbb{R}^3 \to S^2$ by

$$\varphi(x, y, z) = \begin{cases} e & \text{if } \quad |z| \geqq l \\ \omega\left(\dfrac{x}{l^2 - z^2}, \dfrac{y}{l^2 - z^2}\right) & \text{if } \quad |z| < l \end{cases} \tag{3.6}$$

and then set

$$\varphi_n(x, y, z) = \varphi(nx, ny, z) . \tag{3.7}$$

φ_n is smooth on $\mathbb{R}^3 \backslash \{a_+, a_-\}$ and satisfies (3.1) (if n is large enough) and (3.2). Finally, $\varphi_n = e$ outside the set where $z^2 + n(x^2 + y^2)^{1/2} \leqq l^2$. This set (for n large enough) is the N_ε in (b) above. Note that the opening angle of N_ε at a_+ and a_- goes to zero as $\varepsilon \to 0$.

Proof of Theorem 3.1 for Examples 1–3. Let C be a minimal connection. As explained in Sect. II, C can be thought of as a finite collection of strings, each of which is a directed line segment running between pairs of holes and which carries some multiplicity, m. Suppose a string runs between $x \in H$ and $y \in H'$ and has length l. Then the open ball of radius l centered at y does not intersect H and, similarly, the open ball of radius l centered at x does not intersect H'. Thus, for small enough ε, we can insert a basic dipole (of degree m) between H and H'. If two or more different strings intersect at a common point $x \in H$ we can insert the required number of disjoint dipoles if ε is small enough. Inside each N_ε we take φ to be given by (3.7), and we take $\varphi = e$ outside $(\cup N_\varepsilon)$. Then $E(\varphi) \leqq 8\pi L + \varepsilon \cdot$(the number of strings in C). □

Proof of Theorem 3.1 for Example 4. The difference with the previous case is that the strings are now curves instead of line segments and, moreover, they can intersect each other outside of the holes. However, any string between H and H' can be approximated (in length) by a polygonal path in $U \backslash (\cup H_i)$ (not $\bar{U}$). Moreover, we can also assume that any two such polygonal paths intersect at most only at their end points. To imitate the above construction we have to find the analogue of the basic dipole construction for a polygonal path, Γ, with end points $a_\pm$. That is, we want to construct a function φ satisfying (a) $E(\varphi) \leqq 8\pi d|\Gamma| + \varepsilon$; (b) $\varphi = e$ outside $N_\varepsilon(\Gamma)$; (c) $\deg(\varphi, a_\pm) = \pm d$. Here, $N_\varepsilon(\Gamma)$ is contained in an ε neighborhood of Γ and has an ε opening angle at $a_\pm$. Let Γ be the union of line segments $[x_{i-1}, x_i]$ with $x_0 = a_+$, $x_p = a_-$ and all $x_i \in U$. We can, by passing to a refinement if necessary, assume that all $|x_{i-1} - x_i|$ are equal and have the common value $2l$. Think of the points x_i, $i = 1, \ldots, p-1$ as holes of degree zero and construct the function φ as in the end of the above proof, i.e. construct disjoint basic dipoles of degree d, one for each segment $[x_{i-1}, x_i]$. Use the same n in (3.7) for all the intervals. Unfortunately, this function φ is not continuous at the points x_i, $i = 1, \ldots, p-1$. However, φ has degree zero at each x_i, $i = 1, \ldots, p-1$. To remedy the lack of continuity we proceed as follows. Let B_i, $i = 1, \ldots, p-1$ be balls of radius $R < l$ at the x_i and with R small enough so that there are only two basic dipoles in each B_i. We shall modify φ inside the B_i. On ∂B_i there are two disjoint circular caps in which $\varphi \neq e$. These are the intersections with ∂B_i of the two N_ε's of the two dipoles that intersect at x_i. Call the caps K_1 and K_2. There is a unique cylinder, C, with elliptical cross-section, whose intersection with ∂B_i is precisely $K_1 \cup K_2$. If λ is a line in C parallel to the axis of C, then $\varphi(\lambda \cap K_1) = \varphi(\lambda \cap K_2)$. The function $\tilde{\varphi}$, which is the modification of φ and which is continuous, is defined by $\tilde{\varphi}(\lambda \cap B_i) = \varphi(\lambda \cap K_1)$. Outside $\cup B_i$, $\tilde{\varphi} = \varphi$. It is easy to see $E(\tilde{\varphi}) \to E(\varphi)$ as $R \to 0$. □

IV. Lower Bound to the Energy

Again, as in Sect. III, we restrict our attention to $\mathbb{R}^3$. In each of the four examples we have:

Theorem 4.1. *$E \geqq 8\pi L$ with L given by* (2.4).

Proof. Let $H = \bigcup_{i=1}^{k} H_i$ and $\Omega = U \backslash H$. Let φ satisfy the appropriate conditions, namely $\varphi \in C(\Omega; S^2)$, $\nabla\varphi \in L^2(\Omega)$, $\deg(\varphi, H_i) = d_i$ and, in Example 4 only, $\varphi \in C(\bar{U} \backslash H)$ and $\varphi = \text{constant} = e$ on ∂U. As explained in Appendix A, we can also assume that $\varphi \in C^\infty(\Omega)$. We shall show that $E(\varphi) \geqq 8\pi L$.

Construct the vector field $D \in C^\infty(\Omega; \mathbb{R}^3)$ as in Appendix B, namely

$$D = (\varphi \cdot \varphi_y \wedge \varphi_z, \varphi \cdot \varphi_z \wedge \varphi_x, \varphi \cdot \varphi_x \wedge \varphi_y)$$

with $\varphi_x = \partial\varphi/\partial x$, etc.

We claim that a.e. on Ω:

$$|D| \leqq \tfrac{1}{2}|\nabla\varphi|^2 . \tag{4.1}$$

To see this, suppose that $\varphi = (0,0,1)$, $\varphi_x = (a_1, b_1, 0)$, $\varphi_y = (a_2, b_2, 0)$, $\varphi_z = (a_3, b_3, 0)$, using the fact that $\varphi \cdot \varphi_x = 0$, etc. Then $D = A \wedge B$ with $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$. Therefore $|D| \leqq |A||B| \leqq \frac{1}{2}(A^2 + B^2) = \frac{1}{2}|\nabla\varphi|^2$. Equality in (4.1) holds if and only if $A \cdot B = A^2 - B^2 = 0$. Let $\zeta \in C(U)$ with $|\nabla\zeta| \leqq 1$ (in $\mathscr{D}'$) and $\zeta = \zeta_i$ is a constant on each H_i. In Example 3 we also assume that $\zeta \in C(\bar{U})$ and $\zeta = 0$ on ∂U. By (B.16) in Appendix B,

$$E(\varphi) \geqq 2 \int_\Omega |D| \geqq -2 \int_\Omega D \cdot \nabla\zeta = 8\pi \sum_{i=1}^{k} \zeta_i d_i . \tag{4.2}$$

Our goal is to show that with $I(\zeta) = \sum \zeta_i d_i$, $I \equiv \sup_{\zeta \in Z} I(\zeta) = L$, where Z denotes the appropriate above-mentioned class. We only require $I \geqq L$, but it is easily seen that $I(\zeta) \leqq L$. Indeed, in Examples 1–3 (respectively 4), $|\zeta(x) - \zeta(y)| \leqq |x - y|$, [respectively $\text{dist}_G(x, y)$] for all $x, y \in U$ and $\zeta \in Z$, since $|\nabla\zeta| \leqq 1$. Consequently, in all cases $|\zeta_i - \zeta_j| \leqq D(H_i, H_j)$. Since $\sum \zeta_i d_i = \sum_{j=1}^{Q} \zeta(P_j) - \zeta(N_j)$ for any pairing (see Sect. II for notation), $I(\zeta) \leqq L$. Therefore we need only construct some $\zeta \in Z$ with $I(\zeta) = L$.

First, suppose there are k numbers $\{\zeta_i\}$ such that, with $\zeta_0 \equiv 0$ (for Example 3),

$$|\zeta_i - \zeta_j| \leqq D(H_i, H_j), \quad \text{for all } i, j . \tag{4.3}$$

Then we can construct $\zeta \in Z$ such that $\zeta = \zeta_i$ on each H_i. One choice is

$$\begin{aligned} \zeta(x) &= \max_i \{\zeta_i - \text{dist}(x, H_i)\} , \quad \text{Examples 1–3} \\ &= \max_i \{\zeta_i - \text{dist}_G(x, H_i)\} , \quad \text{Example 4} . \end{aligned} \tag{4.4}$$

Here $\operatorname{dist}(x,H)$ [respectively $\operatorname{dist}_G(x,H)]=\inf_{y\in H}|x-y|$ [respectively $\inf_{y\in H}\operatorname{dist}_G(x,y)$]. To see that this $\zeta\in Z$, note that $f_i(x)\equiv\operatorname{dist}(x,H_i)$ [respectively $\operatorname{dist}_G(x,H_i)$] satisfies $|\nabla f_i|\leqq 1$, and hence $|\nabla\zeta|\leqq 1$. Clearly $\zeta(H_i)\geqq\zeta_i$, so we have to check that $\zeta_i\geqq\zeta_j-\operatorname{dist}(x,H_j)$ [respectively $\operatorname{dist}_G(x,H_j)$] for all j and all $x\in H_i$. But $\zeta_j-\zeta_i\leqq D(H_i,H_j)\leqq\operatorname{dist}(x,H_j)$ [respectively $\operatorname{dist}_G(x,H_j)$].

To summarize, we merely have to find k numbers satisfying (4.3) and $\sum\zeta_i d_i=L$. Since $D(H_i,H_j)$ satisfies the triangle inequality, the following lemma establishes the existence of $2Q$ numbers $\{\alpha_i\}$ and $\{\beta_i\}$ such that $\alpha_i=\alpha_j$ (respectively $\beta_i=\beta_j$) if $D(P_i,P_j)=0$ [respectively $D(N_i,N_j)=0$]. With the P's and N's corresponding to holes repeated according to multiplicity, as in Sect. II, we can simply take ζ_i to be the common value of α_i (or β_i) on that hole. □

Lemma 4.2. *Let $P_1,P_2,\ldots,P_Q$ and $N_1,N_2,\ldots,N_Q$ be $2Q$ points and let X be their union. Let D be a semi-metric on X (i.e. a metric without the condition that $D(x,y)=0\Rightarrow x=y$). Let $L=\operatorname{Min}_{\sigma\in S_Q}\sum D(P_i,N_{\sigma i})$, where S_Q is the set of permutations. Then there exist real numbers $\alpha_1,\alpha_2\ldots\alpha_Q$ and $\beta_1,\beta_2\ldots\beta_Q$ such that*

$$\sum_{i=1}^{Q}(\alpha_i-\beta_i)=L\,, \tag{4.5}$$

and for all i,j

$$|\alpha_i-\alpha_j|\leqq D(P_i,P_j)\,,\qquad |\alpha_i-\beta_j|\leqq D(P_i,N_j)\,,\qquad |\beta_i-\beta_j|\leqq D(N_i,N_j)\,. \tag{4.6}$$

Proof. This is a consequence of the Kantorovich theorem (see [10, 15, 20, 29]) and the Birkhoff theorem on doubly stochastic matrices (see [2, 26]). The Kantorovich theorem states that if X is a compact metric space with metric D and μ,ν are two non-negative measures on X such that $\int d\mu=\int d\nu$. Then

$$\operatorname*{Max}_{f\in\mathscr{L}}\left(\int f d\mu-\int f d\nu\right)=\operatorname*{Min}_{m}\iint D(x,y)dm(x,y)\,, \tag{4.7}$$

where $\mathscr{L}=\{f:X\to\mathbb{R}\,|\,|f(x)-f(y)|\leqq D(x,y)\}$, and where m is a non-negative measure on $X\times X$ whose marginals are μ and ν. We apply this to our X and D with $\mu=\sum_{i=1}^{Q}\delta_{P_i}$ and $\nu=\sum_{i=1}^{Q}\delta_{N_i}$.

The measures m whose marginals are μ and ν are precisely of the form

$$m=\sum_{i,j=1}^{Q}a_{ij}\delta_{P_i}\otimes\delta_{N_j}\,,$$

where $A=(a_{ij})$ is a doubly stochastic matrix (denoted by DS), i.e. $a_{ij}\geqq 0$ and

$$\sum_{j=1}^{Q}a_{ij}=\sum_{i=1}^{Q}a_{ij}=1\,,$$

for all i,j. The left side of (4.7) is $\operatorname*{Max}_{\alpha,\beta}\sum_{i=1}^{Q}(\alpha_i-\beta_i)$, where α,β satisfy (4.6). The right side of (4.7) is $\operatorname*{Min}_{A\in DS}\sum a_{ij}D(P_i,N_j)$. Birkhoff's theorem states that every $A\in DS$ is a convex combination of permutation matrices. Therefore the right side of (4.7) is $\operatorname*{Min}_{\sigma\in S_Q}\sum D(P_i,N_{\sigma i})=L$. □

V. The D Problem

If we look back at Sect. IV we see that the lower bound for E was obtained by analyzing a problem that, in principle, is different from the original φ problem about S^2-valued vector fields. In this section we shall explore that auxiliary problem – which will be called the D problem – in more detail. Although the two problems give rise to the same minimal energy E in various cases (which fortunately include the cases of interest to us), the vector fields involved are different. At the end of this section we shall remark about the interrelation of φ of D.

The D problem is defined as follows. It will be defined in $\mathbb{R}^N$ instead of just $\mathbb{R}^3$ because the analysis is independent of N. As before we are given an open set $U \subset \mathbb{R}^N$ and k holes H_i (disjoint compact subsets of U). Let $H = \bigcup_{i=1}^{k} H_i$ and $\Omega = U \backslash H$. Associated with each H_i is a real number d_i (which now need not be an integer). We shall be concerned with L^1 vector fields, D, on Ω and distinguish two cases which we call A and B. Let Q_A denote the linear space of all functions $\zeta \in C(\bar{U})$ with $\nabla\zeta \in L^\infty(U)$, $\zeta = 0$ on ∂U (no condition if $U = \mathbb{R}^N$) and ζ is constant on each H_i. Let Q_B denote the linear space of all functions $\zeta \in C(U)$ with $\nabla\zeta \in L^\infty(U)$ and ζ is constant on each H_i.

Let $\mathscr{A}_A$ (respectively $\mathscr{A}_B$) denote the class of all vector fields $D \in L^1(\Omega; \mathbb{R}^N)$ satisfying

$$-\int_\Omega D \cdot \nabla\zeta = \sigma_N \sum_{i=1}^{k} d_i \zeta(H_i) \quad \text{for all } \zeta \in Q_A \text{ (respectively } Q_B). \tag{5.1}$$

Here σ_N denotes the area of S^{N-1} in $\mathbb{R}^N (\sigma_3 = 4\pi)$.

Note that if $U = \mathbb{R}^N$, then $\mathscr{A}_A$ is not empty if and only if $\sum_{i=1}^{k} d_i = 0$. If $U \neq \mathbb{R}^N$ then $\mathscr{A}_A$ is always non-empty $\left(\text{even if } \sum_{i=1}^{k} d_i \neq 0\right)$. $\mathscr{A}_B$ is non-empty (for any U) if and only if $\sum_{i=1}^{k} d_i = 0$. In this section we shall be concerned with minimizing the energy

$$E(D) \equiv \int_\Omega |D|. \tag{5.2}$$

Let E_A (respectively E_B) denote the infimum of $E(D)$ with D in the class $\mathscr{A}_A$ (respectively $\mathscr{A}_B$). Formally, Case A consists of minimizing $\int |D|$ over vector fields D such that $\operatorname{div} D = 0$ in Ω and $\int_{\partial H_i} D \cdot \nu = \sigma_N d_i$ for each i, where ν is the normal to the surface ∂H_i. Case B consists of minimizing $\int |D|$ over vector fields D such that $\operatorname{div} D = 0$ in Ω, $D \cdot \nu = 0$ on ∂U and $\int_{\partial H_i} D \cdot \nu = \sigma_N d_i$ for each i. (If the holes H_i are points a_i, we have, as in Remarks B.2 and B.3, $\operatorname{div} D = \sigma_N \sum d_i \delta_{a_i}$.)

Case A is relevant for Examples 1–3 of Sect. II while Case B is relevant for Example 4. In the following we shall refer to the distance between holes $D(H, H')$ and we shall adopt the convention that for Case A (respectively Case B), $D(H, H')$ is defined as in Examples 1–3 (respectively 4) of Sect. II. If $N = 3$ and the d_i's are integers, the analysis of Sects. II and IV shows that

$$E_{A,B} = \max\{\sigma_N \sum d_i \zeta(H_i) \mid \|\nabla\zeta\|_{L^\infty} \leqq 1,\ \zeta \in Q_A \text{ (respectively } Q_B)\}. \tag{5.3}$$

In fact, (5.3) is always correct for all N and d_i.

Theorem 5.1. *Equation* (5.3) *holds in all cases. Moreover, if* $d_1, \ldots, d_p > 0$ *and* $d_{p+1}, \ldots, d_q < 0$ *and* $\sum d_i = 0$, *then*

$$E_{A,B} = \sigma_N \inf \left\{ \sum_{i=1}^{p} \sum_{j=p+1}^{q} a_{ij} D(H_i, H_j) \right\}, \tag{5.4}$$

with $a_{ij} \geqq 0$ *and* $\sum_{j=p+1}^{q} a_{ij} = d_i$, $\sum_{i=1}^{p} a_{ij} = |d_j|$.

Proof. Equation (5.3) follows from the duality principles given in Appendix C. Equation (5.4) follows from (5.3) as in Lemma 4.2. □

It is intuitively evident from the variational construction in Sect. II, that a minimizing D for (5.2) often does not exist as an L^1 function. This will be clarified later. However, a minimizing sequence $\{D_n\}$ for (5.2) does have a limit in the sense of measures on $\bar{\Omega}$. More precisely there is a subsequence (which we continue to denote by D_n) such that $D_n \rightharpoonup D$ in the weak * topology of measures on $\bar{\Omega}$. This measure D satisfies

$$\int_{\bar{\Omega}} |D| \leqq E_{A,B}. \tag{5.5}$$

Moreover D satisfies (5.1) except that we have to change the linear spaces Q_A (respectively Q_B) into

$$Q'_{A,B} = \{\zeta \in Q_A \text{ (respectively } \zeta \in Q_B) | \nabla\zeta \in C_c(\bar{U})\},$$

so that, in particular, the expression $\int D \cdot \nabla\zeta$ makes sense. We denote by $\mathscr{A}'_A$ (respectively $\mathscr{A}'_B$) the class of all vector valued measures on $\bar{\Omega}$, $D = (D_1, \ldots, D_N)$ such that $\int |D| < \infty$ and

$$-\int_{\bar{\Omega}} D \cdot \nabla\zeta = \sigma_N \sum_{i=1}^{k} d_i \zeta(H_i) \quad \text{for all } \zeta \in Q'_A \text{ (respectively } Q'_B). \tag{5.6}$$

Our problem is twofold: to establish equality in (5.5) and to identify these limiting measures.

Definition. An open set U is said to be *regular* if the following holds. Let

$$U_\varepsilon = U + B_\varepsilon = \{x + y | x \in U, |y| \leqq \varepsilon\}.$$

We suppose that for any two points $x, y \in U$, their geodesic distance relative to U_ε tends to their geodesic distance relative to U as $\varepsilon \to 0$.

Theorem 5.2.

$$\min \left\{ \int_{\bar{\Omega}} |D| \,\middle|\, D \in \mathscr{A}'_A \right\} = E_A. \tag{5.7}$$

If U is regular then

$$\min \left\{ \int_{\bar{\Omega}} |D| \,\middle|\, D \in \mathscr{A}'_B \right\} = E_B. \tag{5.8}$$

Proof. In view of (5.5) it suffices to prove $\geqq$ in (5.7) [or (5.8)]. Let $D \in \mathscr{A}'_A$; we have

$$\int_{\bar{\Omega}} |D| \geqq -\int_{\bar{\Omega}} D \cdot \nabla\zeta = \sigma_N \sum_{i=1}^{k} d_i \zeta(H_i) \tag{5.9}$$

for all $\zeta \in Q'_A$ with $\|\nabla\zeta\|_{L^\infty} \leqq 1$. Therefore we have to show that the supremum of the right side of (5.9) with $\zeta \in Q'_A$ and $\|\nabla\zeta\|_{L^\infty} \leqq 1$ is given by (5.3). We note in passing that, in general, the supremum is not achieved in the class Q'_A. The situation here is "dual" to that of Theorem 5.1 where E_A is not achieved while the right side of (5.3) is achieved. We recall (see Lemma B.5) that given any $\zeta \in Q_A$ with $\|\nabla\zeta\|_{L^\infty} \leqq 1$ there is a sequence ζ_n such that $\zeta_n \in C_c^\infty(U)$, ζ_n is constant on every H_i, $\|\nabla\zeta_n\|_{L^\infty} \leqq 1$ and $\zeta_n \to \zeta$ uniformly on every compact subset of U. This completes the proof of (5.7).

We turn now to the proof of (5.8). Let $U_\varepsilon = U + B_\varepsilon$ and let $H_{i,\varepsilon} = H_i + B_\varepsilon$. Let E_{B_ε} be the right side of (5.3) for this ε problem. Since U is regular $E_{B_\varepsilon} \to E_B$ as $\varepsilon \to 0$ by Remark 5.1. Let ζ_ε be a maximizer of (5.3) for the ε problem. Without loss of generality we may assume that $\zeta_\varepsilon \in L^\infty(U_\varepsilon)$ (otherwise truncate ζ_ε). Let $\zeta' = J_{\varepsilon/2} * \zeta_\varepsilon$. Clearly $\zeta' \in C^1(\bar{U})$, $\|\nabla\zeta'\|_{L^\infty(U)} \leqq 1$ and $\zeta'(H_i) = \zeta_\varepsilon(H_{i,\varepsilon})$. Finally consider $\zeta_n = (1 + C/n)^{-1}\chi_n\zeta'$, where $\chi_n(x) = \chi(x/n)$ and $\chi \in C_c^\infty(\mathbb{R}^N)$ is any function such that $\chi(x) = 1$ for $|x| < 1$ with $\|\chi\|_{L^\infty} \leqq 1$ and $C = \|\nabla\chi\|_{L^\infty}\|\zeta'\|_{L^\infty}$. Note that $\zeta_n \in Q'_B(U)$, $\|\nabla\zeta_n\|_{L^\infty(U)} \leqq 1$ and $\zeta_n \to \zeta'$ uniformly on every compact subset of $\bar{U}$. □

Remark 5.1. Case B of Theorem 5.2 may fail if U is not regular. Take for example

$$U = \mathbb{R}^3 \backslash \{(x, y, 0) | x \geqq 0,\ y \in \mathbb{R}\}.$$

For this U, the requirement that $\nabla\zeta \in C(\bar{U})$ implies that $Q'_B(U) = Q'_B(\mathbb{R}^3)$, and therefore the supremum of the right side of (5.5) can be less than the right side of (5.3).

We now turn to properties of the minimizing D measures.

1. Properties of the Support

Theorem 5.3. *Let D be any one of the following vector valued measures*

i) *a weak * limit of an L^1 minimizing sequence for* (5.2),

ii) *one of the minima referred to in* (5.7) *or* (5.8).

Let G be the union (which is closed) of all geodesics running between holes (see Sect. II). Then

$$\operatorname{supp} D \subset G. \tag{5.10}$$

Moreover, if all the d_i's are integers, then $\operatorname{supp} D \subset G'$, *where $G' \subset G$ is the union of all minimal connections. Note that the definition of G' depends on the $\{d_i\}$.*

Proof. Let B be a closed set in U such that $B \cap G$ and $B \cap H$ are empty. Consider $V = U \backslash B$. The geodesic distance between holes for the V problem is obviously the same as for the U problem. Consider D restricted to $\bar{V}$ (respectively the $D_n \in L^1$ restricted to V) and call it $\hat{D}$ (respectively $\hat{D}_n$). These vector fields satisfy all the right conditions (for V), so

$$\int_{\Omega(V)} |\hat{D}| \geqq E(V) = E(U) = \int_{\Omega} |D| = \int_{\Omega(V)} |\hat{D}| + \int_B |D|,$$

and thus $D = 0$ on B, which proves (5.9). Note that $E(U) = E(V)$ by virtue of (5.4). A similar reasoning works for the sequence D_n, as well as for the case of integral d's (see Lemma 4.2). □

Remark 5.2. Note that i) holds even if U is not regular.

Remark 5.3. Consider Case A and assume D is a minimizer in (5.7). Then the $|D|$ measure of $(\cup H_i)$ is zero.

Indeed

$$\int_{\mathbb{R}^N\setminus(\cup H_i)} |D| \geqq -\int_{\mathbb{R}^N\setminus(\cup H_i)} D\cdot\nabla\zeta = \sigma_N \sum d_i\zeta(H_i)$$

for all $\zeta\in C^1(\mathbb{R}^N)$ with $|\nabla\zeta|\leqq 1$, $\zeta=0$ on ∂U and ζ is constant near each H_i. Therefore

$$\int_{\mathbb{R}^N\setminus(\cup H_i)} |D| \geqq \sigma_N \operatorname*{Sup}_{\zeta} \sum d_i\zeta(H_i),$$

where ζ runs in the above mentioned class. It follows that

$$\int_{\mathbb{R}^N\setminus(\cup H_i)} |D| \geqq E_A = \int_{\mathbb{R}^N} |D|.$$

Remark 5.4. We conjecture that, for any U (regular or not) we have

$$\min\left\{\int_{\Omega} |D| \,\middle|\, D\in\mathscr{A}'_B,\ \operatorname{supp} D\subset G\right\} = E_B. \tag{5.11}$$

2. The Two Hole Problem

We investigate here two simple cases:

a) Case A with $U=\mathbb{R}^N$ and two disjoint holes H_1 and H_2 with $d_1+d_2=0$.

b) Case B with $U\neq\mathbb{R}^N$ and again two disjoint holes H_1 and H_2 with $d_1+d_2=0$.

In both cases $D(H_1,H_2)=L$.

Let us first analyze the case where there is precisely one geodesic, g, between H_1 and H_2. Let D be as in Theorem 5.3 so that $\operatorname{supp} D\subset g$. By Appendix D we know that D must be a measure of the form

$$D = cD_g. \tag{5.12}$$

On the other hand,

$$\int |D| = \sigma_N |d_1| L, \tag{5.13}$$

where L is the length of g. On the other hand,

$$\int |D_g| = L, \tag{5.14}$$

so that $c=\sigma_N|d_1|$. In particular $|D|$ is the uniform Hausdorff measure on g and the "direction of D is tangent to g." We shall establish similar properties in the general case where there are many geodesics between H_1 and H_2. As before we denote by G the union of all geodesics. In Case A, G is simply a union of line segments of length L which are disjoint except possibly for the end points. In particular every point, x, in $G\setminus(H_1\cup H_2)$ has precisely one geodesic passing through it. We denote its direction (going from H_1 to H_2) by $\mathbf{n}(x)$.

In Case B the situation can be much more complicated. Many geodesics can pass through a single point and the tangent need not be defined at every point of a geodesic. There could also be many geodesics connecting two points.

Theorem 5.4. *Under the assumptions of Theorem 5.3 and in Case A, the vector-valued measure D and the scalar-valued measure $|D|$ are related by*

$$D = \mathbf{n}|D|. \tag{5.15}$$

Note that Eq. (5.12) relied only on the fact that $\operatorname{div} D = 0$. But in Theorem 5.4 we really need the fact that D is minimizing. Consider, for example, the case where H_1 and H_2 are two cubes with parallel faces so that G is a cylinder, C, with a square base. Let g be any curve going from H_1 to H_2 in G. Then $\operatorname{div} D_g = 0$, but (5.15) fails. The requirement that D is minimizing forces the "integral curves" of D to be geodesics.

We believe that a similar result holds in Case **B**.

Conjecture 5.1. Under the assumptions of Theorem 5.3, and in Case **B**, D is such that $\mathbf{n}(x)$ is well defined a.e. $|D|$ and $D = \mathbf{n}(x)|D|$. Here $\mathbf{n}(x)$ is the common tangent – when it exists – to all geodesics through x.

Proof of Theorem 5.4. Let $H_{1,\delta} = \{x \mid \operatorname{dist}(x, H_1) \leqq \delta\}$, and similarly for H_2, with $\delta > 0$ and small. Note that $\operatorname{dist}(H_{1,\delta}, H_{2,\delta}) = L - 2\delta$. Let

$$f(x) = \min\{\operatorname{dist}(x, H_{1,\delta}),\ L - 2\delta\}. \tag{5.16}$$

Let g be a nonnegative C^∞ function with support in a ball of radius one around 0 and $\int g = 1$. Let $g_\varepsilon(x) = \varepsilon^{-N} g(x/\varepsilon)$. Set

$$f_\varepsilon = g_\varepsilon * f \quad \text{for} \quad \varepsilon < \delta, \tag{5.17}$$

so that f_ε is a smooth function which is zero on H_1 and $f_\varepsilon = L - 2\delta$ on H_2. We claim that

$$\nabla f_\varepsilon(x) \to \mathbf{n}(x) \text{ everywhere on } G \backslash (H_{1,\delta} \cup H_{2,\delta}). \tag{5.18}$$

Assuming that (5.18) holds, (5.15) follows easily. Indeed, since f is Lipschitz with constant one, $|\nabla f_\varepsilon| \leqq 1$ and hence

$$\sigma_N L = \int |D| \geqq \int D \cdot \nabla f_\varepsilon = (L - 2\delta)\sigma_N. \tag{5.19}$$

From (5.18), and dominated convergence, we have

$$\int\limits_{G\backslash(H_{1,\delta} \cup H_{2,\delta})} D \cdot \nabla f_\varepsilon \xrightarrow[\varepsilon \to 0]{} \int\limits_{G\backslash(H_{1,\delta} \cup H_{2,\delta})} D \cdot \mathbf{n}, \tag{5.20}$$

and thus, combining (5.19) and (5.20) we obtain

$$\int\limits_{G\backslash(H_{1,\delta} \cup H_{2,\delta})} D \cdot \mathbf{n} \geqq (L - 2\delta)\sigma_N - \int\limits_{(H_{1,\delta} \cup H_{2,\delta})\backslash(H_1 \cup H_2)} |D| \tag{5.21}$$

(note that $f = 0$ near H_1 and H_2). Passing to the limit in (5.21) as $\delta \to 0$ we find

$$\int\limits_{G\backslash(H_1 \cup H_2)} D \cdot \mathbf{n} \geqq L\sigma_N.$$

By Radon-Nikodym, we may write $D = \mathbf{F}|D|$ for some function $\mathbf{F} \in L^\infty(|D|)$ and $|\mathbf{F}| = 1$ a.e. $|D|$. Thus we have $\mathbf{F} \cdot \mathbf{n} = 1$ a.e. $|D|$, and so $\mathbf{F} = \mathbf{n}$ a.e. $|D|$. We turn now to the proof of (5.18). Let $\mathbf{e}$ be any unit vector in $\mathbb{R}^N$. We have

$$\frac{1}{t}[f_\varepsilon(x_0 + t\mathbf{e}) - f_\varepsilon(x_0)] = \frac{1}{t}\int g_\varepsilon(x_0 - y)[f(y + t\mathbf{e}) - f(y)]dy$$

$$= \frac{1}{t}\int g_\varepsilon(x_0 - y)[\operatorname{dist}(y + t\mathbf{e}, H_1) - \operatorname{dist}(y, H_1)]dy$$

for $x_0 \in G \backslash (H_{1,\delta} \cup H_{2,\delta})$ and ε small enough. Given a point z we denote by $a(z)$ any measurable projection of z on H_1. We have

$$\operatorname{dist}(y+t\mathbf{e}, H_1) - \operatorname{dist}(y, H_1) \geqq |y+t\mathbf{e}-a(y+t\mathbf{e})| - |y-a(y+t\mathbf{e})|,$$

and thus

$$\frac{1}{t}[f_\varepsilon(x_0+t\mathbf{e}) - f_\varepsilon(x_0)] \geqq \frac{1}{t}\int g_\varepsilon(x_0-y+t\mathbf{e})\,[|y-a(y)| - |y-t\mathbf{e}-a(y)|].$$

On the other hand

$$\begin{aligned} |y-t\mathbf{e}-a(y)| &= [|y-a(y)|^2 - 2t\mathbf{e}\cdot(y-a(y)) + t^2]^{1/2} \\ &\leqq |y-a(y)|\left\{1 - t\frac{\mathbf{e}\cdot(y-a(y))}{|y-a(y)|^2}\right\} + Ct^2. \end{aligned}$$

Therefore as $t \to 0$ (and fixed ε) we have

$$\nabla f_\varepsilon \cdot \mathbf{e} \geqq \int g_\varepsilon(x_0-y)\frac{\mathbf{e}\cdot(y-a(y))}{|y-a(y)|}\,dy.$$

Finally we observe that

$$\lim_{y \to x_0} \frac{y-a(y)}{|y-a(y)|} = \mathbf{n}(x_0),$$

since $a(y) \to a(x_0)$ because x_0 has a unique projection on H_1. We conclude that

$$\liminf_{\varepsilon \to 0} \nabla f_\varepsilon \cdot \mathbf{e} \geqq \mathbf{e} \cdot \mathbf{n}(x_0).$$

Changing $\mathbf{e}$ into $-\mathbf{e}$ we obtain (5.18). □

As we remarked earlier, when there is only one geodesic g between H_1 and H_2, then $|D|$ is a uniform measure on g. The analogue of this when there are many geodesics is the following

Theorem 5.5. *Let D be a measure as in Theorem 5.3 (for either Case A and Case B). For $0 \leqq \alpha < \beta \leqq L$ consider the slice*

$$S(\alpha, \beta) = \{x \mid \alpha < \operatorname{dist}(x, H_1) \leqq \beta\} \tag{5.22}$$

(with dist = *geodesic distance in Case B). Then*

$$\int_{S(\alpha,\beta)} |D| = \sigma_N |d_1| (\beta - \alpha). \tag{5.23}$$

Proof. Replace H_1 by $H_1 \cup S(0, \alpha)$. For this new problem $E' = \sigma_N |d_1| (L-\alpha)$. But we can use D restricted to $\bar{U} \backslash S(0, \alpha)$ as a variational measure for E' and obtain $E' \leqq \int_{S(\alpha, L)} |D|$. Likewise, replacing H_2 by $H_2 \cup S(\alpha, L)$,

$$\sigma_N |d_1| = E'' \leqq \int_{S(0,\alpha)} |D|.$$

Adding these inequalities we obtain:

$$\sigma_N |d_1| L \leqq \int_{S(\alpha, L)} |D| + \int_{S(0,\alpha)} |D| = \sigma_N |d_1| L.$$

Therefore

$$\int_{S(0,\alpha)} |D| = \sigma_N |d_1| \alpha . \ \square$$

3. *The Many Hole Problem*

We now turn to the description of all minimizing D fields in the general case with many holes.

Suppose there are k holes $H_1, \ldots, H_k$ with degrees $d_1, \ldots, d_k$ (not necessarily integral). Some of these may be zero. We assume $\sum d_i = 0$ because, in Case A, we can assume that the complement of U is also a hole with the appropriate degree. First, let us consider the minimal energy $E_{A,B}$. If $d_1, \ldots, d_p > 0$ are the positive d's and $d_{p+1}, \ldots, d_q < 0$ are the negative d's, (5.4) gives us $E_{A,B}$ in terms of a $p \times r$ matrix $A = \{a_{ij}\}$ (with $r = q - p$). The set of minimizing A's (call it $\mathscr{A}$) is convex, as is the set of minimizing D's.

Recall that $D(H_i, H_j)$ is the geodesic distance (different for Cases A and B). It is realized by a finite sequence of strings (see Sect. II) running between a sequence of holes. More than one sequence may be possible. To be more specific, let $\chi_{ij} = 1$ if there exists a string between H_i and H_j and $\chi_{ij} = 0$ otherwise. For $\chi_{ij} = 1$, define $\mathscr{G}_{ij}$ to be the set of all strings between H_i and H_j, and L_{ij} their common length. Likewise, for $\chi_{ij} = 1$, let $\mathscr{D}_{ij}$ be the set of minimizing D fields (with $d_i = 1$, $d_j = -1$) constructed in the preceding section (the two-hole problem). If $\chi_{ij} = 0$, $\mathscr{G}_{ij}$ (respectively $\mathscr{D}_{ij}$) is a union (respectively sum) of the strings (respectively minimizing D fields) connecting H_i to H_j (respectively with $d_i = +1$, $d_j = -1$).

Now given an $A \in \mathscr{A}$ we can construct a minimizing D field as follows:

$$D = \sum_{i=1}^{p} \sum_{j=p+1}^{q} a_{ij} D^{ij} , \tag{5.24}$$

where $D^{ij} \in \mathscr{D}_{ij}$. Recall that when the d_i's are integral the extreme elements of $\mathscr{A}$ are given by Birkhoff's theorem, namely by a minimal connection in which each $a_{ij} \in \mathbb{Z}^+$.

Theorem 5.6. *For Case A, every minimizing D field is given by* (5.24).

We conjecture that the same is true for Case B.

Proof. Let $M = \{(i,j) | \chi_{ij} = 1\}$. A little thought shows that we can rewrite (5.4) as follows:

$$E_A = \tfrac{1}{2} \sigma_N \min \sum_M |\mu_{ij}| L_{ij} , \tag{5.25}$$

where the minimum is over $\mu_{ij} = -\mu_{ji}$ and $\sum_j \mu_{ij} = d_i$ for $i = 1, \ldots, k$. Pictorially, μ_{ij} can be thought of as the flux from i to j; it is not required that μ_{ij} has any definite sign.

Suppose that $(a,b) \in M$ and $(c,d) \in M$ (all points being distinct) and that geodesics $g_{ab} \in \mathscr{G}_{ab}$, $g_{cd} \in \mathscr{G}_{cd}$ (these are line segments). Suppose also that g_{ab} and g_{cd} intersect at a point P. In this case, we claim that either $\mu_{ab} = 0$ or $\mu_{cd} = 0$. If not, we can assume that $\mu_{cd} \geqq \mu_{ab} > 0$. Clearly, $D(H_a, H_d) < \operatorname{dist}(H_a, P) + \operatorname{dist}(P, H_d)$ and

$D(H_c, H_b) < \mathrm{dist}(H_c, P) + \mathrm{dist}(P, H_b)$. If (a, d) and $(c, b) \in M$, then $D(H_a, H_d) = L_{ad}$ and $D(H_c, H_b) = L_{bc}$, and we can replace the four numbers $\mu_{ab}, \mu_{cd}, \mu_{ad}, \mu_{cb}$ by 0, $\mu_{cd} - \mu_{ab}, \mu_{ad} + \mu_{ab}, \mu_{ab} + \mu_{cb}$ and thereby strictly lower the energy. This construction has to be modified in an obvious way if (a, d) or $(c, b) \notin M$. The conclusion we reach is that whenever $\mathscr{G}_{ab} \cap \mathscr{G}_{cd}$ is not empty, then every minimum of (5.25) has $\mu_{ab} = 0$ or $\mu_{cd} = 0$. The choice $[(a, b)$ or $(c, d)]$ is universal; if $\mu_{ab} \neq 0$ in one minimum and $\mu_{cd} \neq 0$ in another, then by taking the mean (which is still a minimum) we would have a contradiction.

Let $N \subset M$ be the set of (i, j) such that $\mu_{ij} \neq 0$ for some minimizer in (5.25). The families of geodesics $\{\mathscr{G}_{ab}\}$ for $(a, b) \in N$ are disjoint except possibly for the endpoints. Let $G = \bigcup_N \mathscr{G}_{ab}$. Now let D be a minimizer. We claim that $\mathrm{supp}\, D \subset G$. The proof of this is the same as the earlier proof (5.10) that $\mathrm{supp}\, D$ is contained in the geodesics between the positive and negative holes. If $x \notin G$ then remove a small ball around x (thereby creating a degree zero hole). If the ball is small enough nothing changes in (5.25) (recall the strict inequality of the preceding paragraph). Thus E_A does not change, but $D' = (1 - \chi_B) D$, with χ_B being the characteristic function of the removed ball, is an allowed vector field for the new problem, whence $\int_B |D| = 0$.

For $(a, b) \in N$, consider $D_{ab} \equiv F_{ab} D$, where F_{ab} is the characteristic function of $\mathscr{G}_{ab}$. If $\zeta \in C^1(\mathbb{R}^N)$ and $\zeta = 1$ on H_a, $\zeta = 0$ on H_b then, as is easily seen

$$-\int D_{ab} \cdot \nabla \zeta = \sigma_N \alpha_{ab}, \tag{5.26}$$

where α_{ab} is some constant that is independent of ζ. From the defining condition (5.1) on D we see that $\alpha_{ab} = -\alpha_{ba}$ and $\sum_b \alpha_{ab} = d_a$. By (5.25), $_A \leqq \frac{1}{2} \sigma_N \sum_{(a,b) \in N} |\alpha_{ab}| L_{ab}$. On the other hand, $\int |D| = \frac{1}{2} \sum_{(a,b) \in N} \int |D_{ab}|$ [since the $|D|$ measure of the holes is zero (see Remark 5.3)]. Thus $\int |D| \geqq \frac{1}{2} \sigma_N \sum_{(a,b) \in N} |\alpha_{ab}| L_{ab}$ [by (5.26)]. □

Remark 5.5 on the Relation of φ *to* D. Let Ω be an open set in $\mathbb{R}^3$. In Sect. IV, to every $\varphi \in C(\Omega; S^2)$ with $\nabla \varphi \in L^2(\Omega)$ we have associated a D field with the property that $\mathrm{div}\, D = 0$ in $\mathscr{D}'$ [for the generalization to $\mathbb{R}^N$, see (B.7)]. It is a natural question whether any vector field D with $\mathrm{div}\, D = 0$ comes from a φ. The answer is no, as the following, based on a remark of D. Sullivan, shows. Let S be a smooth closed surface in Ω. Let D a smooth vector field with the property that some integral curve of D is dense in S (for example S could be a two-torus, and D restricted to S is an irrational twist of the torus). Then this D can not come from a φ, as we shall now show.

From the definition of D in Sect. IV it follows that

$$(D \cdot \nabla) \varphi = 0, \tag{5.27}$$

since $|\varphi|^2 = 1$ [and thus $\det(\varphi_x, \varphi_y, \varphi_z) = 0$]. It follows that φ is constant on the integral curves of D and in particular φ is constant on S. Therefore $D = 0$ on S since $D = 0$ whenever $\nabla \varphi$ vanishes in two orthogonal directions. This contradicts the fact that $D \not\equiv 0$ on S.

VI. Behavior of Minimizing Sequences for the φ Problem

As before we are given an open set $U \subset \mathbb{R}^3$ and k holes H_i (disjoint compact subsets of U). Let $H = \bigcup_{i=1}^{k} H_i$ and $\Omega = U \backslash H$. Associated with each H_i is an *integer* d_i. We are concerned with the behavior of minimizing sequences for the problem

$$E = \inf \int |\nabla\varphi|^2 \tag{6.1}$$

under the appropriate conditions on φ, namely $\varphi \in C(\Omega; S^2)$, $\nabla\varphi \in L^2(\Omega)$, $\deg(\varphi, H_i) = d_i$ and, in Example 4 only, $\varphi \in C(\bar{U}\backslash H)$ and $\varphi = \text{constant}$ on ∂U. In Example 4 we also assume that U is regular.

Let φ^n be a minimizing sequence for (6.1) and let D^n be the field corresponding to φ^n. By passing to a subsequence we may assume that $D^n \rightharpoonup D$ weakly in the sense of measures. We claim that

$$|\nabla\varphi^n|^2 \rightharpoonup 2|D| \tag{6.2}$$

weakly in the sense of measures. Indeed we have $|D^n| \leqq \frac{1}{2}|\nabla\varphi^n|^2$; let us assume that $|\varphi^n|^2 \rightharpoonup \nu$ weakly in the sense of measures. Then we have

$$2|D| \leqq \nu\,. \tag{6.3}$$

On the other hand, by Theorem 5.2

$$\int |D| \geqq \tfrac{1}{2}E\,. \tag{6.4}$$

Since $\int \nu = E$, we conclude that $\nu = 2|D|$. Again, by Theorem 5.2, D is a minimizer for E_A or E_B, and thus we have the description of D given in Sect. V.

The conclusion of all this is that any minimizing sequence for the φ problem inherits all properties of minimizing sequences for the D problem that we studied in Sect. V. In particular, since D is supported on G, the union of all geodesics running between holes, (6.2) implies that φ^n converges strongly in H^1 to a constant on each connected component of the complement of G. However, the fact that D comes from a D^n, which comes from a φ^n, leads to some additional properties for D beyond those implied by the fact that D is a minimizer for the D problem. To derive these additional properties, Appendix E will play an essential role.

For simplicity we shall restrict our investigation to Examples 1–3 and with the additional assumption that there are only finitely many strings between any two holes.

Theorem 6.1. *There is a minimal connection, C, which we write $C = \cup g_i$, where the g_i's are strings (which are repeated according to their multiplicity in C), such that*

$$D = 4\pi \sum_i D_{g_i}\,. \tag{6.4}$$

D_g is defined in Appendix D. In particular,

$$|D| = 4\pi \sum_i \delta_{g_i}\,, \tag{6.5}$$

where δ_g is the one-dimensional Hausdorff measure on g. Consequently [*by* (6.2), (6.5)],

$$|\nabla\varphi^n|^2 \rightharpoonup 8\pi \sum_i \delta_{g_i}\,. \tag{6.6}$$

Remark 6.1. The point of this theorem is the following. If there is only one minimal connection, then the D problem has a unique minimizer and Theorem 6.1 does not give any additional information beyond that contained in (6.2). The interesting case is where there are several minimal connections, say C_1 and C_2 for example. Let D_1 (respectively D_2) be a minimizer with support in C_1 (respectively C_2). Any convex combination of D_1 and D_2 is also a minimizer, but this cannot happen for the φ problem. $|\nabla\varphi^n|^2$ must converge either to $2|D_1|$ or $2|D_2|$ but cannot converge to $|D_1|+|D_2|$, for example. This is a consequence of the quantization condition of Appendix E.

Proof. We recall that

$$\operatorname{supp} D \subset \bigcup_{(i,j)\in N} g_{ij},$$

where g_{ij} is a string running between H_i and H_j (i.e. it is a line segment). The set N is described in the proof of Theorem 5.6. N has the property that two distinct strings in N can intersect only at a common endpoint. We can write D as

$$D=(\tfrac{1}{2})4\pi \sum_{(i,j)\in N} v_{ij} D_{g_{ij}},$$

where g_{ij} is oriented from H_i to H_j and $v_{ij}=-v_{ji}$. By Theorem E.5 we know that $v_{ij}\in\mathbb{Z}$. Moreover the divergence condition implies that $\sum_j v_{ij}=d_i$ for each $i=1,2,\ldots,k$. The energy is given by

$$\int |D| = \tfrac{1}{2}4\pi \sum_{(i,j)\in N} |v_{ij}| L_{ij},$$

where L_{ij} is the length of g_{ij}. Since the energy is minimal, it follows that v_{ij} is a minimizer for (5.25). We claim that this set of v_{ij} defines a connection (which must be minimal since the energy is minimal). Take any positive hole, say H_i. By the divergence condition there must be at least one $v_{ij}>0$. Go to H_j. If this is a negative hole, then stop and replace v_{ij} by $v_{ij}-1$. If H_j is a zero or positive hole, then keep going until a negative hole is reached. Along this path replace all the v's by $v-1$. By repeating this construction Q times (where Q is the sum of the positive degrees), we obviously have a connection. We claim that the remaining v's are all zero. This follows from the fact that v is a minimizer for (5.25) and that replacing the residual v's by zero would lower the energy in (5.25) and preserve the divergence condition. □

VII. Minimizing the Energy with Specified Boundary Conditions

A problem that we have so far not addressed in this paper is the minimization of the energy when φ is specified on the boundary of a domain (except for the special case where φ is constant on the boundary). Our analysis of the D problem in Sect. V is a useful guide to understanding the solution to certain open problems of this genre. In particular we shall answer the following questions.

Let B be the open unit ball in $\mathbb{R}^3$ and let

$$C_1=\{\varphi\in H^1(B;S^2)\,|\,\varphi(x)=x \text{ on } \partial B\}. \tag{7.1}$$

Let

$$E(\varphi)=\int |\nabla\varphi|^2, \tag{7.2}$$

and

$$E_1 = \inf_{\varphi \in C_1} E(\varphi). \tag{7.3}$$

Question 1. Is $\psi(x) = x/|x|$ a minimizer for E_1?

Answer. Yes (see Theorem 7.1).

Next, let

$$C_2 = \{\varphi \in H^1(B; S^2) | \varphi(x) = g(x) \text{ on } \partial B\},$$

where $g: S^2 \to S^2$ is a given smooth map. Let

$$E_2 = \inf_{\varphi \in C_2} E(\varphi).$$

Question 2. Is $g(x/|x|)$ a minimizer for E_2?

Answer. No, unless g is an isometry (i.e. $+g$ or $-g$ is a rotation) or g is a constant (see Theorems 7.3 and 7.4).

In other words, if g is any smooth map from S^2 to S^2, and if g is extended radially to B, the extension is unstable unless g happens to be the constant map (degree zero) or g is the identity map modulo an isometry (degree ± 1).

We recall that a (smooth) map g from S^2 to S^2 is called harmonic if it satisfies the equation, $-\Delta g = g|\nabla g|^2$, where Δ is the Laplace-Beltrami operator on S^2. Harmonic maps from S^2 to S^2 have been classified (see e.g. [22, 35]) and their form is given in the proof below of Theorem 7.4. They all have the property that they minimize $\int_{S^2} |\nabla g|^2$ subject to the condition that the degree d of g is prescribed. In particular this integral is $8\pi|d|$.

We also recall a result of Schoen-Uhlenbeck [31, 32] that if we take an arbitrary domain Ω and minimize $E(\varphi)$ in $H^1(\Omega; S^2)$ with specified boundary condition, then any minimizing φ has at most finitely many point singularities. Our result implies that these singularities always have degrees ± 1 (see Corollary 7.12). In an earlier work Hardt et al. [19] showed that the degrees of these singularities are bounded by some universal constant. This confirms the observations on liquid crystals that stable point singularities have degree ± 1 [6]. It also confirms numerical studies by Cohen et al. [8] showing that singularities of degree two or more are unstable. Our first result is the following:

Theorem 7.1. *$\psi(x) = x/|x|$ is a minimizer for E_1; in fact, it is the unique minimizer.*

An obvious consequence of Theorem 7.1 is the following:

Corollary 7.2. *Suppose $g(x) = Rx$, where R is a rotation in $SO(3)$. Then $\psi(x) = Rx/|x|$ uniquely minimizes E_2.*

Our second result is:

Theorem 7.3. *If g has degree $+1$, then $g(x/|x|)$ is not a minimizer for E_2 unless $g(x) = Rx$, where R is a rotation in $SO(3)$.*

Our last main result is:

Theorem 7.4. *If g has degree d with $|d| \geqq 2$, then $g(x/|x|)$ is not a minimizer for E_2.*

Proof of Theorem 7.1. Clearly we have

$$E_1 \leqq E(x/|x|) = 8\pi. \tag{7.4}$$

In order to show that $\psi(x) = x/|x|$ is the unique minimizer for E_1 it suffices to show that

$$E(\varphi) > 8\pi \quad \text{for every } \varphi \in C_1,\ \varphi \neq \psi. \tag{7.5}$$

This leads us to the question of finding lower bound for the energy.

A. Lower Bounds for the Energy

There is always a minimizer for E_1 and, if φ_0 is a minimizer, we know from [32] that φ_0 is smooth on $\bar{B}$ except at most at a finite number of points in B. Therefore it suffices to prove (7.5) for φ in the class

$$\tilde{C} = \{\varphi \in C_1 | \varphi \text{ is continuous on } \bar{B}, \text{ except at a finite number of points in } B\}.$$

This will be achieved using the D field associated to φ. [An alternative to using the result of [32] about minimizers is to use a theorem of Bethuel-Zheng (in preparation) which states that $\tilde{C}$ is dense in C_1 for the H^1 norm.]

Let $\varphi \in H^1(B; S^2)$ be smooth on $\bar{B}$ except at a finite number of points in B (we do not assume that $\varphi(x) = x$ on ∂B). Let D be the D field associated with φ as in Sect. IV.

We have

$$\tfrac{1}{2}\int |\nabla\varphi|^2 \geqq \int |D| \geqq \int D \cdot \nabla\zeta = \int_{\partial B} (D \cdot n)\zeta - \int_B (\operatorname{div} D)\zeta$$

for every $\zeta \in C(\bar{B})$ with $\|\nabla\zeta\|_{L^\infty} \leqq 1$. Recall that $D \cdot n$ depends only on the values of φ restricted to ∂B and, more precisely, $D \cdot n = \varphi \cdot \varphi_x \wedge \varphi_y$, where x, y are orthonormal coordinates on S^2. On the other hand

$$\operatorname{div} D = 4\pi \sum_{i=1}^{p} d_i \delta_{a_i}$$

with $d_i \in \mathbb{Z}$ and $a_i \in B$. Consequently

$$\int_{\partial B} (D \cdot n) = 4\pi \deg(\varphi, S^2) \quad \text{and} \quad \sum_{i=1}^{p} d_i = \deg(\varphi, S^2).$$

Therefore we have

$$\int |\nabla\varphi|^2 \geqq 8\pi \max\left\{\frac{1}{4\pi}\int_{\partial B} (D \cdot n)\zeta - \sum_{i=1}^{p} d_i \zeta(a_i) | \zeta \in C(\bar{B}) \text{ with } \|\nabla\zeta\|_{L^\infty} \leqq 1\right\}. \tag{7.6}$$

A basic lower bound for the right side of (7.6) is given by the following

Theorem 7.5. *Let M be a compact metric space with distance $\delta(x, y)$, let μ be a probability measure on M and let $\nu = \sum_{i=1}^{p} d_i \delta_{a_i}$, where $d_i \in \mathbb{Z}$ and $\sum_{i=1}^{p} d_i = 1$.*

Then

$$I(\nu) = \max\{\int \zeta d\mu - \int \zeta d\nu | \|\zeta\|_{\mathrm{Lip}} \leqq 1\} \geqq \min_{c \in M} \int \delta(x, c) d\mu(x), \tag{7.7}$$

where $\|\zeta\|_{\mathrm{Lip}} = \sup_{x \neq y} |\zeta(x) - \zeta(y)|/\delta(x, y)$.

Note that the right side in (7.7) is independent of v and that (7.7) is obvious if $p=1$, namely $v=\delta_a$ [take $\zeta(x)=\delta(x,a)$]. It follows that

$$\min_{v}\max_{\zeta}\{\int \zeta d\mu - \int \zeta dv\} = \min_{c}\int \delta(x,c)d\mu(x)\,.$$

Combining (7.6) and Theorem 7.5 we obtain

Corollary 7.6. *Assume φ restricted to ∂B has degree one and satisfies $D\cdot n \geqq 0$ on ∂B, then*

$$\int |\nabla\varphi|^2 \geqq 2\min_{c\in\bar{B}}\int_{\partial B}|\sigma-c|(D\cdot n)d\sigma\,. \tag{7.8}$$

A generalization of (7.8) is given in Remark 7.5 below.

Proof of Theorem 7.5. An easy approximation argument shows that it suffices to prove (7.7) in the case that $\mu=\sum_{i=1}^{q}\alpha_i\delta_{b_i}$ with $\alpha_i\geqq 0$, $\sum_{i=1}^{q}\alpha_i=1$, and $b_i\in M$. Write $v=\sum_{i=1}^{k}\delta_{p_i}-\sum_{i=1}^{k-1}\delta_{n_i}$ (some points are repeated according to their multiplicity d_i). We shall use induction on k. As we have already indicated, the conclusions is obvious for $k=1$.

As in Sect. IV and V it follows from the Kantorovich theorem [see (5.4)] that

$$I=\min\left\{\sum_{i=1}^{k}\sum_{j=1}^{k-1}t_{ij}\delta(p_i,n_j)+\sum_{i=1}^{k}\sum_{j=1}^{q}s_{ij}\delta(p_i,b_j)\right\}, \tag{7.9}$$

the minimum being taken over the set of constraints $t_{ij}\geqq 0$, $s_{ij}\geqq 0$, $\sum_{i=1}^{k}t_{ij}=1$ for all j $1\leqq j\leqq k-1$, $\sum_{j=1}^{k-1}t_{ij}+\sum_{j=1}^{q}s_{ij}=1$ for all $i, 1\leqq i\leqq k$ and $\sum_{i=1}^{k}s_{ij}=\alpha_j$ for all j, $1\leqq j\leqq q$. Fixing the matrix $S=(s_{ij})$, consider the set τ of all matrices $T=(t_{ij})$ satisfying the above constraints. The set τ is compact and convex, therefore

$$\min_{\tau}\sum\sum t_{ij}\delta(p_i,n_j) \tag{7.10}$$

is achieved by some extremal point of τ. The following lemma, which is a variation of Birkhoff's theorem, gives a useful property of the extremal points of τ.

Lemma 7.7. *Let $\gamma=(c_1,\dots,c_n)$ and $\varrho=(r_1,\dots,r_m)$ be $n+m$ given nonnegative numbers satisfying $\sum c_i=\sum r_i$. Assume $m\leqq 2n$ and let $M_{m,n}(\gamma,\varrho)$ be the set of $m\times n$ matrices with nonnegative entries and having the c_i and r_i as column and row sums. I.e. $T\in M_{m,n}(\gamma,\varrho)$ means $T=\{t_{ij}\}$, $t_{ij}\geqq 0$, $\sum_{i=1}^{m}t_{ij}=c_j$, $\sum_{j=1}^{n}t_{ij}=r_i$. $M_{m,n}(\gamma,\varrho)$ is clearly a closed convex subset of $(\mathbb{R}^+)^{mn}$. If T is an extreme point of $M_{m,n}(\gamma,\varrho)$, then some column of T has $m-2$ zeros, i.e. for some $j\in\{1,\dots,n\}$, $t_{ij}=0$ for at least $m-2$ different i's.*

Proof. We can assume that $m=2n$ simply by adding $2n-m$ rows of zeros. The lemma is trivially true for $n=1$, $m=2$ and we shall use induction on n. Let $n\geqq 2$. If T does not satisfy the lemma then each column of T has at least 3 positive entries. Since T is extremal, it is obvious that every submatrix, A, of T must be extremal (with respect to fixed row and column sums for A). Our goal will be to show that T

has a $k \times n$ submatrix, A, that is not extremal, for some $k \geqq 2$. Let R denote the number of rows of T having $n-1$ zeros. The total number of positive elements of T, call it Σ, satisfies $\Sigma \geqq 3n$. Then $R+(2n-R)n \geqq \Sigma \geqq 3n$. This implies that $R \leqq 2n - [n/(n-1)] < 2n-1$, so $R \leqq 2n-2$. Hence T has $k \geqq 2$ rows with the property that there are at least 2 positive elements in the row. A will be the submatrix of T consisting of those k rows.

We claim that each column of A has at least 2 positive entries. Let $j \in \{1, \ldots, n\}$ label some column of T. Suppose there are 2 rows of T with the property that each row has one positive entry and that entry occurs at a common position j. If this is true we are done, for it suffices to consider the $(2n-2) \times (n-1)$ submatrix, B, of T obtained by deleting those 2 rows and the j^{th} column. By induction, B is extremal and thus has a column with at most 2 positive entries. If s labels this column then column s in T has the same property (because column s had zeros in the 2 deleted rows). This contradicts our assumption that every column of T has $\geqq 3$ positive entries.

Thus, we have found a $k \times n$ submatrix, A, of T with the property that every row and column of A has at least 2 positive entries. This matrix cannot be extremal as we now show. Pick some positive entry of A, walk along the row to another positive entry, walk along that column to another positive entry, and so on until a point (I, J) that has been previously visited is reached. We thus obtain a closed path, starting at (I, J) through positive entries of A. Let F be the matrix that is $+1$ at (I, J), (-1) at the next point in the path and so on. Off the path, $F_{ij} \equiv 0$. Clearly all the row and column sums of F are zero. Moreover, for small ε, $T_{\pm} \equiv T \pm \varepsilon F \in M_{m,n}(\gamma, \varrho)$, so $T = \frac{1}{2}(T_{+} + T_{-})$. $\square$

Proof of Theorem 7.5 Completed. Let $T=(t_{ij})$ be an extreme point of τ that achieves the minimum in (7.10). By Lemma 7.7, there is some j, $1 \leqq j \leqq k-1$ such that $t_{ij} \neq 0$ for at most two values of i. Suppose, for example, $j=1$, $t_{i,1}=0$ when $i \neq 1, 2$, and $t_{21} \leqq t_{11}$. Now fix T and S in (7.9), but replace the point n_1 by p_1. By the triangle inequality, (7.9) is not increased by this replacement. This means that $I(v) \geqq I(\bar{v})$, where $\bar{v}$ is the measure with n_1 replaced by p_1, namely $\bar{v}$ has only $k-1$ positive terms and $k-2$ negative terms. The conclusion follows by induction.

Remark 7.1. One may give an alternative proof of Theorem 7.5 using Graph Theory – more specifically a result of Hamidoune-Las Vergnas [16]. By approximation, we can always assume that $\mu = \sum_j \alpha_j \delta_{b_j}$ with $\alpha_j \geqq 0$, $\sum \alpha_j = 1$, $b_j \in M$, and also $\alpha_j \in \mathbb{Q}$. Therefore, it suffices to consider the case where $\mu = \frac{1}{q} \sum_{j=1}^{q} \delta_{b_j}$ (where the points b_j are not necessarily distinct). As above, write $v = \sum_{i=1}^{k} \delta_{p_i} - \sum_{i=1}^{k-1} \delta_{n_i}$ so that the left side of (7.7) becomes

$$\frac{1}{q} \max \{ \textstyle\int \zeta d\mu' - \int \zeta dv' \,|\, \|\zeta\|_{\mathrm{Lip}} \leqq 1 \},$$

where $\mu' = \sum_{j=1}^{q} \delta_{b_j} + q \sum_{i=1}^{k-1} \delta_{n_i}$ and $v' = q \sum_{i=1}^{k} \delta_{p_i}$. Using the Kantorovich and Birkhoff theorems as in Sect. IV we find that this maximum equals

$$\min_{\sigma} \sum_{l=1}^{kq} \delta(P_l, N_{\sigma l}),$$

where the system (P_l) consists of the points $(p_i)_{i \leqq i \leqq k}$ each repeated q times, and the system (N_l) consists of the points $(n_i)_{1 \leqq i \leqq k-1}$ each repeated q times together with the points $(b_j)_{1 \leqq j \leqq q}$ (counted with multiplicity one). It follows from the result of [16], that in any connection σ, there exists some point p_{i_0} which is joined to *every* $(b_j)_{1 \leqq j \leqq q}$ by disjoint paths. (Two paths are disjoint if they have no strings in common.) In particular we have

$$\sum_{l=1}^{kq} \delta(P_l, N_{\sigma l}) \geqq \sum_{j=1}^{q} \delta(p_{i_0}, b_j) = q \int \delta(x, p_{i_0}) d\mu(x) \geqq q \min_{c \in M} \int \delta(x, c) d\mu(x),$$

which leads to (7.7). $\square$

Remark 7.2. Suppose $M = \bar{B}$ (the unit ball). It is easy to see by going back to the proof of Theorem 7.5 that (7.7) is a *strict* inequality if $\operatorname{Supp}\mu$ is not contained in a single line and ν has at least three atoms.

Proof of Theorem 7.1. From Corollary 7.6 we obtain

$$E_1 \geqq 2 \min_{|c| \leqq 1} \int_{\partial B} |\sigma - c| d\sigma = 8\pi$$

(the minimum is achieved when $c=0$). Next we claim that ψ is the unique minimizer. Let φ_0 be a minimizer for (7.3) and let D_0 be the corresponding D field. In view of Remark 7.2 we know that $\operatorname{div} D_0$ consists of a single Dirac δ_c and c must be zero $\left(\text{otherwise } 2 \int_{\partial B} |\sigma - c| d\sigma > 8\pi\right)$. Therefore, φ_0 has only one singularity with a nonzero degree, and that singularity is at $x=0$. Finally, we have $\partial\varphi_0/\partial r = 0$ because

$$8\pi = \int |\nabla \varphi_0|^2 = \int |\nabla_T \varphi_0|^2 + \int \left|\frac{\partial \varphi_0}{\partial r}\right|^2 \geqq 8\pi + \int \left|\frac{\partial \varphi_0}{\partial r}\right|^2$$

(since φ_0 restricted to every sphere, rS^2, has degree one). $\square$

Corollary 7.8. *Assume $\varphi : \bar{B} \to S^2$ has the following properties:*

$$\varphi(-x) = -\varphi(x) \quad \text{on } \partial B, \tag{7.11}$$

$$D \cdot n = J_\varphi = \varphi \cdot \varphi_x \wedge \varphi_y \geqq 0 \quad \text{on } \partial B, \tag{7.12}$$

and

$$\deg(\varphi, S^2) = 1. \tag{7.13}$$

Then $\int |\nabla \varphi|^2 \geqq 8\pi$.

Proof. We already know, by Corollary 7.6 that

$$\tfrac{1}{2} \int |\nabla \varphi|^2 \geqq \int_{\partial B} |\sigma - c| (D \cdot n) d\sigma \tag{7.14}$$

for some $c \in \bar{B}$. Thus, we also have [by (7.11)]

$$\tfrac{1}{2} \int |\nabla \varphi|^2 \geqq \int_{\partial B} |-\sigma - c| (D \cdot n)(-\sigma) d\sigma = \int_{\partial B} |\sigma + c| (D \cdot n)(\sigma) d\sigma. \tag{7.15}$$

By adding (7.14) and (7.15) we find

$$\tfrac{1}{2} \int |\nabla \varphi|^2 \geqq \int_{\partial B} (D \cdot n) d\sigma = 4\pi. \quad \square$$

Remark 7.3. We conjecture that the conclusion of Corollary 7.8 holds without assumption (7.12).

Corollary 7.9 (Extension of Theorem 7.1). *Let Ω be any bounded domain in $\mathbb{R}^3$, then $\psi(x)=x/|x|$ is the unique minimizer for $\int_\Omega |\nabla\varphi|^2$ under the constraint that $\varphi=\psi$ on $\partial\Omega$.*

Proof. Let B_R be any large ball containing Ω and consider the problem of minimizing $E(\varphi)$ subject to $\varphi(x)=x/|x|$ on ∂B_R. By Theorem 7.1, the minimizing φ for B_R is uniquely $x/|x|$. Now let $\tilde{\varphi}$ be the minimizer for the Ω problem. If $\tilde{\varphi}$ differs from ψ in Ω, then there would be an alternative minimizer for the B_R problem, namely $f(x)=\tilde{\varphi}(x)$ for $x\in\Omega$ and $f(x)=x/|x|$ for $x\notin\Omega$. This would contradict uniqueness. □

Theorem 7.10 (Extension of Theorem 7.5). *Let M be a compact metric space and let μ be a positive measure with total mass $d\in\mathbb{N}$, and let $\nu=\sum_{i=1}^{p} d_i\delta_{a_i}$, where $d_i\in\mathbb{Z}$ and $\sum_{i=1}^{p} d_i=d$.*

Let

$$I(\nu)=\max\{\textstyle\int\zeta d\mu-\int\zeta d\nu \mid \|\zeta\|_{\mathrm{Lip}}\leqq 1\}.$$

Then $\inf I(\nu)$ *(where the infimum runs over all p's, a_i's, and d_i's) is achieved by a measure ν of the form*

$$\nu=\sum_{i=1}^{k+d}\delta_{p_i}-\sum_{j=1}^{k}\delta_{n_j}$$

for some $0\leqq k\leqq d-1$.

Proof. Follow the same argument as in the proof of Theorem 7.5. □

Remark 7.4. For the purpose of Theorem 7.5, it would suffice to have Lemma 7.7 only for the case $m=n+1$. The reason we proved it for $m\leqq 2n$ was that this extended version is needed for Theorem 7.10.

Remark 7.5. Theorem 7.10 gives us a way to compute a lower bound for the problem

$$\min_{\varphi=\varphi_0 \text{ on } \partial B}\int_B|\nabla\varphi|^2,$$

provided $D\cdot n\geqq 0$ on ∂B, but without the assumption that φ_0 has degree 1. As far as the D problem is concerned, it can happen that when $d=2$, for example, the minimum of $I(\nu)$ occurs for three plus points and one minus point (i.e. $k=1$). Just take $D\cdot n$ to be three Dirac masses of strength $\frac{2}{3}$ placed in an equilateral triangle around the equator. The minimizing ν consists of three positive unit masses at the vertices of the triangle and one negative unit mass at the origin.

B. Proof of Theorem 7.3

First note that if $v(x)=g(x/|x|)$ is a minimizer for E_2, then g must be harmonic. Indeed v satisfies the equation $-\Delta v=v|\nabla v|^2$ in B and since v is independent of r, we have $-\Delta g=g|\nabla g|^2$. We shall construct explicitly a map, u, which coincides with g

on ∂B and whose energy is less than 8π. Let $0<a<1$ and let $A=(0,0,a)$. We introduce polar coordinates centered at A with the direction $(0,0,1)$ as the pole. Thus a point x in B has coordinates (r,θ,φ), where r is the distance to A, θ is the polar angle and φ is the azimuthal angle. For a given angle $\theta\in[0,\pi]$, let $R(\theta)$ denote the maximum allowed radius (in $\bar{B}$). The points $(R(\theta),\theta,\varphi)$ with θ fixed and $\varphi\in[0,2\pi)$ all have a common polar angle $\psi(\theta)$ relative to the origin 0. We easily compute that

$$R(\theta)\sin\theta=\sin[\psi(\theta)]\,,\qquad \tan\theta=\sin[\psi(\theta)](\cos[\psi(\theta)]-a)^{-1}\,. \tag{7.16}$$

Our choice for u is

$$u(r,\theta,\varphi)=g(\psi(\theta),\varphi)\,, \tag{7.17}$$

so that its energy is

$$E(u)=\int_0^\pi \sin\theta\,d\theta\int_0^{2\pi}d\varphi\int_0^{R(\theta)}r^2dr\{r^{-2}|\nabla_{\theta,\varphi}u|^2\}\,, \tag{7.18}$$

where $\nabla_{\theta,\varphi}=(\partial/\partial\theta,(\sin\theta)^{-1}\partial/\partial\varphi)$. The r integration gives $R(\theta)\sin\theta$ which equals $\sin[\psi(\theta)]$, so (7.18) becomes

$$E(u)=\int_0^\pi \sin[\psi(\theta)]d\theta\int_0^{2\pi}d\varphi\{|g_2(\psi(\theta),\varphi)|^2[\sin\theta]^{-2}+|g_1(\psi(\theta),\varphi)|^2|\psi'(\theta)|^2\}\,. \tag{7.19}$$

Here g_1 and g_2 mean derivatives with respect to the first and second arguments.

Using (7.16) it is easy to compute

$$\psi'(\theta)=[1-2a\cos\psi(\theta)+a^2]/[1-a\cos\psi(\theta)]\,, \tag{7.20}$$

$$\sin^2[\psi(\theta)]/\sin^2\theta=1-2a\cos\psi(\theta)+a^2\,. \tag{7.21}$$

Inserting this in (7.19) and changing variables from θ,φ to ψ,φ (with Jacobian $|\psi'|^{-1}$), we have

$$\begin{aligned}E(u)=\int_0^\pi \sin\psi\,d\psi\int_0^{2\pi}d\varphi\{&|g_2(\psi,\varphi)|^2[\sin\psi]^{-2}(1-a\cos\psi)\\&+|g_1(\psi,\varphi)|^2[(1-2a\cos\psi+a^2)/(1-a\cos\psi)]\}\,.\end{aligned} \tag{7.22}$$

If we set $a=0$ in (7.22) we obtain $E(g(x/|x|))=8\pi$. To prove the theorem, it therefore suffices to show that $E(u)<8\pi$ for small a. Expanding (7.22) in a near $a=0$, we need to show that

$$\int_0^\pi \sin\psi\,d\psi\int_0^{2\pi}d\varphi\{|g_2(\psi,\varphi)|^2(\sin\psi)^{-2}+|g_1(\psi,\varphi)|^2\}\cos\psi\neq 0\,. \tag{7.23}$$

However, (7.23) can be expressed in coordinate free form as follows. Let $e(\sigma)=|\nabla_T g(\sigma)|^2$, where $\sigma\in S^2$ and ∇_T is the tangential gradient. Then the left side of (7.23) is

$$I(\alpha)=\int_{S^2}e(\sigma)(\alpha\cdot\sigma)d\sigma\,, \tag{7.24}$$

where $\alpha=(0,0,1)$ and $d\sigma$ is the uniform measure on S^2 with $\int d\sigma=4\pi$. It is now clear that $I(\tilde{\alpha})$ is the change of E if we replace $A=a\alpha$ by $\tilde{A}=a\tilde{\alpha}$, $\tilde{\alpha}\in S^2$. Thus, to complete the proof we must show that every harmonic map $g: S^2 \to S^2$, other than $g(x) = \pm Rx$, has the property that, for some $\alpha \in S^2$, $I(\alpha) \neq 0$. In other words, we have to show that for some $i \in \{1, 2, 3\}$,

$$N_i \equiv \int e(\sigma)\sigma_i d\sigma \neq 0 . \tag{7.25}$$

Equation (7.25) means that the center of mass of $e(\sigma)$ is not at the origin.

Let Π denote stereographic projection from $\mathbb{C} \to S^2$. If $z = x + iy$,

$$\Pi(z) = (1 + |z|^2)^{-1}(2x, 2y, 1 - |z|^2) . \tag{7.26}$$

Clearly we have

$$d\sigma = 4(1 + |z|^2)^{-2} dx\, dy , \tag{7.27}$$

and if $h: S^2 \to \mathbb{C}$ and $H \equiv h \circ \Pi : \mathbb{C} \to \mathbb{C}$, then

$$|\nabla_T h|^2 = \tfrac{1}{4} |\nabla H|^2 (1 + |z|^2)^2 . \tag{7.28}$$

If $g: S^2 \to S^2$ and $f \equiv \Pi^{-1} \circ g \circ \Pi : \mathbb{C} \to \mathbb{C}$, then

$$|\nabla_T g|^2 = |\nabla f|^2 (1 + |f|^2)^{-2} (1 + |z|^2)^2 . \tag{7.29}$$

If f happens to be holomorphic, then

$$|\nabla f|^2(z) = 2|f'(z)|^2 ; \tag{7.30}$$

g is harmonic of degree one if and only if

$$f(z) \equiv (\Pi^{-1} \circ g \circ \Pi)(z) = (az + b)/(cz + d) \tag{7.31}$$

for $a, b, c, d \in \mathbb{C}$, see e.g. [22, 35]. By a rotation of S^2 we can assume that $\infty \to \infty$, i.e. $c = 0, d = 1$. By a further rotation, $z \to ze^{iw}$, we can assume $a = \lambda b$ with $\lambda > 0$. Thus we may assume $f(z) = b(z + \lambda)$.

From the above formulas

$$N_i = 8 \int dx\, dy |f'(z)|^2 [1 + |f(z)|^2]^{-2} [1 + |z|^2]^{-1} W_i(z) \tag{7.32}$$

with

$$W_1(z) = 2x , \qquad W_2(z) = 2y , \qquad W_3(z) = 1 - |z|^2 .$$

By symmetry, $N_2 = 0$. If $\lambda > 0$ then $N_1 \neq 0$. To see this, let $K(x, y)$ denote the integrand, and note that for $x > 0$, $K(x, y) < K(-x, y)$ for all y when $\lambda > 0$. Thus, $N_1 = 0$ implies $\lambda = 0$. Finally, it is easy to see that $N_3 = 0$ if and only if $|b| = 1$. But $f(z) = e^{iw} z$ corresponds to $g(x) = Rx$ with R being a rotation by the angle w about the north pole. □

Remark 7.6. The proof of Theorem 7.3 shows something about harmonic maps generally (even those of degree $\neq \pm 1$). If $g(x)$, for $|x| = 1$, is given on the boundary, then $g(x/|x|)$ can never be a minimizer if the center of mass of $e(\sigma)$ is not at the origin, $x = 0$. Here, $e = |\nabla_T g|^2$.

C. Proof of Theorem 7.4

Let d be the degree of g and assume $v(x) = g(x/|x|)$ is a minimizer for E_2. As we remarked, g must be harmonic, and this is the case if and only if $f (= \Pi^{-1} \circ g \circ \Pi)$ is $P(z)/Q(z)$ if $d \geqq 0$ or $P(\bar{z})/Q(\bar{z})$ if $d < 0$, with P and Q being polynomials and with

$|d| = \max\{\deg P, \deg Q\}$. By assumption we have

$$\int_B |\nabla v|^2 \leqq \int_B |\nabla \varphi|^2, \qquad \forall \varphi \in C_2. \tag{7.33}$$

We have clearly

$$\int_B |\nabla v|^2 = \int_{S^2} |\nabla_T g|^2 = 8\pi |d|. \tag{7.34}$$

In order to prove that $|d| \leqq 1$ we shall choose special functions φ of the form described below. Let $\varepsilon \in (0,1)$ and let $\theta : [0,1] \to [0,\infty)$ be any smooth function such that $\theta(1) = 1$, $\theta(t) = 0$ for $t \in [0,\varepsilon]$ and $\theta(t) > 0$ for $t \in (\varepsilon, 1]$. Let

$$\varphi(x) = \Pi \left\{ \frac{1}{\theta(|x|)} f[\Pi^{-1}(x/|x|)] \right\}$$

(with the convention that $0/0 = \infty$). Note that φ equals $N = (0,0,1)$ on the ball $B(0,\varepsilon)$. Moreover φ is smooth on B except at the points εx_i with $g(x_i) = S = (0,0,-1)$. Also $\varphi(x) = g(x)$ for $|x| = 1$. We claim that

$$E(\varphi) = 8\pi |d| (1-\varepsilon) + 16 \int_\varepsilon^1 dr \int_{\mathbb{R}^2} \frac{|\theta'(r)|^2 |f(\zeta)|^2 r^2 d\xi\, d\eta}{(\theta^2(r) + |f(\zeta)|^2)^2 (1+|\zeta|^2)^2}, \tag{7.35}$$

where $\zeta = \xi + i\eta$.

Indeed we have

$$E(\varphi) = \int_{r>\varepsilon} \left(|\nabla_T \varphi|^2 + \left| \frac{\partial \varphi}{\partial r} \right|^2 \right) = 8\pi |d| (1-\varepsilon) + \int_{r>\varepsilon} \left| \frac{\partial \varphi}{\partial r} \right|^2. \tag{7.36}$$

A direct computation shows that

$$\left| \frac{\partial \varphi}{\partial r} \right|^2 = \frac{4|\theta'(r)|^2 |f(\zeta)|^2}{(\theta^2(r) + |f(\zeta)|^2)^2}, \tag{7.37}$$

where $\zeta = \Pi^{-1}(x/|x|)$ and $r = |x|$. In order to compute $\int_{r>\varepsilon} \left| \frac{\partial \varphi}{\partial r} \right|^2$ we change variables and instead of $x = (x_1, x_2, x_3)$ we use the new variables (r, ξ, η), i.e.

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}, \qquad \xi = \frac{x_1}{r - x_3}, \qquad \eta = \frac{x_2}{r - x_3}.$$

Therefore we obtain

$$\int_{r>\varepsilon} \left| \frac{\partial \varphi}{\partial r} \right|^2 dx_1 dx_2 dx_3 = \int_{r>\varepsilon} dr \int_{\mathbb{R}^2} \left| \frac{\partial \varphi}{\partial r} \right|^2 J\, d\xi\, d\eta, \tag{7.38}$$

where J is the Jacobian determinant, i.e.

$$J = \frac{\partial(x_1, x_2, x_3)}{\partial(r, \xi, \eta)} = \frac{4r^2}{(1 + |\xi|^2 + |\eta|^2)^2}. \tag{7.39}$$

Combining (7.36), (7.37), (7.38), and (7.39) we obtain (7.35). Going back to (7.33) and (7.34) with (7.35) we obtain

$$8\pi |d| \leqq 8\pi |d| (1-\varepsilon) + 16 \int_\varepsilon^1 dr \int_{\mathbb{R}^2} \frac{|\theta'|^2 |f|^2 r^2 d\xi\, d\eta}{(\theta^2 + |f|^2)^2 (1+|\zeta|^2)^2}$$

[for simplicity we write f instead of $f(\zeta)$] that is

$$\frac{\pi}{2}|d|\varepsilon \leqq \int_{\varepsilon}^{1} dr \int_{\mathbb{R}^2} \frac{|\theta'(r)|^2|f|^2 r^2 d\xi\, d\eta}{(\theta^2(r)+|f|^2)^2(1+|\zeta|^2)^2}. \tag{7.40}$$

We change variable and set $t=\varepsilon/r$, $\alpha(t)=\theta\left(\frac{\varepsilon}{r}\right)$, $t\in[\varepsilon,1]$. From (7.40) we have

$$\pi\frac{|d|}{2} \leqq \int_{\varepsilon}^{1} dt \int_{\mathbb{R}^2} \frac{|\alpha'(t)|^2|f|^2 d\xi\, d\eta}{(\alpha^2(t)+|f|^2)^2(1+|\zeta|^2)^2}. \tag{7.41}$$

Note that (7.41) holds for any function $\alpha:[\varepsilon,1]\to[0,\infty)$ such that $\alpha(\varepsilon)=1$, $\alpha(1)=0$. Passing to the limit in (7.41) we find

$$\pi\frac{|d|}{2} \leqq \int_{0}^{1} dt \int_{\mathbb{R}^2} \frac{|\alpha'(t)|^2|f|^2 d\xi\, d\eta}{(\alpha^2(t)+|f|^2)^2(1+|\zeta|^2)^2} \tag{7.42}$$

for any function $\alpha:[0,1]\to[0,\infty)$ such that $\alpha(0)=1$, $\alpha(1)=0$. Set

$$F(s)=\int_0^s da\left\{\int_{\mathbb{R}^2}\frac{|f|^2 d\xi\, d\eta}{(a^2+|f|^2)^2(1+|\zeta|^2)^2}\right\}^{1/2}.$$

(It will follow from later computations that $F<\infty$.) We choose now $\alpha(t)=F^{-1}(F(1)(1-t))$ and so we obtain from (7.42), $\pi\frac{|d|}{2}\leqq F(1)^2$, and thus

$$\left[\pi\frac{|d|}{2}\right]^{1/2} \leqq F(1)=\int_0^1 ds\left\{\int_{\mathbb{R}^2}\frac{|f|^2 d\xi\, d\eta}{(s^2+|f|^2)^2(1+|\zeta|^2)^2}\right\}^{1/2}. \tag{7.43}$$

Let $R\in SO(3)$ be a rotation. Set $g_R=R\circ g:S^2\to S^2$, $f_R=\Pi^{-1}\circ g_R\circ\Pi$. Since $u(x)=g(x/|x|)$ is a minimizer, it follows that $u_R(x)=g_R(x/|x|)$ is also a minimizer for the boundary condition g_R, and therefore we have [from (7.34)]

$$\left[\pi\frac{|d|}{2}\right]^{1/2} \leqq \int_0^1 ds\left\{\int_{\mathbb{R}^2}\frac{|f_R|^2 d\xi\, d\eta}{(s^2+|f_R|^2)^2(1+|\zeta|^2)^2}\right\}^{1/2} \tag{7.44}$$

for every $R\in SO(3)$.

We shall average (7.44) over all rotations in $SO(3)$. Let m be the Haar measure left invariant on $SO(3)$. We have by (7.44)

$$\begin{aligned}\left[\pi\frac{|d|}{2}\right]^{1/2} &\leqq \int_0^1 ds \int_{SO(3)} dm(R)\left\{\int_{\mathbb{R}^2}\frac{|f_R|^2 d\xi\, d\eta}{(s^2+|f_R|^2)^2(1+|\zeta|^2)^2}\right\}^{1/2}\\ &\leqq \int_0^1 ds\left\{\int_{\mathbb{R}^2} d\xi\, d\eta \int_{SO(3)}\frac{|f_R|^2 dm(R)}{(s^2+|f_R|^2)^2(1+|\zeta|^2)^2}\right\}^{1/2}.\end{aligned} \tag{7.45}$$

Note that for every function $k:S^2\to\mathbb{R}$ and every $a\in S^2$, we have

$$\int_{SO(3)} k(Ra)\,dm(R)=\frac{1}{4\pi}\int_{S^2}k(\sigma)\,d\sigma \tag{7.46}$$

[clearly the left side of (7.46) is independent of a and so it equals its average on S^2, i.e. $\frac{1}{4\pi}\int_{SO(3)} dm(R) \int_{S^2} k(Ra)da\Big]$. Also note that by changing variables we have

$$\frac{1}{4\pi}\int_{S^2} k(\sigma)d\sigma = \frac{1}{\pi}\int_{\mathbb{R}^2} k(\Pi(Z))\frac{dX\,dY}{(1+|Z|^2)^2} \tag{7.47}$$

$\left(\text{recall that the Jacobian determinant } \frac{\partial\Pi(Z)}{\partial(X,Y)} = \frac{4}{(1+|Z|^2)^2}\right)$. We use (7.46) and (7.47) with

$$k(\sigma) = \frac{|\Pi^{-1}(\sigma)|^2}{(s^2+|\Pi^{-1}(\sigma)|^2)^2} \quad \text{and} \quad a = \Pi\circ f(\zeta),$$

and we obtain, for every ζ,

$$\int_{SO(3)} \frac{f_R(\zeta)^2 dm(R)}{(s^2+|f_R(\zeta)|^2)^2} = \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{|Z|^2 dX\,dY}{(s^2+|Z|^2)^2(1+|Z|^2)^2}. \tag{7.48}$$

A direct computation shows that the right side of (7.48) equals

$$2\frac{(s^2+1)}{(s^2-1)^3}(\ln s) - \frac{2}{(s^2-1)^2} \equiv G(s). \tag{7.49}$$

Going back to (7.45) we obtain

$$\left[\pi\frac{|d|}{2}\right]^{1/2} \leqq \pi^{1/2}\int_0^1 [G(s)]^{1/2} ds,$$

and thus

$$|d| \leqq 2\left\{\int_0^1 G(s)^{1/2} ds\right\}^2.$$

We conclude with the next lemma that $|d|<2$. □

Lemma 7.11. *With* $G(s)$ *defined by* (7.49),

$$\int_0^1 G(s)^{1/2} ds < 1. \tag{7.50}$$

Proof. Note that

$$\int_0^1 G(s)^{1/2} ds = \int_1^\infty G(s)^{1/2} ds$$

[since $G(1/s) = s^4 G(s)$]. Set

$$b(s) = \left\{\frac{s^2+1}{s^2-1}\ln s - 1\right\}^{1/2} \quad \text{for} \quad s>1,$$

so that, for $s>1$,

$$G(s)^{1/2} = \frac{\sqrt{2}}{s^2-1} b(s). \tag{7.51}$$

We claim that the function

$$s \to b(s)/(\ln s) \text{ is decreasing on } (1,\infty). \tag{7.52}$$

Letting $t=s^2$ we have to check that

$$\left(\frac{(t+1)}{2(t-1)}\ln t - 1\right)\Big/(\ln t)^2 \text{ is decreasing},$$

that is

$$(\ln t)^2 \geqq \frac{(1-t^2)}{2t}\ln t + \frac{2}{t}(t-1)^2 . \tag{7.53}$$

Differentiating both sides of (7.53) it suffices to verify that

$$\ln t \geqq \frac{3(t^2-1)}{t^2+4t+1},$$

which holds since

$$\left[\ln t - \frac{3(t^2-1)}{t^2+4t+1}\right]' = \frac{(t-1)^4}{t(t^2+4t+1)^2} \geqq 0 .$$

Thus we have proved (7.52). In particular, we deduce from (7.52) that

$$\frac{b(s)}{\ln s} \leqq b'(1) = \frac{1}{\sqrt{3}} \quad \text{for all } s>1$$

[since $b(1)=0$] and also that

$$\frac{b(s)}{\ln s} \leqq \frac{1}{\sqrt{3}} - \left(\frac{1}{\sqrt{3}} - \frac{b(a)}{\ln a}\right) H(s-a) \tag{7.54}$$

for all $s>1$ and all $a>1$, where H is the Heaviside function [$H(t)=1$ for $t \geqq 0$ and $H(t)=0$ for $t<0$]. It follows from (7.54) that, for all $a>1$,

$$\begin{aligned}
\int_1^\infty \frac{b(s)}{(s^2-1)} ds &\leqq \frac{1}{\sqrt{3}} \int_1^\infty \frac{\ln s}{(s^2-1)} ds - \left(\frac{1}{\sqrt{3}} - \frac{b(a)}{\ln a}\right) \int_a^\infty \frac{\ln s}{(s^2-1)} ds \\
&\leqq \frac{1}{\sqrt{3}} \int_1^\infty \frac{\ln s}{(s^2-1)} ds - \left(\frac{1}{\sqrt{3}} - \frac{b(a)}{\ln a}\right) \int_a^\infty \frac{\ln s}{s^2} ds \\
&= \frac{1}{\sqrt{3}} \int_1^\infty \frac{\ln s}{(s^2-1)} ds - \left(\frac{1}{\sqrt{3}} - \frac{b(a)}{\ln a}\right) \frac{(1+\ln a)}{a} .
\end{aligned}$$

Finally, we recall that

$$\int_1^\infty \frac{\ln s}{(s^2-1)} ds = \frac{\pi^2}{8}$$

$\left[\text{which may be obtained by applying Fubini to } \int_0^\infty \int_0^\infty \frac{dx\, dy}{(1+y)(x^2+y)}\right]$. Thus we find, for all $a>1$,

$$\int_1^\infty G(s)^{1/2} ds \leqq \sqrt{\tfrac{2}{3}}\, \frac{\pi^2}{8} - \sqrt{2}\left(\frac{1}{\sqrt{3}} - \frac{b(a)}{\ln a}\right)\left(\frac{1+\ln a}{a}\right),$$

and we conclude that $\int_1^\infty G(s)^{1/2} ds < 1$ by choosing for example $a=e^2$. □

Remark 7.7. Theorem 7.4 shows that if g has degree $|d| \geqq 2$, then $u(x) = g(x/|x|)$ is not a minimizer. In fact the construction above shows that it is not even a local minimizer.

Corollary 7.12. *Let u be a minimizer for $E(\varphi)$ in a domain Ω with specified boundary condition. Then each point singularity of u has degree ± 1. Moreover, for every singularity x_0 in Ω we have*

$$\lim_{\varepsilon \to 0} u(\varepsilon(x - x_0)) = \pm R(x - x_0)/|x - x_0|,$$

where R is a rotation.

Proof. Without loss of generality we may assume that u has a singularity at $x = 0$. We know from [31, Theorem III] and [33, Sect. 8] that $u(\varepsilon x) \to u_0(x)$ in $H^1(B)$ and uniformly on every compact subset of $B \backslash \{0\}$, where $u_0(x) = g(x/|x|)$ is a non-constant minimizer for E_2. It follows from Theorems 7.3 and 7.4 that g has degree ± 1 and that $\pm g$ is a rotation.

Remark 7.8. The fact that $x/|x|$ is a minimizer for E_1 (but not uniqueness) could also be deduced from Theorems 7.3 and 7.4 and the Schoen-Uhlenbeck result. Indeed let u be any minimizing harmonic map that happens to have a singularity, say at $x = 0$. By [31] we know that $u(\varepsilon x)$ converges (modulo a subsequence) as $\varepsilon \to 0$ to a map $\varphi(x)$ with the properties that: (i) φ is a minimizing harmonic map with a singularity at $x = 0$, (ii) $\varphi(x) = g(x/|x|)$ for some g. Our Theorems 7.3 and 7.4 eliminate all possibilities except $g(x) = \pm Rx$. This shows that $Rx/|x|$ is a minimizing harmonic map and therefore so is $x/|x|$.

VIII. Various Extensions

A. The N-Dimensional Case

A natural generalization is to replace $\mathbb{R}^3$ by $\mathbb{R}^N$ with $N \geqq 2$ and S^2 by S^{N-1}. The quantity which has the homogeneity of a length is now

$$E(\varphi) = \int |\nabla \varphi|^{N-1} \tag{8.1}$$

(and not $|\nabla \varphi|^2$) where φ is a map defined on a subset of $\mathbb{R}^N$ with values into S^{N-1} and

$$|\nabla \varphi|^2 = \sum_{i,j} \left(\frac{\partial \varphi_i}{\partial x_j}\right)^2. \tag{8.2}$$

The analogue of Theorem 1.1 is

Theorem 8.1. *In all four examples*

$$E = \sigma_N (N-1)^{(N-1)/2} L, \tag{8.3}$$

where

$$\sigma_N = 2\pi^{N/2} \Gamma(N/2)^{-1} \tag{8.4}$$

is the area of S^{N-1} in $\mathbb{R}^N$. L is defined in Sect. II.

Proof. As before we construct upper and lower bounds for E. For the lower bound we define D as in (B.7) and note that

$$|D| \leqq (N-1)^{-(N-1)/2} |\nabla \varphi|^{N-1}. \tag{8.5}$$

Indeed, suppose that $\varphi=(0,0,\ldots,1)^t$, then $\varphi_{x_i}=(\alpha_{i,1},\alpha_{i,2}\ldots\alpha_{i,N-1},0)^t$, since φ_{x_i} is orthogonal to φ. The matrix

$$(\varphi_{x_1}, \varphi_{x_2}, \ldots, \varphi_{x_N})$$

has its last row zero. Replace the last row by $(\alpha_1, \alpha_2, \ldots, \alpha_N)$, and call M this new $(N\times N)$ matrix. We have $\det M = \alpha \cdot D$, so that

$$|D| = \sup_{|\alpha|\leqq 1} |\det M|.$$

On the other hand

$$|\det M| \leqq |\alpha| \prod_{j=1}^{N-1} \beta_j,$$

where $\beta_j^2 = \sum_{i=1}^{N-1} \alpha_{i,j}^2$, and thus

$$|\det M|^2 \leqq |\alpha|^2 \prod_{j=1}^{N-1} \beta_j^2 \leqq |\alpha|^2 \left[\frac{1}{(N-1)} \sum_{j=1}^{N-1} \beta_j^2\right]^{N-1} = |\alpha|^2 \frac{1}{(N-1)^{(N-1)}} |\nabla\varphi|^{2(N-1)}.$$

In dimension N, inequality (8.5) replaces the $\mathbb{R}^3$ inequality $|D| \leqq \frac{1}{2}|\nabla\varphi|^2$, and for the remainder of the proof of the lower bound we proceed as in Sect. IV.

For the upper bound we imitate the dipole construction of Sect. III. Let $\Pi : \mathbb{R}^{N-1} \to S^{N-1}$ be stereographic projection, namely

$$\Pi(x) = (\Pi_1(x), \ldots, \Pi_N(x)),$$

$$\Pi_i(x) = 2x_i(1+|x|^2)^{-1} \quad \text{for} \quad i=1,\ldots,N-1 \quad \text{and} \quad \Pi_N(x) = (1-|x|^2)(1+|x|^2)^{-1}.$$

A straightforward computation yields

$$|\nabla \Pi| = 2(N-1)^{1/2}(1+|x|^2)^{-1}. \tag{8.6}$$

Recalling (8.4) we obtain from (8.6)

$$\int_{\mathbb{R}^{N-1}} |\nabla \Pi|^{N-1} = \sigma_N (N-1)^{(N-1)/2}. \tag{8.7}$$

Given $\varepsilon > 0$ we first construct a smooth map $\omega : \mathbb{R}^{N-1} \to S^{N-1}$ such that

$$\int_{\mathbb{R}^{N-1}} |\nabla\omega|^{N-1} \leqq \sigma_N (N-1)^{(N-1)/2} + \varepsilon, \tag{8.8}$$

$$\omega \equiv \text{Const} = e \quad \text{outside the unit ball}, \tag{8.9}$$

$$\deg \omega = 1. \tag{8.10}$$

The idea for constructing ω is the following. Let $v(x) = x/|x|^2$ so that $\Pi \circ v$ satisfies (8.8) with $\varepsilon = 0$ and (8.10). Next, replace v by $\chi v = \tilde{v}$, where $0 \leqq \chi \leqq 1$ and χ has compact support and $\chi = 1$ on a large ball. Finally, replace $\tilde{v}(x)$ by $\tilde{v}(\lambda x)$ with λ large enough. In the general case, $d > 1$, we glue together d maps ω as above (with disjoint supports) and then rescale x. Analogously, for degree -1 we take $v(x) = |x|^{-2}(-x_1, x_2, \ldots, x_{N-1})$. Finally, having constructed ω, the basic dipole is constructed as in (3.6) and (3.7).

Another consequence of the construction in the proof of Theorem 8.1 is the following striking fact.

Theorem 8.2. *For maps* $\varphi : S^{N-1} \to S^{N-1}$, *let*

$$E(\varphi) = \int_{S^{N-1}} |\nabla_T \varphi|^{N-1}.$$

Then

$$\inf_{\deg \varphi = d} E(\varphi) = |d| \sigma_N (N-1)^{(N-1)/2}. \tag{8.11}$$

When $N \geqq 3$, the behavior of minimizing sequences for (8.1) is the same as for $N=3$ as given in Sect. VI, namely if there are only finitely many strings between any two holes and if φ^n is a minimizing sequence then, for a subsequence, $|\nabla \varphi^n|^{N-1}$ converges in the sense of measures to $\sigma_N(N-1)^{(N-1)/2}\delta_C$, where C is a (single) minimal connection. However when $N=2$ the situation is different, as shown by the following example.

Consider four points

$$a_1 = (1,1), \quad a_2 = (1,0), \quad a_3 = (0,0), \quad a_4 = (0,1)$$

with the degrees $d_i = (-1)^i$. Here, we have $E = 2\pi L = 4\pi$ and two minimal connections C_1, C_2 given by $C_1 = [a_2, a_1] \cup [a_4, a_3]$ and $C_2 = [a_2, a_3] \cup [a_4, a_1]$. There exist minimizing sequences φ^n such that, for example, $|\nabla \varphi^n| \rightharpoonup 2\pi(\delta_{C_1} + \delta_{C_2})$. Such a sequence can be obtained as follows. Let $\omega_\pm : \mathbb{R} \to S^1$ be any two maps such that $\omega_\pm(-\infty) = (\pm 1, 0)$, $\omega_\pm(+\infty) = (\mp 1, 0)$, $\omega_\pm$ constant far out and $\int |\nabla \omega_\pm| = \pi$. With $\omega_\pm$ we can associate "half dipoles" which we glue in an appropriate way on each of the intervals $[a_1, a_2]$, $[a_2, a_3]$, $[a_3, a_4]$, $[a_4, a_1]$. The corresponding sequence φ^n has the property that $\varphi^n \to (1,0)$ outside the square $[0,1] \times [0,1]$ and $\varphi^n \to (-1,0)$ inside $[0,1] \times [0,1]$. This lack of quantization in two dimensions is also discussed at the end of Appendix E.

B. Replacing S^{N-1} *by* $\mathbb{R}P^{N-1}$

For physical reasons as explained in Sect. I, it is interesting to replace S^{N-1} by $\mathbb{R}P^{N-1}$ which is the quotient of S^{N-1} by the equivalence relation $x \simeq -x$. The metric on $\mathbb{R}P^{N-1}$ is that induced by S^{N-1}. The energy is still given by (8.1).

The problem we face is to define the degree of a continuous map $\varphi : \Omega \to \mathbb{R}P^{N-1}$ (with $\Omega \subset \mathbb{R}^N$) around a hole in Ω. Unfortunately, $\mathbb{R}P^{N-1}$ is orientable if and only if N is even and therefore the problem will be more difficult when N is odd. The orientability of a manifold implies that the degree can be defined as an integral of a Jacobian. However, the degree for N even (as we shall define it) is in $\frac{1}{2}\mathbb{Z}$, and we shall be able to solve the minimum energy problem only when the given d_i's are integral, except for $N=2$ in which case $\mathbb{R}P^1$ is homeomorphic to S^1 and a special trick allows us to handle all d_i's.

a) N even. Suppose $\Omega \subset \mathbb{R}^N$ and S is a smooth surface in Ω without boundary. Let φ be C^1 in a neighborhood of S with values in $\mathbb{R}P^{N-1}$. The vector field D can be defined as in (B.7). Note that D in (B.7) is uniquely defined for N even because D is not changed by $\varphi \to -\varphi$. We define.

$$d = \sigma_N^{-1} \int_S D \cdot n . \tag{8.12}$$

Since $D \cdot n$ is the Jacobian, $\int_S D \cdot n$ must be an integer times the area of $\mathbb{R}P^{N-1}$, which is $\frac{1}{2}\sigma_N$. Therefore $d \in \frac{1}{2}\mathbb{Z}$.

Thus, given $\varphi \in C(\Omega; \mathbb{R}P^{N-1})$ and $\nabla\varphi \in L^{N-1}(\Omega)$, with $\Omega = U \backslash (\cup H_i)$, we can (by modifying the analysis in Appendix B) define the D field and $\deg(\varphi, H_i) \in \frac{1}{2}\mathbb{Z}$.

For the lower bound to E the D field analysis goes through as before and hence

$$E(\varphi) \geqq \tfrac{1}{2}\sigma_N (N-1)^{(N-1)/2} L(U, \{H_i\}, \{2d_i\}) , \tag{8.13}$$

where L is the length of a minimal connection (with d_i replaced by $2d_i$). Note the factor $\frac{1}{2}$ in (8.13).

For the upper bound we can reproduce the dipole construction of Sect. III when all the $d_i \in \mathbb{Z}$ as will be explained. In this case (8.13) becomes an equality for the infimum, and our problem is solved. Also, the obvious analogue of the results in Sect. VI go through. If some $d_i \notin \mathbb{Z}$ the problem is open.

The reason that $d_i \in \mathbb{Z}$ is special is the following topological fact.

Fact. Let ψ be a continuous map from $X \to \mathbb{R}P^{N-1}$, where X is a simply connected topological space. Then there exists a map $\tilde{\psi} : X \to S^{N-1}$ such that $\psi = P \circ \tilde{\psi}$, where P is the canonical projection of $S^{N-1} \to \mathbb{R}P^{N-1}$. (See the lifting theorem in [34, p. 76].) If X is also connected, there are exactly two choices for $\tilde{\psi}$ related by $\tilde{\psi}_1 = -\tilde{\psi}_2$.

To construct the dipole when $d \in \mathbb{Z}$, first construct the S^{N-1} dipole as in Sect. III and then compose this with P. However, if $d \notin \mathbb{Z}$ we cannot do this because by taking $X = S^{N-1}$ in the above, we would end up with a continuous map $\tilde{\psi} : S^{N-1} \to S^{N-1}$ of degree $d \notin \mathbb{Z}$; this is impossible.

The topological fact also allows us to conclude that if a hole H_i has a neighborhood $\omega \subset U$ such that $\omega \backslash H_i$ is simply connected then necessarily $d_i \in \mathbb{Z}$. Simply take $X = \omega \backslash H_i$. In particular, if H_i is a point and if $N \geqq 4$, $d_i \in \mathbb{Z}$.

b) $N=2$. In this case every hole, even a point hole, can have $d_i \notin \mathbb{Z}$. However $\mathbb{R}P^1$ is homeomorphic to S^1 and we can take advantage of this fact to solve the problem in *all* cases. We identify S^1 with $\{z \in \mathbb{C} \mid |z| = 1\}$. Define $Q : S^1 \to \mathbb{R}P^1$ as follows:

$$Q(z) = P(z') , \quad \text{where} \quad z'^2 = z \tag{8.14}$$

with P being the canonical projection as before.

Clearly $Q(z)$ is independent of the choice of z'. Define $R : \mathbb{R}P^1 \to S^1$ to be

$$R(P(z)) = z^2 \tag{8.15}$$

(again, R is well defined). Note that $R = Q^{-1}$.

Let $\varphi : S^1 \to \mathbb{R}P^1$ be a continuous map. We have

$$\deg \varphi = \tfrac{1}{2} \deg(R \circ \varphi) . \tag{8.16}$$

Given a map φ from Ω into $\mathbb{R}P^1$ (respectively S^1) we have

$$|\nabla(R\circ\varphi)|=2|\nabla\varphi| \quad (\text{respectively } |\nabla(Q\circ\varphi)|=\tfrac{1}{2}|\nabla\varphi|). \tag{8.17}$$

Given reals $d_1, d_2, \ldots, d_k \in \frac{1}{2}\mathbb{Z}$, then

$$E=\pi L(U,\{H_i\},\{2d_i\}). \tag{8.18}$$

c) N odd. Here we shall confine our attention to cases in which $\Omega=U\backslash(\cup H_i)$ is connected and simply connected. This includes the case in which all H_i are points. Given a continuous $\varphi:\Omega\to\mathbb{R}P^{N-1}$, there exists a continuous $\tilde{\varphi}:\Omega\to S^{N-1}$ with $\varphi=P\circ\tilde{\varphi}$. Since there are exactly two choices for $\tilde{\varphi}$ ($\tilde{\varphi}_1=-\tilde{\varphi}_2$), we can define

$$\deg(\varphi,H_i)\equiv|\deg(\tilde{\varphi},H_i)|\in\mathbb{N}. \tag{8.19}$$

(The need for the absolute value is that $\deg(\tilde{\varphi}_1,H_i)=-\deg(\tilde{\varphi}_2,H_i)$ when N is odd.) We also have that $|\nabla\tilde{\varphi}|=|\nabla\varphi|$.

Given nonnegative integers $d_1,\ldots,d_k$, we easily conclude from the above that the infimum satisfies

$$E=\sigma_N(N-1)^{(N-1)/2}\hat{L}, \tag{8.20}$$

where $\hat{L}$ is to be computed as follows:

$$\hat{L}=\min_{\{\varepsilon_i\}} L(U,\{H_i\},\{\varepsilon_i d_i\}), \tag{8.21}$$

where $\varepsilon_i=\pm 1$, all i. In particular, we emphasize that (8.21) solves the minimum energy problem for liquid crystals with point defects and with the simplified energy given by (8.1).

C. Energies with the Homogeneity of an Area

Let $\Gamma\subset\mathbb{R}^3$ be an oriented, rectifiable Jordan curve. Consider the class of maps $\varphi:\mathbb{R}^3\backslash\Gamma\to S^1$ (not S^2) which are continuous. Associated with each φ in this class is an integer $d\in\mathbb{Z}$ defined as follows. Let C be any small circle which links with Γ. On C there is a natural orientation which is consistent with the orientation of Γ. Define

$$d=\deg(\varphi,\Gamma)=\deg(\varphi \text{ restricted to } C).$$

The right side is the usual degree of a map from S^1 to S^1. Note that $\deg(\varphi,\Gamma)$ is independent of the choice of C. The energy

$$E(\varphi)=\int_{\mathbb{R}^3\backslash\Gamma}|\nabla\varphi| \tag{8.22}$$

now has the homogeneity of an area (and not a length).

By analogy with the results of Sects. III and IV we expect that given $d\in\mathbb{Z}$

$$\inf_{\deg(\varphi,\Omega)=d} E(\varphi)=2\pi|d|A, \tag{8.23}$$

where A is the area of a minimal area surface spanned by Γ.

More generally, if M is an oriented manifold without boundary, of dimension m, imbedded in $\mathbb{R}^N$, and $\varphi:\mathbb{R}^N\backslash M\to S^{N-m-1}$ is a continuous map, then one can define (in the same way as above) $\deg(\varphi,M)$. The energy

$$E(\varphi)=\int_{\mathbb{R}^N\backslash M}|\nabla\varphi|^{N-m-1} \tag{8.24}$$

has homogeneity $(m+1)$ and we expect that given $d \in \mathbb{Z}$,

$$\inf_{\deg(\varphi, M)=d} E(\varphi) = c(N, m)|d|V, \tag{8.25}$$

where V is the volume of a "minimal" manifold of dimension $(m+1)$ whose boundary is M. Note that the case $m=0$ corresponds to two point holes and (8.25) reduces to (8.3). We could also consider a finite number of such manifolds $M_1, \ldots, M_k$ and maps $\varphi: \mathbb{R}^N \setminus (\cup M_i)$ into S^{N-m-1} which are continuous except on M_i. It is a natural question to look for $\inf E(\varphi)$ in the class of maps φ such that $\deg(\varphi, M_i) = d_i$ is prescribed. Presumably, the answer is a formula similar to (8.25) where V is a kind of "minimal volume connection" associated with the M_i's and the d_i's.

We have not investigated the validity of (8.23) (or (8.25)) in full generality, and we shall discuss here only the case of a planar curve $\Gamma = \partial U$, where U is some open set in $\mathbb{R}^2$. Again, we split the argument in two parts: the upper bound and the lower bound.

1. *The Upper Bound.* Let ω be any continuous map from $\mathbb{R}$ to S^1 such that

$$\int_{\mathbb{R}} |\omega'| = 2\pi|d|, \tag{8.26}$$

$$\deg \omega = d, \tag{8.27}$$

$$\omega = e \quad \text{outside} \quad [-1, +1]. \tag{8.28}$$

Let $\varphi_n: \mathbb{R}^3 \setminus \Gamma \to S^1$ be defined as follows:

$$\begin{aligned} \varphi_n(x, y, z) &= \omega(nz/l) \quad \text{if} \quad (x, y) \in U, \\ \varphi_n(x, y, z) &= e \qquad\quad\;\; \text{if} \quad (x, y) \notin U, \end{aligned} \tag{8.29}$$

where l denotes the distance of (x, y) to ∂U. Clearly $\deg(\varphi_n, \Gamma) = d$ and $\int |\nabla \varphi_n| \to 2\pi A|d|$, where A is the area of U.

2. *The Lower Bound.* The divergence-free vector field D is now replaced by a curl-free vector field H as follows. To every map φ we associate H defined by

$$H = (\varphi \wedge \varphi_x, \varphi \wedge \varphi_y, \varphi \wedge \varphi_z).$$

An easy computation shows that if φ is smooth on $\mathbb{R}^3 \setminus \Gamma$, then $\operatorname{curl} H = 0$ on $\mathbb{R}^3 \setminus \Gamma$ and, moreover, if $\int |\nabla \varphi| < \infty$, then

$$\operatorname{curl} H = 2\pi d D_\Gamma \quad \text{in} \quad \mathscr{D}'(\mathbb{R}^3), \tag{8.30}$$

where D_Γ is the basic divergence-free vector field over the curve Γ defined in Appendix D. The proof of (8.30) is similar to that of the analogous formula (B.10) for the D field. Moreover, (8.30) extends (by density) to maps φ which are continuous on $\mathbb{R}^3 \setminus \Gamma$ and with $\int |\nabla \varphi| < \infty$. Evidently, we have the inequality

$$|H| \leqq |\nabla \varphi|, \tag{8.31}$$

which plays the same role as $2|D| \leqq |\nabla \varphi|^2$. Therefore we have

$$\int_{\mathbb{R}^3} |\nabla \varphi| \geqq \int_{\mathbb{R}^3} |H| \geqq -\int_{\mathbb{R}^3} H \cdot \operatorname{curl} \zeta = 2\pi d \int D_\Gamma \cdot \zeta \tag{8.32}$$

for every smooth ζ such that $|\operatorname{curl}\zeta| \leqq 1$. On the other hand, by Stokes' theorem

$$\int_{\mathbb{R}^3} D_\Gamma \cdot \zeta = \int_{\Sigma} \operatorname{curl}\zeta \cdot n d\sigma \tag{8.33}$$

for any surface Σ spanned by Γ, where n is the unit normal to Σ. Choosing $\Sigma = U \times \{0\}$ and $\zeta(x,y,z) = \pm(0,x,0)$ we obtain $\operatorname{curl}\zeta = \pm(0,0,1)$ and from (8.32), (8.39) and the fact that $n = (0,0,1)$,

$$\int_{\mathbb{R}^3} |\nabla \varphi| \geqq 2\pi |d| A, \tag{8.34}$$

where A is the area of U. □

Remark 8.1. The upper bound construction presumably extends to nonplanar Γ, at least if the minimal area surface has no self-intersection. M. Gromov has suggested that the lower bound construction might also extend by using Whitney's duality theorem [37].

Appendix A: Approximation by Smooth Functions

Let $\Omega \subset \mathbb{R}^N$ be *any* open set. For the purpose of this paper we are interested in knowing whether we can approximate continuous S^k-valued functions on Ω with derivatives in L^2 by C^∞ S^k-valued functions, both for the uniform norm and energy norm. We present here a result more general than we need.

Lemma A.1. *Assume $u \in C(\Omega; \mathbb{R})$. Then for any $\varepsilon > 0$ there is some $g \in C^\infty(\Omega; \mathbb{R})$ such that*

$$\|g - u\|_{L^\infty} < \varepsilon. \tag{A.1}$$

Moreover if we also assume $\nabla u \in L^{p_i}(\Omega)$ for some finite set $1 \leqq p_1 < p_2 < \ldots < p_m < \infty$ (in the distribution sense), then the above g can also be chosen to satisfy

$$\|\nabla(g - u)\|_{L^{p_i}} < \varepsilon \tag{A.2}$$

for all i.

Proof. This is essentially the same as the Meyers-Serrin theorem (see [25] or [1, p. 52]). The only variation is to note, in the notation of [1], that $\psi_k u \in C_c(\Omega)$ and, therefore, we may choose ε_k such that

$$\|J_{\varepsilon_k} * (\psi_k u) - \psi_k u\|_{L^\infty} \leqq \varepsilon/2^k. \quad \square$$

Lemma A.2. *Assume u satisfies the hypotheses of Lemma A.1 and, moreover, $u \in C(\Omega; S^k)$. Then there is a $g \in C^\infty(\Omega; S^k)$ satisfying* (A.1) *and, if appropriate,* (A.2).

Proof. By Lemma A.1 (applied to each component of u) there is a sequence $\{h_n\}$ in $C^\infty(\Omega; \mathbb{R}^{k+1})$ such that

$$\|h_n - u\|_{L^\infty} \to 0 \quad [\text{and } \|\nabla(h_n - u)\|_{L^{p_i}} \to 0].$$

Assume that $\|h_n - u\|_{L^\infty} < 1/2$, all n. Let $F: \mathbb{R}^{k+1} \to S^k$ be the radial projection, that is $F(x) = x/|x|$. Note that F is smooth for $x \neq 0$. Let $g_n(x) = F(h_n(x))$. Since $h_n \to u$ uniformly, so does g_n [and $\nabla g_n = F'(h_n) \cdot \nabla h_n \to F'(u) \cdot \nabla u$ in L^p, since $F'(h_n) \to F'(u)$ uniformly and $\nabla h_n \to \nabla u$ in L^{p_i}]. □

Remark. In Lemma A.2 it is essential that u is continuous. Suppose that $\Omega=\{x\in\mathbb{R}^3\mid |x|<1\}$ and $k=2$ and $u(x)=\frac{x}{|x|}$. This u has $\nabla u\in L^2$. However, there is no sequence $\{g_n\}$ with $g_n\in C(\Omega;S^2)\cap H^1(\Omega;S^2)$ such that $g_n\to u$ a.e. and $\nabla g_n\to\nabla u$ in L^2. See [32].

Appendix B: Generalities About Degrees of Maps

Let $U\subset\mathbb{R}^N$ be an open set and let $H\subset U$ be a compact subset (called a *hole*). Let $\varphi: U\backslash H\to\mathbb{R}^N$ be a continuous map such that $\varphi(x)\neq 0$, all $x\in U\backslash H$. We shall define $\deg(\varphi,H)$ as follows. Let

$$H_t=\{x\mid \mathrm{dist}(x,H)<t\}\,, \tag{B.1}$$

and assume ε is small enough so that $H_{4\varepsilon}\subset U$.

First, let ψ be any function in $C(U;\mathbb{R}^N)$ such that $\psi=\varphi$ on $U\backslash H_{3\varepsilon}$. (Such functions certainly exist. For example let $\chi\in C(U)$ be such that

$$\chi=\begin{cases}0 & \text{on} \quad H_\varepsilon\\ 1 & \text{on} \quad U\backslash H_{3\varepsilon},\end{cases}$$

then take $\psi=\chi\varphi$).

From the general theory of degrees of maps (see e.g. Nirenberg [27] or Lloyd [24]) the integer

$$d=\deg(\psi,H_{3\varepsilon},0) \tag{B.2}$$

is well defined. Part of this general theory is that d depends only on ψ restricted to $\partial H_{3\varepsilon}$, but this is independent of the choice of ψ (by construction). Conceivably d could depend on ε. However, it does not depend on ε (because if $\varepsilon_1<\varepsilon_2$ and ψ_1 corresponds to ε_1 we may take $\psi_2=\psi_1$).

Hence we are entitled to define

$$\deg(\varphi,H)\equiv\deg(\psi,H_{3\varepsilon},0)\,. \tag{B.3}$$

It follows from standard properties of degrees of maps that if $\varphi_n\to\varphi$ uniformly on every compact subset of $U\backslash H$, then $\deg(\varphi_n,H)\to\deg(\varphi,H)$.

Let us note some explicit formulas for d in (B.2). We can easily construct ψ such that $\psi\in C^1(H_{2\varepsilon};\mathbb{R}^N)$ and $\psi\neq 0$ in $U\backslash H_\varepsilon$. For such ψ,

$$d=\int_{H_\varepsilon} f(\psi(x))J_\psi(x)dx\,, \tag{B.4}$$

where $f:\mathbb{R}^N\to\mathbb{R}$ is any continuous function with compact support contained in the connected component of 0 in $\mathbb{R}^N\backslash\psi(\partial H_\varepsilon)$ and such that $\int_{\mathbb{R}^N} f(y)dy=1$. Here

$$J_\psi(x)=\det(\partial\psi_j/\partial x_i) \tag{B.5}$$

is the Jacobian determinant of ψ. Another formula for d can be obtained if one chooses ψ with the aforementioned properties and additionally $\psi=0$ at only finitely many points $x_1, \ldots, x_m$ in H_ε and $J_\psi \neq 0$ at these points. (Such a ψ exists by Sard's lemma.) Then

$$d = \sum_{i=1}^{m} \operatorname{sgn} J_\psi(x_i). \tag{B.6}$$

Examples. $U = \{x/|x| < 1\}$ and $H = \{0\}$. Let $\varphi_1(x) = x$ and $\varphi_2(x) = x/|x|$. Then $\deg(\varphi_1, \{0\}) = \deg(\varphi_2, \{0\}) = 1$.

Now suppose that $\varphi \in C(U \backslash H; \mathbb{R}^N)$ and $\nabla\varphi \in L^{N-1}(U \backslash H)$ (in $\mathscr{D}'$). To such a φ we associate a vector field $D \in L^1(U \backslash H; \mathbb{R}^N)$, with components D_j, as follows.

$$D_j = \det\left(\frac{\partial\varphi}{\partial x_1}, \ldots, \frac{\partial\varphi}{\partial x_{j-1}}, \varphi, \frac{\partial\varphi}{\partial x_{j+1}}, \ldots, \frac{\partial\varphi}{\partial x_N}\right), \tag{B.7}$$

which is obviously in $L^1(U \backslash H)$. If, in addition, we assume that $\nabla\varphi \in L^N(U \backslash H)$, then J_φ, given by (B.5), is in $L^1(U \backslash H)$ and

$$\operatorname{div} D = N J_\varphi \quad \text{in} \quad \mathscr{D}'(U \backslash H). \tag{B.8}$$

(This is clear when φ is C^2; the general case follows by density, using Appendix A.)

Now suppose that $\varphi \in C(U \backslash H; S^{N-1})$ and $\nabla\varphi \in L^{N-1}(U \backslash H)$, but we do not assume $\nabla\varphi \in L^N(U \backslash H)$. Then

$$\operatorname{div} D = 0 \quad \text{in} \quad \mathscr{D}'(U \backslash H). \tag{B.9}$$

[Reason: By Appendix A, we can approximate φ by C^2 functions φ_n with $\|\varphi_n - \varphi\|_{L^\infty} \to 0$, $\|\nabla(\varphi_n - \varphi)\|_{L^{N-1}} \to 0$ and $\varphi_n(x) \in S^{N-1}$. Note that $J_{\varphi_n} = 0$, since $\varphi \cdot \varphi = 1 \Rightarrow \varphi \cdot \partial\varphi/\partial x_i = 0 \Rightarrow$ the N vectors $\partial\varphi/\partial x_i$ are linearly dependent. By (B.8), $\operatorname{div} D(\varphi_n) = 0$, but $D(\varphi_n) \to D(\varphi) = D$ in L^1.]

Theorem B.1. *Assume $\varphi \in C(U \backslash H; S^{N-1})$ and $\nabla\varphi \in L^{N-1}(U \backslash H)$, (in $\mathscr{D}'$). Then*

$$-\int_{U \backslash H} D \cdot \nabla\zeta = \sigma_N \deg(\varphi, H) \tag{B.10}$$

for every $\zeta \in \operatorname{Lip}(U)$ with compact support in U and $\zeta \equiv 1$ on some neighborhood of H. Here σ_N denotes the area of S^{N-1} in $\mathbb{R}^N$ ($\sigma_3 = 4\pi$).

Proof. By Lemma A.2 we can assume that $\varphi \in C^\infty(U \backslash H; S^{N-1})$. Clearly we may also assume that $\zeta \in C^\infty(U)$. With $I(\zeta)$ denoting the left side of (B.10) we first prove that $I(\zeta)$ is independent of ζ, and thus that it suffices to prove (B.10) for one ζ. Indeed,

$$I(\zeta_1) - I(\zeta_2) = -\int_{U \backslash H} D \cdot \nabla(\zeta_1 - \zeta_2) = \int_{U \backslash H} (\operatorname{div} D)(\zeta_1 - \zeta_2) = 0 \tag{B.11}$$

(because $\zeta_1 - \zeta_2$ has compact support in $U \backslash H$).

Now observe that for all $\theta \in C^\infty(U \backslash H)$

$$N J_{\theta\varphi} = D \cdot \nabla\theta^N + N\theta^N J_\varphi = D \cdot \nabla\theta^N, \tag{B.12}$$

which follows from a trivial calculation. Hence

$$\begin{aligned} N \int_U J_{(1-\zeta)\varphi} &= \int_U D \cdot \nabla(1-\zeta)^N \\ &= \int_U D \cdot \nabla[(1-\zeta)^N - (1-\zeta)] + \int_U D \cdot \nabla(1-\zeta) = -\int_{U \backslash H} D \cdot \nabla\zeta, \end{aligned} \tag{B.13}$$

where we have used that $(1-\zeta)^N-(1-\zeta)$ is a C^∞ function of compact support in $U\backslash H$ [cf. (B.9)]. Take ζ with the properties that $0\leqq\zeta\leqq 1$ and $\zeta=0$ on $U\backslash H_\varepsilon$. Then the left side of (B.13) is $I\equiv\sigma_N\int_{H_\varepsilon} f(\psi)J_\psi$, where $\psi=(1-\zeta)\varphi$ and $f(x)=N/\sigma_N$ for $|x|\leqq 1$ and $f(x)=0$ for $|x|>1$. (Recall that $J_\varphi\equiv 0$ on $U\backslash H_\varepsilon$.) Since $\int f=1$ and $|\psi|=1$ on ∂H_ε, we can apply (B.4) together with an approximation argument using dominated convergence, to conclude that $I=\sigma_N\deg(\varphi,H)$. □

Remark B.1. Let $U\subset\mathbb{R}^N$ be open, let $H\subset U$ be compact and let $\varphi\in C^1(U\backslash H; S^{N-1})$. Let V be open with $\bar{V}\subset U$ and with $H\subset V$. Assume that V is bounded and that ∂V is (piecewise) smooth. Then

$$\int_{\partial V} D\cdot\nu=\sigma_N\deg(\varphi,H), \tag{B.14}$$

where ν is the outward normal to ∂V. [To prove this, apply (B.10) to any $\zeta\in C_c^\infty(U)$ with $\zeta\equiv 1$ on V. Integrate by parts and use (B.9).] Equation (B.14) is the classical formula for the degree. Note that

$$D\cdot\nu=\det(\varphi,\varphi_{x_1},\ldots,\varphi_{x_{N-1}}), \tag{B.15}$$

where $x_1,\ldots,x_{N-1}$ are orthonormal coordinates in the tangent space to ∂V. On the other hand, we can think of φ restricted to ∂V as a map from the $N-1$ dimensional manifold $M\equiv\partial V$ to S^{N-1}. This map has a Jacobian determinant, which is nothing other than the right side of (B.15). Thus $\int D\cdot\nu$ can be identified as the right side of (B.4) [with $f(\psi)=1$ for $|\psi|\leqq 1$] with the integrating being over M, and not over V. Alternatively, $\int D\cdot\nu/\sigma_N$ is the number of times (including sign) that φ covers S^{N-1}.

Here are some consequences of Theorem B.1:

Theorem B.2. *Let $U\subset\mathbb{R}^N$ be open and let $H_1,H_2,\ldots,H_k$ be disjoint holes in U and let $H=\bigcup_{i=1}^k H_i$. Let $\varphi\in C(U\backslash H; S^{N-1})$ with $\nabla\varphi\in L^{N-1}(U\backslash H)$. Then*

$$-\int_{U\backslash H} D\cdot\nabla\zeta=\sigma_N\sum_{i=1}^k\zeta(H_i)\deg(\varphi,H_i) \tag{B.16}$$

for every $\zeta\in C(\bar{U})$ with $\nabla\zeta\in L^\infty(U)$ (in the distributional sense), $\zeta=0$ on ∂U and $\zeta=\zeta(H_i)$ is a constant on each H_i.

Theorem B.3. *Let U, H_i, and H be as in Theorem B.2. Let $\varphi\in C(\bar{U}\backslash H; S^{N-1})$ with $\nabla\varphi\in L^{N-1}(U\backslash H)$. Assume also that φ is constant on ∂U. Then* (B.16) *holds for every $\zeta\in C(U)$ with $\nabla\zeta\in L^\infty(U)$ (in the distributional sense) and $\zeta=\zeta(H_i)$ is a constant on each H_i. Note that here we do not assume that $\zeta=0$ on ∂U.*

The proofs rely on the following lemma.

Lemma B.4. *Let $V\subset\mathbb{R}^N$ be open and let $F\subset\mathbb{R}^N$ be closed with $F\subset V$. F need not be compact. Let $\varphi\in C(\bar{V}\backslash F; S^{N-1})$ with $\nabla\varphi\in L^{N-1}(V\backslash F)$. Assume that φ is constant on ∂V (no assumption is made if $V=\mathbb{R}^N$). Then*

$$\int_{V\backslash F} D\cdot\nabla\zeta=0 \tag{B.17}$$

for every $\zeta\in C(V)$ with $\nabla\zeta\in L^\infty(V)$ and $\zeta=0$ on F.

Proof. The intuitive reason that (B.17) holds is clear. Indeed, set $\Omega = V\backslash F$; we write

$$\int_\Omega D\cdot \nabla\zeta = \int_{\partial\Omega} (D\cdot\nu)\zeta - \int_\Omega (\operatorname{div} D)\zeta,$$

where ν denotes the outward normal on $\partial\Omega$. However, $\partial\Omega$ consists of two disjoint parts, namely ∂V and ∂F. On ∂V we have $D\cdot\nu = 0$ (since φ is constant on ∂V), while on ∂F we have $\zeta = 0$. On the other hand, $\operatorname{div} D = 0$ on Ω [by (B.9)].

Since, in general, we do not assume that ∂V and ∂F are regular, the integration by parts is not justified and the proof becomes more delicate. First, without loss of generality, we can assume that $\zeta\in L^\infty(V)$. Otherwise, consider

$$\zeta_n(x) = \begin{cases} \zeta(x) & \text{if } |\zeta(x)|\leqq n \\ n\operatorname{sgn}\zeta(x) & \text{if } |\zeta(x)| > n. \end{cases}$$

Clearly $\int_\Omega D\cdot\nabla\zeta_n \to \int_\Omega D\cdot\nabla\zeta$ (by dominated convergence).

Second, we can also assume that ζ vanishes outside a large ball. Otherwise, consider a sequence $\zeta_n = \alpha_n\zeta$, where $\alpha_n(x) = 1$ for $|x|\leqq n$, $\alpha_n(x) = 2-(|x|/n)$ for $n\leqq |x|\leqq 2n$ and $\alpha_n(x) = 0$ for $|x|\geqq 2n$. Again, $\int_\Omega D\cdot\nabla\zeta_n\to\int_\Omega D\cdot\nabla\zeta$ since $D\in L^1(\Omega)$.

Next, we can also assume that $\zeta = 0$ on a neighborhood of F and that φ is constant on a neighborhood of ∂V. Indeed let $g:\mathbb{R}\to\mathbb{R}$ be a smooth function such that $g(t) = 0$ for $|t|\leqq 1$ and $g(t) = t$ for $|t|\geqq 2$. Consider $\zeta_n(x) = \frac{1}{n} g(n\zeta(x))$. It is clear that $\zeta_n\in C(V)\cap L^\infty(V)$, ζ_n vanishes outside a large ball, $\|\nabla\zeta_n\|_{L^\infty}\leqq C\|\nabla\zeta\|_{L^\infty}$, $\zeta_n = 0$ on some neighborhood of F (namely $\{x \mid |\zeta(x)| < 1/n\}$) and $\nabla\zeta_n\to\nabla\zeta$ a.e. on V. We proceed in the same way with φ. Let $G:\mathbb{R}^N\to\mathbb{R}^N$ be defined by $G(v)_i = g(v_i)$ for all i (g as above). Let e be the value of φ on ∂V. Consider $\psi_n(x) = \frac{1}{n} G(n(\varphi(x)-e)) + e$. It is easy to check that $\psi_n\in C(\bar{V}\backslash F;\mathbb{R}^N)$, $\psi_n = e$ in a neighborhood of ∂V [namely $\{x \mid |\varphi(x)-e| < 1/n\}$], $\|\psi_n-\varphi\|_{L^\infty}\leqq C/n$ and $\nabla\psi_n\to\nabla\varphi$ in $L^{N-1}(V\backslash F)$.

Finally, we choose $\varphi_n = \psi_n/|\psi_n|$ (for n large enough), so that φ_n satisfies the same properties as ψ_n and, moreover, φ_n takes its values in S^{N-1}. Clearly $D_n = D(\varphi_n)\to D(\varphi) = D$ in $L^1(V\backslash F)$ and therefore $\int_\Omega D_n\cdot\nabla\zeta_n\to\int_\Omega D\cdot\nabla\zeta$ (by dominated convergence).

In conclusion, it suffices to establish (B.17) with the additional assumptions that $\zeta = 0$ outside a large ball, $\zeta = 0$ on a neighborhood of F and $D = 0$ on a neighborhood of ∂V. Since $K = \operatorname{Supp} D\cap\operatorname{Supp}\zeta$ is a compact subset of Ω we may fix a function $\alpha\in C_c^\infty(\Omega)$ such that $\alpha\equiv 1$ on some neighborhood of K. By (B.9) we have $\int_\Omega D\cdot\nabla(\alpha\zeta) = 0$, and on the other hand, $D\cdot\nabla(\alpha\zeta) = D\cdot\nabla\zeta$ a.e. on Ω (from the definition of α). □

Proofs of Theorems B.2 and B.3. If ζ_1 and ζ_2 are two admissible functions with the same values $\zeta(H_i)$ for every i, then by Lemma B.4, applied to $\zeta = \zeta_1 - \zeta_2$ we have $\int_{U\backslash H} D\cdot\nabla\zeta_1 = \int_{U\backslash H} D\cdot\nabla\zeta_2$ [choose $V = \mathbb{R}^3$ and $F = H\cup({}^cU)$ for Theorem B.2 and $V = U$, $F = H$ for Theorem B.3]. Thus, it suffices to prove (B.16) for one admissible ζ. Take $\zeta = \sum_{i=1}^k \zeta_i$ with each $\zeta_i = \zeta(H_i)$ near H_i and $\operatorname{Supp}\zeta_i$ is contained in a small neighborhood of H_i. Then apply Theorem B.1. □

Remark B.2. Let $U \subset \mathbb{R}^N$ be open. Assume that all the holes H_i are points a_i in U. Let D be *any* vector field in $L^1(U; \mathbb{R}^N)$. Let d_i be *any* real numbers. Then the relation

$$-\int D \cdot \nabla\zeta = \sigma_N \sum_{i=1}^{k} d_i \zeta(a_i) \tag{B.18}$$

for every $\zeta \in C(\bar{U})$ with $\nabla\zeta \in L^\infty(U)$ (in the distributional sense), $\zeta = 0$ on ∂U, is equivalent to the relation

$$\operatorname{div} D = \sigma_N \sum_{i=1}^{k} d_i \delta_{a_i} \quad \text{in} \quad \mathscr{D}'(U), \tag{B.19}$$

where δ_a is the Dirac measure at $a \in \mathbb{R}^N$. [In particular the D field in (B.16) satisfies (B.19) for point holes.] Equation (B.19) looks weaker than (B.18) because the class of testing functions for (B.19) is more restrictive, namely $C_c^\infty(U)$. The equivalence of (B.18) and (B.19) follows from the following general density lemma.

Lemma B.5. *Suppose ζ is a function in $C(\bar{U})$ with $\nabla\zeta \in L^\infty(U)$ (in $\mathscr{D}'(U)$), $\zeta = 0$ on ∂U and ζ is a constant on each H_i. Then there exists a sequence ζ_n in $C_c^\infty(U)$ such that $\zeta_n \to \zeta$ uniformly on every compact subset of $\bar{U}$, $\|\nabla\zeta_n\|_{L^\infty} \leqq \|\nabla\zeta\|_{L^\infty}$, $\nabla\zeta_n \to \nabla\zeta$* a.e. *on U and ζ_n is a constant on each H_i.*

The proof uses the same techniques as in the proof of Lemma B.4 and therefore we shall omit it.

Remark B.3. Assume the same conditions as in Remark B.2 except that (B.18) holds for every $\zeta \in C(U)$ with $\nabla\zeta \in L^\infty(U)$, as in the setup of Theorem B.3. Then, the analogue of Remark B.2 is that (B.18) is equivalent to

$$\operatorname{div} D = \sigma_N \sum_{i=1}^{k} d_i \delta_{a_i} \quad \text{in} \quad \mathscr{D}'(U), \qquad D \cdot \nu = 0, \quad \text{on} \quad \partial U, \tag{B.20}$$

where ν is the normal to ∂U. The relation $D \cdot \nu = 0$ has to be interpreted in a formal sense since ∂U need not be smooth and since D is only L^1.

Appendix C: Duality for Vector Fields

We recall a classical abstract duality principle (see [12, 30, 36]).

Theorem C.1. *Let E be a Banach space and E^* its dual. Let $M \subset E$ be a linear subspace (not necessarily closed) and let Φ be a convex function from E into $(-\infty, +\infty]$ such that $\Phi(0) \neq +\infty$ and Φ is continuous at 0. Let Φ^* be the conjugate function on E^*, namely*

$$\Phi^*(f) = \sup\{\langle f, u\rangle - \Phi(u) \mid u \in E\}. \tag{C.1}$$

Then

$$\inf_M \Phi = -\min_{M^\perp} \Phi^*, \tag{C.2}$$

where

$$M^\perp = \{f \in E^* \mid \langle f, u\rangle = 0 \text{ for all } u \in M\}. \tag{C.3}$$

The following lemma will also be used.

Lemma C.2. *Let E be a separable Banach space and let N be a linear subspace in E^* that is sequentially closed in the weak * topology. Let*

$$N^\perp = \{u \in E \mid \langle f, u\rangle = 0 \text{ for all } f \in N\}. \tag{C.4}$$

Then $(N^\perp)^\perp = N$.

Proof. It follows easily from the Hahn-Banach theorem that $(N^\perp)^\perp$ is the weak * closure of N. To prove the lemma, therefore, it suffices to show that N is weak * closed. In view of the theorem of Banach, Dieudonné, Krein, and Smulian (see e.g. [11, Theorem V.5.7]) we have only to check that $\tilde{N} = N \cap B$ is weak * closed, where B is the unit ball in E^*. But $\tilde{N}$ is metrizable for the weak * topology (see e.g. [3, Theorem III.25]) so it suffices to note that $\tilde{N}$ is sequentially weak * closed. □

Theorem C.1 will be applied in the following two cases (A and B). In the notation of Sect. V, we take

$$E = L^1(\Omega; \mathbb{R}^N), \qquad E^* = L^\infty(\Omega; \mathbb{R}^N), \tag{C.5}$$

and

$$M_{A,B} = \{D \in E \mid \textstyle\int D \cdot \nabla\zeta = 0 \text{ for all } \zeta \in Q_A \text{ (respectively } Q_B)\}. \tag{C.6}$$

Fix any $D^0 \in \mathscr{A}_A$ (respectively $\mathscr{A}_B$) and let

$$\Phi(D) = \int_\Omega |D + D^0|. \tag{C.7}$$

Clearly,

$$E_{A,B} = \inf\{\Phi(D) \mid D \in M_A \text{ (respectively } M_B)\}, \tag{C.8}$$

and, for every $f \in E^*$,

$$\Phi^*(f) = -\int f \cdot D^0 + \sup_{D\in E}\{\textstyle\int f \cdot D - \int |D|\} = \begin{cases} -\int f \cdot D^0 & \text{if } \|f\|_{L^\infty} \leqq 1 \\ +\infty & \text{if } \|f\|_{L^\infty} > 1. \end{cases} \tag{C.9}$$

Lemma C.3.

$$M^\perp_{A,B} = \{\nabla\zeta \mid \zeta \in Q_A \text{ (respectively } Q_B)\}. \tag{C.10}$$

Proof. We shall omit the A, B subscript. Let $N \subset E^*$ be the right side of (C.10). By the definition of M, $N^\perp = M$ so that $(N^\perp)^\perp = M^\perp$. We claim that N is sequentially weak * closed, whence, by Lemma C.2, $N = (N^\perp)^\perp = M^\perp$, which is precisely (C.10). To check that N is sequentially weak * closed, let ζ_n be a sequence in Q such that $\nabla\zeta_n \to f \in E^*$ in the weak * topology. We want to prove that $f = \nabla\zeta$ for some $\zeta \in Q$. By the uniform boundedness principle we know that $\|\nabla\zeta_n\|_{L^\infty} \leqq C$. We can always assume $\zeta_n(x_0) = 0$ for some fixed $x_0 \in U$. By Ascoli's theorem $\zeta_{n_k} \to \zeta$ uniformly on compact subsets of $\bar{U}$ (respectively U) in case A (respectively B). Clearly, $\zeta \in Q$ and $f = \nabla\zeta$. □

Applying Theorem C.1 and Lemma C.3, we find that

$$\begin{aligned} E_{A,B} &= \max\{\textstyle\int \nabla\zeta \cdot D^0 \mid \|\nabla\zeta\|_{L^\infty} \leqq 1,\ \zeta \in Q_A \text{ (respectively } Q_B)\} \\ &= \max\{\sigma_N \textstyle\sum d_i \zeta(H_i) \mid \|\nabla\zeta\|_{L^\infty} \leqq 1,\ \zeta \in Q_A \text{ (respectively } Q_B)\}. \end{aligned} \tag{C.11}$$

This is precisely the statement of Theorem 5.1.

Appendix D: The Basic Divergence-Free Vector Field on a Curve

Let g be a rectifiable curve in $\mathbb{R}^N$ with no self-intersection and end points a and b, $a \neq b$. Let L be its length. To be more precise the curve can be parametrized by a Lipschitz function $X(t):[0,1] \to \mathbb{R}^N$ and we can always assume that $\dot{X}(t) \neq 0$ a.e. Among the choices for $X(t)$ there is a canonical constant speed choice denoted by $X_0(t)$, so that $|\dot{X}_0(t)| = L$ a.e.

Now consider the problem of finding an $\mathbb{R}^N$-valued measure, D, on $\mathbb{R}^N$ such that

$$\operatorname{supp} D \subset g, \tag{D.1}$$

$$\operatorname{div} D = \delta_a - \delta_b \quad \text{in} \quad \mathscr{D}'(\mathbb{R}^N). \tag{D.2}$$

Theorem D.1. *There is precisely one solution to the above problem, namely*

$$\langle D_g, \varphi \rangle \equiv \int_0^1 \varphi(X(t)) \cdot \dot{X}(t) dt \tag{D.3}$$

for all $\varphi \in C_c(\mathbb{R}^N; \mathbb{R}^N)$. Here $X(t)$ denotes any parametrization of g and (D.3) *is independent of the choice of the parametrization. Moreover, $|D_g|$ is the one-dimensional Hausdorff measure of g, denoted by δ_g. In particular*

$$\int_{\mathbb{R}^N} |D_g| = L. \tag{D.4}$$

Proof. It is obvious that D_g given by (D.3) is independent of parametrization and satisfies (D.1).

Let us check that D_g satisfies (D.2). Choose $\zeta \in C_c^\infty(\mathbb{R}^N)$. We have

$$\langle D_g, \nabla\zeta \rangle = \int_0^1 \nabla\zeta(X(t)) \cdot \dot{X}(t) dt = \int_0^1 \frac{d}{dt} \zeta(X(t)) dt = \zeta(b) - \zeta(a). \tag{D.5}$$

The last equality follows from the fact that Lipschitz functions are absolutely continuous. Next, we establish uniqueness. Consider $D - D_g$ and call it D, so that D satisfies

$$\operatorname{supp} D \subset g, \tag{D.6}$$

$$\operatorname{div} D = 0 \quad \text{in} \quad \mathscr{D}'(\mathbb{R}^N). \tag{D.7}$$

We have to show that $D = 0$. It follows from (D.6) that there is an $\mathbb{R}^N$-valued measure, m, on $[0,1]$, such that

$$\langle D, \varphi \rangle = \int_0^1 \varphi(X(t)) \cdot dm(t) \tag{D.8}$$

for all $\varphi \in C_c(\mathbb{R}^N; \mathbb{R}^N)$. The existence of m follows from the fact that for any continuous function, α, on $[0,1]$ there exists some $\varphi \in C_c(\mathbb{R}^N; \mathbb{R}^N)$ with $\varphi(X(t)) = \alpha(t)$ and $\|\varphi\| = \|\alpha\|$. Thus, D can be viewed as an element of the dual of $C([0,1]; \mathbb{R}^N)$, but these are measures.

Next, we claim that

$$\int_0^T \nabla\zeta(X(t)) \cdot dm(t) = 0 \tag{D.9}$$

for every $\zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R})$ and a.e. T. Assuming that (D.9) holds we conclude easily that $m=0$ (and so $D=0$). Indeed, by differentiating (D.9) in the sense of distributions we find

$$\nabla\zeta(X(t))\cdot m(t)=0 \quad \text{in} \quad \mathscr{D}'(0,1).$$

Choosing $\zeta(x)=x_i\theta(x)$, where $\theta\in C_c^\infty(\mathbb{R}^N)$ and $\theta\equiv 1$ on some neighborhood of g, we see that $m=0$.

To establish (D.9) we fix $T\in(0,1)$ such that

$$m(\{T\})=0, \tag{D.10}$$

$$\text{and} \quad \dot X(T) \quad \text{exists and} \quad \dot X(T)\neq 0. \tag{D.11}$$

For any $\varepsilon>0$ (small enough) let $A=X([0,T])$ and $B_\varepsilon=X([T+\varepsilon,1])$. Set $d_\varepsilon=\operatorname{dist}(A,B_\varepsilon)$. There exists a function $F_\varepsilon\in C_c^\infty(\mathbb{R}^N)$ such that

$$F_\varepsilon=1 \quad \text{near } A, \qquad F_\varepsilon=0 \quad \text{near } B, \tag{D.12}$$

$$|\nabla F_\varepsilon|\leqq C/d_\varepsilon, \tag{D.13}$$

where C is a constant independent of ε. By (D.7) we have

$$0=\langle D,\nabla(\zeta F_\varepsilon)\rangle=\langle D,\zeta\nabla F_\varepsilon\rangle+\langle D,F_\varepsilon\nabla\zeta\rangle=I_1+I_2 \tag{D.14}$$

with

$$\begin{aligned} I_1=\langle D,\zeta\nabla F_\varepsilon\rangle &= \int_T^{T+\varepsilon}\zeta(X(t))\nabla F_\varepsilon(X(t))\cdot dm(t)\\ &=\int_T^{T+\varepsilon}(\zeta(X(t))-\zeta(X(T)))\nabla F_\varepsilon(X(t))\cdot dm(t)+\zeta(X(T))\int_T^{T+\varepsilon}\nabla F_\varepsilon(X(t))\cdot dm(t).\end{aligned}$$

The last integral is $\langle D,\nabla F_\varepsilon\rangle=0$. We claim that $I_1\to 0$ as $\varepsilon\to 0$. Indeed

$$|I_1|\leqq C\int_T^{T+\varepsilon}|X(t)-X(T)|\,|\nabla F_\varepsilon(X(t))|\,|dm(t)|\leqq C\frac{\varepsilon}{d_\varepsilon}\int_T^{T+\varepsilon}|dm(t)|.$$

Since $\int_T^{T+\varepsilon}|dm(t)|\to 0$ as $\varepsilon\to 0$ (by dominated convergence) it suffices to check that $\varepsilon/d_\varepsilon$ remains bounded as $\varepsilon\to 0$. Suppose not. Then there exists a sequence $\varepsilon_n\to 0$ such that $d_{\varepsilon_n}/\varepsilon_n\to 0$. Thus, there are sequences $t_n\geqq T+\varepsilon_n$ and $s_n\leqq T$ such that $|X(t_n)-X(s_n)|/\varepsilon_n\to 0$. Clearly $t_n\to T$ and $s_n\to T$ since X is one to one. Observe that $X(t_n)-X(T)=(t_n-T)\dot X(T)+o(t_n-T)$ and similarly $X(s_n)-X(T)=(s_n-T)\dot X(T)+o(T-s_n)$. Thus,

$$\frac{1}{\varepsilon_n}(X(t_n)-X(s_n))=\frac{t_n-s_n}{\varepsilon_n}\left(\dot X(T)+\frac{o(t_n-T)+o(T-s_n)}{(t_n-T)+(T-s_n)}\right).$$

Since $t_n-s_n\geqq\varepsilon_n$ and $\dot X(T)\neq 0$ we have a contradiction. Therefore $I_1\to 0$.

Next,

$$I_2=\langle D,F_\varepsilon\nabla\zeta\rangle=\int_0^T\nabla\zeta(X(t))\cdot dm(t)+\int_T^{T+\varepsilon}F_\varepsilon(X(t))\nabla\zeta(X(t))\cdot dm(t).$$

The last integral is bounded by $C \int_T^{T+\varepsilon} |dm(t)|$ which goes to zero. This establishes (D.9) and hence (D.3). To prove (D.4) we use $X_0(t)$ in (D.3). First, we have

$$|\langle D_g, \varphi \rangle| \leqq \int_0^1 |\varphi(X_0(t))| L dt = \langle \delta_g, |\varphi| \rangle,$$

and hence $|D_g| \leqq \delta_g$. On the other hand, $\int |D_g| = L$. Indeed we have

$$\int |D_g| = \operatorname{Sup}\left\{ \int_0^1 \varphi(X_0(t)) \cdot \dot{X}_0(t) dt \mid \varphi \in C_c(\mathbb{R}^N; \mathbb{R}^N) \text{ with } \|\varphi\|_{L^\infty} \leqq 1 \right\}.$$

Let $Y_n \in C([0,1]; \mathbb{R}^N)$ be a sequence of functions such that $\|Y_n\|_{L^\infty} \leqq L$ and $Y_n \to \dot{X}_0$ in $L^2(0,1)$. There exists a sequence of functions, $\psi_n \in C_c(\mathbb{R}^N; \mathbb{R}^N)$ such that $\psi_n(X_0(t)) = Y_n(t)$ and $\|\psi_n\|_{L^\infty} \leqq L$. Letting $\varphi_n = (1/L)\psi_n$ we have

$$\int_0^1 \varphi_n(X_0(t)) \cdot \dot{X}_0(t) dt = (1/L) \int_0^1 Y_n(t) \cdot \dot{X}_0(t) dt \to L,$$

and therefore $\int |D_g| \geqq L$. □

Corollary D.2. *Let everything be as in Theorem D.1 except that hypothesis* (D.2) *is replaced by*

$$\operatorname{div} D = 0 \quad in \quad \mathscr{D}'(\mathbb{R}^N \backslash \{a, b\}). \tag{D.15}$$

Then, there exists a constant c and two vectors A and B in $\mathbb{R}^N$ *such that*

$$D = c D_g + A\delta_a + B\delta_b \quad in \quad \mathscr{D}'(\mathbb{R}^N). \tag{D.16}$$

Proof. From (D.15) and a standard result about distributions with support on a point we have

$$\operatorname{div} D = \sum_\alpha c_\alpha \partial^\alpha \delta_a + \sum_\alpha c'_\alpha \partial^\alpha \delta_b \quad \text{in} \quad \mathscr{D}'(\mathbb{R}^N), \tag{D.17}$$

where the sums are finite. Since D is a measure, the right side of (D.17) contains only zeroth and first order derivatives. Since $\int \operatorname{div} D = 0$, the zeroth order terms have to be equal and opposite, namely $c(\delta_a - \delta_b)$. Therefore,

$$\operatorname{div}(D - cD_g) = \operatorname{div}(A\delta_a) + \operatorname{div}(B\delta_a) \tag{D.18}$$

for some vectors A and B. Transposing the right side of (D.18) to the left side and then using the uniqueness part of Theorem (D.1) we derive (D.16). □

Finally, we mention another corollary which will be used in Appendix E. Let g be a rectifiable curve in $\mathbb{R}^N$ without self-intersection and end points a and b, $a \neq b$. Let Ω be an open set such that $g \backslash \{a, b\} \subset \Omega$. Let D be an $\mathbb{R}^N$-valued bounded measure on Ω such that $\operatorname{supp} D \subset g$, and $\operatorname{div} D = 0$ in $\mathscr{D}'(\Omega)$.

Corollary D.3. *Under the above assumptions there exists a constant c such that*

$$D = c D_g \quad in \quad \mathscr{D}'(\Omega).$$

Proof. Extend D to all of $\mathbb{R}^N$ by 0 outside Ω. Let $\tilde{D}$ be the extension. We claim that $\operatorname{div} \tilde{D} = 0$ in $\mathscr{D}'(\mathbb{R}^N \backslash \{a, b\})$. Let $\zeta \in C_c^\infty(\mathbb{R}^N \backslash \{a, b\})$ and let $\theta \in C_c^\infty(\Omega)$ with $\theta = 1$ on a neighborhood of $g \cap \operatorname{Supp}\zeta$. We have

$$\int \tilde{D} \cdot \nabla\zeta = \int_\Omega D \cdot \nabla\zeta = \int_\Omega D \cdot \nabla(\theta\zeta) = 0,$$

since $\theta\zeta \in C_c^\infty(\Omega)$. We may now apply Corollary D.2 to $\tilde{D}$. □

Appendix E: Quantization and Weak Limits of Vector Fields

Let V be an open set in $\mathbb{R}^N$ with $N \geqq 1$. Let φ be a map from V into S^N such that $\nabla\varphi \in L^N(V)$. Set

$$\Delta = \det(\varphi, \varphi_{x_1}, \ldots, \varphi_{x_N}). \tag{E.1}$$

Similarly let φ^n be a sequence of such maps and set

$$\Delta^n = \det(\varphi^n, \varphi^n_{x_1}, \ldots, \varphi^n_{x_N}). \tag{E.2}$$

We are concerned with the following situation. Suppose $\varphi^n \to \varphi$ a.e. and $\nabla\varphi^n$ is *bounded* in $L^N(V)$. Then Δ^n is bounded in $L^1(V)$ so that, by passing to a subsequence, we can assume that Δ^n tends to some measure μ in the weak * topology of measures. In general $\mu \neq \Delta$ unless $\nabla\varphi^n \to \nabla\varphi$ *strongly* in L^N.

If one merely assumes that $\varphi^n \to \varphi$ a.e. and $\nabla\varphi^n$ is bounded in $L^N(V)$ and if one replaces Δ^n by $|\nabla\varphi^n|^N$, for example, then we may still assume that $|\nabla\varphi^n|^N$ tends weakly to some measure ν. However, in this case one can say virtually nothing about $\nu - |\nabla\varphi|^N$. It is a striking fact that despite the lack of strong convergence it is possible to say something precise about $\mu - \Delta$. This is due to the fact that Δ has a geometric significance. Lions [23] considered maps φ with values in $\mathbb{R}^{N+1}$ instead of S^N and proved that $\mu - \Delta$ is a sum (possibly infinite) of Dirac masses but with arbitrary weights. Our result, Theorem E.1, uses the geometry of S^N and shows that there can only be finitely, many Dirac masses and that they have integer weights. Our proof is completely different from that of Lions.

A typical example is the following. Let ψ be a smooth map from $\mathbb{R}^N$ into S^N which is a constant C far out. Let $\varphi^n(x) = \psi(nx)$ so that $\varphi^n \to \varphi = C$ a.e. and $\nabla\varphi^n$ is bounded in L^N. Note that $\Delta^n \rightharpoonup \alpha\delta_0$, where $\alpha = \int_{\mathbb{R}^N} \det(\psi, \psi_{x_1}, \ldots, \psi_{x_N})dx$ and α/σ_{N+1} belongs to $\mathbb{Z}$, since α/σ_{N+1} is the degree of ψ (cf. Appendix B). This example displays a quantization feature which holds in the general setting.

Theorem E.1. *Assume $\varphi^n \to \varphi$* a.e., *$\nabla\varphi^n$ is bounded in $L^N(V)$ and $\Delta^n \rightharpoonup \mu$. Then there exist p integers $d_1, d_2, \ldots, d_p \in \mathbb{Z}$ and p points $a_1, a_2, \ldots, a_p$ in V such that*

$$\mu - \Delta = \sigma_{N+1} \sum_{i=1}^{p} d_i \delta_{a_i}. \tag{E.3}$$

The proof relies on three lemmas.

Lemma E.2. *Assume Q is a cube in $\mathbb{R}^N$ and let $\varphi, \bar{\varphi}$ be two maps from Q to S^N such that $\nabla\varphi, \nabla\bar{\varphi} \in L^N(Q)$ and $\varphi, \bar{\varphi}$ restricted to ∂Q belong to $W^{1,N}(\partial Q)$, so that, in particular, φ and $\bar{\varphi} \in C(\partial Q)$. Then there is an integer d such that*

$$\left| \int_Q (\Delta - \bar{\Delta}) - \sigma_{N+1} d \right| \leqq C \|\varphi - \bar{\varphi}\|_{L^\infty(\partial Q)}, \tag{E.4}$$

where C depends only on the norms of φ and $\bar{\varphi}$ in $W^{1,N-1}(\partial Q)$.

Proof. Consider the cylinder $Q \times (0,1)$ in $\mathbb{R}^{N-1}$ and its boundary Γ. Γ consists of three pieces $\Gamma_0 = \bar{Q} \times \{0\}$, $\Gamma_1 = \bar{Q} \times \{1\}$, and $\Gamma_2 = \partial Q \times [0,1]$. We recall that it follows from the density result of [32] that if $\theta \in W^{1,N}(\Gamma, S^N)$, then

$$\frac{1}{\sigma_{N+1}} \int_\Gamma \det(\theta, \theta_{x_1}, \ldots, \theta_{x_N}) \in \mathbb{Z}, \tag{E.5}$$

where $x_1, x_2, \ldots, x_N$ are orthonormal coordinates in the tangent space to Γ [cf. (B.14)]. Let $\tilde{\theta}(x,t) = t\varphi(x) + (1-t)\bar{\varphi}(x)$, $x \in \bar{Q}$, $t \in [0,1]$, and let $\theta = \tilde{\theta}/|\tilde{\theta}|$. Note that θ is well defined, at least if $\|\varphi - \bar{\varphi}\|_{L^\infty(\partial Q)} < 1/2$; otherwise, the conclusion is trivial. Also, $|\tilde{\theta}| > 1/2$ everywhere on Γ. Clearly,

$$\int_{\Gamma_0 \cup \Gamma_1} \det(\theta, \theta_{x_1}, \ldots, \theta_{x_N}) = \int_Q (\Delta - \bar{\Delta}).$$

Now we estimate $\int_{\Gamma_2} \det(\theta, \theta_{x_1}, \ldots, \theta_{x_N})$. Observe that

$$\det(\theta, \theta_{x_1}, \ldots, \theta_{x_N}) = \frac{1}{|\tilde{\theta}|^{N+1}} \det(\tilde{\theta}, \tilde{\theta}_{x_1}, \ldots, \tilde{\theta}_{x_N})$$

and $\tilde{\theta}_t = \varphi - \bar{\varphi}$. Since we are now on Γ_2, one of the x_j may be taken to be t. Therefore,

$$\left| \int_{\Gamma_2} \det(\theta, \theta_{x_1}, \ldots, \theta_{x_N}) \right| \leqq C \|\varphi - \bar{\varphi}\|_{L^\infty(\partial Q)},$$

where C depends only on $W^{1,N-1}(\partial Q)$. □

Remark E.1. Clearly, Lemma E.2 extends to domains other than cubes under appropriate assumptions on the regularity of the boundary.

For every $h > 0$, set

$$Q_h = \{x \in \mathbb{R}^N \mid |x_i| < h/2,\ i = 1, \ldots, N\}.$$

Lemma E.3. *Let f_n be a sequence of functions on V which is bounded in $L^1(V)$. Let $h > 0$. Then, for* a.e. *$a \in \mathbb{R}^N$ there is a subsequence f_{n_k} (depending on a) such that f_{n_k} restricted to $(a + \partial Q_h) \cap V$ is bounded in $L^1((a + \partial Q_h) \cap V)$.*

Proof. We consider only the case where $N = 2$ since the argument is the same in the general case. Extend f_n by zero outside V and for a.e. $y \in \mathbb{R}$ set

$$g_n(y) = \int_{\mathbb{R}} (|f_n(x,y)| + |f_n(x, y+h)|)\,dx.$$

Note that

$$\int_0^h g_n(y)\,dy \leqq \int_V |f_n(x,y)|\,dx\,dy \leqq C.$$

Applying Fatou's lemma we deduce that $\liminf g_n(y) < \infty$ for a.e. $y \in \mathbb{R}$. Similarly, if we reverse x and y. Therefore, for a.e. $a \in \mathbb{R}^2$, there is a subsequence f_{n_k} (depending on a) such that f_{n_k} restricted to $a + \partial Q_h$ is bounded in $L^1(a + \partial Q_h)$. □

Lemma E.4. *Let λ_n be a sequence of measures on V such that $\lambda_n \rightharpoonup \lambda$ and $|\lambda_n| \rightharpoonup \nu$ weakly in the sense of measures. Let Q be an open cube such that $\bar{Q} \subset V$ and $\nu(\partial Q) = 0$. Then $\lambda_n(Q) \to \lambda(Q)$.*

The proof is straightforward; approximate characteristic functions by continuous functions. □

Proof of Theorem E.1. Without loss of generality we may assume that $|\Delta^n| \rightharpoonup \nu$ weakly in the sense of measures. We shall say that an open cube Q is a *good* cube if $\bar{Q} \subset V$ and Q satisfies the following properties:

(i) there is a subsequence φ^{n_k} (depending on Q) such that $\nabla\varphi^{n_k}$ restricted to ∂Q is bounded in $L^N(\partial Q)$,

(ii) $\nu(\partial Q)=0$,

(iii) $\varphi^n \to \varphi$ a.e. on ∂Q.

The proof consists of three steps.

Step 1: For every good cube Q one has

$$\frac{1}{\sigma_{N+1}}\left(\mu(Q)-\int_Q \Delta\right)\in \mathbb{Z}.$$

Indeed, $\nabla\varphi^{n_k}$ is bounded in $L^N(\partial Q)$ and therefore $\varphi^{n_k}\to\varphi$ in $L^\infty(\partial Q)$ (by the Morrey-Sobolev imbedding theorem). Applying Lemma E.2 we see that there exists a sequence of integers d_k such that

$$\left|\int_Q (\Delta^{n_k}-\Delta)-\sigma_{N+1}d_k\right|\to 0.$$

The conclusion follows since, by Lemma E.4, we have $\int_Q \Delta^{n_k}\to\mu(Q)$.

Step 2: $\frac{1}{\sigma_{N+1}}\mu(\{a\})\in\mathbb{Z}$ for every $a\in V$. Let Q_j be a sequence of good cubes such that $a\in Q_j$ for all j and $|Q_j|\to 0$. Such a sequence exists by Lemma E.3 applied to $f_n=|\nabla\varphi^n|^N$ [for (ii) and (iii) the argument is standard]. We know from Step 1 that, for all j,

$$\frac{1}{\sigma_{N+1}}\left(\mu(Q_j)-\int_{Q_j}\Delta\right)=d_j\in\mathbb{Z}.$$

Finally, we let $j\to\infty$ and conclude, using the fact that $\int_{Q_j}\Delta\to 0$.

It follows from Step 2 that μ has only finitely many atoms. The atomic part of μ will be denoted by $\sigma_{N+1}\sum_{i=1}^{p} d_i\delta_{a_i}$ with $d_i\in\mathbb{Z}$ and $a_i\in V$.

Step 3. Let $m=\mu-\Delta-\sigma_{N+1}\sum_{i=1}^{p} d_i\delta_{a_i}$. We claim that $m=0$.

Indeed, by Step 1, we know that $\sigma_{N+1}^{-1}m(Q)\in\mathbb{Z}$ for every good cube Q. Let V' be an open set with compact closure in V. Since m has no atoms there is some $\varepsilon>0$ such that $m(Q)=0$ for every good cube, Q, with $|Q|<\varepsilon$ and $Q\cap V'\neq\emptyset$ (the argument is by contradiction). Let $h>0$ be such that $h^N<\varepsilon$ and $h<\operatorname{dist}(V',\partial V)$. Then $\chi_{Q_h}*m=0$ in $\mathscr{D}'(V')$, since $m(x-Q_h)=0$ for a.e. $x\in V'$ (note that $x-Q_h$ is a good cube for a.e. $x\in V'$, by Lemma E.3).

On the other hand, $h^{-N}\chi_{Q_h}*m\rightharpoonup m$ as $h\to 0$ and therefore $m=0$ in V'. □

Corollary E.5. *Let φ^n be a sequence of maps from S^N into S^N satisfying the same assumptions as in Theorem E.1. Then the same conclusion,* (E.3), *holds. (In* (E.1) *and* (E.2) *one has to interpret the x_i as orthonormal coordinates on S^N.)*

Proof. Use two stereographic projections (for example north and south poles) and note that the measure Δdx is invariant under diffeomorphisms. □

Finally, we consider the situation in which there is a sequence of continuous maps φ^n from $\Omega \subset \mathbb{R}^N$ to S^{N-1} $(N \geqq 3)$ with $\nabla \varphi^n \in L^{N-1}(\Omega)$. Associated with each φ^n is a vector field D^n given by (B.7). Let us suppose that $\nabla \varphi^n$ remains bounded in $L^{N-1}(\Omega)$ so that D^n is bounded in $L^1(\Omega)$, and thus we may assume that $D^n \rightharpoonup D$ weakly in the sense of measure. Let us suppose that

$$\operatorname{supp} D \subset g, \tag{E.6}$$

where g is a rectifiable curve in Ω without self-intersections. I.e. there is a Lipschitz map $X:[0,1] \to \bar{\Omega}$ which is injective and such that $X((0,1)) \subset \Omega$. Since $\operatorname{div} D^n = 0$ [see (B.9)] it follows that $\operatorname{div} D = 0$ in $\mathscr{D}'(\Omega)$ and thus, by Corollary D.3,

$$D = cD_g, \tag{E.7}$$

where D_g is given by (D.3). Appendix D only tells us that c in (E.7) is some constant, but the fact that φ^n takes values in S^{N-1} leads to the following

Theorem E.5. *Under the conditions on φ^n just stated, the constant c in* (E.7) *is an integer multiple of σ_N.*

Proof. Without loss of generality we may assume that $|D^n| \rightharpoonup \nu$ weakly in the sense of measures (in general, $\operatorname{supp} \nu$ need not be contained in g). Consider, as in Appendix D, the canonical parametrization, $X(t)$, of g and fix some $T \in (0,1)$ such that $\dot{X}(T)$ exists, $v \equiv \dot{X}(T) \neq 0$ and also $\nu(\{X(T)\}) = 0$. Set $a = X(T)$.

We wish to find a hyperplane Π through a with the following properties:

(i) $v \notin \Pi - a$,

(ii) $|\nabla \varphi^n|$ restricted to Π is uniformly bounded in $L^{N-1}(\Pi \cap \Omega)$,

(iii) $\nu(\Pi) = 0$.

This construction is possible – indeed (i), (ii), and (iii) hold for almost every Π. Using (i) we can find $r > 0$ (small enough) so that

$$g \cap \Pi \cap B(a,r) = \{a\}. \tag{E.8}$$

Indeed suppose not; then there exists a sequence $t_n \in (0,1)$ such that $X(t_n) \in \Pi$, $X(t_n) \neq a$, and $X(t_n) \to a$. We may always assume that $t_n \to t \in [0,1]$ and, since X is injective, we must have $t = T$. On the other hand, $(t_n - T)^{-1}(X(t_n) - X(T)) \in \Pi - a$, and at the limit we find $v \in \Pi - a$; this contradicts (i). Further, we may also assume that $B(a,r) \subset \Omega$. Let ζ be a smooth function such that $\zeta = 1$ on $B(a, r/2)$ with support in $B(a,r)$. Let H be the open half-space determined by Π and which contains $a - v$, and let $\vec{\nu}$ be the outward normal to H. We have

$$\int_H D^n \cdot \nabla \zeta = \int_\Pi (D^n \cdot \vec{\nu}) \zeta. \tag{E.9}$$

Using (ii) and Theorem E.1, we know that (for some subsequence still denoted D^n)

$$D^n \cdot \vec{\nu} \rightharpoonup f + \sigma_N \sum d_i \delta_{a_i} \tag{E.10}$$

with $f \in L^1(\Pi \cap \Omega)$ and $d_i \in \mathbb{Z}$. The reason that we can apply Theorem E.1 is the following. Since $|\nabla \varphi^n|$ restricted to Π is bounded in $L^{N-1}(\Pi \cap \Omega)$ and $N \geqq 3$, it follows that, for some subsequence, φ^n converges a.e. (on $\Pi \cap \Omega$) to some limit ψ and $\nabla \psi \in L^{N-1}(\Pi \cap \Omega)$. Note that this may fail when $N=2$. (The case $N=2$ is special and will be examined subsequently.) We may always choose r so small that $B(a,r)$ contains at most one a_i, namely a. Let d be the coefficient of δ_a in (E.10). From (E.9) we have

$$\int_H D^n \cdot \nabla \zeta \to \sigma_N d + \int_\Pi f \zeta . \tag{E.11}$$

On the other hand, by (iii) and Lemma E.4 we see that

$$\int_H D^n \cdot \nabla \zeta \to \int_H D \cdot \nabla \zeta . \tag{E.12}$$

We claim that

$$\int_H D \cdot \nabla \zeta = c, \tag{E.13}$$

where c is the constant introduced in (E.7). To prove this, let us assume there exists a $0 < T_1 < T$ and a radius r such that

$$\begin{array}{lll} \text{(i)} & X(t) \in H \quad \text{for} \quad T_1 \leqq t < T, \\ \text{(ii)} & X(t) \notin B(a,r) \cap H \quad \text{if} \quad t \notin [T_1, T). \end{array} \tag{E.14}$$

If this is so then, with $\tau = \{t \,|\, X(t) \in B(a,r) \cap H\}$, it is easy to see that

$$\begin{aligned} \int_H D \cdot \nabla \zeta &= c \int_\tau \nabla \zeta(X(t)) \cdot \dot{X}(t) dt \\ &= c \int_{T_1}^{T} \nabla \zeta(X(t)) \cdot \dot{X}(t) dt = c[\zeta(X(T)) - \zeta(X(T_1))] = c . \end{aligned} \tag{E.15}$$

The theorem follows from (E.15) and (E.11) by letting $r \to 0$, so that the integral in (E.11) goes to zero.

Now to prove that (E.14) can be satisfied observe that X is differentiable at T so that $X(t) = X(T) + v(t-T) + o(t-T)$, so that (i) is satisfied for $t < T$ and $T - t < \alpha$ for some α. Likewise, if $\beta > t - T > 0$ then $X(t) \notin H$. The curve $X(t)$ for $1 \geqq t \geqq \beta + T$ is closed and therefore has a positive distance from the point a. Call it δ_+. Likewise $|X(t) - a| \geqq \delta_- > 0$ for $0 \leqq t \leqq T - \alpha$. Choose $r < \min(\delta_+, \delta_-)$. For $t \geqq T$ either $X(t) \notin H$ or $|X(t) - a| > r$. For $t < T$, either $X(t) \in H$ or $|X(t) - a| > r$. This accomplishes (E.14). □

We turn now to the case $N=2$ which is not covered by Theorem E.5. Suppose φ^n is a sequence of continuous maps from $\Omega \subset \mathbb{R}^2$ to S^1 with $\nabla \varphi^n \in L^1(\Omega)$. Let us suppose that $\nabla \varphi^n$ remains bounded in $L^1(\Omega)$ so that D^n is bounded in $L^1(\Omega)$, and thus we may assume that $D^n \rightharpoonup D$ weakly in the sense of measures. Let us suppose, as above that $\operatorname{Supp} D \subset g$, and therefore, for some constant, c, we have

$$D = c D_g . \tag{E.16}$$

Theorem E.6. *Under the conditions on φ^n just stated, and also that $\varphi^n \to C$* a.e. *on Ω, where C is a constant, then the constant c in* (E.16) *is an integer multiple of $\sigma_2 = 2\pi$.*

The proof is the same as the proof of Theorem E.5 and we shall omit it. The assumption $\varphi^n \to C$ a.e. is essential, as the following simple case shows. Let Ω be the disk $\{x \in \mathbb{R}^2 \mid |x| < 1\}$ and let $g = \{(x_1, x_2) \mid x_1 = 0, |x_2| \leqq 1\}$ be a diameter. Let $f: \mathbb{R} \to \mathbb{R}$ be any smooth function with $f' \in L^1(\mathbb{R})$. The sequence $\varphi^n((x_1, x_2)) = (\cos f(nx_1), \sin f(nx_1))$ has all the right properties except that φ^n converges to two *different* constants for $x_1 > 0$ and $x_1 < 0$ [provided $f(+\infty) - f(-\infty)$ is not an integer multiple of 2π]. On the other hand, the limiting D field is cD_g with $c = f(+\infty) - f(-\infty)$.

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