Modelling of multilayered piezoelectric composites

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This chapter is the continuation of my Ph. D. [12], [5], [3]. It generalizes the multilayered shell model to piezoelectric structures.
In a first time, it was natural to consider the related plate model; thanks to satisfying results, we then considered the shell model.

1 Multilayered piezoelectric plates

We hereafter study the modelling of composite multilayered piezoelectric plates. Our theory is based on an hybrid approach, where the mechanical and electrical continuity conditions are satisfied, as well as the boundary conditions on the top and bottom surfaces of the structure.
The accuracy of the proposed theory is assessed through investigation of significant problems, for which an exact three-dimensional solution is known.

1.1 Introduction

The development of the so-called "smart-structures", e.g. made of piezoelectric composites, require nowadays more and more precision in their design and sizing. The importance of efficient models has, so far, led to numerous theories.
The first kind of approach is generally based on the assumption according to which the multilayered piezoelectric structure behaves as a single-layered one. Tiersten [47], Mindlin [33], Lee et al. [31], [32], have applied Kirchhoff-Love's theory to piezoelectricity. Yet, only thin structures are concerned.
First order theories can be found, for single-layered structures, for instance, in Chandrashekhara et al. [15].
However, the electric field induced by mechanical efforts is, generally, not taken into account.

Vatal'yan et al [56] devoted themselves to bilayered ceramic plates.
Thanks to expansions into series of the thickness coordinate, Lee et al. [32] obtained the two dimensional equations of motion, for plates made of piezoelectric crystals.
Yang et al. [58], Yong et al. [59] have generalized these results to multilayered plates.
The problem is that those models do not generally take into account the coupling that occurs in the equations of motion.
Hybrid theories, with a " single-layered type " approach, and a modelling " per layer " of the electrostatic potential, have been proposed: Mitchell et al. [34] used a third-order theory for the displacement field, while taking into account the variations of the electrostatic potential through the thickness of the plate; Fernandes et al. [21], [22] developed a model that takes into account refinements of the shear terms, as in Touratier [51].
Other " layerwise models " have been developed, in a first time by Pauley [42]. Pai et al. [39] have proposed an induced-strain model of multilayered piezoelectric plate model.
Heyliger et al. [25], [26] developed an exact three-dimensional theory, for composite multilayered plates.
Yet, those models do not simultaneously take into account the continuity conditions for the mechanical and electrical quantities. One can of course find asymptotic theories, which enable us to satisfy both kinds of conditions, but they only apply to thin structures [17].
We propose in the following, a two-dimensional theory, more accurate, that

enables us to model thick multilayered piezoelectric structures.
This theory generalizes to piezoelectricity our composite multilayered model, developed in [12], [3], [5], relating our displacement approach, to a " single-layered " type approach, continuous at layer interfaces, to quadratic variations through the thickness of the considered structure, of the electrostatic potential, which is also continuous. The transverse shear stresses, as well as the electric displacement, under a constant strain, are continuous. Refinements of the shear and membrane terms are taken into account, by means of trigonometric functions.
Eventually, the conditions on the frontier of the structure, or at layer interfaces, are satisfied.
The accuracy of the plate model thus obtained is assessed through investigation of significant problems, for which a three-dimensional solution is known [25], [26].

Consider a multilayered piezoelectric plate, made of an arrangement of N layers.
Denote by a the length of the plate, by b its width, by h its thickness, and by V the volume occupied by the plate (see figure 1).

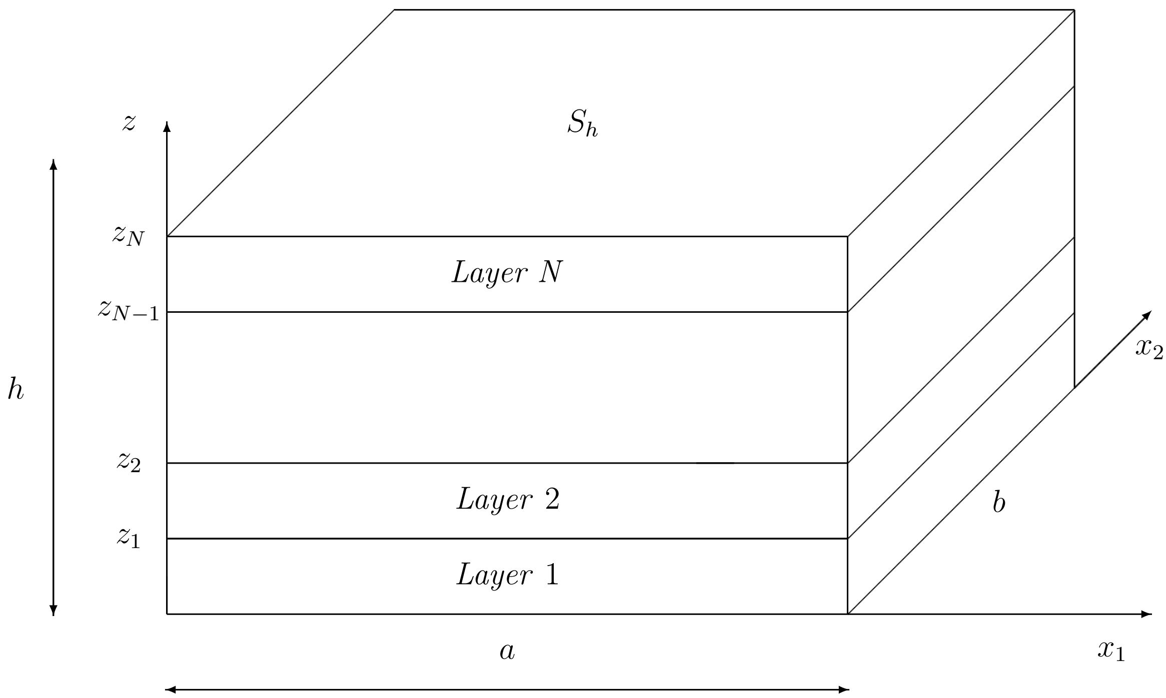


Figure 1: The multilayered piezoelectric plate

Notation. The frontier of the plate is made of the reunion of its bottom surface S_0, its top surface S_h, and its lateral surface $\mathcal{A}$.
Denote by S_i the interface between the i^{th} and $(i+1)^{th}$ layers, and by z_i the distance between S_0 and S_i.

The reference surface coïncides with the bottom surface S_0.

Assumptions 1.1. *The transverse shear stresses are supposed equal to zero on the top and bottom surfaces of the plate.*

Notation. The Einsteinian summation convention applies to repeated indices, where Latin indices range from 1 to 3 while Greek indices range from 1 to 2.

Notation.

V	Volume occupied by the plate
h	Total thickness of the plate
S_0	Bottom surface of the plate
S_h	Top surface of the plate
S_i	Bottom surface of the i^{th} layer
z_i	Distance between S_0 et S_i
z_{i0}	Distance between S_0 and the mid-surface of the i^{th} layer
$\mathcal{A}$	Lateral surface of the plate
(x_i)	Cartesian coordinates
L_i	Lamé coefficients
s_k, $k = 1, \ldots, 6$	Strains
σ_k, $k = 1, \ldots, 6$	Stresses
${C_{mnpq}}^{(i)}$, or ${C_{KL}}^{(i)}$	Components of the elastic strain tensor under a constant electric field constant
$'$	Differentiation with respect to z
$_{,i}$	Differentiation with respect to x_i
$\dot{}$	Differentiation with respect to time t
δ	Variational operator
φ	Electrostatic potential o
φ^{1B}	Electrostatic potential on S_0
$\varphi^{N+1,B}$	Electrostatic potential on S_h
φ^{iB}	Electrostatic potential on S_i
φ^{iM}	Electrostatic potential on the midsurface of the i^{th} layer
φ^{iT}	Electrostatic potential on S_{i+1}
E_l, $l = 1, \ldots, 3$	Components of the electrical field $\overrightarrow{E}$
D_k, $k = 1, \ldots, 3$	Components of the electric field
${e_{kl}}^{(i)}$	piezoelectric constants, under a constant strain, of the i^{th} layer
${\varepsilon_{kl}}^{(i)}$	dielectric constants, under a constant strain, of the i^{th} layer (components of the dielectric tensor of the i^{th} layer)
ρ	Mass density
ε_0	Permittivity of vacuum [1]

[1] $\varepsilon_0 = 8.85\,10^{-12}\,F\,/\,m$

1.2 Mechanical study

1.2.1 Kinematic assumptions

Assumptions 1.2. *The displacement field* $\overrightarrow{U}$ *of any point* $M(x_\alpha, z)$ *of the structure is determined by its components* (U_α, U_z) *in the basis* $\left(\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}\right)$, *which are approximated under the following form:*

$$\left\{ \begin{array}{lcl} U_\alpha & = & u_\alpha + z\,\eta_\alpha + f(z)\,\psi_\alpha + g(z)\,\gamma_\alpha^0 + \sum_{m=1}^{N-1}(z - z_m)\,u_{m\alpha}\,H(z - z_m) \\ U_z & = & w \end{array} \right. \tag{1}$$

where

$$f(z) = \frac{h}{\pi}\,\sin\left(\frac{\pi\,z}{h}\right) \quad , \quad g(z) = \frac{h}{\pi}\,\cos\left(\frac{\pi\,z}{h}\right) \tag{2}$$

and where H *denotes the Heaviside step function.*

Remark 1.1.

i. The Heaviside step function has been previously used, among others, by Di Sciuva [18] and He [24].

ii. The use of the *sine* and *cosine* functions can be justified as in Touratier [49], by a discrete layer theory from the three-dimensional modelling of Cheng [16] for thick plates.

Definition 1.1. The u_α are membrane displacements, the γ_α^0 are the components of the transverse shear stress at $z = 0$, w is the deflection, the ψ_α and $u_{m\alpha}$ are *a priori unknown functions*, which are to be determined using the boundary conditions on the top and bottom surfaces, as well as at layer interfaces.

1.2.2 Voigt notation

The anisotropy of the piezoelectric materials requires using the *two indices Voigt notation*; we thus use the correspondance:

$$(\mathit{1},\ \mathit{1}) \equiv 1 \ ,\ (\mathit{2},\ \mathit{2}) \equiv 2 \ ,\ (\mathit{3},\ \mathit{3}) \equiv 3$$

$$(\mathit{2},\ \mathit{3}) \equiv 4 \ ,\ (\mathit{3},\ \mathit{2}) \equiv 4$$

$$(\mathit{1},\ \mathit{3}) \equiv 5 \ ,\ (\mathit{3},\ \mathit{1}) \equiv 5$$

$$(\mathit{1},\ \mathit{2}) \equiv 6 \ ,\ (\mathit{2},\ \mathit{1}) \equiv 6$$

Notation. In the following, capital letters denote a couple of latin indices.

1.2.3 Uncoupled constitutive equations

Denote by ${C_{mnkl}}^{(i)}$ (or ${C_{KL}}^{(i)}$, Voigt notation) the components of the elastic strain tensor, under a electric field, related to the i^{th} layer, and by s_{kl} the components of the strain tensor (or s_I, Voigt notation).

Assumptions 1.3. *We use, in the following, the assumption of* small strains, *which yields:*

$$s_{kl} = \frac{U_{k,l} + U_{l,k}}{2} \tag{3}$$

Remark 1.2. The piezoelectric theory thus obtained is *linear.*

The uncoupled constitutive equations are:

$${\sigma_{mn}}^{(i)} = {C_{mnkl}}^{(i)}\, s_{kl} \tag{4}$$

or, Voigt notation:

$${\sigma_J^{(i)}}^d = {C_{JK}}^{(i)}\, s_K \tag{5}$$

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = C \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \end{pmatrix} \tag{6}$$

1.3 Boundary conditions

i. Conditions on the top and bottom surfaces of the plate:

The nullity of the uncoupled transverse shear stress on S_0 and S_h yields:

$$\left\{ \begin{array}{rcl} {\sigma_{6-\alpha}^{(1)}}^d & = & 0 \\ {\sigma_{6-\alpha}^{(N)}}^d & = & 0 \end{array} \right. \tag{7}$$

The latter system yields:

$$\left\{ \begin{array}{rcl} {C_{6-\alpha}}^{(1)} \, \{U_{\alpha,3}(x_\alpha, 0; t) + w_{,\alpha}\} & = & 0 \\ {C_{6-\alpha}}^{(N)} \, \{U_{\alpha,3}(x_\alpha, h; t) + w_{,\alpha}\} & = & 0 \end{array} \right. \tag{8}$$

i. e.:

$$\left\{ \begin{array}{rcl} \eta_\alpha & = & -w_{,\alpha} - \sum_{m=1}^{N-1} u_{m\alpha} \\ \psi_\alpha & = & \frac{1}{2} \sum_{m=1}^{N-1} u_{m\alpha} \end{array} \right. \tag{9}$$

ii. Layer interfaces conditions:

The continuity of the uncoupled shear stress at layer interface between i^{th} and $(i+1)^{th}$ layers can be written as:

$${\sigma_{6-\alpha}^{(i)}}^d(x_\alpha, z_i) = {\sigma_{6-\alpha}^{(i+1)}}^d(x_\alpha, z_i) \tag{10}$$

i. e.:

$${C_{6-\alpha,6-\alpha}}^{(i)} \left[g'(z_i)\, \gamma_\alpha^0 + \frac{1}{2} \sum_{m=1}^{i-1} (f'(z_i) - 1)\, u_{m\alpha} \right] = {C_{6-\alpha,6-\alpha}}^{(i+1)} \left[g'(z_i)\, \gamma_\alpha^0 + \frac{1}{2} \sum_{m=1}^{i} (f'(z_i) - 1)\, u_{m\alpha} \right] \tag{11}$$

The linear system thus obtained enables us to express the $u_{m\alpha}$, $m = 1, \; \ldots \;, N$, as functions of the transverse shears γ_α^0:

$$u_{m\alpha} = \lambda_{m\alpha} \, \gamma_\alpha^0 \tag{12}$$

where the $\lambda_{m\alpha}$ are real constants, given by the resolution of the above system.

1.4 Final form of the kinematic field

Proposition 1.4. *The kinematic field* $\overrightarrow{U}$ *of a point* $M(x_\alpha, z)$ *of the plate is given by:*

$$\left\{ \begin{array}{lcl} U_\alpha & = & u_\alpha - z\, w_{,\alpha} + f(z)\, \psi_\alpha + \sum_{m=1}^{N-1} u_{m\alpha}\, h_\alpha(z) \\ U_z & = & w \end{array} \right. \tag{13}$$

where the h_α *are functions of the global thickness parameter* z*, given by:*

$$h_\alpha(z) = g(z) + \sum_{m=1}^{N-1} \left\{ \frac{1}{2}\,(f(z) - z) + (z - z_m)\, H(z - z_m) \right\} \lambda_{m\alpha} \tag{14}$$

This kinematic field has been developed in [12], [3], [5].

1.5 Electrical study

1.5.1 Electrical assumptions

Assumptions 1.5. *The electrostatic potential is approximated under the form:*

$$\phi(x_\alpha, z; t) = \sum_{i=1}^{N-1} \varphi_i(x_\alpha, z; t)\, \chi_i(z) \tag{15}$$

where the φ_i *denote the* potentials per layer*, and the* χ^i *the* characteristic functions per layer*:*

$$\chi^i(z) = \left\{ \begin{array}{ll} 1 & si\, z \in [z_i, z_{i+1}[\\ 0 & else \end{array} \right. \tag{16}$$

Notation. Let us introduce, for each layer:

i. the thickness coordinate ξ_i, defined by:

$$\xi_i = \frac{2\,(z_i - z_i^0)}{h_i} \tag{17}$$

(see figure 2).

ii. the electrostatic potential on the bottom face, S_i : ${\varphi^i}^B = \varphi^i(x_\alpha, z_i; t)$;

iii. the electrostatic potential on the mid-surface: ${\varphi^i}^M = \varphi^i(x_\alpha, \frac{z_{i+1}-z_i}{2}; t)$;

iv. the electrostatic potential on the top face, S_{i+1} : ${\varphi^i}^T = \varphi^i(x_\alpha, z_{i+1}; t)$;

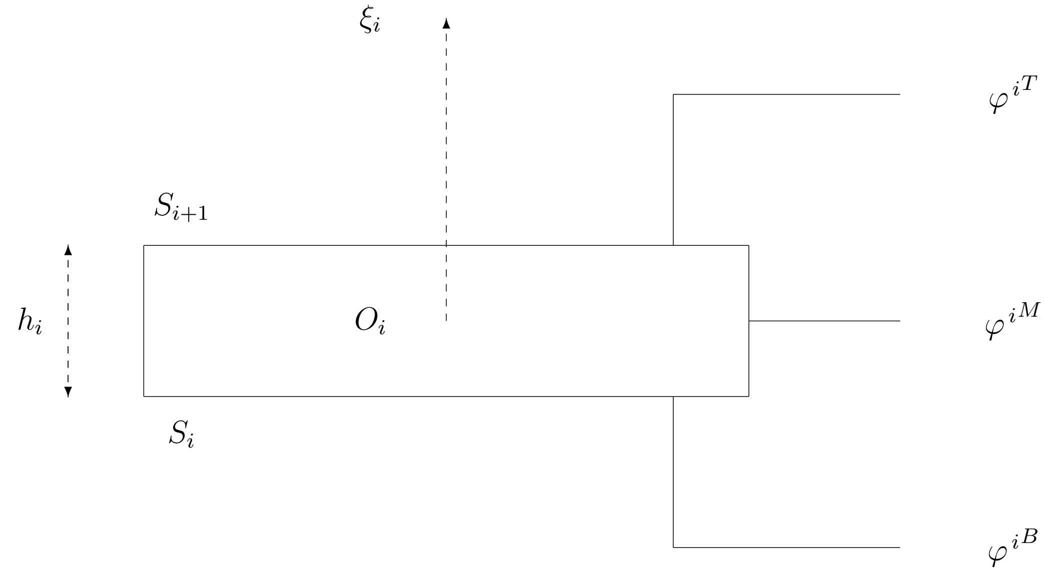


Figure 2: The i^{th} layer

Assumptions 1.6. *The " potentials per layer " are assumed of the following form:*

$$\varphi^i(x_\alpha, z; t) = \frac{1}{2}\,\xi_i\,(\xi_i - 1)\,{\varphi^i}^B(x_\alpha; t) + (1 - {\xi_i}^2)\,{\varphi^i}^M(x_\alpha; t) + \frac{1}{2}\,\xi_i\,(\xi_i + 1)\,{\varphi^i}^T(x_\alpha; t) \tag{18}$$

Remark 1.3. In so far as: ${\varphi^i}^B(x_\alpha, z_{i+1}; t) = {\varphi^i}^T(x_\alpha, z_i; t)$, the continuity of the electrostatic potential at layer interfaces of the potential is thus automatically satisfied.

For this reason, we choose to keep, as unknowns quantities, the ${\varphi^i}^B$ and ${\varphi^i}^M$.

For future applications, we shall suppose that p values of the ${\varphi^i}^B$ are known.

1.5.2 Uncoupled piezoelectric constitutive law

The uncoupled piezoelectric constitutive law takes the form:

$$D_3 = -\varepsilon_{33}\, \varphi_{,3} + e_{3j}\, s_j \tag{19}$$

1.5.3 Electric boundary conditions

Proposition 1.7. *The electrostatic potential can be written as:*

$$\varphi(x_\alpha, z; t) = \sum_{i_k \in I} Q_{i_k}(z;t)\, \varphi^{i_k B} + \sum_{j_l \in J} Q_{j_l}(z;t)\, \varphi^{j_l M} \tag{20}$$

where the Q_{i_k}, Q_{j_l} are polynomial functions of the global thickness coordinate z.

Proof. The continuity of ${D_3}^d$ on the top and bottom surfaces, and at the p layer interfaces where the value of electrostatic potential is known, lead to a linear system of $N+p-1$ equations, which enable us to eliminate some of the φ^{iB} and φ^{iM}.

□

Notation. Denote by $\varphi^{i_k B}$, $i_k \in I$, $\varphi^{j_l M}$, $j_l \in J$, I and J being finite subsets $\mathbb{N}$, the remaining unknowns quantities.

1.6 How to take the coupling into account The need of some correction factors

The resolution of the boundary problem enables us to obtain the values of the generalized mechanical and electrical unknowns.
The sole quantity that cannot be obtained without an *a posteriori* treatment is the coupled *electric displacement.*

We recall the coupled constitutive piezoelectric law:

$$D_3 = -\varepsilon_{33}\, \varphi_{,3} + e_{3j}\, s_j \tag{21}$$

The *dielectric* constants $\varepsilon_{33}\, \varphi_{,3}$ are very small compared to the *mechanical* ones $e_{3j}\, s_j$.

So, if the final value of the electric displacement is not corrected *a posteriori*, *mechanical quantities* $e_{3j}\, s_j$ are going to be predominant.
Those terms are not continued at layer interfaces (see former section).

Proposition 1.8. *The final electrical displacement is obtained, after correction, as:*

$$D_3{}^{final}(x_\alpha, z; t) = -\varepsilon_{33}\, \varphi_{,3} + e_{3j}\, s_j - \sum_{k=1}^{N} [e_{3j}{}^{(k)}\, s_j{}^{(k)}(z_k)\, \chi^k(z) - e_{3j}{}^{(k+1)}\, s_j{}^{(k+1)}(z_k)\, \chi^{k+1}(z)] \tag{22}$$

1.7 The linear piezoelectric constitutive law

1.7.1 Tiersten equations

Notation. We recall that s denotes the strain tensor, $\overrightarrow{E}$ the electrical field, C the elastic strain tensor, under a constant electrical field, e the tensor of piezoelectric coefficients, under constant strain, and ε the tensor of dielectric constants, under a constant strain.
i, j vary from 1 to 6, k, l from 1 to 3.

Properties. Tensors C, e, ε satisfy the following symmetry properties:

$$C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk} \tag{23}$$

$$e_{ijk} = e_{ikj} \tag{24}$$

$$\varepsilon_{ij} = \varepsilon_{ji} \tag{25}$$

Proposition 1.9. *The linear piezoelectric constitutive law is given by:*

$$\left\{ \begin{array}{rcl} \sigma_{ij} & = & C_{ijkl}\, s_{kl} - e_{kij}\, E_k \\ D_k & = & \varepsilon_{ij}\, E_j + e_{ijk}\, s_{jk} \end{array} \right. \tag{26}$$

or, under double indexation:

$$\left\{ \begin{array}{rcl} \sigma_I & = & C_{IJ}\, s_J - e_{kI}\, E_k \\ D_k & = & \varepsilon_{kl}\, E_l + e_{kI}\, s_I \end{array} \right. \tag{27}$$

Remark 1.4. For orthotropic piezoelectric materials, as PVDF polymers, part of the piezoelectric coefficients vanish; the tensors e and ε can thus be written as:

$$e = \begin{pmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{32} \\ 0 & 0 & e_{33} \\ 0 & e_{24} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{28}$$

$$\varepsilon = \begin{pmatrix} \varepsilon 11 & 0 & 0 \\ 0 & \varepsilon 22 & 0 \\ 0 & 0 & \varepsilon 33 \end{pmatrix} \tag{29}$$

Remark 1.5. Lots of piezoelectric materials, as PZT ceramics, or ZnO crystals, ..., are transversally isotropic.
By choosing the orthotropy axis parallel to the vertical axis, the resulting simplifications occur:

$$e_{31} = e_{32} \ \ , \ \ e_{24} = e_{15} \ \ , \ \ \varepsilon_{11} = \varepsilon_{22} \tag{30}$$

1.7.2 Two-dimensional coefficients

Assumptions 1.10. *In the following, the normal stress is assumed to be equal to zero.*

Proposition 1.11. *Under the assumption of nullity of the normal stress, the linear piezoelectric constitutive law is given by:*

$$\left\{ \begin{array}{rcl} \sigma_I & = & {C_{IJ}}^{2D} \, s_J - {e_{kI}}^{2D} \, E_k \\ D_k & = & {\varepsilon_{kl}}^{2D} \, e_l + {e_{kI}}^{2D} \, s_I \end{array} \right. \tag{31}$$

where:

$$\left\{ \begin{array}{rcl} {C_{IJ}}^{2D} & = & C_{IJ} - C_{I3} \, \frac{C_{3J}}{C_{33}} \\ {e_{kI}}^{2D} & = & e_{kI} - C_{I3} \, \frac{e_{k3}}{C_{33}} \\ {\varepsilon_{kl}}^{2D} & = & \varepsilon_{kl} + e_{l3} \, \frac{e_{k3}}{C_{33}} \end{array} \right. \tag{32}$$

Proof 1.12. *The assumption of nullity of the normal stress:*

$$\sigma_3 = 0 \tag{33}$$

can be written as:

$$C_{3J}\, s_J - e_{k3}\, E_k = 0 \tag{34}$$

which enables us to eliminate the normal strain s_3:

$$s_3 = \frac{e_{k3}}{C_{33}}\, E_k - \frac{C_{3\alpha}}{C_{33}}\, s_\alpha - \frac{C_{3,6-\alpha}}{C_{33}}\, s_{6-\alpha} - \frac{C_{36}}{C_{33}}\, s_6 \tag{35}$$

Substituting this expressions into (27), we obtain:

$$\begin{aligned}
\sigma_I &= C_{IJ}\, s_J - e_{kI}\, E_k \\
&= C_{I\alpha}\, s_\alpha + C_{I,6-\alpha}\, s_{6-\alpha} + C_{I,6}\, s_6 + C_{I3}\, s_3 - e_{kI}\, E_k \\
&= C_{I\alpha}\, s_\alpha + C_{I3}\left\{\frac{e_{k3}}{C_{33}}\, E_k - \frac{C_{3,6-\alpha}}{C_{33}}\, s_{6-\alpha} - \frac{C_{36}}{C_{33}}\, s_6\right\} - e_{kI}\, E_k \\
&= \left\{ C_{I\alpha} - C_{I3}\,\frac{C_{3\alpha}}{C_{33}} \right\} s_\alpha + \left\{ C_{I,6-\alpha} - C_{I3}\,\frac{C_{3,6-\alpha}}{C_{33}} \right\} s_{6-\alpha} \\
&+ \left\{ C_{I,6} - C_{I3}\,\frac{C_{3,6}}{C_{33}} \right\} s_6 - \left\{ e_{kI} - C_{I3}\,\frac{e_{k3}}{C_{33}} \right\} E_k
\end{aligned} \tag{36}$$

1.8 The two-dimensional boundary problem

1.8.1 Variational formulation

Hamilton's Principle yields:

$$\int_0^t \left\{ \int_V \{\sigma_i\, \delta s_i + D_i\, \delta\varphi_i\} dV + \int_V \{\overrightarrow{f_v} \cdot \delta\overrightarrow{U} + W\, \delta\varphi\}\, dV + \int_{\mathcal{A}} \overrightarrow{f_s} \cdot \delta\overrightarrow{U}\, dA + \int_{S_0} (p_0 - p_h)\, dS \right\} dt = 0 \tag{37}$$

where δ denotes a variational operator, $\overrightarrow{f_v}$ the density of external volumic forces, $\overrightarrow{f_s}$ density of external lateral forces, p_0 and p_h the pressures respectively acting on the top and bottom surfaces of the structure, W the density of external surface electrical forces.

Notation. Let us introduce:

i. the generalized forces:

$$\left\{\begin{array}{rcl} {N_1}^{\alpha} &=& -\int_0^h \{C_{\alpha j}\, s_j - e_{3\alpha}\, E_3\}\, dz \\ {N_2}^{\alpha\beta} &=& -\int_0^h C_{66}\, s_6\, (1-\delta_{\alpha\beta})\, dz \\ {N_3}^{\alpha} &=& -\int_0^h \{C_{\alpha j}\, s_j - e_{3\alpha}\, E_3\}\, h_\alpha(z)\, dz \\ {N_4}^{\alpha\beta} &=& -\int_0^h C_{66}\, s_6 (1-\delta_{\alpha\beta})\, h_\alpha(z)\, dz \\ {N_5}^{\alpha} &=& -\int_0^h \frac{1}{2}\, \{C_{6-\alpha,6-\alpha}\, s_{6-\alpha} - e_{2,6-\alpha}\, E_2\} dz \end{array}\right. \tag{38}$$

$$\left\{\begin{array}{rcl} \mathcal{N}^{iB} &=& -\int_0^h E_3\, \varepsilon_{33}\, Q^{iB'}(z) dz \ \ , \ i \in I \\ \mathcal{N}^{jM} &=& -\int_0^h E_3\, \varepsilon_{33}\, Q^{jM'}(z)\, dz \ \ , \ j \in J \end{array}\right. \tag{39}$$

ii. the generalized momentums:

$$\left\{\begin{array}{rcl} {M_1}^{\alpha} &=& -\int_0^h \{C_{\alpha j}\, s_j - e_{3\alpha}\, E_3\}\, z\, dz \\ {M_2}^{\alpha\beta} &=& -\int_0^h C_{66}\, s_6 (1-\delta_{\alpha\beta})\, z dz \end{array}\right. \tag{40}$$

$$\left\{\begin{array}{rcl} \mathcal{M}^{iB\alpha} &=& -\int_0^h \{E_\alpha\, \varepsilon_{\alpha\alpha} + e_{k\alpha}\, s_\alpha\}\, Q^{iB}(z)\, dz \ \ , \ i \in I \\ \mathcal{M}^{jM\alpha} &=& -\int_0^h \{E_\alpha\, \varepsilon_{\alpha\alpha} + e_{k\alpha}\, s_\alpha\}\, Q^{iM}(z)\, dz \ \ , \ j \in J \end{array}\right. \tag{41}$$

iii. the generalized mechanical forces:

$$\left\{\begin{array}{rcl} {F_\nu}^{1}{}_{\alpha} &=& -\int_0^h f_{\nu\alpha}\, dz \\ {F_\nu}^{2}{}_{\alpha} &=& -\int_0^h f_{\nu\alpha}\, z\, dz \\ {F_\nu}^{3}{}_{\alpha} &=& -\int_0^h f_{\nu\alpha}\, h_\alpha(z)\, dz \\ {F_\nu}^{3} &=& -\int_0^h f_{\nu\alpha}\, dz \\ P &=& p_0 - p_h \end{array}\right. \tag{42}$$

iv. the generalized electrical forces:

$$\left\{\begin{array}{rcl} W^{iB} &=& -\int_0^h W\, Q^{iB}(z)\, dz \ \ , \ i \in I \\ W^{jM} &=& -\int_0^h W\, Q^{iB}(z)\, dz \ \ , \ j \in J \end{array}\right. \tag{43}$$

iv. the inertia terms:

$$\left\{ \begin{array}{lcl} I_1 & = & -\int_0^h \rho\, dz \\ I_2 & = & -\int_0^h \rho\, z\, dz \\ I_3 & = & -\int_0^h \rho\, h_\alpha(z)\, dz \\ I_4 & = & -\int_0^h \rho\, z^2\, dz \\ I_5 & = & -\int_0^h \rho\, z\, h_\alpha(z)\, dz \\ I_6 & = & -\int_0^h \rho\, z\, {h_\alpha}^2(z)\, dz \end{array} \right. \tag{44}$$

Proposition 1.13. *The movement equations are given by:*

$$\left\{ \begin{array}{rcl} {N_1}^\alpha{}_{,\alpha} + {N_2}^{\alpha\beta}{}_{,\beta} & = & {I_1}^\alpha\, \ddot{u}_\alpha - I_2\, \ddot{w}_{,\alpha} + {I_3}^\alpha\, \ddot{\gamma}^0_\alpha \\ {M_1}^\alpha{}_{,\alpha\alpha} + {M_2}^{\alpha\beta}{}_{,\alpha\beta} & = & {I_2}^\alpha\, \ddot{u}_{\alpha,\alpha} - I_4\, \ddot{w}_{,\alpha\alpha} + {I_5}^\alpha\, \ddot{\gamma}^0_{\alpha,\alpha} + I_1\, \ddot{w} \\ {N_3}^\alpha{}_{,\alpha} + {N_4}^{\alpha\beta}{}_{,\beta} + {N_5}^\alpha & = & {I_3}^\alpha\, \ddot{u}_\alpha - I_5\, \ddot{w}_{,\alpha} + {I_6}^\alpha\, \ddot{\gamma}^0_\alpha \\ \mathcal{N}^{iB} + {\mathcal{M}^{iB}}^\alpha{}_{,\alpha} & = & 0 \ \ ,\ i \in I \\ \mathcal{N}^{jM} + {\mathcal{M}^{jm}}^\alpha{}_{,\alpha} & = & 0 \ \ ,\ j \in J \end{array} \right. \tag{45}$$

Proof 1.14. *The movement equations are deduced from Hamilton's Principle, in conjunction with the kinematic (13), and the constitutive equations (27), by integration through the thickness of the plate.*

Proposition 1.15. *The boundary conditions leading to a regular problem are given by:*

$$\left\{ \begin{array}{ccccc} {N_1}^\alpha\, n_\alpha\ + {N_2}^{\alpha\beta}\, n_{\alpha\beta} & = & {F_\nu}^1{}_\alpha & ou & \delta u_\alpha = 0 \\ {M_1}^\alpha{}_{,\alpha} + {M_2}^{\alpha\beta}{}_{,\beta} & = & {F_\nu}^2{}_\alpha & ou & \delta w = 0 \\ {N_4}^{\alpha\beta}\, n_\beta & = & {F_\nu}^3{}_\alpha & ou & \delta {\gamma^0}_\alpha = 0 \\ {M_1}^\alpha\, n_\alpha + {M_2}^{\alpha\beta}\, n\beta & = & F_{\nu 3} & ou & \delta w_{,\alpha} = 0 \\ {\mathcal{N}^{iB}}^\alpha\, n_\alpha & = & W^{iB} & ou & \delta\varphi^{iB} = 0\ \ ,\ i \in I \\ {\mathcal{M}^{jM}}^\alpha\, n_\alpha & = & W^{jM} & ou & \delta\varphi^{jM} = 0\ \ ,\ j \in J \end{array} \right. \tag{46}$$

1.9 Numerical validation of the piezoelectric plate model

1.9.1 Free vibrations of a single-layered plate

Consider, in the following, a single-layered plate ($N = 1$), made of PZT4 ceramic, simply supported, in closed circuit (which means that the electrostatic potential on its top and bottom surfaces is equal to zero: $\varphi^{1^B} = \varphi^{1^T} = 0$) (see figure 3).

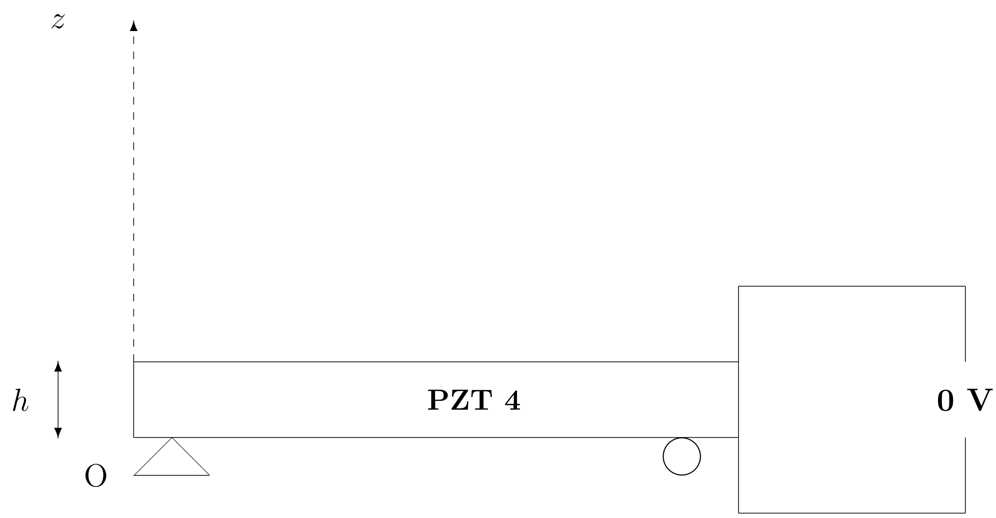


Figure 3: Single-layered piezoelectric plate, in closed circuit

Data 1.16. *Material constants of the ceramic PZT4 can be found in tables 1, 2.*

	C_{11}	C_{22}	C_{33}	C_{12}	C_{13}	C_{23}	C_{44}	C_{55}	C_{66}
PZT 4	139	139	115	77.8	74.3	74.3	25.6	25.6	30.6

Table 1: Independent elastic constants of the PZT4 ceramic (in GPa)

	e_{31}	e_{32}	e_{33}	e_{15}	ε_{11}	ε_{22}	ε_{33}
PZT 4	-5.2	-5.2	15.1	12.7	13.06	13.06	11.51

Table 2: Independent piezoelectric and dielectric constants of the PZT4 ceramic (e_{ij} in C/m^2, ε_{ii} in nF/m)

Proposition 1.17. *The simply supported boundary conditions yield:*

$$w(x_\alpha = 0, z; t) = w(x_\alpha = a, z; t) = 0 \tag{47}$$

Proposition 1.18. *The electrostatic potential (20) is approximated as (18)*

$$\varphi^1(x_\alpha, z; t) = (1 - {\xi_1}^2)\, {\varphi^1}^M(x_\alpha; t) \tag{48}$$

The generalized mechanical unknowns are the membrane displacements u_α, the deflection w, and the transverse shear stresses ${\gamma_\alpha}^0$.
The generalized electrical unknown is ${\varphi^1}^M$.

Assumptions 1.19. *The solution is searched under the following form, which characterizes the propagation of two-dimensional plane waves:*

$$\left\{ \begin{array}{rcl} u_1 &=& A_1\, e^{j\,\omega\,t} \cos(\frac{\pi\,x_1}{a}) \sin(\frac{\pi\,x_2}{b}) \\ u_2 &=& A_2\, e^{j\,\omega\,t} \sin(\frac{\pi\,x_1}{a}) \cos(\frac{\pi\,x_2}{b}) \\ w &=& B\, e^{j\,\omega\,t} \sin(\frac{\pi\,x_1}{a}) \sin(\frac{\pi\,x_2}{b}) \\ {\gamma_1}^0 &=& C_1\, e^{j\,\omega\,t} \cos(\frac{\pi\,x_1}{a}) \sin(\frac{\pi\,x_2}{a}) \\ {\gamma_2}^0 &=& C_2\, e^{j\,\omega\,t} \sin(\frac{\pi\,x_1}{a}) \cos(\frac{\pi\,x_2}{a}) \end{array} \right. \quad , \quad {\varphi^1}^M = \Phi_1\, e^{j\,\omega\,t} \sin(\frac{\pi\,x_1}{a}) \sin(\frac{\pi\,x_2}{b}) \tag{49}$$

and which enables us to satisfy the simply supported boundary conditions (47).

By substituting these expressions into the equations of motion given by equations (13), in conjunction with the boundary conditions (45), the constitutive law (31) and the displacement field (13), we obtain a linear system in A_α, B, C_α, Φ_1, under the form:

$$[K_1 - \omega^2\, M_1] \begin{pmatrix} A_1 \\ A_2 \\ B \\ C_1 \\ C_2 \\ \Phi_1 \end{pmatrix} = 0 \tag{50}$$

The detail of coefficients of the matrices K_1, M_1 is given in [7].

The system admits a non trivial solution if its determinant vanishes:

$$det[K_1 - \omega^2 M_1] = 0 \tag{51}$$

A symbolic calculus tool (*Mathematica* or *Maple*) enables us to calculate the free pulsations of the considered plate, and to compare them to Heyliger's analytical solution (see Table 3).

	Model	**Exact solution**
$\frac{a}{h} = 4$	194297.10^2	194255.10^2
	335054.10^2	327663.10^2
$\frac{a}{h} = 10$	7771870	7770210
	134022.10^2	133695.10^2
$\frac{a}{h} = 50$	15 54370	1554040
	2680440	26828.10^2

Table 3: Free pulsations of the single-layered piezoelectric plate (in $rad.s^{-1}$)

Figures 4, 5 display the variations, as functions of the global thickness coordinate, of the normalized electrostatic potential:

$$\overline{\varphi}(z;t) = \frac{\varphi(x_\alpha, z; t)}{\varphi(x_\alpha = \frac{a}{2}, z = \frac{h}{2}; t)} \tag{52}$$

for significative values of the plate's thickness: $a = 4\,h$, and $a = 10\,h$.

Results perfectly fit the exact solution.

1.9.2 Free vibrations of a 5-layered plate

Consider, in the following a 5-layered plate ($N = 5$), the external layers of which are made of PZT 4 ceramic, with a core made of an arrangement 0 / 90 / 0 of epoxy resin, simply supported, in closed circuit (which means that the electrostatic potential on its top and bottom surfaces is equal to zero: $\varphi^{1^B} = \varphi^{3^T} = 0$) (see figure 6).

Data 1.20. *The respective thicknesses of the external layers are* $h_1 = h_5 = \frac{h}{10}$, *those of each of the epoxy resin,* $h_2 = h_3 = h_4 = \frac{1}{3}\,\frac{8\,h}{10}$.

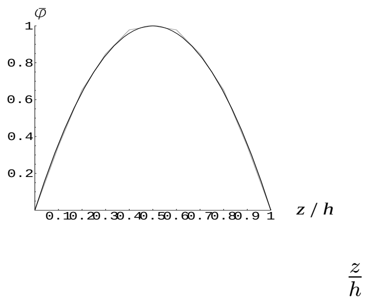


Figure 4: Electrostatic normalized potential, single-layered plate, $a = 4\,h$. In gray: exact solution. In black: our model.

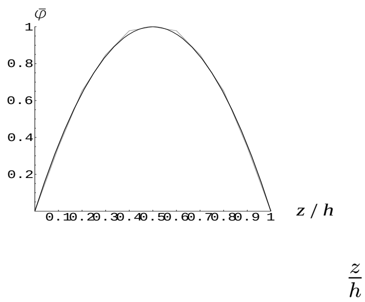


Figure 5: Electrostatic normalized potential, single-layered plate, $a = 10\,h$. In black: our model.

Data 1.21. *Material constants of the ceramic PZT4 and the epoxy resin are given in tables 4, 5.*

	C_{11}	C_{22}	C_{33}	C_{12}	C_{13}	C_{23}	C_{44}	C_{55}	C_{66}
PZT 4	139	139	115	77.8	74.3	74.3	25.6	25.6	30.6
Epoxy	134.6	14.352	14.352	5.1563	7.1329	3.606	5.654	5.654	5.654

Table 4: Independent elastic constants of PZT4 and epoxy resin (in GPa)

Proposition 1.22. *The simply supported boundary conditions yield:*

$$w(x_\alpha = 0, z; t) = w(x_\alpha = a, z; t) = 0 \tag{53}$$

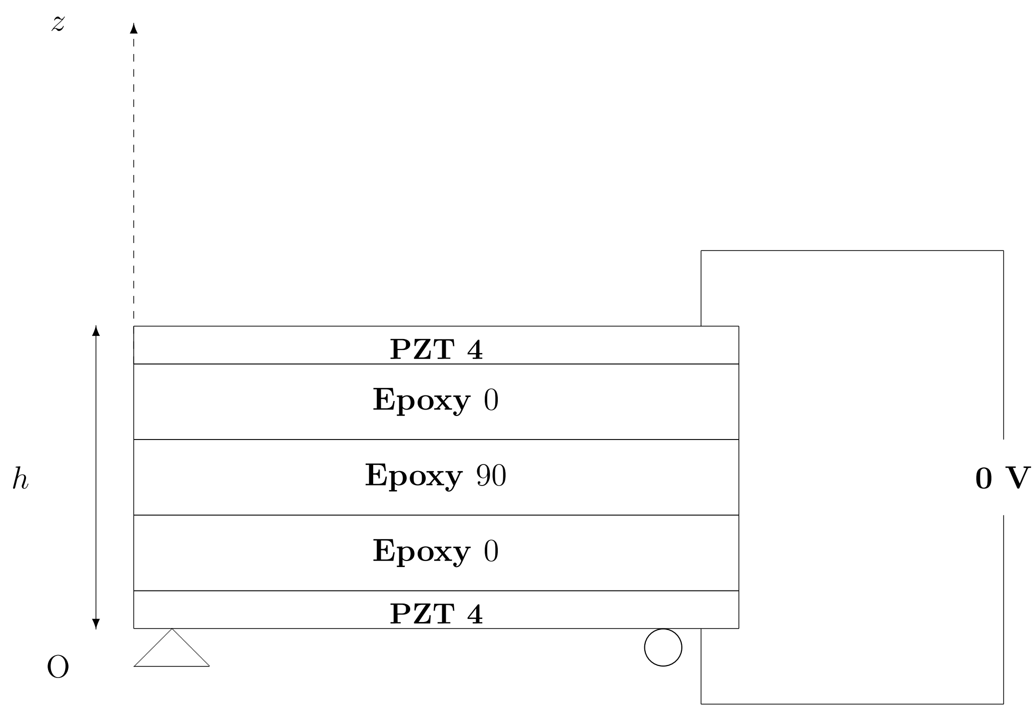


Figure 6: 5-layered piezoelectric plate

	e_{31}	e_{32}	e_{33}	e_{15}	ε_{11}	ε_{22}	ε_{33}
PZT 4	-5.2	-5.2	15.1	12.7	13.06	13.06	11.51
Epoxy	0	0	0	12.7	0.031	0.0266	0.0266

Table 5: Independent piezoelectric and dielectric constants of PZT4 and epoxy resin (e_{ij} in C/m^2, ε_{ii} in nF/m)

Lemma 1.23. *Electrical and dielectric properties of the 3 layers of the core of the plate being identical, they behave, under an electrical point of view, as a single layer.*

Lemma 1.24. *We thus take into account, for the 5-layered plate, a 3-layered type modelling, with an elastic core of epoxy resin.*

Proposition 1.25. *The electrostatic potential (20) is approximated as:*

$$\varphi(x_\alpha, z; t) = Q_{2M}(z)\, \varphi^{2^M} \tag{54}$$

where:

$$\begin{array}{rcl} Q_{2M}(z) & = & \left[(1-{\xi_1}^2)\,\lambda^{1M,2M^M} + \frac{1}{2}\,\xi_1\,(\xi_1+1)\,\lambda^{2B,2M}\right]\chi^1(z) \\ & + & \left[\frac{1}{2}\,\xi_2\,(\xi_2-1)\,\lambda^{2B,2M} + (1-{\xi_2}^2)\; + \frac{1}{2}\,\xi_2\,(\xi_2+1)\,\lambda^{2B,2M}\right]\chi^2(z) \\ & + & \left[\frac{1}{2}\,\xi_3\,(\xi_3-1)\,\lambda^{2B,2M} + (1-{\xi_3}^2)\,\lambda^{1M,2M}\right]\chi^3(z) \end{array} \tag{55}$$

Proof 1.26. *Symmetries of the problem yield:*

$$\varphi^{2^B}(x_\alpha;t) = \varphi^{3^B}(x_\alpha;t)\ \ ,\ \ \varphi^{1^M}(x_\alpha;t) = \varphi^{3^M}(x_\alpha;t) \tag{56}$$

The approximation of the electrostatic potential (20), (18) becomes thus:

$$\begin{array}{rcl} \varphi(x_\alpha,z;t) & = & \left[(1-{\xi_1}^2)\,\varphi^{1^M}(x_\alpha;t) + \frac{1}{2}\,\xi_1\,(\xi_1+1)\,\varphi^{1^T}(x_\alpha;t)\right]\chi^1(z) \\ & + & \left[\frac{1}{2}\,\xi_2\,(\xi_2-1)\,\varphi^{2^B}(x_\alpha;t) + (1-{\xi_2}^2)\,\varphi^{2^M}(x_\alpha;t) + \frac{1}{2}\,\xi_2\,(\xi_2+1)\,\varphi^{2^B}(x_\alpha;t)\right]\chi^2(z) \\ & + & \left[\frac{1}{2}\,\xi_3\,(\xi_3-1)\,\varphi^{2^B}(x_\alpha;t) + (1-{\xi_3}^2)\,\varphi^{1^M}(x_\alpha;t)\right]\chi^3(z) \end{array} \tag{57}$$

The continuity of the uncoupled electrical displacement at layer interfaces can be written as:

$$\left\{ \begin{array}{rcl} -{\varepsilon_{33,1}}^1\,{\varphi_{,3}}^1(x_\alpha,z_1;t) & = & -{\varepsilon_{33,1}}^2\,{\varphi_{,3}}^2(x_\alpha,z_1;t) \\ -{\varepsilon_{33,1}}^2\,{\varphi_{,3}}^1(x_\alpha,z_2;t) & = & -{\varepsilon_{33,1}}^1\,{\varphi_{,3}}^3(x_\alpha,z_2;t) \end{array} \right. \tag{58}$$

We thus have a linear system, which enables us to express φ^{1^M} and φ^{2^B} as functions of φ^{2^M}, under the following form:

$$\left\{ \begin{array}{rcl} \varphi^{1^M} & = & \lambda^{1M,2M^M}\,\varphi^{2^M} \\ \varphi^{2^B} & = & \lambda^{2B,2M^M}\,\varphi^{2^M} \end{array} \right. \tag{59}$$

The generalized mechanical unknowns are membrane displacements u_α, the deflection w, and the transverse shear stresses ${\gamma_\alpha}^0$.
The generalized electrical unknown is φ^{2^M}.

Assumptions 1.27. *In the same way as for the single-layered plate, the solution is searched under the form (167):*

$$\left\{\begin{array}{lcl} u_1 & = & A_1\, e^{j\,\omega\,t}\,\cos(\frac{\pi\,x_1}{a})\,\sin(\frac{\pi\,x_2}{b}) \\ u_2 & = & A_2\, e^{j\,\omega\,t}\,\sin(\frac{\pi\,x_1}{a})\,\cos(\frac{\pi\,x_2}{b}) \\ w & = & B\, e^{j\,\omega\,t}\,\sin(\frac{\pi\,x_1}{a})\,\sin(\frac{\pi\,x_2}{b}) \\ {\gamma_1}^0 & = & C_1\, e^{j\,\omega\,t}\,\cos(\frac{\pi\,x_1}{a})\,\sin(\frac{\pi\,x_2}{a}) \\ {\gamma_2}^0 & = & C_2\, e^{j\,\omega\,t}\,\sin(\frac{\pi\,x_1}{a})\,\cos(\frac{\pi\,x_2}{a}) \end{array}\right. \quad , \; \varphi^{2M} = \Phi_2\, e^{j\,\omega\,t}\,\sin(\frac{\pi\,x_1}{a})\,\sin(\frac{\pi\,x_2}{b}) \tag{60}$$

which enables us to satisfy the simply supported boundary conditions (53).

By substituting these expressions into the equations of motion given by equations (13), in conjunction with the boundary conditions (45), the constitutive law (31) and the displacement field (13), we obtain a linear system in A_α, B, C_α, Φ_1:

$$[K_5 - \omega^2\, M_5] \begin{pmatrix} A_1 \\ B \\ C_1 \\ \Phi_2 \end{pmatrix} = 0 \tag{61}$$

Detail of coefficients of matrices K_5 and M_5 is given in [7].

The system admits a non trivial solution if its determinant vanishes:

$$det[K_5 - \omega^2\, M_5] = 0 \tag{62}$$

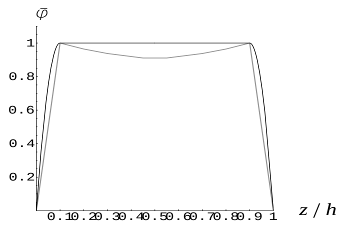
A symbolic calculus tool (*Mathematica* or *Maple*) enables then us to calculate the free pulsations of the considered plate, and to compare them with Heyliger's analytical solution [25] (see 6).

	Model	**Exact solution**
$\frac{a}{h} = 4$	194903.10^2	191301.10^2
	251763.10^2	250769.10^2
$\frac{a}{h} = 10$	1559230	1568100
	209479.10^2	209704

Table 6: Free pulsations of the 5-layered piezoelectric (in $rad.s^{-1}$)

Figure 7 displays the variations, as a function of the global thickness coordinate z, of the normalized electrostatic potential:

$$\overline{\varphi}(z;t) = \frac{\varphi(x_\alpha, z; t)}{\varphi(x_\alpha = \frac{a}{2}, z = \frac{h}{2}; t)} \tag{63}$$

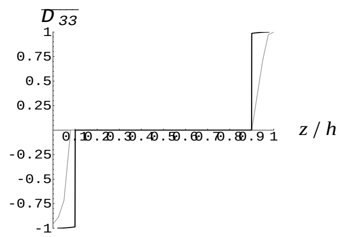


$\frac{z}{h}$

Figure 7: Normalized electrostatic potential, 5-layered plate, $a = 4\,h$ In gray: the exact solution. In black: our model.

Figure 8 displays the variations, as a function of the global thickness coordinate z, of the normalized electric displacement:

$$\overline{D}_3(z;t) = \frac{D_3(x_\alpha, z; t)}{D_3(x_1 = 0, x_2 = \frac{a}{2}, z = h; t)} \tag{64}$$

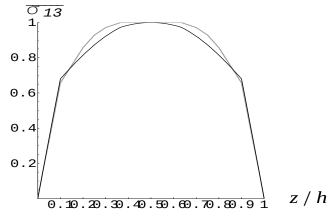


$\frac{z}{h}$

Figure 8: Normalized electric displacement, 5-layered plate, $a = 4h$. In gray: the exact solution. In black: our model.

Figure 9 displays the variations, as a function of the global thickness coordinate z, of the normalized transverse shear stress:

$$\overline{\sigma}_{13}(z;t) = \frac{\sigma_{13}(x_\alpha, z; t)}{\sigma_{13}(x_1 = 0, x_2 = \frac{a}{2}, z = \frac{h}{2}; t)} \tag{65}$$

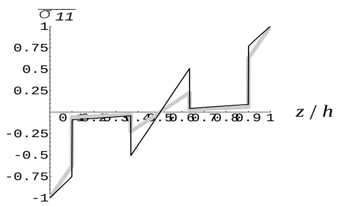


$\frac{z}{h}$

Figure 9: Normalized transverse shear stress, 5-layered plate, $a = 4\,h$. In gray: the exact solution. In black: our model.

Figure 10 displays the variations, as a function of the global thickness coordinate z, of the normalized longitudinal shear stress:

$$\overline{\sigma}_{11}(z;t) = \frac{\sigma_{11}(x_\alpha, z; t)}{\sigma_{11}(x_\alpha = \frac{a}{2}, z = 0; t)} \tag{66}$$



$\frac{z}{h}$

Figure 10: Normalized longitudinal shear stress, 5-layered plate, $a = 4\,h$. In gray: the exact solution. In black: our model.

1.10 Applications of the piezoelectric plate model

In the following, we present applications of the piezoelectric plate model; more results can be found in [7].

1.10.1 Bimorph plate under mechanical loading

Consider a rectangular plate, of width a, bimorph [1], supposed of infinite length, made of PZT4 ceramic, under cylindrical bending, simply supported, submitted to a force density p on its top face, in closed circuit (which means that the electrostatic potential on the top and bottom surfaces is equal zero): $\varphi^{1^B} = \varphi^{3^T} = 0$) (see figure 11).

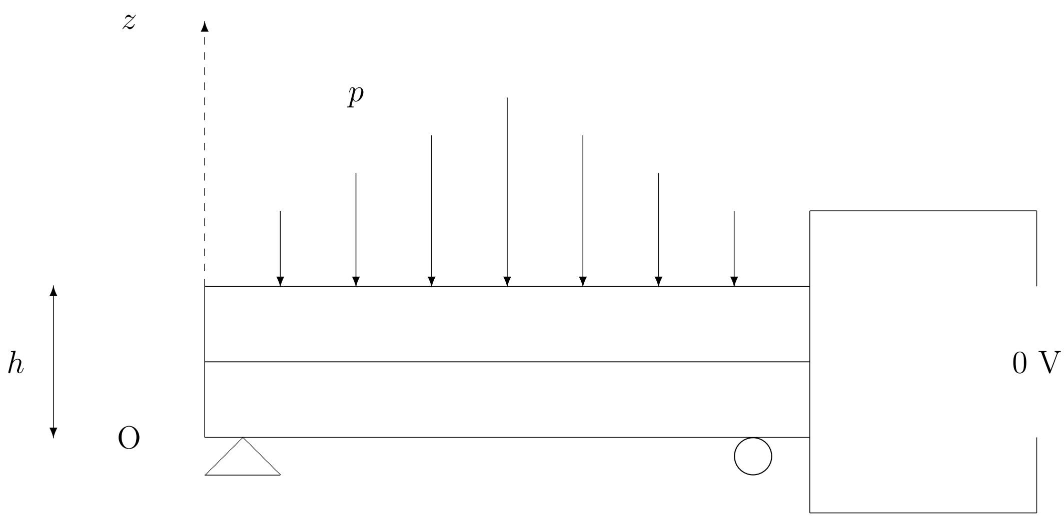


Figure 11: Bimorph plate under mechanical loading

Proposition 1.28. *The plate being supposed of infinite length, mechanical and electrical quantities (stresses, strains, displacements, electric field, electrostatic potential) do not depend on x_2.*

Assumptions 1.29. *Since the component U_2 of the displacement field does not play any part, we shall take:*

$$U_2 = 0$$

Data 1.30. *Material constants of PZT4 ceramic are given in tables 1, 2.*

Proposition 1.31. *The simply supported boundary conditions can be written as:*

$$w(x_1 = 0, z; t) = w(x_1 = a, z; t) = 0 \tag{67}$$

[1] A *bimorph* plate is a structure made of two identical materials, but with opposed polarization axes: ${e_{3\alpha}}^2 = -{e_{3\alpha}}^1$, ${e_{15}}^2 = -{e_{15}}^1$, ${e_{24}}^2 = -{e_{24}}^1$

Assumptions 1.32. *The force density p is assumed to be simply sinusoidal:*

$$p\,(x_\alpha, z; t) = p_0\, e^{j\,\omega\,t}\, \sin(\frac{\pi\, x_1}{a}) \tag{68}$$

where $p_0 = 0.05 MPa$.

Proposition 1.33. *The electrostatic potential (20) can be written as:*

$$\varphi(x_\alpha, z; t) = Q_{1M}(z)\,{\varphi^1}^M \tag{69}$$

where:

$$Q_{1M}(z) = (1 - {\xi_1}^2)\,\chi^1(z) - (1 - {\xi_2}^2)\,\chi^2(z) \tag{70}$$

Proof 1.34. *The antisymmetry of the problem yields:*

$${\varphi^2}^M = -{\varphi^1}^M \tag{71}$$

Remark 1.6. The piezoelectric coefficients of both layers being identical, no continuity conditions are required for the electrical displacement.

The generalized mechanical unknowns are the membrane displacements u_α, the deflection w, and the transverse shear stresses ${\gamma_\alpha}^0$.
The generalized electrostatic unknown is ${\varphi^1}^M$.

Assumptions 1.35. *In the same way as for the single layered plate, the solution is searched as (167):*

$$\left\{ \begin{array}{rcl} u_1 & = & A_1\, e^{j\,\omega\,t}\,\cos(\frac{\pi\,x_1}{a})\,\sin(\frac{\pi\,x_2}{b}) \\ w & = & B\, e^{j\,\omega\,t}\,\sin(\frac{\pi\,x_1}{a})\,\sin(\frac{\pi\,x_2}{b}) \\ {\gamma_1}^0 & = & C_1\, e^{j\,\omega\,t}\,\cos(\frac{\pi\,x_1}{a})\,\sin(\frac{\pi\,x_2}{a}) \end{array} \right. \quad , \quad {\varphi^1}^M = \Phi_1\, e^{j\,\omega\,t}\,\sin(\frac{\pi\,x_1}{a})\,\sin(\frac{\pi\,x_2}{b}) \tag{72}$$

which enables us to satisfy the simply supported boundary conditions (53).

By substituting these expressions into the equations of motion given by equations (13), in conjunction with the boundary conditions (45), the constitutive law (31) and the displacement field (13), we obtain a linear system in A_1, B, C_1, Φ_1, of the form:

$$K_2 \begin{pmatrix} A_1 \\ B \\ C_1 \\ \Phi_1 \end{pmatrix} = B_2 \tag{73}$$

Detail of the coefficients of the matrices K_2 and of the vector B_2 is given in [7].

Figure 12 displays the variations, as a function of the global thickness parameter z, of the normalized electrostatic potential:

$$\overline{\varphi}(z;t) = \frac{\varphi(x_\alpha, z; t)}{\varphi(x_\alpha = \frac{a}{2}, z = \frac{h}{2}; t)} \tag{74}$$

Figure 13 displays the variations, as a function of the global thickness parameter z, of the normalized transverse shear stress:

$$\overline{\sigma}_{13}(z;t) = \frac{\sigma_{13}(x_\alpha, z; t)}{\sigma_{13}(x_1 = 0, x_2 = \frac{a}{2}, z = \frac{h}{2}; t)} \tag{75}$$

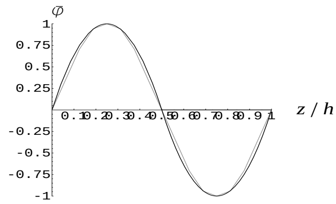


Figure 12: Normalized electrostatic potential, bimorph plate under mechanical loading, $a = 5\,h$. In gray: exact solutio. In black: our model.

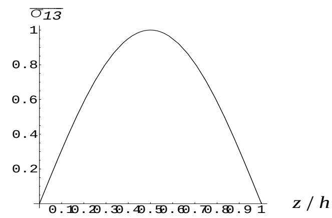


Figure 13: Normalized transverse shear stress, bimorph plate under mechanical loading, $a = 5\,h$.

Figure 14 displays the variations, as a function of the global thickness parameter z, of the normalized longitudinal stress:

$$\overline{\sigma}_{11}(z;t) = \frac{\sigma_{11}(x_\alpha, z; t)}{\sigma_{11}(x_\alpha = \frac{a}{2}, z = 0; t)} \tag{76}$$

Figure 15 displays the variations, as a function of the global thickness parameter z, of the normalized longitudinal displacement:

$$\overline{U}_1(z;t) = \frac{U_1(x_\alpha, z; t)}{U_1(x_\alpha = \frac{a}{2}, z = 0; t)} \tag{77}$$

1.10.2 Bimorph plate with imposed potentials

Consider a rectangular plate, of width a, bimorph, supposed of infinite length, made of PZT4 ceramic, under cylindrical bending, simply supported, submitted, on its top face, to an electrostatic potential $+V$, and, on its bottom face, to an electrostatic potential $-V$ (see figure 16).

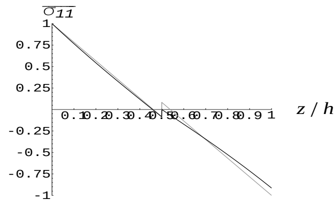


Figure 14: Normalized longitudinal stress, bimorph plate under mechanical loading, $a = 5\,h$. In gray: exact solution. In black: our model.

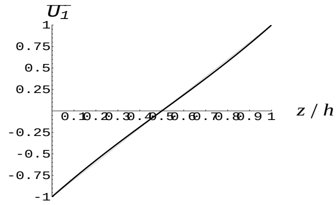


Figure 15: Normalized longitudinal displacement, bimorph plate under mechanical loading, $a = 5\,h$. In gray: exact solution. In black: our model.

Proposition 1.36. *The plate being supposed of infinite length, the mechanical and electrical quantities (stresses, strains, displacements, electric field, electrostatic potential), do not depend on* x_2.

Assumptions 1.37. *Since the component* U_2 *of the kinematic field does not play any part, we shall take:*

$$U_2 = 0$$

Data 1.38. *Material constants of the PZT4 ceramic are given in tables 1, 2.*

Proposition 1.39. *The simply supported boundary conditions yield:*

$$w(x_1 = 0, z; t) = w(x_1 = a, z; t) = 0 \tag{78}$$

Assumptions 1.40. *The potential* V *is assumed to be simply sinusoidal:*

$$V\,(x_\alpha, z; t) = V_0\, e^{j\,\omega\,t}\, \sin\big(\frac{\pi\, x_1}{a}\big) \tag{79}$$

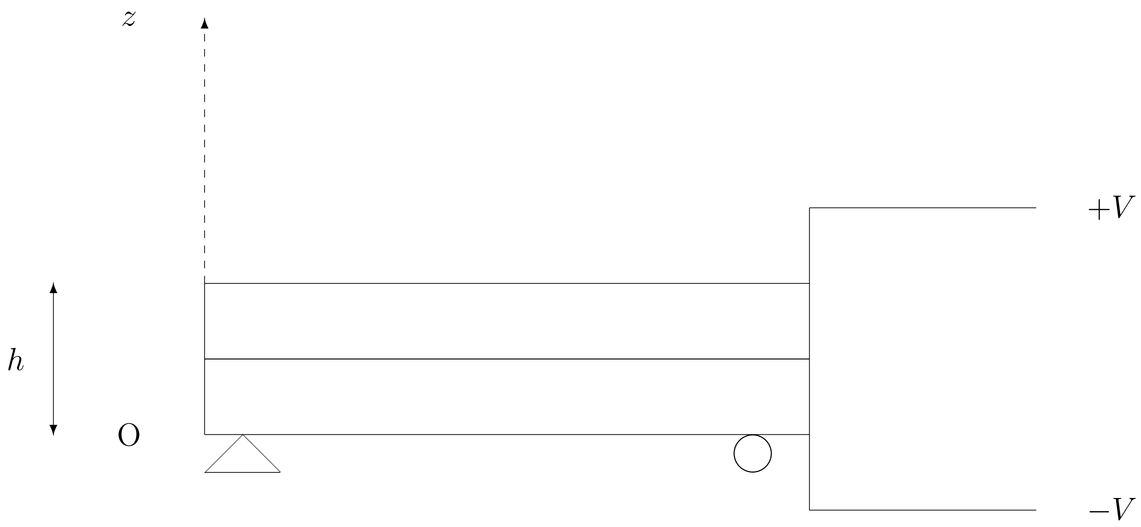


Figure 16: Bimorph plate, with imposed potentials

Proposition 1.41. *The electrostatic potential (20) can be written as:*

$$\varphi(x_\alpha, z; t) = \left[-\frac{1}{2}\,\xi_1\,(\xi_1{-}1)\,V{+}(1{-}{\xi_1}^2)\,{\varphi^1}^M(x_\alpha; t){+} \right] \chi^1(z) \left[\frac{1}{2}\,\xi_2\,(\xi_2{+}1)\,V{-}(1{-}{\xi_2}^2)\,{\varphi^1}^M(x_\alpha; t) \right] \chi^2(z) \tag{80}$$

Proof 1.42. *The antisymmetry of the problem yields:*

$$ {\varphi^2}^M = -{\varphi^1}^M \tag{81}$$

Remark 1.7. The dielectric coefficients of both layers being identical, no continuity conditions are required for the electrical displacement.

The generalized mechanical unknowns are membrane displacements u_α, w, and the transverse shears ${\gamma_\alpha}^0$.
The generalized electrical unknown is ${\varphi^1}^M$.

Assumptions 1.43. *As well as for the single-layered plate, the solution is searched as (167):*

$$\left\{\begin{array}{lcl} u_1 & = & A_1\, e^{j\,\omega\,t}\,\cos(\frac{\pi\,x_1}{a})\,\sin(\frac{\pi\,x_2}{b}) \\ w & = & B\, e^{j\,\omega\,t}\,\sin(\frac{\pi\,x_1}{a})\,\sin(\frac{\pi\,x_2}{b}) \\ {\gamma_0}^1 & = & C_1\, e^{j\,\omega\,t}\,\cos(\frac{\pi\,x_1}{a})\,\sin(\frac{\pi\,x_2}{a}) \end{array}\right. \quad , \quad \varphi^{1M} = \Phi_1\, e^{j\,\omega\,t}\,\sin(\frac{\pi\,x_1}{a})\,\sin(\frac{\pi\,x_2}{b}) \tag{82}$$

which enables us to satisfy the simply supported boundary conditions (??).

By substituting these expressions into the equations of motion given by equations (13), in conjunction with the boundary conditions (45), the constitutive law (31) and the displacement field (13), we obtain a linear system in A_1, B, C_1, Φ_1:

$$K_2 \begin{pmatrix} A_1 \\ B \\ C_1 \\ \Phi_1 \end{pmatrix} = B_2 \tag{83}$$

The detail of coefficients of the matrices K_2 and of the vector B_2 is given in [7].

Figure 17 displays the variations, as a function of the global thickness parameter z, of the normalized electrostatic potential:

$$\overline{\varphi}(z;t) = \frac{\varphi(x_\alpha, z; t)}{\varphi(x_\alpha = \frac{a}{2}, z = \frac{h}{2}; t)} \tag{84}$$

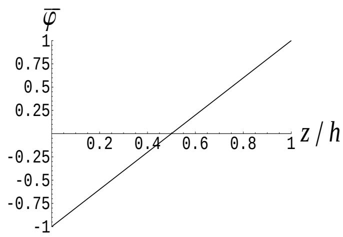


Figure 17: Normalized electrostatic potential, bimorph plate with imposed potentials, $a = 4\,h$. In gray: exact solution. In black: our model.

Figure 18 displays the variations, as a function of the global thickness parameter z, of the normalized transverse shear stress:

$$\overline{\sigma}_{13}(z;t) = \frac{\sigma_{13}(x_\alpha, z; t)}{\sigma_{13}(x_1 = 0, x_2 = \frac{a}{2}, z = \frac{h}{2}; t)} \tag{85}$$

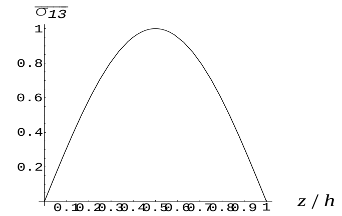


Figure 18: Normalized transverse shear stress, bimorph plate with imposed potentials, $a = 5\,h$.

Figure 19 displays the variations, as a function of the global thickness parameter z, of the normalized longitudinal stress:

$$\overline{\sigma}_{11}(z;t) = \frac{\sigma_{11}(x_\alpha, z; t)}{\sigma_{11}(x_\alpha = \frac{a}{2}, z = 0; t)} \tag{86}$$

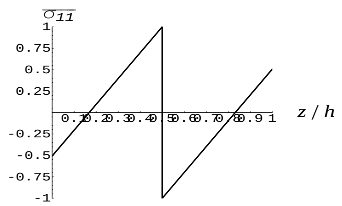


Figure 19: Normalized longitudinal stress, bimorph plate with imposed potentials, $a = 5\,h$.

2 Multilayered piezoelectric shells modelling

We hereafter study the modelling of multilayered piezoelectric shells. Our theory is based on the same hybrid approach as previously: the electrical and mechanical continuity conditions at layer interfaces are satisfied, as well as the boundary conditions on the top and bottom surfaces of the shell.
The accuracy of our theory is assessed through investigation of significant problems, for which an exact three-dimensional solution is known.

2.1 Introduction

The modeling of piezoelectric shells mostly concerns cases attached to specific geometries (cylindrical, spherical).
Toupin [48] studied the static response of a radially polarized spherical piezoelectric shell.
Adelman et al.[13], [14] examined cases involving hollow piezoelectric cylinders.
Sun et al. [45], Karlash [27] studied wave propagation in layered piezoelectric cylinders.
Paul et al. [40], [41] examined free vibration problems.
Siao et al. [44]] proposed a semi-analytic model for layered piezoelectric cylinders taking into account a layerwise behavior of the composite.
Analytic solutions for laminated piezoelectric cylinders were proposed by Mitchell et al. [35], Xu et al. [57], Heyliger [26], Dumir et al. [20], Drumheller et al. [19]. For this purpose, Drumheller [19] used classical shell theory for free vibrations of shells of revolution.
Haskins et al. [23] proposed the development of electrical and mechanical quantities as expansions of the thickness variable.
Tzou et al. [52] proposed the development of electrical and mechanical quantities as expansions of the thickness variable.
It was done by Tzou et al. [53], this time with a shear-deformation theory.
Other piezoelectric shell models and finite element approximations, based on single-layer models were also developped by Tzou et al. [54].
A Reissner-Mindlin shear-deformation shell finite element with surface bonded piezoelectric layers was developped by Lammering [29].
Koconis et al. [28] used a Ritz method for three-layered shells with embedded piezoelectric actuators. Tzou et al. [54] proposed a coupled theory where the piezoelectric shells are considered as a layerwise assembly of curvilinear solid piezoelectric triangular elements.

Heyliger et al. [26] developed a finite-element for laminated piezoelectric shells.
Saravanos [43] used a coupled mixed theory for curvilinear composite piezoelectric laminates with the first-order shear deformation theory hypothesis and a layerwise approximation of the electrostatic potential, along with the corresponding finite element for piezoelectric shells.

We presently extend our piezoelectric plate model to shells.
As previously, we associate our displacement type approach, which is a " single-layered " one, continuous at layer interfaces, to quadratic variations through the thickness of the electrostatic potential, also continuous. The transverse shear stresses, under a constant electrical field, as well as the electrical displacement, under a constant strain, are also continuous. Refinements of the membrane and transverse shear stresses are taken into account by means of trigonometric functions
Also, the conditions at layer interfaces, where values of the electrostatic potential can be imposed, are satisfied.
Finally, the piezoelectric boundary value problem is constructed using the consistant coupled constitutive law, in conjunction with the above displacements and electrostatic potential fields. The proposed piezoelectric shell model is evaluated for significant problems, for which the exact three-dimensional solution is known [26].

2.2 Mechanical study

Consider an undeformed laminated shell of constant thickness h. The space occupied by the shell will be denoted V. The boundary of the shell is the reunion of the upper surface S_h , the lower surface S_0 , and the edge faces A. consisting of an arrangement of a finite number N of piezoelectric layers.
a denotes the length, b the width, h its thickness, and V the volume occupied by the shell (see figure 2.2).

Notation. The frontier of the shell is constituted by the reunion of its bottom surface S_0, its top surface S_h, and its lateral surface $\mathcal{A}$.
S_i denotes the interface between the i^{th} and $(i+1)^{th}$ layers, and z_i the distance between S_0 and S_i.

The reference surface coïncides with the bottom surface S_0.

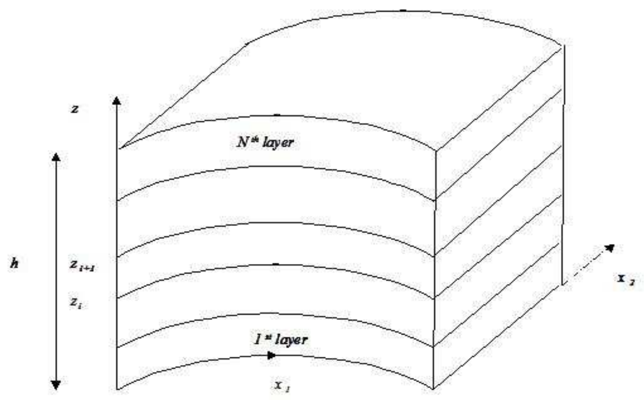


Figure 20: The multilayered piezoelectric shell.

Notation. The Einsteinian summation convention applies to repeated indices, where Latin indices range from 1 to 3 while Greek indices range from 1 to 2.

Notation.

V	Volume occupied by the shell
h	Total thickness of the shell
R	Radiius of curvature of the shell
R_1, R_2	Main radii of curvature of the reference surface
S_0	Bottom surface of the shell
S_h	Top surface of the shell
S_i	Bottom surface of the i^{th} layer
z_i	Distance between S_0 and S_i
z_{i0}	Distance between S_0 and the mid surface of the i^{th} layer
$\mathcal{A}$	Lateral surface of the shell
$(\overrightarrow{a_i})$	Covariant basis
$(\overrightarrow{g_i})$	Covariant basis
$(\overrightarrow{a^i})$	Contravariant basis
$(\overrightarrow{g^i})$	Contravariant basis
${\delta_\alpha}^\beta$	Kronecker symbol
$b_\alpha\beta$	Covariant components of the curvature tensor
${b_\alpha}^\beta$	Mix components of the curvature tensor
(x_i)	Curvilinear coordinatess
L_i	Lamé coefficients
s_k, $k = 1, \ldots, 6$	Strains
σ_k, $k = 1, \ldots, 6$	Stresses
${C_{mnpq}}^{(i)}$, or ${C_{KL}}^{(i)}$	Components of the elastic stiffness tensor under a constant electrical field
$'$	Differentiation with respect to z
$_{\|i}$	Covariant derivation with respect to x_i
$\cdot$	Differentiation with respect to time t
δ	Variational operator
φ	Electrostatic potential
${\varphi^1}^B$	Electrostatic potential on S_0
${\varphi^{N+1}}^B$	Electrostatic potential on S_h
${\varphi^i}^B$	Electrostatic potential on S_i
${\varphi^i}^M$	Electrostatic potential on the midsurface of the i^{th} layer
${\varphi^i}^T$	Electrostatic potential sur S_{i+1}
E_l, $l = 1, \ldots, 3$	Components of the electric field $\overrightarrow{E}$
D_k, $k = 1, \ldots, 3$	Components of the electric displacement
${e_{kl}}^{(i)}$	Piezoelectric constants, under a constant strain, of the i^{th} layer
${\varepsilon_{kl}}^{(i)}$	Dielectric constants, under a constant strain, of the i^{th} layer (or components of the tensor of permittivities of the i^{th} layer)
ρ	Mass density
ε_0	Permittivity of vacuum [1]

[1] $\varepsilon_0 = 8.85\,10^{-12}\,F\,/\,m$

2.2.1 Geometric considerations for shells

A point M out of the reference surface being given, let us denote by P the point of the reference surface closest to M.
Covariant base vectors $(\overrightarrow{a_i})$, $(\overrightarrow{g_i})$, and contravariant base vectors $(\overrightarrow{a^i})$, $(\overrightarrow{g^i})$, are defined by:

$$\overrightarrow{a_\alpha} = P_{,\alpha} \quad , \quad \overrightarrow{a_3} = \frac{\overrightarrow{a_1} \wedge \overrightarrow{a_2}}{\|\overrightarrow{a_1} \wedge \overrightarrow{a_2}\|} \quad , \quad \left(\overrightarrow{a_1} \wedge \overrightarrow{a_2}\right) \cdot \overrightarrow{a_3} > 0 \tag{87}$$

$$\begin{cases} \overrightarrow{a_\alpha} \cdot \overrightarrow{a^\beta} &= {\delta_\alpha}^\beta \\ \overrightarrow{a^3} &= \overrightarrow{a_3} \end{cases} \tag{88}$$

$$\overrightarrow{g_i} = M_{,i} \quad , \quad \left(\overrightarrow{g_1} \wedge \overrightarrow{g_2}\right) \cdot \overrightarrow{g_3} > 0 \tag{89}$$

$$\begin{cases} \overrightarrow{g_\alpha} \cdot \overrightarrow{g^\beta} &= {\delta_\alpha}^\beta \\ \overrightarrow{g^3} &= \overrightarrow{g_3} \end{cases} \tag{90}$$

Thus:

$$P = M + z\,\overrightarrow{a_3} \tag{91}$$

Notation. Let us introduce:

$$a_{\alpha\beta} = \overrightarrow{a_\alpha} \cdot \overrightarrow{a_\beta} \quad , \quad g_{\alpha\beta} = \overrightarrow{g_\alpha} \cdot \overrightarrow{g_\beta} \tag{92}$$

Proposition 2.1.

$$\overrightarrow{a_\beta} = a_{\alpha\beta}\,\overrightarrow{a^\beta} \quad , \quad \overrightarrow{a^\beta} = a^{\alpha\beta}\,\overrightarrow{a_\beta} \tag{93}$$

$$\overrightarrow{g_\alpha} = {\mu^\beta}_\alpha\,\overrightarrow{a_\beta} = g_{\alpha\beta}\,\overrightarrow{g^\beta} \quad , \quad \overrightarrow{g^\alpha} = -{{\mu^\alpha}_\beta}^{-1}\,\overrightarrow{a^\beta} = g^{\alpha\beta}\,\overrightarrow{g_\beta} \quad , \quad \overrightarrow{g^3} = \overrightarrow{a^3} = \overrightarrow{a_3} \tag{94}$$

Definition 2.1. The mixed components of the *shifter tensor* are given by:

$$\mu^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} - z\, b^{\alpha}{}_{\beta} \tag{95}$$

The covariant components of the *curvature tensor* are given by:

$$b_{\alpha\beta} = \overrightarrow{a_{\alpha,\beta}} \cdot \overrightarrow{a^3} \tag{96}$$

The mixed components of the *curvature tensor* are given by:

$$b^{\alpha}{}_{\beta} = -\overrightarrow{a_{3,\beta}} \cdot \overrightarrow{a^{\alpha}} \tag{97}$$

Assumptions 2.2. *In the following, the curvilinear coordinates (or shell coordinates) are assumed to be orthogonal, and are such that the curves x_1 = constant, x_2 = constant are lines of curvature on the reference surface.*
The curves z = constant are straight lines perpendicular to the surface S_0.
R_1, R_2 denote the principal radii of curvature of the reference surface.

Proposition 2.3. *The distance between two points $P(x_1, x_2, 0)$ and $P'(x_1 + dx_1, x_2 + dx_2, 0)$ of the reference surface is given by:*

$$ds^2 = {\alpha_1}^2\, {dx_1}^2 + {\alpha_2}^2\, {dx_2}^2 \tag{98}$$

where α_1 and α_2 are the coefficients of metrics, given by:

$${\alpha_l}^2 = \left(\frac{\partial P}{\partial x_l}\right) \left(\frac{\partial P}{\partial x_l}\right) \quad l = 1,\, 2 \tag{99}$$

Proposition 2.4. *The distance between two points $M(x_1, x_2, x_3)$ and $M'(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$, out of the reference surface, is given by:*

$$ds^2 = {L_1}^2\, {dx_1}^2 + {L_2}^2\, {dx_2}^2 + {L_3}^2\, {dx_3}^2 \tag{100}$$

where L_1, L_2 and L_3 denote the Lamé coefficients, given by:

$$L_1 = \alpha_l \left(1 + \frac{z}{R_1}\right) \;,\;\; L_2 = \alpha_l \left(1 + \frac{z}{R_2}\right) \;,\;\; L_3 = 1 \tag{101}$$

2.3 Kinematic assumptions

Assumptions 2.5. *The displacement field $\overrightarrow{U}$ of a point $M(x_\alpha, z)$ of the shell, is defined by its components (U_α, U_z) in the covariant basis $\left(\overrightarrow{g^\alpha}, \overrightarrow{g^3}\right)$, approximated under the form:*

$$\left\{ \begin{array}{lcl} U_\alpha & = & u_\alpha + z\,\eta_\alpha + f(z)\,\psi_\alpha + g(z)\,\gamma^0_\alpha + \sum_{m=1}^{N-1}(z - z_m)\,u_{m\alpha}\,H(z - z_m) \\ U_z & = & w \end{array} \right. \tag{102}$$

where:

$$f(z) = \frac{h}{\pi}\,\sin\left(\frac{\pi\,z}{h}\right)\ ,\ \ g(z) = \frac{h}{\pi}\,\cos\left(\frac{\pi\,z}{h}\right) \tag{103}$$

where H denotes the Heaviside step function.

As in the previous part, the use of the *sine* and *cosine* functions can be justified as in Touratier [49], by a discrete-layer approach, from the three-dimensional modelling of Cheng [16] for thick plates.

The u_α are membrane displacements, the γ^0_α the components of the transverse shear stress at $z = 0$, w the deflection, ψ_α and $u_{m\alpha}$ *a priori unknown functions*, which will be determined thanks to the conditions at layer interfaces as well as on the top and bottom surfaces.

2.4 Uncoupled constitutive law

Let us denote by ${C_{mnkl}}^{(i)}$ (or ${C_{KL}}^{(i)}$, double indexation) the components of the elastic stiffness tensor, under a constant electric field, of the i^{th} layer, and by s_{kl} the components of the stress tensor (or s_I, double indexation).

Assumptions 2.6. *We use, in the following, the hypothesis of* small pertubations,*which yields:*

$$s_{kl} = \frac{V_{k\|l} + V_{l\|k}}{2} \tag{104}$$

where:

$$\left\{ \begin{array}{lcl} V_{\alpha\|\beta} & = & {\mu^\nu}_\alpha\,[\,U_{\nu|\beta} - b_{\nu\beta}\,w\,] \\ V_{\alpha\|3} & = & {\mu^\nu}_\alpha\,U_{\nu,3} \\ V_{3\|3} & = & U_{3,3} \end{array} \right. \tag{105}$$

Remark 2.1. Equations (105) yield:

$$\left\{ \begin{array}{lcl} V_{\alpha\|\beta} & = & \mu^{\nu}{}_{\alpha}\,[\,u_{\nu|\beta} + z\,\eta_{\nu|\beta} + f(z)\,\psi_{\nu|\beta} + g(z)\,\gamma^{0}{}_{\nu|\beta} + \sum_{m=1}^{N-1}(z-z_m)\,u_{m\nu|\beta}\,H(z-z_m) - b_{\nu\beta}\,w\,] \\ V_{\alpha\|3} & = & \mu^{\nu}{}_{\alpha}\,[\eta_{\nu} + f'(z)\,\psi_{\nu} + g'(z)\,\gamma^{0}_{\alpha}\nu + \sum_{m=1}^{N-1}\,u_{m\nu|}\,H(z-z_m)\,] \\ V_{3\|\alpha} & = & w_{|\alpha} + b^{\nu}{}_{\alpha}\,[u_{\nu} + z\,\eta_{\nu} + f(z)\,\psi_{\nu} + g(z)\,\gamma^{0}{}_{\nu} + \sum_{m=1}^{N-1}(z-z_m)\,u_{m\nu}\,H(z-z_m)] \\ V_{3\|3} & = & w_{,3} \end{array} \right. \tag{106}$$

Especially:

$$\begin{array}{lcl} s_{\alpha 3} & = & \frac{1}{2}\,[\,V_{\alpha\|3} + V_{3\|\alpha}\,] \\ & = & \frac{1}{2}\left\{\mu^{\nu}{}_{\alpha}\,[\eta_{\nu} + f'(z)\,\psi_{\nu} + g'(z)\,\gamma^{0}_{\alpha}\nu + \sum_{m=1}^{N-1}\,u_{m\nu}\,H(z-z_m)\,]\right. \\ & + & \left. w_{|\alpha} + b^{\nu}{}_{\alpha}\,[u_{\nu} + z\,\eta_{\nu} + f(z)\,\psi_{\nu} + g(z)\,\gamma^{0}{}_{\nu} + \sum_{m=1}^{N-1}(z-z_m)\,u_{m\nu}\,H(z-z_m)]\,\right\} \end{array} \tag{107}$$

The uncoupled constitutive law yields:

$$\sigma_{mn}{}^{(i)} = C_{mnkl}{}^{(i)}\,s_{kl} \tag{108}$$

i.e., under double indexation:

$$\sigma_{J}^{(i)\,d} = C_{JK}{}^{(i)}\,s_{K} \tag{109}$$

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = C \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \end{pmatrix} \tag{110}$$

2.5 Boundary conditions

i. *Conditions on the top and bottom surfaces:*

Proposition 2.7. *The zero value of the transverse shear stress on S_0 and S_h, under a constant electric field, yields:*

$$\eta_{\alpha} = -\psi_{\alpha} - w_{|\alpha} - b^{\nu}{}_{\alpha}\,[u_{\nu} + \frac{h}{\pi}\,\gamma^{0}{}_{\nu}] \tag{111}$$

and:

$$\psi_\alpha = d^\beta{}_\alpha \, \gamma^0{}_\beta + \sum_{m=1}^{N-1} f_m{}^\beta{}_\alpha \, u_{m\beta} \tag{112}$$

where:

$$\left\{ \begin{array}{rcl} [\, d^\beta{}_\alpha \,] & = & [\, h\, b^\beta{}_\alpha - 2\, \delta^\beta{}_\alpha \,]^{-1} \, [\, \frac{2\,h}{\pi} \, b^\beta{}_\alpha \,] \\ [\, f_m{}^\beta{}_\alpha \,] & = & [h\, b^\beta{}_\alpha - 2\, \delta^\beta{}_\alpha]^{-1} \, [\delta^\beta{}_\alpha - b^\beta{}_\alpha \, z_m] \end{array} \right. \tag{113}$$

Proof. The nullity of the transverse shear stress on S_0 and S_h, under a constant electric field, can be written as:

$$\left\{ \begin{array}{rcl} \sigma_{6-\alpha}{}^{(1)^d} & = & 0 \\ \sigma_{6-\alpha}{}^{(N)^d} & = & 0 \end{array} \right. \tag{114}$$

which yields:

$$\left\{ \begin{array}{rcl} s_{\alpha 3}(z=0) & = & 0 \\ s_{\alpha 3}(z=h) & = & 0 \end{array} \right. \tag{115}$$

i.e., according to (107):

$$s_{\alpha 3}(z=0) = \frac{1}{2} \left[-\delta^\nu{}_\alpha \, [\eta_\nu + \frac{h}{\pi} \, \psi_\nu \,] + w_{|\alpha} + b^\nu{}_\alpha \, [u_\nu + \frac{h}{\pi} \, \gamma^0{}_\nu] \right] \tag{116}$$

and:

$$s_{\alpha 3}(z=h) = \frac{1}{2} \left[\mu^\nu{}_\alpha \, [\eta_\nu - \psi_\nu + \sum_{m=1}^{N-1} u_{m\nu} \,] + w_{|\alpha} + b^\nu{}_\alpha \, [u_\nu + h\, \eta_\nu - \frac{h}{\pi} \, \gamma^0{}_\nu + \sum_{m=1}^{N-1} (h - z_m) \, u_{m\nu}] \right] \tag{117}$$

since:

$$f(0) = g'(0) = f(h) = g'(h) = 0 \ , \ \ f'(0) = -f'(h) = 1 \ , \ \ g(0) = -g(h) = \frac{h}{\pi} \tag{118}$$

$$\mu^\nu{}_\alpha = \delta^\nu{}_\alpha - z\, b^\nu{}_\alpha \tag{119}$$

we thus have:

$$s_{\alpha 3}(z=0)=\frac{1}{2}\left[\,\eta_\alpha+\psi_\alpha+w_{|\alpha}+b^\nu{}_\alpha\,[u_\nu+\frac{h}{\pi}\,\gamma^0{}_\nu]\,\right] \tag{120}$$

and:

$$s_{\alpha 3}(z=h)=\frac{1}{2}\left[\,\mu^\nu{}_\alpha\,[\eta_\nu-\psi_\nu+\sum_{m=1}^{N-1}u_{m\nu}\,]+w_{|\alpha}+b^\nu{}_\alpha\,[u_\nu+h\,\eta_\nu-\frac{h}{\pi}\,\gamma^0{}_\nu+\sum_{m=1}^{N-1}(h-z_m)\,u_{m\nu}]\,\right] \tag{121}$$

$$\begin{array}{rcl} s_{\alpha 3} & = & \frac{1}{2}\,[\,V_{\alpha\|3}+V_{3\|\alpha}\,] \\ & = & \frac{1}{2}\,\{\mu^\nu{}_\alpha\,[\eta_\nu+f'(z)\,\psi_\nu+g'(z)\,\gamma^0_\nu+\sum_{m=1}^{N-1}u_{m\nu}\,H(z-z_m)\,] \\ & + & w_{|\alpha}+b^\nu{}_\alpha\,[u_\nu+z\,\eta_\nu+f(z)\,\psi_\nu+g(z)\,\gamma^0{}_\nu+\sum_{m=1}^{N-1}(z-z_m)\,u_{m\nu}\,H(z-z_m)]\,\} \end{array} \tag{122}$$

Hence:

$$\eta_\alpha=-\psi_\alpha-w_{|\alpha}-b^\nu{}_\alpha\,[u_\nu+\frac{h}{\pi}\,\gamma^0{}_\nu] \tag{123}$$

and:

$$-2\,\psi_\alpha+b^\nu{}_\alpha\,\psi_\nu n-2\,\frac{h}{\pi}\,\gamma^0{}_\nu+\sum_{m=1}^{N-1}(\delta^\nu{}_\alpha-b^\nu{}_\alpha\,z_m)\,u_{m\nu}=0 \tag{124}$$

which can be written as:

$$\psi_\alpha=d^\beta{}_\alpha\,\gamma^0{}_\beta+\sum_{m=1}^{N-1}f_m{}^\beta{}_\alpha\,u_{m\beta} \tag{125}$$

where:

$$\left\{\begin{array}{rcl} [\,d^\beta{}_\alpha\,] & = & [\,h\,b^\beta{}_\alpha-2\,\delta^\beta{}_\alpha\,]^{-1}\,[\,\frac{2\,h}{\pi}\,b^\beta{}_\alpha\,] \\ [\,f_m{}^\beta{}_\alpha\,] & = & [h\,b^\beta{}_\alpha-2\,\delta^\beta{}_\alpha]^{-1}\,[\delta^\beta{}_\alpha-b^\beta{}_\alpha\,z_m] \end{array}\right. \tag{126}$$

By substituting these expressions in (123), we deduce:

$$\eta_\alpha=-d^\beta{}_\alpha\,\gamma^0{}_\beta-\sum_{m=1}^{N-1}f_m{}^\beta{}_\alpha\,u_{m\beta}-w_{|\alpha}-b^\nu{}_\alpha\,[u_\nu+\frac{h}{\pi}\,\gamma^0{}_\nu] \tag{127}$$

$\square$

We then have, thanks to (122):

$$\begin{array}{rcl} s_{\alpha 3} & = & \frac{1}{2}\left\{[\,\mu^\nu{}_\alpha + z\,b^\nu{}_\alpha\,]\,\eta_\nu + [\,\mu^\nu{}_\alpha\;f'(z) + b^\nu{}_\alpha\,f(z)\,]\,\psi_\nu\right. \\ & + & [\mu^\nu{}_\alpha\;g'(z) b^\nu{}_\alpha\,g(z)\,]\,\gamma^0_\alpha\nu \\ & + & \sum_{m=1}^{N-1}[\mu^\nu{}_\alpha\; + b^\nu{}_\alpha\,(z - z_m)\,]\,u_{m\nu}\,H(z - z_m) \\ & + & \left. w_{|\alpha} + b^\nu{}_\alpha\,u_\nu\,\right\} \end{array} \tag{128}$$

i.e.:

$$\begin{array}{rcl} s_{\alpha 3} & = & \frac{1}{2}\left\{[\,\mu^\nu{}_\alpha + z\,b^\nu{}_\alpha\,]\,\{-d^\beta{}_\nu\,\gamma^0{}_\beta - \sum_{m=1}^{N-1} f_m{}^\beta{}_\nu\,u_{m\beta} - w_{|\nu} - b^\lambda{}_\nu\,[u_\lambda + \frac{h}{\pi}\,\gamma^0{}_\lambda]\}\right. \\ & + & [\,\mu^\nu{}_\alpha\;f'(z) + b^\nu{}_\alpha\,f(z)\,]\,\{\,d^\lambda{}_\nu\,\gamma^0{}_\lambda + \sum_{m=1}^{N-1} f_m{}^\lambda{}_\nu\,u_{m\beta}\,\} \\ & + & [\mu^\nu{}_\alpha\;g'(z) b^\nu{}_\alpha\,g(z)\,]\,\gamma^0_\nu \\ & + & \sum_{m=1}^{N-1}[\mu^\nu{}_\alpha\; + b^\nu{}_\alpha\,(z - z_m)\,]\,u_{m\nu}\,H(z - z_m) \\ & + & \left. w_{|\alpha} + b^\nu{}_\alpha\,u_\nu\,\right\} \end{array} \tag{129}$$

ii. Conditions at layer interfaces:

The continuity of the uncoupled transverse shear stress between the i^{th} and $(i+1)^{th}$ layer can be written as:

$$\sigma_{6-\alpha}{}^{(i)^d}(x_\alpha, z_i) = \sigma_{6-\alpha}{}^{(i+1)^d}(x_\alpha, z_i) \tag{130}$$

Thanks to (129), we then obtain a linear system of $N-1$ equations, which enables us to express the $u_{m\alpha}$, $m = 1, \ldots, N-1$ as functions of the transverse shears γ^0_α:

$$u_{m\alpha} = \lambda_{m\alpha}\,\gamma^0_\alpha \tag{131}$$

where the $\lambda_{m\alpha}$ are real constants, given by the resolution of the latter system.

2.6 Final form of the displacement field

Proposition 2.8. *The displacement field $\overrightarrow{U}$ of any point $M(x_\alpha, z)$ of the structure is given by:*

$$\left\{ \begin{array}{rcl} U_\alpha & = & \mu^\beta{}_\alpha\,u_\beta - z\,w_{|\alpha} + h^\beta{}_\alpha\,\gamma^0{}_\beta \\ U_z & = & w \end{array} \right. \tag{132}$$

where the $h^\beta{}_\alpha$ are functions of the global thickness variable z, given by:

$$h^\beta{}_\alpha(z) = g(z)\,\delta^\beta{}_\alpha - z\frac{h}{\pi}\,b^\beta{}_\alpha + [f(z)-z]\,d^\beta{}_\alpha + \sum_{m=1}^{N-1}\left\{ f_{(m)}{}^\beta{}_\alpha + (z-z_m)\,H(z-z_m)\,\delta^\beta{}_\alpha \right\}\lambda_{m\beta} \tag{133}$$

This displacement field has been developed in [12], [5], [3].

2.7 Electrical study; the linear piezoelectric constitutive law

This section refers to the same results as in the case of the plate (see above)

2.8 The two-dimensional boundary-value problem

2.8.1 Variational formulation

Hamilton's Principle yields:

$$\int_0^t \left\{ \int_V \{\sigma_i\,\delta s_i + D_i\,\delta\varphi_i\} dV + \int_V \{\overrightarrow{f_v}\cdot\delta\overrightarrow{U} + W\,\delta\varphi\}\,dV + \int_{\mathcal{A}} \overrightarrow{f_s}\cdot\delta\overrightarrow{U}\,dA + \int_{S_0}(p_0 - p_h)\,dS \right\} dt = 0 \tag{134}$$

δ being a variational operator, $\overrightarrow{f_v}$ the volumic density of body forces, $\overrightarrow{f_s}$ the surface density of body forces on the lateral surface of the shell, p_0 and p_h the prescribed components of traction on the top and bottom surfaces, and W the density of electric forces.

Notation. μ denotes the value of the determinant of *shifter tensor* $[\mu^\beta{}_\alpha]$ *at* $z=0$.
Let us ntroduce:

i. the generalized stresses:

$$\left\{\begin{array}{rcl}
N_1{}^{\alpha\beta} &=& -\int_0^h \{C_{\lambda j}\,s_j - e_{3\alpha}\,E_3\}\,\mu^\nu{}_\lambda\,\mu^\alpha{}_\nu\,\mu\,dz \\
N_2{}^{\alpha\beta} &=& -\int_0^h C_{66}\,s_6\,(1-\delta_{\lambda\beta})\,\mu^\nu{}_\lambda\,\mu^\alpha{}_\nu\,\mu\,dz \\
N_3 &=& -\int_0^h \left[\{C_{\alpha j}\,s_j - e_{3\alpha}\,E_3\}\,\mu^\nu{}_\alpha\,b_{\nu\alpha} + C_{66}\,s_6\,(1-\delta_{\alpha\beta})\,\mu^\nu{}_\alpha\,b_{\nu\beta}\right]\mu\,dz \\
N_4{}^\alpha &=& -\int_0^h \left[C_{6-\lambda,6-\lambda}\,s_{6-\lambda,6-\lambda} - e_{k,6-\lambda}\,E_k\right]\{\,\mu^\nu{}_\lambda\,h^\alpha{}_{\nu,3} + b^\nu{}_\lambda\,h^\alpha{}_\nu\,\}\,\mu\,dz \\
N_5{}^\alpha &=& -\int_0^h \{C_{\lambda j}\,s_j - e_{3\alpha}\,E_3\}\,\mu^\nu{}_\lambda\,h^\alpha{}_\nu(z)\,\mu\,dz \\
N_6{}^\alpha &=& -\int_0^h C_{66}\,s_6(1-\delta_{\lambda\beta})\,\mu^\nu{}_\lambda\,h^\alpha{}_\nu(z)\,\mu\,dz
\end{array}\right. \tag{135}$$

$$\left\{\begin{array}{lll} \mathcal{N}^{iB} & = & -\int_0^h E_3\,\varepsilon_{33}\,Q^{iB'}(z)\,\mu\,dz \ \ , \, i\in I \\ \mathcal{N}^{jM} & = & -\int_0^h E_3\,\varepsilon_{33}\,Q^{jM'}(z)\,\mu\,dz \ \ , \, j\in J \end{array}\right. \tag{136}$$

ii. the generalized momentums:

$$\left\{\begin{array}{lll} {M_1}^{\alpha\beta} & = & -\int_0^h \left[\,\{\,C_{\alpha j}\,s_j - e_{3\alpha}\,E_3\,\}\,{\mu^\nu}_\alpha + C_{66}\,s_{6|\nu\beta}\,(1-\delta_{\alpha\beta})\,{\mu^\nu}_\alpha\,\right]z\,\mu\,dz \\ {M_2}^{\alpha\beta} & = & -\int_0^h C_{66}\,s_6\,(1-\delta_{\alpha\beta})\,z\,\mu\,dz \end{array}\right. \tag{137}$$

$$\left\{\begin{array}{lll} \mathcal{M}^{iB\alpha} & = & -\int_0^h \{E_\alpha\,\varepsilon_{\alpha\alpha} + e_{k\alpha}\,s_\alpha\}\,Q^{iB}(z)\,\mu\,dz \ \ , \, i\in I \\ \mathcal{M}^{jM\alpha} & = & -\int_0^h \{E_\alpha\,\varepsilon_{\alpha\alpha} + e_{k\alpha}\,s_\alpha\}\,Q^{iM}(z)\,\mu\,dz \ \ , \, j\in J \end{array}\right. \tag{138}$$

iii. the generalized external mechanical forces :

$$\left\{\begin{array}{lll} {F_\nu}^{1\beta}{}_\alpha & = & -\int_0^h f_{\nu\alpha}\,{\mu^\beta}_\alpha\,\mu\,dz \\ {F_\nu}^{2}{}_\alpha & = & -\int_0^h f_{\nu\alpha}\,z\,\mu\,dz \\ {F_\nu}^{3\beta}{}_\alpha & = & -\int_0^h f_{\nu\alpha}\,{h^\beta}_\alpha(z)\,\mu\,dz \\ {F_\nu}^{3} & = & -\int_0^h f_{\nu\alpha}\,\mu\,dz \\ P & = & p_0 - p_h \end{array}\right. \tag{139}$$

iv. the generalized external electrostatic forces :

$$\left\{\begin{array}{lll} W^{iB} & = & -\int_0^h W\,Q^{iB}(z)\,\mu\,dz \ \ , \, i\in I \\ W^{jM} & = & -\int_0^h W\,Q^{iB}(z)\,\mu\,dz \ \ , \, j\in J \end{array}\right. \tag{140}$$

iv. the inertia terms:

$$\left\{\begin{array}{lll} I_1 & = & -\int_0^h \rho\,\mu\,dz \\ I_2 & = & -\int_0^h \rho\,z\,\mu\,dz \\ I_3 & = & -\int_0^h \rho\,h_\alpha(z)\,\mu\,dz \\ I_4 & = & -\int_0^h \rho\,z^2\,\mu\,dz \\ I_5 & = & -\int_0^h \rho\,z\,h_\alpha(z)\,\mu\,dz \\ I_6 & = & -\int_0^h \rho\,z\,{h_\alpha}^2(z)\,\mu\,dz \end{array}\right. \tag{141}$$

Proposition 2.9. *The equations of motion are given by:*

$$\left\{\begin{array}{rcl} {N_1^{\alpha\beta}}_{|\beta} + {N_2^{\alpha\beta}}_{|\beta} & = & {I_1}^{\alpha}\, \ddot{u}_\alpha - I_2\, \ddot{w}_{|\alpha} + {I_3}^{\alpha}\, \ddot{\gamma}^0_\alpha \\ {M_1^{\alpha\beta}}_{|\alpha\beta} + {M_2^{\alpha\beta}}_{|\alpha\beta} & = & {I_2}^{\alpha}\, \ddot{u}_{\alpha|\alpha} - I_4\, \ddot{w}_{|\alpha\alpha} + {I_5}^{\alpha}\, \ddot{\gamma}^0_{\alpha|\alpha} + I_1\, \ddot{w} \\ {N_5^{\alpha}}_{|\alpha} + {N_6^{\alpha\beta}}_{|\beta} + {N_4}^{\alpha} & = & {I_3}^{\alpha}\, \ddot{u}_\alpha - I_5\, \ddot{w}_{|\alpha} + {I_6}^{\alpha}\, \ddot{\gamma}^0_\alpha \\ \mathcal{N}^{iB} + {\mathcal{M}^{iB^\alpha}}_{|\alpha} & = & 0 \ , \ i \in I \\ \mathcal{N}^{jM} + {\mathcal{M}^{jm^\alpha}}_{|\alpha} & = & 0 \ , \ j \in J \end{array}\right. \tag{142}$$

Proof 2.10. *The equations of motion are deduced from Hamilton's Principle, in conjunction with the kinematics (132), including the constitutive law given by equations (27), by integration through the thickness of the shell.*

Proposition 2.11. *The boundary conditions leading to a " regular problem " are:*

$$\left\{\begin{array}{rcccc} {N_1}^{\alpha\beta} n_\beta \ + {N_2}^{\alpha\beta} n_\beta & = & {F_\nu}^{1\alpha}{}_{\beta} & ou & \delta u_\alpha = 0 \\ {M_1^{\alpha\beta}}_{|\beta} + {M_2^{\alpha\beta}}_{|\beta} & = & {F_\nu}^{2}{}_{\alpha} & ou & \delta w = 0 \\ {N_4}^{\alpha\beta} n_\beta & = & {F_\nu}^{3\alpha}{}_{\beta} & or & \delta {\gamma^0}_\alpha = 0 \\ {M_1}^{\alpha\beta} n_\beta + {M_2}^{\alpha\beta} n_\beta & = & F_{\nu 3} & ou & \delta w_{|\alpha} = 0 \\ \mathcal{N}^{iB^\alpha}\, n_\alpha & = & W^{iB} & ou & \delta\varphi^{iB} = 0 \ , \ i \in I \\ \mathcal{M}^{jM^\alpha}\, n_\alpha & = & W^{jM} & ou & \delta\varphi^{jM} = 0 \ , \ j \in J \end{array}\right. \tag{143}$$

Proof 2.12. *The equations of motions can be derived from Principe de Hamilton's Principle, the kinematics (132), and the constitutive law (27), through integration on the thickness of the shell.*

2.9 Numerical validation of the piezoelectric shell model

2.9.1 Associated plate model

The plate being a degenerated shell, our shell model is, in a first time, validated by the related plate model.

2.9.2 Free vibrations of an orthotropic cylindrical panel

Consider an orthotropic cylindrical panel, supposed of infinite length, made of PZT4 ceramic, under cylindrical bending, simply supported, submitted to a

surface force density p, in closed circuit (which means the electrostatic potential on the top and bottom surfaces is equal to zero: $\varphi^{1^B} = \varphi^{1^T} = 0$).
R denotes the radius of the cylinder, h its thickness, α its central angle, and θ the angular coordinate (see figure 21).

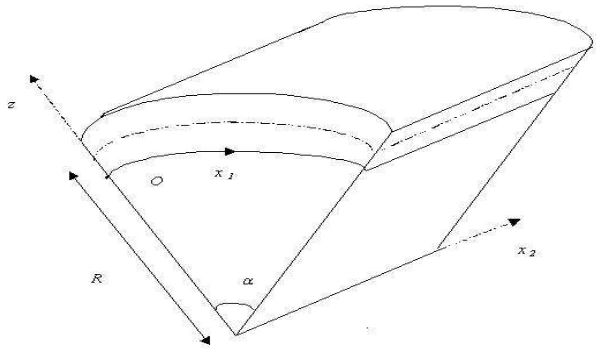


Figure 21: The cylindrical panel.

Data 2.13. *Material constants of PZT4 ceramic are given in tables 7, 8.*

	C_{11}	C_{22}	C_{33}	C_{12}	C_{13}	C_{23}	C_{44}	C_{55}	C_{66}
PZT 4	139	139	115	77.8	74.3	74.3	25.6	25.6	30.6

Table 7: Independent elastic constants of PZT4 ceramic (in GPa)

	e_{31}	e_{32}	e_{33}	e_{15}	ε_{11}	ε_{22}	ε_{33}
PZT 4	-5.2	-5.2	15.1	12.7	13.06	13.06	11.51

Table 8: Independent piezoelectric and dielectric constants of PZT4 ceramic (e_{ij} in C/m^2, ε_{ii} in nF/m)

Proposition 2.14. *The simply supported boundary conditions yield:*

$$w(\theta = 0, z; t) = w(\theta = \alpha, z; t) = 0 \tag{144}$$

Proposition 2.15. *The panel being supposed of infinite length, the mechanical and electrical quantities (stresses, strains, displacements, electric field, electrostatic potential), do not depend on* x_2.

Assumptions 2.16. *Since the component* U_2 *of the displacement field does not play any part, we shall take:*

$$U_2 = 0$$

Assumptions 2.17. *The force density* p *is assumed to be simply sinusoidal, of the form:*

$$p\,(x_\alpha, z; t) = p_0\, e^{j\,\omega\,t}\, \sin(\frac{\pi\, x_1}{\alpha}) \tag{145}$$

where $p_0 = 10\, N/m^2$.

Proposition 2.18. *The electrostatic potential (20) is approximated as (18):*

$$\varphi^1(x_\alpha, z; t) = (1 - {\xi_1}^2)\, {\varphi^1}^M(x_\alpha; t) \tag{146}$$

The mechanical generalized displacements remaining unknowns the membrane displacement u_1, w, and the transverse shear ${\gamma_1}^0$.
The electrical generalized unknown is ${\varphi^1}^M$.

Assumptions 2.19. *The solution is searched under the following form,which characterizes the propagation of harmonic plane-waves:*

$$\left\{ \begin{array}{ccc} u_1 & = & A_1\, e^{j\,\omega\,t}\, \cos(\frac{\pi\, x_1}{\alpha}) \\ w & = & B\, e^{j\,\omega\,t}\, \sin(\frac{\pi\, x_1}{\alpha}) \\ {\gamma_1}^0 & = & C_1\, e^{j\,\omega\,t}\, \cos(\frac{\pi\, x_1}{\alpha}) \end{array} \right. \quad , \quad {\varphi^1}^M = \Phi_1\, e^{j\,\omega\,t}\, \sin(\frac{\pi\, x_1}{\alpha}) \tag{147}$$

which enable us to satisfy the simply supported boundary conditions (159).

By substituting these expressions in the equations of the equilibrium (142), in conjunction with the boundary conditions (143), the constitutive law (31) and the displacement field (132), we obtain a linear system in A_1, B, C_1, Φ_1, of the form:

$$K_1 \begin{pmatrix} A_1 \\ B \\ C_1 \\ \Phi_1 \end{pmatrix} = B_1 \tag{148}$$

Detail of coefficients of the matrix K_1 and of the vector B_1 can be found in [8].

Figure 22 displays the variations, as a function of the global thickness variable z, of the normalized electrostatic potential:

$$\overline{\varphi}(z;t) = \frac{\varphi(\theta = \frac{\alpha}{2}, z; t)}{\varphi(\theta = \frac{\alpha}{2}, z = \frac{h}{2}; t)} \tag{149}$$

for a significative value of $\frac{R}{h}$.

Results (in black) are compared to the exact solution of Dumir [20] (in gray).

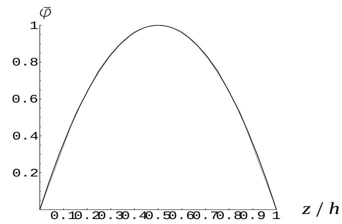


Figure 22: Normalized electrostatic potential, single-layered panel, $R = 4\,h$. In black: our model. In gray: exact solution.

Results obtained by our model perfectly fit the exact solution.

Figures 23, 24, 25 display the variations, as functions of the global thickness coordinate z, of the normalized transverse shear stress:

$$\overline{\sigma}_{13}(z;t) = \frac{\sigma_{13}(\theta = \frac{\alpha}{2}, z; t)}{\sigma_{13}(\theta = \frac{\alpha}{2}, z = \frac{h}{2}; t)} \tag{150}$$

for significative values of $\frac{R}{h}$.

Results (in black) are compared to the exact solution of Dumir [20] (in gray).

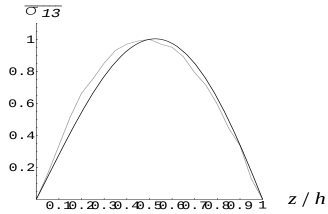


Figure 23: Normalized transverse shear stress, single-layered panel, $R = 4\,h$. In black: our model. In gray: exact solution.

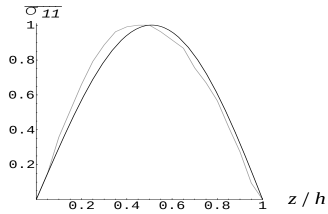


Figure 24: Normalized transverse shear stress, single-layered panel, $R = 10\,h$. In black: our model. In gray: exact solution.

The model appears to be in good agreement with the exact solution.
As expected, as the shell grows thinner, the results of our model get closer to those of the exact solution.

Figures 26, 27, 28 display the variations, as functions of the global thickness coordinate z, of the normalized longitudinal stress:

$$\overline{\sigma}_{11}(z;t) = \frac{\sigma_{11}(\theta = \frac{\alpha}{2}, z; t)}{\sigma_{11}(\theta = \frac{\alpha}{2}, z = \frac{h}{2}; t)} \tag{151}$$

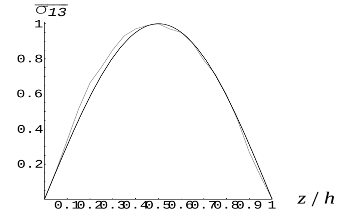


Figure 25: Normalized transverse shear stress, single-layered panel, $R = 100\,h$. In black: our model. In gray: exact solution.

for significative values of the quotient $\frac{R}{h}$.

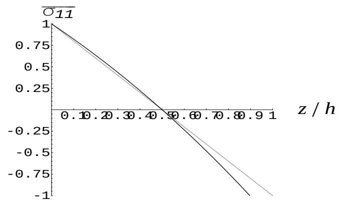


Figure 26: Normalized longitudinal stress, single-layered circular cylindrical panel, $R = 4\,h$. In black: our model. In gray: exact solution.

As for the transverse shear stress, the model appears to be in good agreement with the exact solution.

2.10 Applications of the multilayered piezoelectric shell model

We present, in the following, applications of our multilayered piezoelectric shell model. For the considered problems, there is no exact three-dimensional solution. Our results can be interpreted as generalizations, to the " shell case ", of Fernandes's [21], [22], in the case of piezoelectric plates.

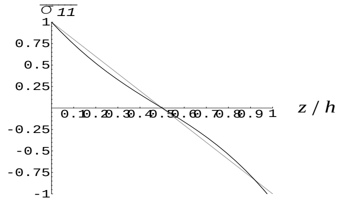


Figure 27: Normalized longitudinal stress, single-layered circular cylindrical panel, $R = 10\,h$. In black: our model. In gray: exact solution.

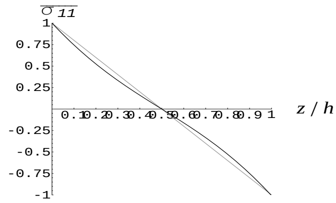


Figure 28: Normalized longitudinal stress, single-layered circular cylindrical panel, $R = 100\,h$. In black: our mode. In gray: exact solution.

2.10.1 Bimorph shell

Consider a bimorph circular cylindrical panel, supposed of infinite length, made of PZT4 ceramic, under cylindrical bending, simply supported, submitted, on its top face, to a potential $+V$, while, on its lower face, to a potential $-V$ (see figure 29).
As previously, R denotes the radius of the cylinder, h its thickness, α its central angle, and θ the angular coordinate (see figure 21).

Data 2.20. *Material constants of PZT4 ceramic can be found in tables 1, 2.*

Proposition 2.21. *The simply supported boundary conditions yield:*

$$w(\theta = 0, z; t) = w(\theta = \alpha, z; t) = 0 \tag{152}$$

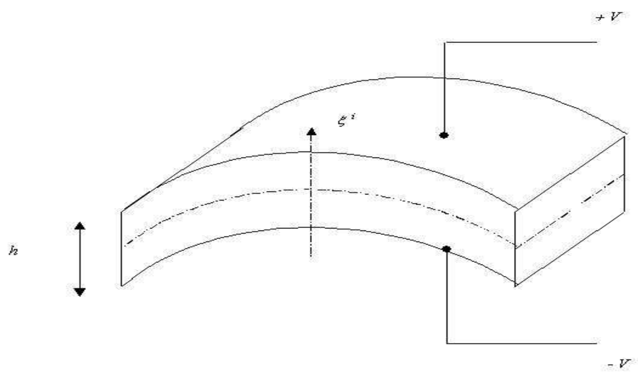


Figure 29: Bimorph shell, with imposed potentials.

Proposition 2.22. *The panel being supposed of infinite length, the mechanical and electrical quantities (stresses, strains, displacements, electric field, electrostatic potential), do not depend on* x_2.

Assumptions 2.23. *Since the component* U_2 *of the displacement field does not paly any part, we shall take:*

$$U_2 = 0$$

Assumptions 2.24. *The potential* V *is assumed to be simply sinusoidal:*

$$V(x_\alpha, z; t) = V_0\, e^{j\,\omega\,t}\, \sin(\frac{\pi\, x_1}{a}) \tag{153}$$

Proposition 2.25. *The electrostatic potential (20) is approximated as (18):*

$$\begin{array}{rcl} \varphi(x_1, z; t) & = & \left[(1 - {\xi_1}^2)\, {\varphi^1}^M(x_1; t) - \frac{1}{2}\, \xi_1\, (\xi_1 - 1)\, V(x_1; t)\right] \chi^1(z) \\ & + & \left[(1 - {\xi_2}^2)\, {\varphi^2}^M(x_1; t) + \frac{1}{2}\, \xi_2\, (\xi_2 + 1)\, V(x_1; t)\right] \chi^2(z) \end{array} \tag{154}$$

Remark 2.2. The dielectric coefficients of the two layers that constitute the bimorph being identical, no continuity conditions at layer interfaces are requested for the electric displacement.

The mechanical generalized displacements remaining unknowns are the membrane displacement u_1, w, and the transverse shear ${\gamma_1}^0$.

The electrical generalized unknown is φ^{1M}.

Assumptions 2.26. *The solution is searched under the following form, which characterizes the propagation of harmonic plane-waves:*

$$\left\{\begin{array}{lcl} u_1 & = & A_1\, e^{j\,\omega\,t}\, \cos(\frac{\pi\, x_1}{\alpha}) \\ w & = & B\, e^{j\,\omega\,t}\, \sin(\frac{\pi\, x_1}{\alpha}) \\ {\gamma_1}^0 & = & C_1\, e^{j\,\omega\,t}\, \cos(\frac{\pi\, x_1}{\alpha}) \end{array}\right. \quad , \ \varphi^{1M} = \Phi_1\, e^{j\,\omega\,t}\, \sin(\frac{\pi\, x_1}{\alpha}) \tag{155}$$

which enable us to satisfy the simply supported boundary conditions(159).

By substituting these expressions in the equations of the equilibrium (142), in conjunction with the boundary conditions (143), the constitutive law (31) and the displacement field (132), we obtain a linear system in A_1, B, C_1, Φ_1, of the form:

$$K_1 \begin{pmatrix} A_1 \\ B \\ C_1 \\ \Phi_1 \end{pmatrix} = B_1 \tag{156}$$

Detail of coefficients of the matrix K_1 and of the vector B_1 is given in [8].

Figure 30 displays the variations, as a function of the global thickness coordinate z, of the normalized electrostatic potential:

$$\overline{\varphi}(z;t) = \frac{\varphi(\theta = \frac{\alpha}{2}, z; t)}{\varphi(\theta = \frac{\alpha}{2}, z = \frac{h}{2}; t)} \tag{157}$$

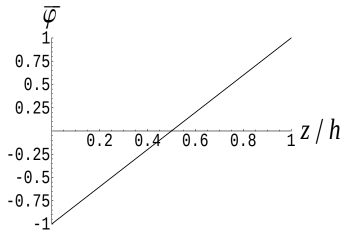


Figure 30: Normalized electrostatic potential, bimorph cylindrical panel, $R = 4\,h$.

Figure 31 displays the variations, as a function of the global thickness coordinate z, of the normalized transverse shear stress:

$$\overline{\sigma}_{13}(z;t) = \frac{\sigma_{13}(\theta = \frac{\alpha}{2}, z; t)}{\sigma_{13}(\theta = \frac{\alpha}{2}, z = \frac{h}{2}; t)} \tag{158}$$

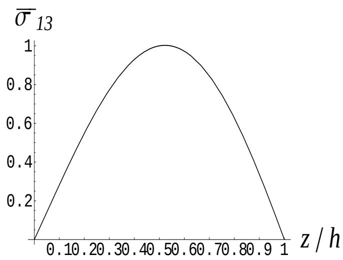


Figure 31: Normalized transverse shear stress, bimorph cylindrical panel, $R = 4\,h$.

2.10.2 Three-layered shell, submitted to a force density

Consider a symmetric 3-layered circular cylindrical panel, of infinite length, a circular cylindrical panel, the external layers of which are made of ZnO oxyde, with a silicium core, under cylindrical bending, simply supported, submitted to a force density p on its top face, and in closed circuit (which means that the electrostatic potential on its top and bottom surfaces is equal to zero: $\varphi^{1^B} = \varphi^{1^T} = 0$) (see figure 32).
As previously, R denotes the radius of the cylinder, h its thickness, α its central angle, and θ the angular coordinate (see figure 21).

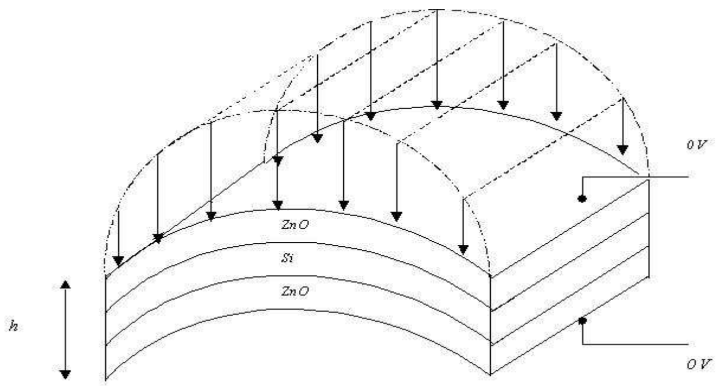


Figure 32: Three-layered shell.

Data 2.27. *Material constants of ZnO oxyde and silicium Si can be found in tables 9, 10.*

	C_{11}	C_{22}	C_{33}	C_{12}	C_{13}	C_{23}	C_{44}	C_{55}	C_{66}
ZnO	209.7	209.7	210.9	121.1	105.1	42.5	42.5	25.6	30.6
Si	166	166	166	63.9	63.9	79.6	5.654	5.654	5.654

Table 9: Independent mechanical constants of ZnO and silicium Si (in GPa)

	e_{31}	e_{32}	e_{33}	e_{15}	ε_{11}	ε_{22}	ε_{33}
ZnO	-0.61	-0.61	1.14	-0.59	13.06	13.06	11.51
Si	0	0	0	12.7	0.01045	0.01045	0.01045

Table 10: Independent piezoelectric and dielectric constants of ZnO and silicium Si (e_{ij} in C/m^2, ε_{ii} in nF/m)

Proposition 2.28. *The simply supported boundary conditions yield:*

$$w(\theta = 0, z; t) = w(\theta = \alpha, z; t) = 0 \tag{159}$$

Proposition 2.29. *The panel being supposed of infinite length, the mechanical and electrical quantities (stresses, strains, displacements, electric field, electrostatic potential), do not depend on* x_2.

Assumptions 2.30. *Since the component* U_2 *of the displacement field does not play any part, we shall take:*

$$U_2 = 0$$

Assumptions 2.31. *The force density* p *is supposed to be simply sinusoidal, of the form:*

$$p\,(x_\alpha, z; t) = p_0\, e^{j\,\omega\,t}\, \sin(\frac{\pi\, x_1}{\alpha}) \tag{160}$$

where $p_0 = 10N/m^2$.

Proposition 2.32. *The electrostatic potential (20) is approximated as (18):*

$$\varphi(x_\alpha, z; t) = Q_{2M}(z)\, \varphi^{2^M} \tag{161}$$

where:

$$\begin{array}{rcl} Q_{2M}(z) & = & \left[(1-{\xi_1}^2)\, \lambda^{1M,2M^M} + \frac{1}{2}\,\xi_1\,(\xi_1+1)\,\lambda^{2B,2M}\right] \chi^1(z) \\ & + & \left[\frac{1}{2}\,\xi_2\,(\xi_2-1)\,\lambda^{2B,2M} + (1-{\xi_2}^2) \,+ \frac{1}{2}\,\xi_2\,(\xi_2+1)\,\lambda^{2B,2M}\right] \chi^2(z) \\ & + & \left[\frac{1}{2}\,\xi_3\,(\xi_3-1)\,\lambda^{2B,2M} + (1-{\xi_3}^2)\,\lambda^{1M,2M}\right] \chi^3(z) \end{array} \tag{162}$$

where λ^{i^M}, λ^{i^B} *(*$i = 1,\, 2$*) are real constants, determined by means of the interface continuity conditions.*

Proof. Symmetries of the problem lead to:

$$\varphi^{2^B}(x_\alpha; t) = \varphi^{3^B}(x_\alpha; t) \;\;,\;\; \varphi^{1^M}(x_\alpha; t) = \varphi^{3^M}(x_\alpha; t) \tag{163}$$

which enable us to simplify the electrostatic potential:

$$\begin{array}{rcl} \varphi(x_\alpha, z; t) & = & \left[(1-{\xi_1}^2)\,{\varphi^1}^M(x_\alpha;t) + \frac{1}{2}\,\xi_1\,(\xi_1+1)\,{\varphi^1}^T(x_\alpha;t)\right]\chi^1(z) \\ & + & \left[\frac{1}{2}\,\xi_2\,(\xi_2-1)\,{\varphi^2}^B(x_\alpha;t) + (1-{\xi_2}^2)\,{\varphi^2}^M(x_\alpha;t) + \frac{1}{2}\,\xi_2\,(\xi_2+1)\,{\varphi^2}^T(x_\alpha;t)\right]\chi^2(z) \\ & + & \left[\frac{1}{2}\,\xi_3\,(\xi_3-1)\,{\varphi^2}^B(x_\alpha;t) + (1-{\xi_3}^2)\,{\varphi^1}^M(x_\alpha;t)\right]\chi^3(z) \end{array} \tag{164}$$

The continuity of the uncoupled electric displacement at layer interfaces yields:

$$\left\{ \begin{array}{rcl} -{\varepsilon_{33,1}}^1\,{\varphi_{,3}}^1(x_\alpha, z_1; t) & = & -{\varepsilon_{33,1}}^2\,{\varphi_{,3}}^2(x_\alpha, z_1; t) \\ -{\varepsilon_{33,1}}^2\,{\varphi_{,3}}^1(x_\alpha, z_2; t) & = & -{\varepsilon_{33,1}}^1\,{\varphi_{,3}}^3(x_\alpha, z_2; t) \end{array} \right. \tag{165}$$

We thus obtain a linear system, which enables us to express ${\varphi^1}^M$ and ${\varphi^2}^B$ as functions of ${\varphi^2}^M$, under the form:

$$\left\{ \begin{array}{rcl} {\varphi^1}^M & = & {\lambda^{1M,2M}}^M\,{\varphi^2}^M \\ {\varphi^2}^B & = & {\lambda^{2B,2M}}^M\,{\varphi^2}^M \end{array} \right. \tag{166}$$

□

The generalized mechanical unknowns are the membrane displacements u_1, the deflection w, and the transverse shear stresses ${\gamma_1}^0$.
The generalized electrical unknown is ${\varphi^2}^M$.

Assumptions 2.33. *The solution is searched under the following form, which characterizes the propagation of harmonic plane waves:*

$$\left\{ \begin{array}{rcl} u_1 & = & A_1\,e^{j\,\omega\,t}\,\cos(\frac{\pi\,x_1}{\alpha}) \\ w & = & B\,e^{j\,\omega\,t}\,\sin(\frac{\pi\,x_1}{\alpha}) \\ {\gamma_0}^1 & = & C_1\,e^{j\,\omega\,t}\,\cos(\frac{\pi\,x_1}{\alpha}) \end{array} \right. \quad , \quad {\varphi^2}^M = \Phi_2\,e^{j\,\omega\,t}\,\sin(\frac{\pi\,x_1}{\alpha}) \tag{167}$$

which enables us to satisfy the simply supported boundary conditions (159).

By substituting these expressions in the equations of the equilibrium (142), in conjunction with the boundary conditions (143), the constitutive law (31) and the displacement field (132), we obtain a linear system in A_1, B, C_1, Φ_1, of the form:

$$K_2 \begin{pmatrix} A_1 \\ B \\ C_1 \\ \Phi_2 \end{pmatrix} = B_2 \tag{168}$$

detail of the coefficients of the matrix K_2 and of the vector B_2 is given in [8].

Figure 33 displays the variations, as a function of the global thickness parameter z, of the normalized electrostatic potential:

$$\overline{\varphi}(z;t) = \frac{\varphi(\theta = \frac{\alpha}{2}, z; t)}{\varphi(\theta = \frac{\alpha}{2}, z = \frac{h}{2}; t)} \tag{169}$$

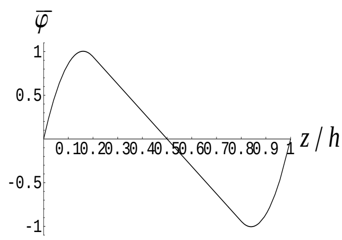


Figure 33: Normalized electrostatic potential, three-layered cylindrical panel, $R = 4\,h$.

Figure 34 displays the variations, as a function of the global thickness parameter z, of the normalized transverse shear stress:

$$\overline{\sigma}_{13}(z;t) = \frac{\sigma_{13}(\theta = \frac{\alpha}{2}, z; t)}{\sigma_{13}(\theta = \frac{\alpha}{2}, z = \frac{h}{2}; t)} \tag{170}$$

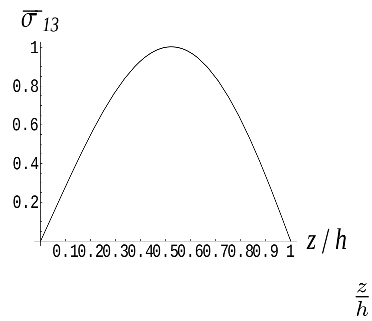


Figure 34: Normalized transverse shear stress, three-layered cylindrical panel, $R = 4\,h$.

Figure 35 displays the variations, as a function of the global thickness parameter z, of the normalized longitudinal stress:

$$\overline{\sigma}_{11}(z;t) = \frac{\sigma_{11}(\theta = \frac{\alpha}{2}, z; t)}{\sigma_{11}(\theta = \frac{\alpha}{2}, z = \frac{h}{2}; t)} \qquad (171)$$

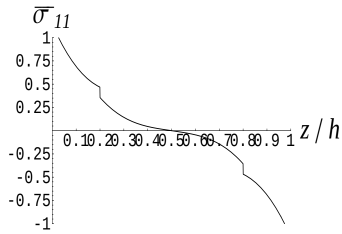


Figure 35: Normalized longitudinal stress, three-layered cylindrical panel, $R = 4\,h$.

2.11 Conclusion

The latter results constitute, as expected, a generalization of similar ones obtained in the case of plates by Fernandes [21], [22].
The general aspect of the curves is the same, with, of course, an influence of the curvature.
The fitting between those results, and the fact that the considered problems correspond to thick shells ($R = 4\,h$), show that our model can efficiently represent piezoelectric structures, submitted to different kinds of loadings, electric or mechanic.

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