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LIE GROUP STUDY OF FINITE DIFFERENCE SCHEMES

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Abstract. Differential equations arising in fluid mechanics are usually derived from the intrinsic properties of mechanical systems, in the form of conservation laws, and bear symmetries, which are not generally preserved by a finite difference approximation, and lead to inaccurate numerical results. This paper deals with the analysis of symmetry group of finite difference equations, which is based on the differential approximation. We develop a new scheme, the related differential approximation of which is invariant under the symmetries of the original differential equations. A comparison of numerical performance of this scheme, with standard ones and a higher order one has been realized for the Burgers equation.

1. **Introduction.** Lie groups were introduced by Sophus Lie in 1870 in order to study the symmetries of differential equations, yielding thus analytical solutions. Literature provides substantial works and applications, [3], [4]. Symmetry groups can be determined by an automatic procedure, but it often turns out to be tedious and induces errors. A large amount of packages using symbolic manipulations of mathematical expressions have been written. We mention here some of those works: Schwartz [17], Vu and Carminati[14], Herod [15], Baumann [16], Cantwell [5].
In this paper we are interested in the application of the theory of Lie group to numerical analysis.
Finite difference equations used to approximate the solutions of a differential equation generally do not respect the symmetries of the original equation, and can lead to inaccurate numerical results. Various techniques, that enable us to build a scheme preserving the symmetries of the original differential equation, have been studied. One of these techniques consists in constructing an invariant scheme from a given one by applying the method of the moving frame in [7], [8]. Another one consists in constructing an invariant scheme with the help of the discret invariants of its symmetry group [9], [10], [11], [12], [13] and provides the building of symmetry-adapted meshes, in preserving the differential equation symmetries. This technique is based on a direct study of the symmetries of difference equations and lattices.
Yanenko [2] and Shokin [1], proposed to apply the Lie group theory to finite difference equations by means of the differential approximation. Thus, they have set down conditions under which the differential representation of a finite difference scheme preserves the group of continuous symmetries of the original differential equation. They performed numerical simulations, which show that the resulting scheme is more accurate than non-invariant ones, after a frame transformation. The approach based on the differential approximation has been used in [6]. Ames

and al. [6] show that, in specific cases, the produced scheme is as accurate as higher order numerical methods.
In this paper, we focus on the latter approach. We have developed what can be called a *"semi-invariant scheme"*, in so far as the invariance condition is weaker than the one of the direct method, which acts on difference variables and enables one to determine the equations of an invariant mesh. A comparison is made between the numerical solutions of the Burgers equation for some standard schemes and the *"semi-invariant"* one.
The paper is organized as follows. Definitions and invariance condition for differential equations are provided in section 2. Section 3 recalls the approach of Yanenko and Shokin. Section 4 concentrates on classical schemes. In section 5, we present a method that enables us to build the semi-invariant scheme.

2. **Definitions and invariance condition for differential equations.** A r-parameter Lie group G_r of point transformations in the Euclidean space $\mathcal{E}(x,u)$ can be written under the form:

$$G_r = \{x_i^* = \phi_i(x,u,a);\ u_j^* = \varphi_j(x,u,a),\ i=1,\dots,m;\ j=1,\dots,n\} \tag{1}$$

Consider a system of l^{th}-order differential equations:

$$\mathcal{F}^\lambda\big(x,u,u^{(k_1)},u^{(k_1,k_2)},\dots,u^{(k_1\dots k_l)}\big)=0,\quad \lambda=1,\dots,q \tag{2}$$

Denote by $u^{(k_1\dots k_p)}$ the vector, the components of which are partial derivatives of order p, namely, $u_j^{(k_1\dots k_p)}=\frac{\partial^p u_j}{\partial x_{k_1}\dots\partial x_{k_p}}$ $j=1,\dots,n$ and $k_1,\dots,k_p\in\{1,\dots,m\}$.
Denote by $x=(x_1,\dots,x_m)$ the independent variables, $u=(u_1,\dots,u_n)$ the dependent variables, and $(x_{k_1}\dots x_{k_p})$ a set of elements of the independent variables.
Equation (2) is a subset of the Euclidean space $\mathcal{E}\big(x,u,u^{(k_1)},\dots,u^{(k_1\dots k_l)}\big)$. In order to take into account the derivative terms involved in the differential equation, the action of the group G_r of transformations in the space $\mathcal{E}\big(x,u\big)$ needs to be extended to the space of the derivatives of the dependent variables.
Denote by $\widetilde{G}_r^{(l)}$ a r-parameter Lie group of point transformation in the space $\mathcal{E}\big(x,u,u^{(k_1)},\dots,u^{(k_1\dots k_l)}\big)$ of the independent variables, dependent variables and the derivative of the dependent variables with respect to the independent ones.
The l^{th}-prolongation operator of G_r is:

$$\widetilde{\mathbf{L}}_\alpha^{(l)}=\xi_i^\alpha(x,u)\frac{\partial}{\partial x_i}+\eta_j^\alpha(x,u)\frac{\partial}{\partial u_j}+\sigma_j^{\alpha,(k_1)}\frac{\partial}{\partial u_j{}^{(k_1)}}+\dots+\sigma_j^{\alpha,(k_1\dots k_l)}\frac{\partial}{\partial u_j{}^{(k_1\dots k_l)}},\qquad i=1,\dots,m;\ j=1,\dots,n;\ \alpha=1,\dots,r. \tag{3}$$

ξ_i^α, η_j^α, $\sigma_j^{\alpha,(k_1)}$ and $\sigma_j^{\alpha,(k_1\dots k_o)}$ are given by:

$$\xi_i^\alpha=\frac{\partial\phi_i}{\partial a_\alpha}\Big|_{a=0},\quad \eta_j^\alpha=\frac{\partial\varphi_j}{\partial a_\alpha}\Big|_{a=0},\quad \sigma_j^{\alpha,(k_1)}=\frac{\mathcal{D}\eta_j^\alpha}{\mathcal{D}x_{k_1}}-\sum_{i=1}^m\frac{\partial u_j}{\partial x_i}\frac{\mathcal{D}\xi_i^\alpha}{\mathcal{D}x_{k_1}}$$

$$\sigma_j^{\alpha,(k_1\dots k_o)}=\frac{\mathcal{D}\sigma_j^{\alpha,(k_1\dots k_{o-1})}}{\mathcal{D}x_{k_o}}-\sum_{i=1}^m\frac{\partial^o u_j}{\partial x_i\partial x_{k_1}\dots\partial x_{k_{o-1}}}\frac{\mathcal{D}\xi_i^\alpha}{\mathcal{D}x_{k_o}},\quad o=2,\dots,l$$

where: $\dfrac{\mathcal{D}}{\mathcal{D}x_k}=\dfrac{\partial}{\partial x_k}+\sum_{j=1}^n\dfrac{\partial u_j}{\partial x_k}\dfrac{\partial}{\partial u_j}$

The system of l^{th}-order differential equations is invariant under the group $\widetilde{G}_r^{(l)}$ if and only if:

$$\widetilde{\mathbf{L}}_\alpha^{(l)}\mathcal{F}^\lambda\Big|_{\mathcal{F}^\lambda=0}=0,\quad \alpha=1,\dots,r;\ \lambda=1,\dots,q \tag{4}$$

3. **Lie group for the differential approximation.** The finite difference scheme, which approximates the differential system (2), can be written as:

$$\Lambda^{\lambda}(x,u,h,Tu)=0,\quad \lambda=1,\dots,q \tag{5}$$

where $h=(h_1,h_2,\dots,h_m)$ denotes the space step vector, and $T=(T_1,T_2,\dots,T_m)$ the shift-operator along the axis of the independent variables, defined by:

$$T_i[u](x_1,x_2,\dots,x_{i-1},x_i,x_{i+1},\dots,x_m)=u(x_1,x_2,\dots,x_{i-1},x_i+h_i,x_{i+1},\dots,x_m). \tag{6}$$

Definition 1. The differential equation:

$$\begin{aligned}\mathcal{P}^{\lambda}\big(x,u,u^{(k_1)},\dots,u^{(k_1\dots k_{l'})}\big) &= \mathcal{F}^{\lambda}\big(x,u,u^{(k_1)},\dots,u^{(k_1\dots k_l)}\big)\\ &\quad+\sum_{\beta=1}^{s}\sum_{i=1}^{m}(h_i)^{l_\beta}\mathcal{R}_i^{\lambda}(x,u,u^{(k_1)},\dots,u^{(k_1\dots k_{l'\lambda,i})}),\\ &\quad\lambda=1,\dots,q;\ l'=max_{(\lambda,i)}l'_{\lambda,i}\end{aligned} \tag{7}$$

is called the s^{th}-order differential approximation of the finite difference scheme (5). In the specific case $s=1$, the above equation is called the first differential approximation.

Equation (7) is obtained from equation (5) by applying Taylor series expansion to the components of Tu about the point $x=(x_1,\ \dots,\ x_m)$ and truncating the expansion to a given finite order. Denote by G'_r a group of transformations in the space $\mathcal{E}(x,u,h)$:

$$G'_r=\{x_i^*=\phi_i(x,u,a);\ u_j^*=\varphi_j(x,u,a);h_i^*=\psi_i(x,u,h,a),\ i=1,\dots,m;\ j=1,\dots,n\} \tag{8}$$

by $\mathbf{L}_\alpha{}'$ the basis infinitesimal operator of G'_r:

$$\mathbf{L}_\alpha{}'=\xi_i^{\alpha}(x,u)\frac{\partial}{\partial x_i}+\eta_j^{\alpha}(x,u)\frac{\partial}{\partial u_j}+\zeta_i^{\alpha}(x,u,h)\frac{\partial}{\partial h_i},\quad \alpha=1,\dots,r \tag{9}$$

where $\zeta_i^{\alpha}=\frac{\partial\psi_i}{\partial a_\alpha}\Big|_{a=0}$, $\alpha=1,\dots,r$

and by $\widetilde{G}_r^{(l')}$ a group of transformation in the space $\mathcal{E}(x,u,h,u^{(k_1)},\dots,u^{(k_1\dots k_{l'})})$. The l'^{th}-prolongation operator of G'_r, $\widetilde{\mathbf{L}}_\alpha^{(l')}$ can be written as:

$$\widetilde{\mathbf{L}}_\alpha^{(l')}=\mathbf{L}_\alpha{}'+\sum_{j=1}^{n}\sum_{p=1}^{l'}\sigma_j^{\alpha,(k_1\dots k_p)}\frac{\partial}{\partial u_j^{(k_1\dots k_p)}} \tag{10}$$

Theorem 1. *The differential approximation (7) is invariant under the group $\widetilde{G}_r^{(l')}$ if and only if*

$$\widetilde{\mathbf{L}}_\alpha^{(l')}\mathcal{P}^{\lambda}\big((x,u,u^{(k_1)},\dots,u^{(k_1\dots k_{l'})}\big)\Big|_{\mathcal{P}^{\lambda}=0} = 0,\quad \alpha=1,\dots,r;\ \lambda=1,\dots,q \tag{11}$$

$$or\ \Big[\widetilde{\mathbf{L}}_\alpha^{(l)}\mathcal{F}^{\lambda}+\widetilde{\mathbf{L}}_\alpha^{(l')}\Big(\sum_{\beta=1}^{s}\sum_{i=1}^{m}(h_i)^{l_\beta}\mathcal{R}_i^{\lambda}\Big)\Big]\Big|_{\mathcal{P}^{\lambda}=0} = 0,\quad \alpha=1,\dots,r;\ \lambda=1,\dots,q \tag{12}$$

4. **The specific case of the Burgers equation.**

4.1. **Symmetries of the Burgers equation.** The Burgers equation can be written as:

$$\mathcal{F}(x,t,u,\nu,u_x,u_t,u_{xx})=u_t+u\ u_x-\nu\ u_{xx}=0 \tag{13}$$

where $\nu\geq 0$ is the dynamic viscosity.

Let us denote by G a group of transformations of the Burgers equation in the space $\mathcal{E}(x,t,u,\nu)$ of the independent variables (x,t), the dependent variable u, and the viscosity ν. The viscosity is taken as a symmetry variable in order to enable us to take into account variations of the Reynolds number.

G is a set of transformations acting smoothly on the space $\mathcal{E}(x,t,u,\nu)$.

The six-dimensional Lie algebra of the group G is generated by the following operators:

$$\mathbf{L}_1 = \frac{\partial}{\partial x},\ \mathbf{L}_2 = \frac{\partial}{\partial t},\ \mathbf{L}_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}$$
$$\mathbf{L}_4 = xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} + (-ut+x)\frac{\partial}{\partial u},\ \mathbf{L}_5 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u},\ \mathbf{L}_6 = -t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + \nu\frac{\partial}{\partial \nu} \quad (14)$$

which respectively correspond to:

- the space translation : $(x,t,u,\nu) \longmapsto (x+\epsilon_1, t, u, \nu)$;
- the time translation : $(x,t,u,\nu) \longmapsto (x, t+\epsilon_2, u, \nu)$;
- the dilatation : $(x,t,u,\nu) \longmapsto (\epsilon_3 x, \epsilon_3^2 t, \epsilon_3^{-1} u, \nu)$;
- the projective transformation : $(x,t,u,\nu) \longmapsto \Big(\frac{x}{1-\epsilon_4 t}, \frac{t}{1-\epsilon_4 t}, x\epsilon_4 + u(1-\epsilon_4 t), \nu\Big)$;
- the Galilean transformation : $(x,t,u,\nu) \longmapsto (x+\epsilon_5\ t, t, u+\epsilon_5, \nu)$;
- the dilatation : $(x,t,u,\nu) \longmapsto (x, \epsilon_6^{-1} t, \epsilon_6 u, \epsilon_6 \nu)$.

$(\epsilon_i)_{i=1,\dots,6}$ are constants.

4.2. **Symmetries of first differential approximations.** Denote by h the mesh size, τ the time step, N_x the number of mesh points, N_t the number of time steps, and $u_i^n,\ i \in \{0,\dots,N_t\},\ n \in \{0,\dots,N_x\}$ the discrete approximation of $u(ih, n\tau)$. In order to shorten the size of the finite difference scheme expressions, we use the following notations introduced by Hildebrand in [18]:

$$\delta(u_i^n) = \frac{u_{i+\frac{1}{2}}^n - u_{i-\frac{1}{2}}^n}{h}, \qquad \mu(u_i^n) = \frac{u_{i+\frac{1}{2}}^n + u_{i-\frac{1}{2}}^n}{2}$$
$$\delta^+(u_i^n) = \frac{u_{i+1}^n - u_i^n}{h}, \qquad \delta^-(u_i^n) = \frac{u_i^n - u_{i-1}^n}{h}, \qquad E^\alpha u_i^n = u_{i+\alpha}^n$$

The Burgers equation can be discretized by means of:

- **the FTCS (forward-time and centered-space) scheme**:

$$\frac{u_i^{n+1} - u_i^n}{\tau} + \frac{\mu\delta}{h}\big(\frac{u^2}{2}\big)_i^n - \nu\frac{\delta^2}{h^2}u_i^n = 0$$

- **the Lax-Wendroff scheme**:

$$\frac{u_i^{n+1} - u_i^n}{\tau} + \frac{\mu\delta}{h}\big(\frac{u^2}{2}\big)_i^n - \nu\frac{\delta^2}{h^2}u_i^n + A_i^n = 0$$

where:

$$\begin{aligned} A_i^n = \ & - \frac{\tau}{2h^2}\Big[E^{\frac{1}{2}}u_i^n\ \delta^+\big(\frac{u^2}{2}\big)_i^n - E^{-\frac{1}{2}}u_i^n\ \delta^-\big(\frac{u^2}{2}\big)_i^n\Big] - \frac{\nu^2\tau}{2}\Big[\frac{\delta^4}{h^4}u_i^n\Big] \\ & + \frac{\nu\tau}{2h^3}\Big[E^{\frac{1}{2}}u_i^n\ \delta^2(E^{\frac{1}{2}}u_i^n) - E^{-\frac{1}{2}}u_i^n\ \delta^2(E^{-\frac{1}{2}}u_i^n)\Big] + \frac{\nu\tau}{2}\Big[\frac{\mu\delta^3}{h^3}\big(\frac{u^2}{2}\big)_i^n\Big] \end{aligned}$$

- **the Crank-Nicolson scheme**:

$$\frac{u_i^{n+1} - u_i^n}{\tau} + \frac{\mu\delta}{h}\Big[\big(\frac{u^2}{2}\big)_i^{n+1} + \big(\frac{u^2}{2}\big)_i^n\Big] - \nu\frac{\delta^2}{h^2}[u_i^{n+1} + u_i^n] = 0$$

The linear stability properties and the related orders of approximation are:

- **the FTCS scheme**: $S \leq \frac{1}{2}$, $CFL \leq 1$; $\mathcal{O}(\tau, h^2)$
- **the Lax-Wendroff scheme**: $S^* \leq \frac{1}{2}$, $CFL \leq 1$; $\mathcal{O}(\tau^2, h^2)$
- **the Crank-Nicolson scheme**: unconditional stability; $\mathcal{O}(\tau^2, h^2)$

where $CFL = \frac{a\tau}{h}$, $S = \frac{\nu\tau}{h^2}$ and $S^* = \big(\nu + \frac{ahCFL}{2}\big)\frac{\tau}{h^2}$.
Consider ${u_i}^n$ as a function of the time step τ, and of the mesh size h, expand it at a given order by means of its Taylor series, and neglect the $o(\tau^\alpha)$ and $o(h^\beta)$ terms, where α and β depend on the order of the schemes. This yields the differential representation of the finite difference equation.
The following differential representations are obtained:

- **for the FTCS scheme:**

$$u_t + \frac{1}{2}(u^2)_x - \nu\ u_{xx} + \frac{\tau}{2}g_2 + \frac{h^2}{12}(u^2)_{xxx} - \frac{\nu h^2}{12}u_{xxxx} = 0$$

- **for the Lax-Wendroff scheme:**

$$u_t + \frac{1}{2}(u^2)_x - \nu\ u_{xx} + \frac{\tau^2}{6}g_3 + \frac{h^2}{12}(u^2)_{xxx} - \frac{\nu h^2}{12}u_{xxxx} = 0$$

- **for the Crank-Nicolson scheme:**

$$u_t + \frac{1}{2}(u^2)_x - \nu u_{xx} + \tau^2\big(\frac{g_3}{6} + \frac{1}{4}(g_1^2 + ug_2)_x - \frac{\nu}{4}(g_2)_{xx}\big) + \frac{h^2}{12}(u^2)_{xxx} - \frac{\nu h^2}{12}u_{xxxx} = 0$$

where $g_1 = -\big(\frac{u^2}{2}\big)_x + \nu u_{xx}$, $g_2 = \big(-g_1 u\big)_x + \nu\big(g_1\big)_{xx}$, $g_3 = \big(-g_2 u - g_1^2\big)_x + \nu\big(g_2\big)_{xx}$

Denote by G' the group of transformations of a first differential approximation in the space $\mathcal{E}(x,t,u,h,\tau,\nu)$ of the independent variables (x,t) and the dependent variable u, the step size variables (h,τ) and the viscosity ν.
The l'^{th}-prolongation of G' can be written as:

$$\widetilde{\mathbf{L}}_\alpha^{\prime(l')} = \xi_1^\alpha\frac{\partial}{\partial x} + \xi_2^\alpha\frac{\partial}{\partial t} + \eta^\alpha\frac{\partial}{\partial u} + \sum_{p=1}^{l'}\sigma_j^{\alpha,(k_1\ldots k_p)}\frac{\partial}{\partial u_j^{(k_1\ldots k_p)}} + \zeta_1^\alpha\frac{\partial}{\partial h} + \zeta_2^\alpha\frac{\partial}{\partial \tau} + \theta^\alpha\frac{\partial}{\partial \nu} \quad (15)$$

where l' has been defined in **definition 1**.
Theorem 1 enables us to obtain the necessary and sufficient condition of invariance of the first differential approximation $\mathcal{P}$:

$$\widetilde{\mathbf{L}}_\alpha^{\prime(l')}\mathcal{P}\Big|_{\mathcal{P}=0} = 0 \quad (16)$$

Theorem 1 is applied to the differential representations of the above schemes. The resolution of the determining equations of each first differential approximation yields the 4-parameter group (see [19]):

$$\begin{array}{lll} \xi_1^\alpha = a + b\ x, & \xi_2^\alpha = c + (2b-d)\ t, & \eta^\alpha = (-b+d)\ u \\ \zeta_1^\alpha = b\ h, & \zeta_2^\alpha = (2b-d)\ \tau, & \theta^\alpha = e\nu \end{array} \quad (17)$$

The 4-dimensional Lie algebra of G' is generated by:

$$\begin{array}{l} \mathbf{L}_1 = \dfrac{\partial}{\partial x},\ \ \mathbf{L}_2 = \dfrac{\partial}{\partial t},\ \ \mathbf{L}'_3 = x\dfrac{\partial}{\partial x} + 2t\dfrac{\partial}{\partial t} - u\dfrac{\partial}{\partial u} + h\dfrac{\partial}{\partial h} + 2\tau\dfrac{\partial}{\partial \tau} \\ \mathbf{L}'_4 = -t\dfrac{\partial}{\partial t} + u\dfrac{\partial}{\partial u} - \tau\dfrac{\partial}{\partial \tau} + \nu\dfrac{\partial}{\partial \nu} \end{array} \quad (18)$$

These operators are respectively related to:

- the space translation : $(x,t,u,h,\tau,\nu) \longmapsto (x+\epsilon_1,t,u,h,\tau,\nu)$;
- the time translation : $(x,t,u,h,\tau,\nu) \longmapsto (x,t+\epsilon_2,u,h,\tau,\nu)$;
- the dilatation : $(x,t,u,h,\tau,\nu) \longmapsto (\epsilon_3 x,\epsilon_3^2 t,\epsilon_3^{-1}u,\epsilon_3 h,\epsilon_3^2\tau,\nu)$;
- the dilatation : $(x,t,u,h,\tau,\nu) \longmapsto (x,\epsilon_4^{-1}t,\epsilon_4 u,h,\epsilon_4^{-1}\tau,\epsilon_4\nu)$;

where $(\epsilon_i)_{i=1,\ldots,4}$ are constants.
The above finite difference equations are preserved by the space translation, the time translation and both dilatations.
Approximating the Burgers equation by the above finite difference equations results in the loss of the projective and Galilean transformations.

5. **The semi-invariant scheme.**

5.1. **Semi-invariant scheme construction.** The direct method, introduced by Dorodnitsyn in [9], provides an exact symmetry analysis of finite difference schemes and leads to the definition of meshes whose geometrical structure is preserved by the entire group. The method proposed by Yanenko [2] and Shokin [1] consists of a symmetry study of the differential approximation. Although the last method is not fully exact, the numerical results in [1] and [6] has proved its effectiveness. This approach gives a recevable symmetry analysis of finite difference schemes.
The scheme proposed below is associated to an uniform orthogonal mesh.
We propose to approximate the Burgers equation by the finite difference scheme:

$$\frac{u_i^{n+1}-u_i^n}{\tau}+\frac{1}{h}(\mu\delta-\frac{\mu\delta^3}{6})(\frac{u^2}{2})_i^n-\nu\frac{1}{h^2}(\delta^2-\frac{\delta^4}{12})(u_i^n)-h\Big(\Omega^n_{i+\frac{1}{2}}\delta^+-\Omega^n_{i-\frac{1}{2}}\delta^-\Big)u_i^n=0 \quad (19)$$

where $\Omega_i^n=\Omega(x_i,t_n,u_i^n)$ is defined next so that the related differential representation is preserved by the symmetries of the Burgers equation. The scheme has second-order accuracy in space and first-order accuracy in time. The derivatives $(u^2)_x$ and u_{xx} are approximated by fourth order accuracy difference expressions:

$$(\frac{\mu\delta}{h}-\frac{\mu\delta^3}{6h})(u_i^n)=\big(u_x-\frac{h^4}{30}u_{5x}\big)_i^n+\mathcal{O}(h^6),\ (\frac{\delta^2}{h^2}-\frac{\delta^4}{12h^2})(u_i^n)=\big(u_{xx}-\frac{h^4}{90}u_{6x}\big)_i^n+\mathcal{O}(h^6) \quad (20)$$

The truncation error of the difference scheme (19) can be written as:

$$\epsilon \;=\; \frac{\tau}{2}u_{tt}-h^2\Big(\Omega u_x\Big)_x+\mathcal{O}(\tau^2)+\mathcal{O}(h^4)$$

u_{tt} is replaced by an expression involving partial derivatives with respect to x, by using the Burgers equation. Replacing the obtained expression in the truncation error leads to:

$$\epsilon \;=\; \Big(Cu_x\Big)_x-\frac{\nu\tau}{2}\Big(uu_{xx}\Big)_x-\frac{\nu\tau}{2}(\frac{u^2}{2})_{xxx}+\frac{\nu^2\tau}{2}u_{xxxx}+\mathcal{O}(\tau^2)+\mathcal{O}(h^4)$$

where $C=\frac{\tau}{2}u^2-h^2\Omega$.
It is convenient for the calculation of C that the truncation error is reduced to:

$$\epsilon \;=\; \Big(Cu_x\Big)_x+\mathcal{O}(\tau^2)+\mathcal{O}(h^4)$$

The related finite difference scheme is the following first order accuracy in time and second order accuracy in space:

$$\begin{aligned}\frac{u_i^{n+1}-u_i^n}{\tau}+\frac{1}{h}(\mu\delta-\frac{\mu\delta^3}{6})(\frac{u^2}{2})_i^n-\nu\frac{1}{h^2}(\delta^2-\frac{\delta^4}{12})(u_i^n)-h\Big(\Omega^n_{i+\frac{1}{2}}\delta^+-\Omega^n_{i-\frac{1}{2}}\delta^-\Big)u_i^n\\ +\frac{\nu\tau}{2}\Big(u^n_{i+\frac{1}{2}}\frac{\mu\delta^2}{h^2}(u^n_{i+\frac{1}{2}})-u^n_{i-\frac{1}{2}}\frac{\mu\delta^2}{h^2}(u^n_{i-\frac{1}{2}})\Big)-\frac{\nu^2\tau}{2}\frac{\delta^4}{h^4}u_i^n+\frac{\nu\tau}{2}\frac{\mu\delta^3}{h^3}(\frac{u^2}{2})_i^n=0\end{aligned} \quad (21)$$

and the differential approximation can be written as:

$$\mathcal{P}(x,t,u,\nu,u_x,u_t,u_{xx})=u_t+u\ u_x-\nu\ u_{xx}+(Cu_x)_x=0 \quad (22)$$

The von Neumann stability analysis of scheme (21) under a linearized form provides the following necessary conditions for S, CFL and $\Omega_\tau=\Omega\tau$:

$$CFL^2-2S-2\Omega_\tau\le 0,\quad 0\le(4S)/3-2S^2+\Omega_\tau\le 1/2 \quad (23)$$

If Ω is sufficiently close to zero, these conditions become then sufficient for the linear formulation.

5.2. **Calculation of the artificial viscosity term.** We now describe the method for determining the artificial viscosity term $(Cu_x)_x$, which is constructed in such a way that the differential approximation (22) is preserved by the symmetries of the Burgers equation. It is worth noting that the related semi-invariant scheme (21) is not nessarily an invariant manifold with respect to the infinitesimal operator prolonged to the difference variables. That is, the semi-invariant scheme (21) expressed with respect to the transformed difference variables may be not valid and may produce additional terms, which will depend on group parameters. We are not aiming at recovering full invariance for the semi-invariant scheme. We are interested

in the invariance of equation (22) under all the symmetries. C is a function of the variables (x,t,u,τ,h), and also depends on the partial derivatives of u with respect to x: u_x and u_{xx}. $C = C(x,t,h,\tau,u,u_x,u_{xx})$. The necessary and sufficient condition for the differential approximation to be an invariant of the Burgers equation symmetry group is:

$$\widetilde{\mathbf{L}}_\alpha^{(2)}(u_t + u\ u_x - \nu\ u_{xx})\Big|_{\mathcal{P}=0} + \widetilde{\mathbf{L}}_\alpha^{(3)}((Cu_x)_x)\Big|_{\mathcal{P}=0} = 0 \tag{24}$$

Equation (24) involves partial derivatives of the unknown C. The infinitesimal functions of G', which are coordinates of the vector field $\mathbf{L}_\alpha$ are given by:

$$\begin{array}{lll} \xi_1^\alpha = a + b\ x + c\ t + d\ tx, & \xi_2^\alpha = e + d\ t^2 + (2b - f)\ t, & \zeta_1^\alpha = b\ h, \\ \zeta_2^\alpha = (2b - f)\ \tau, & \eta^\alpha = c + d\ x + (-b - d\ t + f)\ u, & \theta^\alpha = f\nu \end{array} \tag{25}$$

The determination of C is realized for each subgroup of G':

- the space translation $C = C_1(t,h,\tau,u,u_x,u_{xx})$;
- the time translation $C = C_2(x,h,\tau,u,u_x,u_{xx})$;
- the dilatation $C = C_3(\frac{t}{x^2}, \frac{h}{x}, ux, \frac{\tau}{x^2}, u_x, u_{xx})$;
- the projective transformation $C = C_4(h,\tau,\frac{2+tu_x}{t})$;
- the Galilean transformation $C = C_5(\frac{ut-x}{t}, t, h, \tau, u, u_x, u_{xx})$;
- the dilatation $C = \frac{1}{t}C_6(x,h,\frac{\tau}{t})$.

5.3. **Numerical application.** The numerical resolution of the Burgers equation has been implemented for scheme (21), the standard schemes (cf. section 4.2) and a scheme with second-order accuracy in time and fourth-order accuracy in space, which is obtained from the semi-invariant scheme when $C = 0$. The solutions are calculated in the reference frame $(F1)$, where the mesh is uniform and orthogonal, and in the frame $(F2)$ resulting from the Galilean transformations $(x,t,u,\nu) \longmapsto (x+t,t,u+1,\nu)$, where the orthogonality of the mesh is broken.
The artificial viscosity has the following expression:

$$C = -0.01t(tu - x)^2(u_x)^2 \tag{26}$$

$\Omega = \frac{1}{h^2}(\frac{\tau}{2}u^2 - C)$ is in a sufficiently small neighborhood of zero, which enables us to have the sufficiency of conditions (23) for the linear formulation.
The problem consists in solving the following differential system:

$$u_t + uu_x - \nu u_{xx} = 0,\ x \in [0,40],\ t \in [0,20],\ u(x,0) = f(x),\ u(0,t) = g(t),\ u(40,t) = h(t)$$

The initial and boundary conditions, f, h, and g are provided by an exact solution of the Burgers equation:

$$u(x,t) = (((x-2t)/(t+0.1))/(1 + \nu^2\sqrt{t+0.1}\exp((x-2t)^2/(4\nu(t+0.1))))) + 2 \tag{27}$$

Figures 1, 3 and 5 show the time evolution of the L^2-norm of the error for the considered schemes, for specific values of the CFL number and the mesh Reynolds number Re_h. Figures 2, 4 and 7 display the variations, as functions of the space variable, of the numerical solutions of the considered schemes for the specific value $t = 5$. In each frame, the numerical solutions are compared to the exact one.
The error analysis of the semi-invariant scheme in the reference frame through the features of the truncation error and the graphical representation of the norms of the error (cf. Figures 1, 3 and 5) allows to say that the semi-invariant scheme is dissipative and slightly dispersive.
The presence of the dissipative term $(Cu_x)_x$ in the differential representation of the semi-invariant scheme and the presence of the higher order error terms involving the even-order derivative u_{6x} (cf. Equation (20)) show that the scheme produces numerical damping. Particularly, the amplitudes are not correctly represented for high frequencies, since the solution is subjected to rather rough variation during the

first iterations. The dissipation is stronger for $Re_h = 2$, $CFL = 0.08$ in the reference frame (see Figure 4). Moreover, the presence of higher order error terms involving the odd-order derivative u_{5x} corresponds to a phase error. The non-invariant schemes are more altered by the change of the frame than the semi-invariant one. Moreover, the semi-invariant scheme appears to be as accurate as the higher order one in the frame (F2).

6. **Conclusion.** If, on the one hand, the differential approximation is invariant with respect to the prolonged infinitesimal operator of the group of the continuous equations, the related *semi-invariant* scheme is not, on the other hand, an invariant manifold with respect to the group isomorphic to the group of the continuous equations. Moreover, this local analysis can not enable one to build a mesh invariant under the group of the original equations. However, our method enables us to have an idea of the group properties of the considered finite difference schemes. Also the *semi-invariant* scheme gives better results in the transformed frame.
In the next future, we will focus on direct methods of group analysis of finite difference schemes. Especially, we will concentrate on the approach developped by Dorodnitsyn [9]. This non-local analysis is complex, in so far as invariance conditions are stronger. This approach will be retained as long as its application to model equations of aeronautics engineering is tractable.

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REFERENCES

[1] Yu. I. Shokin (1983), *The method of differential approximation*, Springer-Verlag, Berlin, Heidelberg, New-York, Tokyo.
[2] N. N. Yanenko and Yu. I. Shokin (1976), *Group classification of difference schemes for a system of one-dimensional equations of gas dynamics*, Amer. Math. Soc. Transl., **2** no. 104, 259–265.
[3] P. J. Olver (1986), *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New-York.
[4] N. H. Ibragimov (1994), *Symmetries exact solutions and conservation laws*, CRC Handbook of Lie Group Analysis of Differential Equations, vol. 1.
[5] B. J. Cantwell (2002), *Introduction to symmetry analysis*, Cambridge University Press.
[6] W. F. Ames, F. V. Postell and E. Adams (1992), *Optimal numerical algorithms*, Applied Numerical Mathematics, **10**, 235–259.
[7] P. J. Olver (2001), *Geometric foundations of numerical algorithms and symmetry*, Appl. Alg. Engin. Comp. Commun, **11**, 417–436.
[8] P. Kim (2004), *Invariantization of numerical schemes using moving frames*, Mathematical Physics Seminar.
[9] V. A. Dorodnitsyn (1991), *Transformation groups in the net spaces*, J. Sov. Math., **55**, 1490–1517.
[10] V. A. Dorodnitsyn (1994), *Finite difference models entirely inheriting continuous symmetry of original differential equations*, Int. J. Mod. Phys., serie C, **5**, no. 4, 723–734.
[11] V. A. Dorodnitsyn and R. Kozlov (2003), *A heat transfert with a source: the complete set of invariant difference schemes*, J. Non. Math. Phys., **10**, no. 1, 16–50.
[12] M. I. Bakirova, V. A. Dorodnitsyn and R. Kozlov(1997), *Symmetry-preserving difference schemes for some heat transfert equations*, J. Phys. A: Math. Gen., **30**, 8139–8155.
[13] F. Valiquette and P. Winternitz (2005), *Discretization of partial differential equations preserving their physical symmetries*, J. Phys. A: Math. Gen., **38**, 9765–9783.
[14] K. Vu and J. Carminati (2000), *Symbolic computation and differential equations: Lie symmetries*, J. Symbolic Computation, **29**, 95-116.
[15] S. Herod (1992), *MathSym: a Mathematica program for computing Lie symmetries*, Preprint, Program in Applied Mathematics, Boulder, Colorado, The University of Colorado.

[16] G. Baumann (1992), *Lie symmetries of differential equations: A mathematica program to determine Lie symmetries*, Wolfram Research Inc., Champaign, Illinois, MathSource 0202-622.
[17] F. Schwarz (1982), *A REDUCE package for determining Lie symmetries of ordinary and partial differential equations*, Comput. Phys. Commun., **27**, 179–186.
[18] F. B. Hildebrand (1956), *Introduction to Numerical Analysis*, New York: McGraw-Hill.
[19] E. Hoarau and C. David, *Lie group computation of finite difference schemes*, Dynamics of Continuous, Discrete and Impulsive Systems (DCDIS), Series A. *To appear in 2007*.

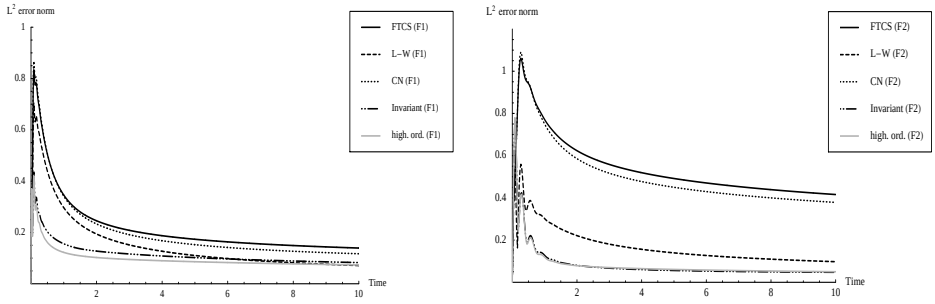

FIGURE 1. Evolution of the error L^2-norm in F1, F2. $Re_h = 2$, $CFL = 0.04$

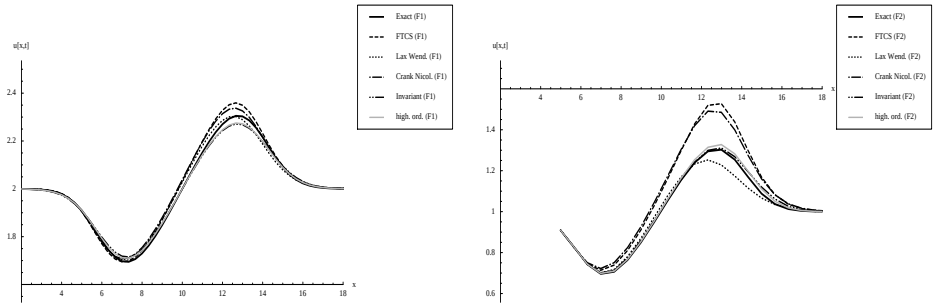

FIGURE 2. Space variation of the numerical and exact solutions in F1, F2. $Re_h = 2$, $CFL = 0.04$

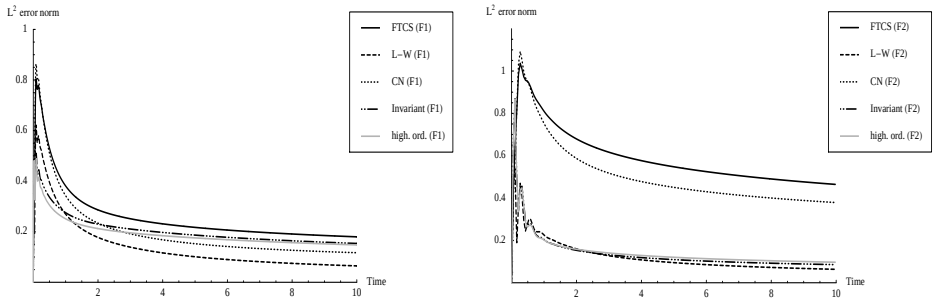

FIGURE 3. Evolution of the error L^2-norm in F1, F2. $Re_h = 2$, $CFL = 0.08$

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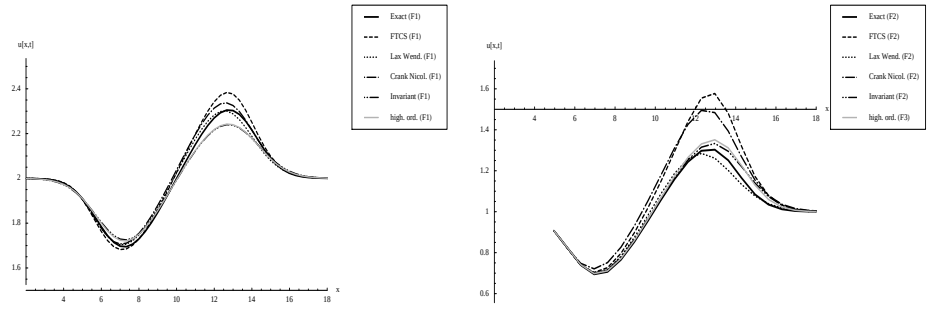

FIGURE 4. Space variation of the numerical and exact solutions in F1, F2. $Re_h = 2$, $CFL = 0.08$

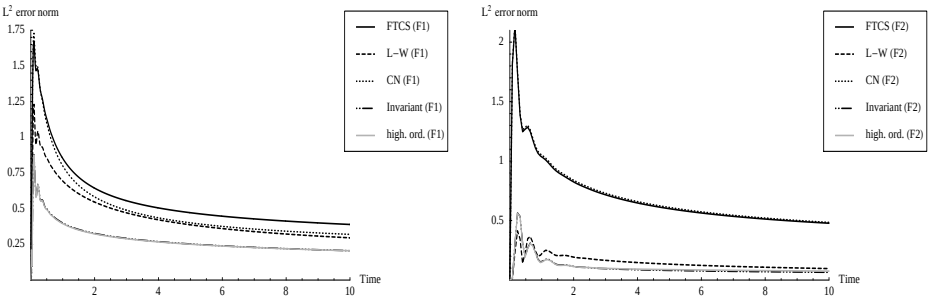

FIGURE 5. Evolution of the error L^2-norm in F1, F2. $Re_h = 3$, $CFL = 0.08$

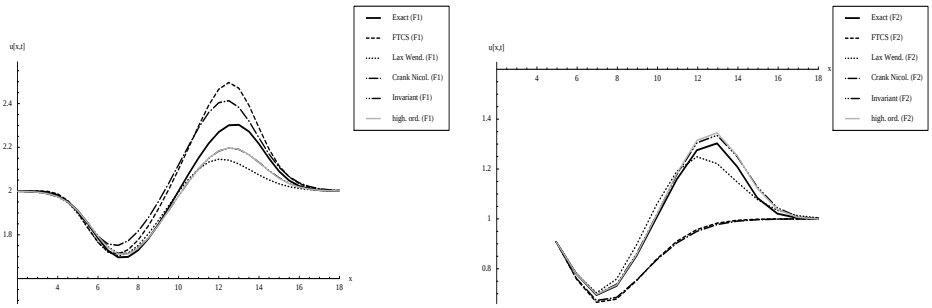

FIGURE 6. Space variation of the numerical and exact solutions in F1, F2. $Re_h = 3$, $CFL = 0.08$

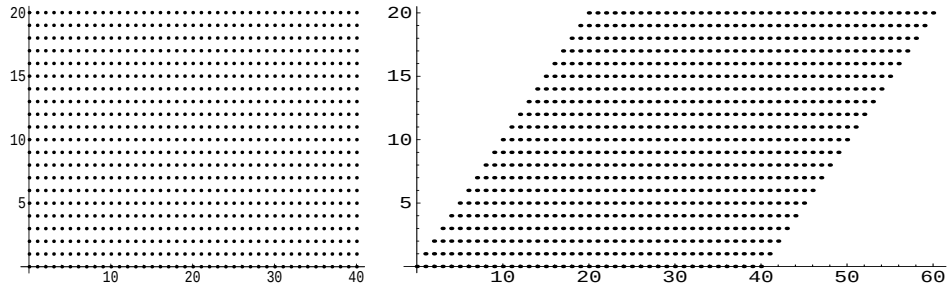

FIGURE 7. Original (F1) and transformed (F2) meshes for large step size variables $h = 1$, $\tau = 1$