Spurious solitons and structural stability of finite difference schemes for nonlinear wave equations

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Abstract

The goal of this work is to determine classes of traveling solitary wave solutions for a differential approximation of a finite difference scheme by means of a hyperbolic ansatz. It is shown that spurious solitary waves can occur in finite-difference solutions of nonlinear wave equation. The occurance of such a spurious solitary wave, which exhibits a very long life time, results in a non-vanishing numerical error for arbitrary time in unbounded numerical domain. Such a behavior is referred here to has a structural instability of the scheme, since the space of solutions spanned by the numerical scheme encompasses types of solutions (solitary waves in the present case) that are not solution of the original continuous equations.

1 Introduction

The Burgers equation:

$$u_t + c\,u\,u_x - \mu\,u_{xx} = 0, \tag{1}$$

c, μ being real constants, plays a crucial role in the history of wave equations. It was named after its use by Burgers [1] for studying turbulence in 1939.

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A finite difference scheme for the Burgers equation can be written under the following general form:

$$F({u_l}^m, h, \tau) = 0, \tag{2}$$

where the discrete solution is denoted

$${u_l}^m = u\,(l\,dx, m\,dt) \tag{3}$$

$l \in \{i-1,\, i,\, i+1\}$, $m \in \{n-1,\, n,\, n+1\}$, $j = 1, \ldots, n_x$, $n = 1, \ldots, n_t$, h, τ denoting respectively the mesh size and time step, and σ the Courant-Friedrichs-Lewy number (cfl) coefficient, defined as $\sigma = c\,\tau/h$.

A numerical scheme is specified by selecting appropriate expression of the function F in equation (2).

Considering the ${u_l}^m$ terms as functions of the mesh size h and time step τ, expanding them at a given order by means of their Taylor series expansion, and neglecting the $o(\tau^p)$ and $o(h^q)$ terms, for given values of the integers p, q, leads to a differential approximation of the Burgers equation:

$$\mathcal{F}(u, \frac{\partial^r u}{\partial x^r}, \frac{\partial^s u}{\partial t^s}, h, \tau) = 0, \tag{4}$$

r, s being integers.

For sake of simplicity, a non-dimensional form of Eq. (4) will be used:

$$\widetilde{\mathcal{F}}(\tilde{u}, \frac{\partial^r \tilde{u}}{\partial \tilde{x}^r}, \frac{\partial^s \tilde{u}}{\partial \tilde{t}^s}) = 0, \tag{5}$$

Depending on this differential approximation (4), solutions, as solitary waves, may arise.

The paper is organized as follows. Two specific schemes are exhibited in section 2. The general method is exposed in Section 3. In Section 4, it is shown that out of the two studied schemes, only one leads to solitary waves. A related class of traveling wave solutions of equation (4) is thus presented, by using a hyperbolic ansatz. The stability of this class of solutions is discussed in the same section.

2 Analysis of some usual finite-difference schemes

2.1 Finite-difference second-order centered scheme in space, Euler-time scheme

For the finite second-order accurate centered scheme in space and Euler-time scheme, the function F in Eq. (2) takes the form:

$$F({u_l}^m, h, \tau) = \frac{{u_i}^{n+1} - {u_i}^n}{\tau} + c\, {u_i}^n\, \frac{{u_{i+1}}^n - {u_{i-1}}^n}{2\,h} - \mu\, \frac{{u_{i+1}}^n - 2\,{u_i}^n + {u_{i-1}}^n}{h^2} = 0 \quad (6)$$

Consider ${u_i}^{n+1}$ as a function of the time step τ, and expand it at the second order by means of its Taylor series:

$${u_i}^{n+1} = u\,(i\,h, (n+1)\,\tau) = u\,(i\,h, n\,\tau) + \tau\, u_t\,(i\,h, n\,\tau) + \frac{\tau^2}{2}\, u_{tt}\,(i\,h, n\,\tau) + o(\tau^2) \quad (7)$$

It ensures:

$$\frac{{u_i}^{n+1} - {u_i}^n}{\tau} = u_t\,(i\,h, n\,\tau) + \frac{\tau}{2}\, u_{tt}\,(i\,h, n\,\tau) + o(\tau) \quad (8)$$

In the same way, consider ${u_{i+1}}^n$ and ${u_{i-1}}^n$ as functions of the mesh size h, and expand them at the fourth order by means of their Taylor series expansion:

$$\begin{aligned} {u_{i+1}}^n &= u\,((i+1)\,h, n\,\tau) \\ &= u\,(i\,h, n\,\tau) + h\, u_x\,(i\,h, n\,\tau) + \frac{h^2}{2}\, u_{xx}\,(i\,h, n\,\tau) + \frac{h^3}{3!}\, u_{xxx}\,(i\,h, n\,\tau) + \frac{h^4}{4!}\, u_{xxxx}\,(i\,h, n\,\tau) + o(h^4) \end{aligned} \quad (9)$$

$$\begin{aligned} {u_{i-1}}^n &= u\,((i-1)\,h, n\,\tau) \\ &= u\,(i\,h, n\,\tau) - h\, u_x\,(i\,h, n\,\tau) + \frac{h^2}{2}\, u_{xx}\,(i\,h, n\,\tau) - \frac{h^3}{3!}\, u_{xxx}\,(i\,h, n\,\tau) + \frac{h^4}{4!}\, u_{xxxx}\,(i\,h, n\,\tau) + o(h^4) \end{aligned} \quad (10)$$

It ensures:

$$\frac{{u_{i+1}}^n - 2\,{u_i}^n + {u_{i-1}}^n}{h^2} = u_{xx}\,(i\,h, n\,\tau) + \frac{2\,h^2}{4!}\, u_{xxxx}\,(i\,h, n\,\tau) + o(h^2) \quad (11)$$

and:

$$\frac{{u_{i+1}}^n - {u_{i-1}}^n}{2\,h} = u_x\,(i\,h, n\,\tau) + \frac{h^2}{3!}\, u_{xxx}\,(i\,h, n\,\tau) + o(h^3) \quad (12)$$

Equation (4) can thus be written as:

$$\begin{aligned} & u_t\,(i\,h,n\,\tau)+\frac{\tau}{2}\,u_{tt}\,(i\,h,n\,\tau)+o(\tau) \\ & +c\,u\,(i\,h,n\,\tau)\left[u_x\,(i\,h,n\,\tau)+\frac{h^2}{3!}\,u_{xxx}\,(i\,h,n\,\tau)+o(h^3)\right] \\ & -\mu\left[u_{xx}\,(i\,h,n\,\tau)+\frac{2\,h^2}{4}\,u_{xxxx}\,(i\,h,n\,\tau)+o(h^2)\right]=0 \end{aligned} \tag{13}$$

i. e., at $x = i\,h$ and $t = n\,\tau$:

$$\left[u_t+\frac{\tau}{2}\,u_{tt}+o(\tau)+c\,u\left[u_x+\frac{h^2}{3!}\,u_{xxx}+o(h^3)\right]-\mu\left[u_{xx}+\frac{2\,h^2}{4!}\,u_{xxxx}+o(h^2)\right]\right]_{(x,t)}=0 \tag{14}$$

The first differential approximation of the Burgers equation (1) is thus obtained neglecting the $o(\tau)$ and $o(h^2)$ terms, yielding:

$$\left[u_t+\frac{\tau}{2}\,u_{tt}+c\,u\left[u_x+\frac{h^2}{3!}\,u_{xxx}\right]-\mu\left[u_{xx}+\frac{h^2}{12}\,u_{xxxx}\right]\right]_{(x,t)}=0 \tag{15}$$

that we will keep as:

$$u_t+c\,u\,u_x-\mu\,u_{xx}+\frac{\tau}{2}\,u_{tt}+\frac{h^2}{6}\,u\,u_{xxx}-\mu\,\frac{h^2}{12}\,u_{xxxx}=0 \tag{16}$$

For sake of simplicity, this latter equation can be adimensionalized through in the following way:
set:

$$\left\{\begin{array}{rcl} u & = & U_0\,\tilde{u} \\ t & = & \tau_0\,\tilde{t} \\ x & = & h_0\,\tilde{x} \end{array}\right. \tag{17}$$

where:

$$U_0=\frac{h_0}{\tau_0} \tag{18}$$

In the following, Re_h will denotes the mesh Reynolds number, defined as:

$$Re_h=\frac{U_0\,h}{\mu} \tag{19}$$

The change of variables (17) leads to:

$$\left\{\begin{array}{rcl} u_t & = & \frac{U_0}{\tau_0}\,\tilde{u}_{\tilde{t}} \\ u_{x^k} & = & \frac{U_0}{h_0^k}\,\tilde{u}_{\tilde{x}^k} \end{array}\right. \tag{20}$$

Multiplying (16) by $\frac{\tau_0}{U_0}$ yields:

$$\tilde{u}_{\tilde{t}} + c\,\frac{U_0\,\tau_0}{h_0}\,\tilde{u}\,\tilde{u}_{\tilde{x}} - \mu\,\frac{\tau_0}{h_0^2}\,\tilde{u}_{\tilde{x}\tilde{x}} + \frac{\tau}{2\,\tau_0}\,\tilde{u}_{\tilde{t}\tilde{t}} + \frac{h^2\,U_0\tau_0}{6\,h_0^3}\,\tilde{u}\,\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} - \mu\,\frac{h^2\,\tau_0}{12\,h_0^4}\,\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} = 0 \tag{21}$$

Relations (18) and (19) ensure:

$$\tilde{u}_{\tilde{t}} + c\,\tilde{u}\,\tilde{u}_{\tilde{x}} - \frac{h}{h_0\,Re_h}\,\tilde{u}_{\tilde{x}\tilde{x}} + \frac{\tau}{2\,\tau_0}\,\tilde{u}_{\tilde{t}\tilde{t}} + \frac{h^2}{6\,h_0^2}\,\tilde{u}\,\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} - \frac{h^3}{12\,Re_h\,h_0^3}\,\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} = 0 \tag{22}$$

For $h = h_0$, due to $\sigma = \frac{U_0\,\tau}{h}$, Eq. (23) becomes:

$$\tilde{u}_{\tilde{t}} + c\,\tilde{u}\,\tilde{u}_{\tilde{x}} - \frac{1}{Re_h}\,\tilde{u}_{\tilde{x}\tilde{x}} + \sigma\,\tilde{u}_{\tilde{t}\tilde{t}} + \frac{1}{6}\,\tilde{u}\,\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} - \frac{1}{12\,Re_h}\,\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} = 0 \tag{23}$$

2.2 The Lax-Wendroff scheme

For the Lax-Wendroff scheme, the function F of (2) takes the form:

$$F({u_l}^m, h, \tau) = \frac{{u_i}^{n+1} - {u_i}^n}{\tau} + c\,{u_i}^n\,\left\{\frac{{u_{i+1}}^n - {u_{i-1}}^n}{2\,h}\right\} - (\mu + \frac{c^2\,\tau}{2})\,\left\{\frac{{u_{i+1}}^n - 2\,{u_i}^n + {u_{i-1}}^n}{h^2}\right\} = 0 \tag{24}$$

$\frac{{u_i}^{n+1} - {u_i}^n}{\tau}$ is expressed by means of (8), and $\frac{{u_{i+1}}^n - 2\,{u_i}^n + {u_{i-1}}^n}{h^2}$ by means of (11), leading to:

$$\frac{{u_{i+1}}^n - 2\,{u_i}^n + {u_{i-1}}^n}{h^2} = u_{xx}\,(i\,h, n\,\tau) + \frac{2\,h^2}{4!}\,u_{xxxx}\,(i\,h, n\,\tau) + o(h^2) \tag{25}$$

Equation (11) also yields:

$$\frac{{u_{i+1}}^n - {u_{i-1}}^n}{2\,h} = u_x\,(i\,h, n\,\tau) + \frac{h^2}{3!}\,u_{xxx}\,(i\,h, n\,\tau) + o(h^3) \tag{26}$$

Equation (24) can thus be written as:

$$\begin{aligned} & u_t\,(i\,h, n\,\tau) + \frac{\tau}{2}\,u_{tt}\,(i\,h, n\,\tau) + o(\tau) \\ & + \alpha\,u\,(i\,h, n\,\tau)\left[u_x\,(i\,h, n\,\tau) + \frac{h^2}{3!}\,u_{xxx}\,(i\,h, n\,\tau) + o(h^3)\right] \\ & - (\mu + \frac{c^2\,\tau}{2})\left[u_{xx}\,(i\,h, n\,\tau) + \frac{2\,h^2}{4!}\,u_{xxxx}\,(i\,h, n\,\tau) + o(h^2)\right] = 0 \end{aligned} \tag{27}$$

i. e., at $x = i\,h$ and $t = n\,\tau$:

$$\Big[\, u_t + \frac{\tau}{2}\, u_{tt} + o(\tau) + c\, u \,\Big[u_x + \frac{h^2}{3!}\, u_{xxx} + o(h^3) \Big] - (\mu + \frac{c^2\,\tau}{2})\,\Big[u_{xx} + \frac{2\,h^2}{4!}\, u_{xxxx} + o(h^2) \Big]\,\Big]_{(x,t)} = 0 \qquad (28)$$

The first differential approximation of the Burgers equation (1) is thus obtained neglecting the $o(\tau)$ and $o(h^2)$ terms:

$$\Big[\, u_t + \frac{\tau}{2}\, u_{tt} + c\, u \,\Big[u_x + \frac{h^2}{3!}\, u_{xxx} \Big] - (\mu + \frac{c^2\,\tau}{2})\,\Big[u_{xx} + \frac{h^2}{12}\, u_{xxxx} \Big]\,\Big]_{(x,t)} = 0 \qquad (29)$$

that we will keep as:

$$u_t + c\, u\, u_x - (\mu + \frac{c^2\,\tau}{2\,h^2})\, u_{xx} + \frac{\tau}{2}\, u_{tt} + \frac{h^2}{6}\, u\, u_{xxx} - (\mu + \frac{c^2\,\tau}{2})\,\frac{h^2}{12}\, u_{xxxx} = 0 \qquad (30)$$

Equation (30) is adimensionalized as in section 2.1, leading to:

$$\tilde{u}_{\tilde{t}} + c\, \tilde{u}\, \tilde{u}_x - (\frac{1}{Re_h} + \frac{c^2\,\sigma}{2})\, \tilde{u}_{\tilde{x}\tilde{x}} + \frac{\tau}{2}\, \tilde{u}_{\tilde{t}\tilde{t}} + \frac{1}{6}\, \tilde{u}\, \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} - (\frac{1}{Re_h} + \frac{c^2\,\sigma}{2})\,\frac{1}{12}\, \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} = 0 \qquad (31)$$

3 Solitary waves

Approximated solutions of the Burgers equation (1) by means of the difference scheme (2) strongly depend on the values of the time and space steps. For specific values of τ and h, equation (5) can, for instance, exhibit traveling wave solutions which can represent great disturbances of the searched solution.
We presently aim at determining the conditions, depending on the values of the parameters τ and h, which give birth to traveling wave solutions of (16).
Following Feng [2] and our previous work [3], in which traveling wave solutions of the cBKDV equation were exhibited as combinations of bell-profile waves and kink-profile waves, we aim at determining traveling wave solutions of (5).
Following [2], we assume that equation (5) has traveling wave solutions of the form

$$\tilde{u}(\tilde{x}, \tilde{t}) = \tilde{u}(\xi), \quad \xi = \tilde{x} - v\, \tilde{t} \qquad (32)$$

where v is the wave velocity. Substituting (32) into equation (5) leads to:

$$\widetilde{\mathcal{F}}_\xi(\tilde{u}, \tilde{u}^{(r)}, (-v)^s\, \tilde{u}^{(s)}) = 0, \qquad (33)$$

Performing an integration of (33) with respect to ξ and setting the integration constant to zero leads to an equation of the form:

$$\widetilde{\mathcal{F}}_\xi^{\mathcal{P}}(\tilde{u}, \tilde{u}^{(r)}, (-v)^s\, \tilde{u}^{(s)}) = 0, \qquad (34)$$

which will be the starting point for the determination of solitary waves solutions.

4 Traveling Solitary Waves

4.1 Hyperbolic Ansatz

The discussion in the preceding section provides us useful informations when we construct traveling solitary wave solutions for equation (33). Based on these results, in this section, a class of traveling wave solutions of the equivalent equation (16) is searched as a combination of bell-profile waves and kink-profile waves of the form

$$\tilde{u}(\tilde{x},\tilde{t}) = \sum_{i=1}^{n} \left(U_i \tanh^i \left[C_i(\tilde{x} - v\,\tilde{t}) \right] + V_i \operatorname{sech}^i \left[C_i(\tilde{x} - v\,\tilde{t} + x_0) \right] \right) + V_0 \qquad (35)$$

where the $U_i's$, $V_i's$, $C_i's$, $(i = 1, \cdots, n)$, V_0 and v are constants to be determined. In the following, c is taken equal to 1.

4.2 Theoretical analysis

Substitution of (35) into equation (33) leads to an equation of the form

$$\sum_{i,j,k} A_i \tanh^i\big(C_i\,\xi\big) \operatorname{sech}^j\big(C_i\,\xi\big) \sinh^k\big(C_i\,\xi\big) = 0 \qquad (36)$$

the A_i being real constants.
The difficulty for solving equation (36) lies in finding the values of the constants U_i, V_i, C_i, V_0 and v by solving the over-determined algebraic equations. Following [2], after balancing the higher-order derivative term and the leading nonlinear term, we deduce $n = 1$. Then, following [3] we replace $\operatorname{sech}(C_1\,\xi)$ by $\frac{2}{e^{C_1\,\xi}+e^{-C_1\,\xi}}$, $\sinh(C_1\,\xi)$ by $\frac{e^{C_1\,\xi}-e^{-C_1\,\xi}}{2}$, $\tanh(C_1\,\xi)$ by $\frac{e^{C_1\,\xi}-e^{-C_1\,\xi}}{e^{C_1\,,\xi}+e^{-C_1\,\xi}}$, and multiply both sides by $\left(e^{\xi\,C_1} + e^{-\xi\,C_1}\right)^5 e^{5\,\xi\,C_1}$, so that equation (36) can be rewritten in the following form:

$$\sum_{k=0}^{10} P_k(U_1,\ V_1,\ C_1,\ v,\ V_0)\, e^{k\,C_1\,\xi} \ = \ 0, \qquad (37)$$

where the P_k $(k = 0, ..., 10)$, are polynomials of U_1, V_1, C_1, V_0 and v.

4.3 Numerical scheme analysis

4.3.1 Finite-difference second-order centered scheme in space, explicit Euler-time integration

Equation (33) is presently given by:

$$-v\,\tilde{u}'(\xi)+c\,\tilde{u}(\xi)\,\tilde{u}'(\xi)-\frac{1}{Re_h}\,\tilde{u}''(\xi)+v^2\,\frac{\tau}{2}\,\tilde{u}''(\xi)+\frac{1}{6}\,\tilde{u}(\xi)\,\tilde{u}^{(3)}(\xi)-\frac{1}{Re_h}\,\frac{1}{12}\,\tilde{u}^{(4)}(\xi)=0 \tag{38}$$

Performing an integration of (38) with respect to ξ and setting the integration constant to zero yields:

$$-v\,\tilde{u}(\xi)+\frac{c}{2}\,\tilde{u}^2(\xi)+(v^2\,\frac{\sigma}{2}-\frac{1}{Re_h})\,\tilde{u}'(\xi)+\frac{1}{6}\,\left\{\tilde{u}(\xi)\,\tilde{u}^{(2)}(\xi)-\frac{1}{2}\,\tilde{u}'^2(\xi)\right\}-\frac{1}{12\,Re_h}\,\tilde{u}^{(3)}(\xi)=0 \tag{39}$$

The related system (37) has consistent solutions, which are given in Tables 1.
For sake of simplicity, we use ε to denote 1 or -1.

Table 1:

	σ	v	U_1	V_1	C_1	V_0
Sets 1, 2	$\frac{484\,Re_h}{729}$	$\varepsilon\,\frac{108}{11\sqrt{11}\,Re_h}$	$\varepsilon\,\frac{108}{5\sqrt{11}\,Re_h}$	0	$-\varepsilon\,\frac{6}{\sqrt{11}}$	$-\varepsilon\,\frac{108}{5\sqrt{11}\,Re_h}$
Set 3	$\frac{5\,Re_h\left(17\,C_1^2-12\right)}{6C_1^2\left(4C_1^2-9\right)^2}$	$-\frac{2\left(4C_1^3-9C_1\right)}{5\,Re_h}$	$-\frac{18C_1}{5\,Re_h}$	0	$\in\mathbb{R}$	$\frac{18C_1}{5\,Re_h}$
Set 4	$-\frac{Re_h\left(64C_1^6-384C_1^4+551C_1^2-156\right)}{6C_1^2\left(4C_1^2-9\right)^2}$	$\frac{-\frac{5C_1}{13-8C_1^2}-C_1}{Re_h}$	$-\frac{2\left(8C_1^3-9C_1\right)}{Re_h\left(8C_1^2-13\right)}$	0	$\in\mathbb{R}$	$\frac{18C_1}{Re_h\left(8C_1^2-13\right)}$

In the following, we shall denote:

$$\left\{\begin{array}{rcl}\sigma_{1,2} & = & \frac{484\,Re_h}{729}\\ \sigma_3 & = & \frac{5\,Re_h\left(17\,C_1^2-12\right)}{6C_1^2\left(4C_1^2-9\right)^2}\\ \sigma_4 & = & -\frac{Re_h\left(64C_1^6-384C_1^4+551C_1^2-156\right)}{6C_1^2\left(4C_1^2-9\right)^2} = -\frac{Re_h\,(C_1^2-4)\,(8\,C_1^2-13)\,(8\,C_1^2-3)}{6C_1^2\left(4C_1^2-9\right)^2}\end{array}\right. \tag{40}$$

4.3.2 The Lax-Wendroff scheme

Equation (33) is then given by:

$$-v\,\tilde{u}'(\xi)+c\,\tilde{u}(\xi)\,u'(\xi)-\left(\frac{1}{Re_h}+\frac{c^2\,\sigma}{2}\right)\tilde{u}''(\xi)+v^2\,\frac{\sigma}{2}\,\tilde{u}''(\xi)+\frac{1}{6}\,\tilde{u}(\xi)\,u^{(3)}(\xi)-(\frac{1}{Re_h}+\frac{c^2\,\sigma}{2})\,\frac{1}{12}\,u^{(4)}(\xi)=0 \tag{41}$$

Performing an integration with respect to ξ and setting the integration constant to zero yields:

$$-v\,\tilde{u}(\xi)+\frac{c}{2}\,\tilde{u}^2(\xi)+(v^2\,\frac{\sigma}{2}-(\frac{1}{Re_h}+\frac{c^2\,\sigma}{2}))\,u'(\xi)+\frac{1}{6}\left[\tilde{u}(\xi)\,\tilde{u}^{(2)}(\xi)-\frac{1}{2}\,\tilde{u}'^2(\xi)\right]-\left(\frac{1}{Re_h}+\frac{c^2\,\sigma}{2}\right)\frac{1}{12}\,\tilde{u}^{(3)}(\xi)=0 \tag{42}$$

The related system (37) does not admit consistent solutions.

5 Conclusions

The analysis of the nonlinear equivalent differential equation for finite-differenced Burgers equation has beeen carried out. It is shown that some finite-difference schemes can lead to the occurance of spurious travelling solitary waves, which are not solutions of the exact continuous original equation. It is proposed to refer these schemes as structurally instable schemes. Such spurious solitary waves have constant energy, and therefore the numerical error norm does not vanish at arbitrary long integration times on unbounded numerical domains.

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