Chapter 1

TOWARDS NEW SCHEMES: A LIE-GROUP APPROACH OF THE BURGERS EQUATION

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Abstract

The aim of this paper is to propose methods that enable us to build new numerical schemes, which preserve the Lie symmetries of the original differential equations. To this purpose, the particular case of the Burgers equation is examined, and the resulting semi-invariant scheme is exposed.

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1. Introduction

Finite difference equations used to approximate the solutions of a differential equation generally do not respect the symmetries of the original equation, and can lead to inaccurate numerical results. Usually, specific equations are considered, for which the authors build a scheme preserving the symmetries of the original differential equation. Yet, it is more interesting to directly consider a class of differential equations, in order to obtain more general results.

Using the work of Yanenko [**?**] and Shokin [**?**], who applied the Lie group theory to finite difference equations by means of the differential approximation, in conjunction with the approach of Ames et al. [**?**], we generalize results developed in [**?**], [**?**], and expose the invariance condition for a differential approximation of the considered equation.

2. Notion of symmetry

Though it has been the object of numerous works, it is still interesting to go back to the original definitions and tools that enable one to deal with Lie symmetries, which are then applied to finite difference schemes, by means of their equivalent equation.

2.1. Definitions

Definition 2..1. A r-parameter Lie group is a r-dimensional smooth manifold G_r, which has the group properties, such that the group operation of multiplication and inversion are smooth maps.

Especially, a Lie group G_r is defined as a group of continuous transformations which act on an open subset of the Euclidean space $\mathbb{R}^k$ of variables, which change under the action of G_r. We presently concentrate on a local group, the transformations of which are close to the identity transformation.

Definition 2..2. A r-parameter Lie group G_r is a group of point transformations, which acts on $X \times U$, the space of the independent variables and the dependent ones:

$$G_r = \{x_i^* = \phi_i(x,u,a);\ u_j^* = \varphi_j(x,u,a),\ i = 1,\dots,m;\ j = 1,\dots,n; a = (a_1,\dots,a_r)\} \tag{1}$$

where $x \in X \subset \mathbb{R}^m$ and $u \in U \subset \mathbb{R}^n$.

G_r locally satisfies the group axioms: existence of an identity element, associativity, inversibility, closure under the binary composition operation. The transformation corresponding to a zero parameter is the identity transformation.
Expand the transformations by means of a Taylor series at the zero value of the parameter a:

$$\begin{aligned} x_i^* &= x_i + a_\alpha \frac{\partial \phi_i}{\partial a_\alpha}\Big|_{a=0} + O(a_\alpha^2),\ \alpha = 1,\dots,r \\ u_j^* &= u_j + a_\alpha \frac{\partial \varphi_j}{\partial a_\alpha}\Big|_{a=0} + O(a_\alpha^2),\ \alpha = 1,\dots,r \end{aligned} \tag{2}$$

The derivatives of ϕ_i and φ_j with respect to the parameter a_α are smooth functions, called *infinitesimals of the group* G_r. Denote by ξ_i^α and η_j^α the infinitesimals of G_r.
The point transformation group G_r can be represented by means of the operator $\mathbf{L}_\alpha$:

$$\mathbf{L}_\alpha = \xi_i^\alpha(x,u)\frac{\partial}{\partial x_i} + \eta_j^\alpha(x,u)\frac{\partial}{\partial u_j}, \quad i=1,\dots,m;\ j=1,\dots,n;\ \alpha=1,\dots,r \tag{3}$$

The operators $\mathbf{L}_\alpha$, $\alpha = 1,\dots,r$ are called the *infinitesimal operators* of G_r.
$\{\mathbf{L}_\alpha,\ \alpha = 1,\dots,r\}$ represents the set of tangent vectors to the manifold G_r, when the zero value is assigned to the parameter a. The set is a basis of the Lie-algebra of the infinitesimal operators of G_r, the dimension of which is the same as the one of the Lie group G_r.
The knowledge of the $\mathbf{L}_\alpha$ enables us to determine the point transformations of the group G_r by solving the equations:

$$\frac{\partial x_i^*}{\partial a_\alpha} = \xi_i^\alpha(x^*,u^*),\ \ \frac{\partial u_j^*}{\partial a_\alpha} = \eta_j^\alpha(x^*,u^*),\ \ i=1,\dots,m;\ j=1,\dots,n;\ \alpha=1,\dots,r \tag{4}$$

in conjunction with the initial conditions:

$$x_i^*\big|_{a=0} = x_i;\ \ u_j^*\big|_{a=0} = u_j \tag{5}$$

Example 2..3. The Galilean transformation group
The Galilean transformations correspond to time dependent translations of a reference frame :

$$G = \{\mathcal{T} : (x,t,u) \mapsto (x^*,t^*,u^*) = (x+\varepsilon t, t, u+\varepsilon)\}$$

where ε is the translation constant velocity.
$\mathcal{T}$ and its inverse function are continuous. The infinitesimals functions of the group are $(\xi_x^\varepsilon, \xi_t^\varepsilon, \eta_u^\varepsilon) = (t,0,1)$

2.2. Symmetry properties of differential equations

The notion of symmetry is a tool for generating new solutions of differential equations. Let us review the main aspects of the application of the Lie group theory to differential equations.
Consider a system of l^{th}-order differential equations:

$$\mathcal{F}^\lambda\left(x,u,u^{(k_1)},u^{(k_1,k_2)},\dots,u^{(k_1\dots k_l)}\right) = 0,\ \ \lambda = 1,\dots,q \tag{6}$$

Denote by $u^{(k_1\dots k_p)}$ the vector, the components of which are partial derivatives of order p, namely, $u_j^{(k_1\dots k_p)} = \frac{\partial^p u_j}{\partial x_{k_1}\dots\partial x_{k_p}}$ $j = 1,\dots,n$ and $k_1,\dots,k_p \in \{1,\dots,m\}$.
Denote by $x = (x_1,\dots,x_m)$ the independent variables, $u = (u_1,\dots,u_n)$ the dependent variables, and $(x_{k_1}\dots x_{k_p})$ a set of elements of the independent variables.
Equation (**??**) is a subset of $X \times U^{(l)}$, a prolongation of the space $X \times U$ to the space of the partial derivatives of u with respect to x up to order l. $X \times U^{(l)}$, which is a smooth manifold, is called the l-th order jet space of $X \times U$. In order to take into account the derivative terms involved in the differential equation, the action of G_r on $X \times U$ needs to be prolonged to the space of the derivatives of the dependent variables.

Denote by $\widetilde{G}_r^{(l)}$ a r-parameter Lie group of point transformations acting on an open subset $M^{(l)}$ of the l-th order jet space $X \times U^{(l)}$ of the independent variables x, dependent variables u and the partial derivatives of u with respect to x.
The l^{th}-prolongation operator of G_r is:

$$\widetilde{\mathbf{L}}_\alpha^{(l)} = \xi_i^\alpha(x,u)\frac{\partial}{\partial x_i} + \eta_j^\alpha(x,u)\frac{\partial}{\partial u_j} + \sigma_j^{\alpha,(k_1)}\frac{\partial}{\partial {u_j}^{(k_1)}} + \cdots + \sigma_j^{\alpha,(k_1\ldots k_l)}\frac{\partial}{\partial {u_j}^{(k_1\ldots k_l)}}, \quad i=1,\ldots,m;\ j=1,\ldots,n;\ \alpha=1,\ldots,r. \tag{7}$$

The infinitesimal functions ξ_i^α, η_j^α, $\sigma_j^{\alpha,(k_1)}$ and $\sigma_j^{\alpha,(k_1\ldots k_o)}$ are given by:

$$\xi_i^\alpha = \frac{\partial \phi_i}{\partial a_\alpha}\Big|_{a=0}, \quad \eta_j^\alpha = \frac{\partial \varphi_j}{\partial a_\alpha}\Big|_{a=0}, \quad \sigma_j^{\alpha,(k_1)} = \frac{\mathcal{D}\eta_j^\alpha}{\mathcal{D}x_{k_1}} - \sum_{i=1}^m \frac{\partial u_j}{\partial x_i}\frac{\mathcal{D}\xi_i^\alpha}{\mathcal{D}x_{k_1}} \tag{8}$$

$$\sigma_j^{\alpha,(k_1\ldots k_o)} = \frac{\mathcal{D}\sigma_j^{\alpha,(k_1\ldots k_{o-1})}}{\mathcal{D}x_{k_o}} - \sum_{i=1}^m \frac{\partial^o u_j}{\partial x_i \partial x_{k_1}\ldots\partial x_{k_{o-1}}}\frac{\mathcal{D}\xi_i^\alpha}{\mathcal{D}x_{k_o}}, \quad o=2,\ldots,l$$

where: $\dfrac{\mathcal{D}}{\mathcal{D}x_k} = \dfrac{\partial}{\partial x_k} + \sum_{j=1}^n \dfrac{\partial u_j}{\partial x_k}\dfrac{\partial}{\partial u_j}$

Theorem 2..4. *The system of l^{th}-order differential equations is invariant under the group $\widetilde{G}_r^{(l)}$ if and only if it satisfies the following infinitesimal invariance criterion:*

$$\widetilde{\mathbf{L}}_\alpha^{(l)}\mathcal{F}^\lambda\Big|_{\mathcal{F}^\lambda=0} = 0, \quad \alpha=1,\ldots,r;\ \lambda=1,\ldots,q \tag{9}$$

2.3. The case of the Burgers equation

The one-dimensional Burgers equation can be written as:

$$\mathcal{F}(q) = u_t + \left(\frac{u^2}{2}\right)_x - \nu u_{xx} = 0 \tag{10}$$

where $q=(x,t,u,\nu)$, $(x,t)\in X$ being the independant variable, $u \in U$ the dependant one, and ν the constant dynamic viscosity.
The infinitesimal generators of the equation are:

$$\begin{aligned}
\xi &= \alpha_1 + \alpha_3 t - \alpha_4 x - \alpha_5 xt\\
\vartheta &= \alpha_2 - 2\alpha_4 t - \alpha_5 t^2\\
\eta &= \alpha_3 + \alpha_4 u + \alpha_5(tu - x)
\end{aligned}$$

The Lie algebra $\mathcal{G}$ of these transformations, under which the Burgers equation remains invariant, is generated by the 5 following vector fields:

- the spatial translation
$$\mathbf{L_1} = \partial_x$$
- the time translation
$$\mathbf{L_2} = \partial_t$$

- the scale change

$$\mathbf{L_3} = x\partial_x + 2t\partial_t - u\partial_u$$

- the Galilean transformation

$$\mathbf{L_4} = t\partial_x + \partial_u$$

- the projective transformation

$$\mathbf{L_5} = -xt\partial_x + t^2\partial_t + (tu - x)\partial_u$$

3. Lie group of a differential approximation

3.1. The theoretical point of view

The symmetry group analysis of the differential approximation uses the techniques of the Lie group theory applied to differential equations. The differential approximations involve step size variable which change under the action of the group.
The technique of symmetry analysis is not directly applied to the finite difference schemes. It is based on the differential approximation, which describes approximately the numerical solution behavior at a reference point of the mesh. Thus the concept of differential approximation is a local object, which can not systematically detect a mesh change. Despite the local behavior of the symmetry group analysis, the invariant method based on the differential approximation has enabled one to establish interesting properties of symmetry group of finite difference schemes.
The finite difference scheme, which approximates the differential system (**??**), can be written as:

$$\Lambda^{\lambda}(x,u,h,Tu) = 0, \quad \lambda = 1,\ldots,q \tag{11}$$

where $h = (h_1, h_2, \ldots, h_m)$ denotes the space step vector, and $T = (T_1, T_2, \ldots, T_m)$ the shift-operator along the axis of the independent variables, defined by:

$$T_i[u](x_1,x_2,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_m) = u(x_1,x_2,\ldots,x_{i-1},x_i+h_i,x_{i+1},\ldots,x_m). \tag{12}$$

Definition 3..1. The differential system:

$$\begin{aligned} \mathcal{P}^{\lambda}\big(x,u,u^{(k_1)},\ldots,u^{(k_1\ldots k_{l'})}\big) &= \mathcal{F}^{\lambda}\big(x,u,u^{(k_1)},\ldots,u^{(k_1\ldots k_l)}\big) \\ &+\sum_{\beta=1}^{s}\sum_{i=1}^{m}(h_i)^{l_\beta}\mathcal{R}_i^{\lambda}(x,u,u^{(k_1)},\ldots,u^{(k_1\ldots k_{l'}{}_{\lambda,i})}), \\ &\lambda = 1,\ldots,q;\ l' = max_{(\lambda,i)}l'_{\lambda,i} \end{aligned} \tag{13}$$

is called the s^{th}-order differential approximation of the finite difference scheme (**??**). In the specific case $s = 1$, the above system is called the first differential approximation.

The differential system (**??**) is obtained from the algebraic system (**??**) by applying Taylor series expansion to the components of Tu about the point $x = (x_1,\ \ldots,\ x_m)$ and truncating the expansion to a given finite order.
Denote by G'_r a group of transformations acting on an open subset M' of $X \times U \times H$ the space of the independent variables, the dependent variables and the step size variables :

$$G'_r = \{x_i^* = \phi_i(x,u,a);\ u_j^* = \varphi_j(x,u,a); h_i^* = \psi_i(x,u,h,a),\ i = 1,\ldots,m;\ j = 1,\ldots,n\} \tag{14}$$

by $\mathbf{L}_\alpha{}'$ the basis infinitesimal operator of G_r':

$$\mathbf{L}_\alpha{}' = \xi_i^\alpha(x,u)\frac{\partial}{\partial x_i} + \eta_j^\alpha(x,u)\frac{\partial}{\partial u_j} + \zeta_i^\alpha(x,u,h)\frac{\partial}{\partial h_i}, \quad \alpha = 1,\dots,r \tag{15}$$

where $\zeta_i^\alpha = \frac{\partial\psi_i}{\partial a_\alpha}\Big|_{a=0}$, $\alpha = 1,\dots,r$

and by $\widetilde{G}_r^{(l')}$ a group of transformation acting on an open subset $M^{(l')}$ of the space of the independent variables, the dependent variables and the step size variables and the partial derivatives involved in the differential system . The l'^{th}-prolongation operator of G_r', $\widetilde{\mathbf{L}}_\alpha^{(l')}$ can be written as:

$$\widetilde{\mathbf{L}}_\alpha^{(l')} = \mathbf{L}_\alpha{}' + \sum_{j=1}^{n}\sum_{p=1}^{l'} \sigma_j^{\alpha,(k_1\dots k_p)} \frac{\partial}{\partial u_j^{(k_1\dots k_p)}} \tag{16}$$

Theorem 3..2. *The differential approximation (**??**) is invariant under the group $\widetilde{G}_r^{(l')}$ if and only if*

$$\widetilde{\mathbf{L}}_\alpha^{(l')} \mathcal{P}^\lambda\big((x,u,u^{(k_1)},\dots,u^{(k_1\dots k_{l'})}\big)\Big|_{\mathcal{P}^\lambda=0} \quad = \quad 0, \quad \alpha = 1,\dots,r;\ \lambda = 1,\dots,q \tag{17}$$

$$or\ \Big[\widetilde{\mathbf{L}}_\alpha^{(l)} \mathcal{F}^\lambda + \widetilde{\mathbf{L}}_\alpha^{(l')}\Big(\sum_{\beta=1}^{s}\sum_{i=1}^{m}(h_i)^{l_\beta}\mathcal{R}_i^\lambda\Big)\Big]\Big|_{\mathcal{P}^\lambda=0} \quad = \quad 0, \quad \alpha = 1,\dots,r;\ \lambda = 1,\dots,q \tag{18}$$

3.2. The case of classical schemes

The Burgers equation (**??**) can be discretized by means of the following numerical schemes:

- **the FTCS (*Forward-Time Centered-Space*) scheme:**

$$\frac{u_i^{n+1} - u_i^n}{\tau} + \frac{\mu\delta}{h}\Big(\frac{u^2}{2}\Big)_i^n - \nu\frac{\delta^2}{h^2}(u_i^n) = 0 \tag{19}$$

- **The Lax-Wendroff scheme:**

$$\frac{u_i^{n+1} - u_i^n}{\tau} + \frac{\mu\delta}{h}\Big(\frac{u^2}{2}\Big)_i^n - \nu\frac{\delta^2}{h^2}(u_i^n) + A_i^n = 0 \tag{20}$$

where

$$A_i^n = \quad - \quad \frac{\tau}{2h^2}\Big[u_{i+\frac{1}{2}}^n\delta^+\Big(\frac{u^2}{2}\Big)_i^n - u_{i-\frac{1}{2}}^n\delta^-\Big(\frac{u^2}{2}\Big)_i^n\Big] \tag{21}$$

$$+ \quad \frac{\nu\tau}{2}\Big[\frac{\mu\delta^3}{h^3}\Big(\frac{u^2}{2}\Big)_i^n\Big] - \frac{\nu^2\tau}{2}\Big[\frac{\delta^4}{h^4}u_i^n\Big] \tag{22}$$

- **the Crank-Nicolson scheme**

$$\frac{u_i^{n+1} - u_i^n}{\tau} + \frac{\mu\delta}{2h}\Big(\Big(\frac{u^2}{2}\Big)_i^{n+1} + \Big(\frac{u^2}{2}\Big)_i^n\Big) - \frac{\nu\delta^2}{2h^2}(u_i^{n+1} + u_i^n) = 0 \tag{23}$$

where we recall the Hildebrand notations:

$$\begin{array}{ll} \delta(u_i^n) = u^n_{i+\frac{1}{2}} - u^n_{i-\frac{1}{2}} & , \ \mu(u_i^n) = \dfrac{u^n_{i+\frac{1}{2}} + u^n_{i-\frac{1}{2}}}{2} \\ \delta^-(u_i^n) = u^n_{i+1} - u^n_i & , \ \delta^+(u_i^n) = u^n_i - u^n_{i-1} \end{array}$$

Numerical approximations of the velocity $u_{i+\alpha}^{n+\beta}$ are expanded by means of their Taylor series at the reference point x_i^n of the stencil. Those expansions are the truncated at a given order, and substituted in the numerical scheme, leading to what is then called the Γ **form** of the equivalent equation. It consists in the original Burgers equation, and of the related truncature error, which is a function of (h^p, τ^q), $(p,q) \in \mathbb{N}^2$, and of the time and space derivatives of u.
The Π **form** of the equivalent equation, which is of simpliest use, is obtained by expressing all derivatives as space derivatives, by means of the original Burgers equation (**??**).

The equivalent equations of the above schemes can be written under their Γ or Π form respectively as:

- **for the FTCS scheme:**

$$u_t + \frac{1}{2}(u^2)_x - \nu\, u_{xx} + \frac{\tau}{2} u_{tt} + \frac{h^2}{12}(u^2)_{xxx} - \frac{\nu h^2}{12} u_{xxxx} = 0 \quad (\Gamma\,\text{form}) \tag{24}$$

$$u_t + \frac{1}{2}(u^2)_x - \nu\, u_{xx} + \frac{\tau}{2} g_2 + \frac{h^2}{12}(u^2)_{xxx} - \frac{\nu h^2}{12} u_{xxxx} = 0 \quad (\Pi\,\text{form}) \tag{25}$$

- **for the Lax-Wendroff scheme:**

$$u_t + \frac{1}{2}(u^2)_x - \nu\, u_{xx} + \frac{\tau^2}{6} u_{ttt} + \frac{h^2}{12}(u^2)_{xxx} - \frac{\nu h^2}{12} u_{xxxx} = 0 \quad (\Gamma\,\text{form}) \tag{26}$$

$$u_t + \frac{1}{2}(u^2)_x - \nu\, u_{xx} + \frac{\tau^2}{6} g_3 + \frac{h^2}{12}(u^2)_{xxx} - \frac{\nu h^2}{12} u_{xxxx} = 0 \quad (\Pi\,\text{form}) \tag{27}$$

- **for the Crank-Nicolson scheme:**

$$u_t + \frac{1}{2}(u^2)_x - \nu u_{xx} + \tau^2 \Big(\frac{u_{ttt}}{6} + \frac{1}{4}(u_t^2 + u u_{tt})_x - \frac{\nu}{4} u_{xxtt} \Big) \tag{28}$$

$$+ h^2 \Big(\frac{1}{6} (\frac{u^2}{2})_{xxx} - \frac{\nu}{12} u_{xxxx} \Big) = 0 \quad (\Gamma\,\text{form}) \tag{29}$$

Table 1. Scheme invariance for the Burgers equation

Symmetries	FTCS	Lax-Wendroff	Crank-Nicolson
Spatial translation	yes	yes	yes
Time translation	yes	yes	yes
Galilean transformation	**no**	**no**	**no**
Scale change	yes	yes	yes
projective transformation projective	**no**	**no**	**no**

$$u_t + \frac{1}{2}(u^2)_x - \nu u_{xx} + \tau^2\Big(\frac{g_3}{6} + \frac{1}{4}(g_1^2 + ug_2)_x - \frac{\nu}{4}(g_2)_{xx}\Big) \tag{30}$$

$$+h^2\Big(\frac{1}{6}(\frac{u^2}{2})_{xxx} - \frac{\nu}{12}u_{xxxx}\Big) = 0 \quad (\Pi\,\text{form}) \tag{31}$$

where $g_1 = -\big(\frac{u^2}{2}\big)_x + \nu u_{xx}$, $g_2 = \big(-g_1 u\big)_x + \nu\big(g_1\big)_{xx}$, $g_3 = \big(-g_2 u - g_1^2\big)_x + \nu\big(g_2\big)_{xx}$

Table **??** displays the results obtained for the above classical schemes.
The prolongations of the vector fields to the derivatives that appear in the equivalent equations are:

- the space translation: $\mathbf{L_1^{(6)}} = \partial_x;$
- the time translation: $\mathbf{L_2^{(6)}} = \partial_t;$
- the scale change:

$$\begin{aligned}\mathbf{L_3^{(6)}} &= x\partial_x + 2t\partial_t + h\partial_h + 2\tau\partial_\tau - u\partial_u - 3u_t\partial_{u_t} - 2u_x\partial_{u_x} \\ &\quad - 3u_{xx}\partial_{u_{xx}} - 4u_{3x}\partial_{u_{3x}} - 5u_{4x}\partial_{u_{4x}} - 6u_{5x}\partial_{u_{5x}} - 7u_{6x}\partial_{u_{6x}}\end{aligned}$$

- the Galilean transformation: $\mathbf{L_4^{(6)}} = t\partial_x + \partial_u - u_x\partial_{u_t} - 2u_{xt}\partial u_{tt};$
- the projective transformation:

$$\begin{aligned}\mathbf{L_5^{(6)}} &= xt\partial_x + t^2\partial_t + ht\partial_h + \tau(2t+\tau)\partial_\tau + (x - tu)\partial_u + (1 - 2tu_x)\partial_{u_x} \\ &\quad + (-u - 3tu_t - xu_x)\partial_{u_t} - 3tu_{xx}\partial_{u_{xx}} - 4tu_{3x}\partial_{u_{3x}} - 5tu_{4x}\partial_{u_{4x}} \\ &\quad - 6tu_{5x}\partial_{u_{5x}} - 7tu_{6x}\partial_{u_{6x}} + (-4u_t - 5tu_{tt} - 2xu_{xt})\partial_{u_{tt}}.\end{aligned}$$

The symmetry group of each of the equivalent equations is determined using the symbolic calculus tool Mathematica®.
The invariance condition based on the equivalent equation shows that they do not remain invariant under the Galilean and projective transformations.

The invariance condition under the group of galilean transformations is applied, first, to the Γ form, and, the, to the Π form.
The non invariance under the Galilean transformation of the FTCS scheme can be written as:

$$\begin{aligned} \tilde{\mathbf{L}}'^{(4)}_4 \Lambda^\lambda = & \ \tfrac{h^2}{6} u_{xxx} - \tau u_{xt} \\ = & \ \tfrac{h^2}{6} u_{xxx} + \tau \left(\tfrac{u^2}{2} \right)_x - \tau \nu u_{xxx} + \dots \end{aligned}$$

Under the action of the projective transformation, the invariance condition is not satisfied:

$$\tilde{\mathbf{L}}'^{(4)}_5 \Lambda^\lambda = -3tu_t - 2tuu_x + 3\nu t u_{xx} - xu_x + O(\tau, h^2)$$

4. Determination of the infinitesimal functions

The calculation of Lie groups of differential equations with pencil and paper is tedious and may induce errors. The size of related equations increases with the number of the symmetry variables, and the order of the differential equations. A large amount of packages have been created using software programs with symbolic manipulations, such as Mathematica, MACSYMA, Maple, REDUCE, AXIOM, MuPAD. Schwarz [**?**] wrote algorithms for REDUCE and AXIOM computer algebra systems, Vu and Carminati [**?**] worked on DESOLVE, a Maple program, Herod [**?**] and Baumann [**?**] developed Mathematica programs.
The authors have implemented a Mathematica package [**?**], for the determination of the Lie group of the differential approximation of one dimensional model equations.
Theorems **??** and **??**, respectively give the algorithmic procedure for the determination of Lie group of any differential system and differential approximation.
The theorem infinitesimal invariance criteria involved the independent variables x, the dependent ones u, products of the partial derivatives of u with respect to x, the unknown infinitesimal functions ξ_i^α, and η_j^α, $i = 1,\dots,m;\ j = 1,\dots,n$ and their partial derivatives with respect to x and u.
The partial derivatives of the infinitesimal functions with respect to x and u come from the coefficients of prolonged Lie algebra vector field (**??**).
Equations (**??**) and (**??**) are simplified by means of the conditions (**??**) and (**??**). This simplification manipulation eliminates some derivatives of u. These equations are then solved algebrically with respect to the partial derivatives of the dependent variables, handled as independent variables. Denote by w the vector, the components of which are these variables. Since the whole equation holds for all the w components, each coefficient in front of the products of the w components has to be zero. This leads to a linear overdetermined system of partial differential equation, with respect to the infinitesimal functions, called the *determining equations* of the Lie group of the related differential system .
The solution of the overdetermined system can be found either by using elementary methods of the theory of linear partial differential equations or by using a polynomial form for

the infinitesimals. The last technique provides a linear system of algebraic equations with respect to the polynomial coefficients. But it can not find infinitesimals with transcendal functions.
The resolution of the determining equations yields explicitly the expression of ξ_i^α, η_j^α, $\alpha = 1,\ldots,r,\ i=1,\ldots,m,\ j=1,\ldots,n.$ Then relations (**??**) and (**??**) provide the calculation of group transformations from the infinitesimal expression.

5. The specific case of the Burgers equation

The Yanenko [**?**] and Shokin [**?**] symmetry analysis is applied to finite difference schemes for solving the Burgers equation. Thus, the symmetries of the Burgers equation, which are finally broken by the finite difference discretization, are determined. The techniques exposed in [**?**] and [**?**] enable one to construct differential approximations, which preserve the symmetries of the original differential system. We call the related finite difference scheme a *semi-invariant* scheme, in so far as the invariance condition is weaker than the one of the other invariance methods, defined as direct invariance methods. Indeed, the approach in [**?**] and [**?**] does not deal with the invariance of the algebraic equations, which govern the mesh evolution.
The differential equation has provided important characteristics for numerical schemes, in the study of numerical stability, dissipation and dispersive property. In [**?**], [**?**] and [**?**], the differential approximation has been revealed as a practical and admissible tool for symmetry analysis of finite difference scheme.
A comparison is made between the numerical solutions of the Burgers equation for some standard schemes and the semi-invariant one.

5.1. A semi-invariant scheme

By means of the equivalent equation, Yanenko [**?**] and Shokin [**?**] build a method, which preserves the Lie symmetries. It is developped in [**?**], for the resolution of the compressible Euler equations:

$$q_t + (f(q))_x = 0 \ (\Sigma) \tag{32}$$

A semi-implicit 6-points discretization, with a uniform and time-constant mesh, leads to the following equivalent equation:

$$q_t + (f(q))_x = (C\,q_x)_q \ \ (\Sigma') \tag{33}$$

which is obtained by means of truncated Taylor expansions of the values of the solution at a reference point of the stencil, taking into account only the first terms of the error.
Moreover, the artificial viscosity term is choosen in order that the differential system remains invariant under the symmetries of (Σ), which is done by means of the Lie group tools described in the above.

C depends of the spatial variables vector x, of the dependant ones q, and of the space step h.

In Hoarau et al. [**?**], a similar method is developped for the numerical resolution of the Burgers equation, with an explicit 6-points discretization, with a uniform and time-constant stencil:

$$\frac{u_j^{n+1}-u_j^n}{\tau}+\frac{1}{h}\left(\mu\,\delta-\mu\,\frac{\delta^3}{6}\right)\left(\frac{u^2}{2}\right)_j^n-\frac{\nu}{h^2}\left(\delta^2-\mu\,\frac{\delta^4}{12}\right)u_j^n=0\ (H_0) \tag{34}$$

The resulting equivalent equation is:

$$\mathcal{P}(x,t,u,\nu,u_x,u_t,u_{xx})=u_t+\left(\frac{u^2}{2}\right)_x-\nu\,u_{xx}-(C\,u_x)_x \tag{35}$$

where C is the artificial viscosity term.
The related numerical method is of first order in time, and second order in space:

$$\begin{aligned}\frac{u_j^{n+1}-u_j^n}{\tau}+\frac{1}{h}\left(\mu\,\delta-\mu\,\frac{\delta^3}{6}\right)\left(\frac{u^2}{2}\right)_j^n-\frac{\nu}{h^2}\left(\delta^2-\mu\,\frac{\delta^4}{12}\right)u_j^n\\ -h\left(\Omega^n_{i+\frac{1}{2}}\,\delta^+-\Omega^n_{i-\frac{1}{2}}\,\delta^-\right)u_j^n\\ +\frac{\nu\,\tau}{2}\left(u^n_{i+\frac{1}{2}}\,\mu\,\frac{\delta^2}{h^2}-u^n_{i-\frac{1}{2}}\,\mu\,\frac{\delta^2}{h^2}\right)-\frac{\nu^2\,\tau}{2}\,\frac{\delta^4}{h^4}\,u_j^n-\frac{\nu\,\tau}{2}\,\frac{\mu\,\delta^3}{h^3}\left(\frac{u^2}{2}\right)_j^n=0\end{aligned} \tag{36}$$

where:

$$C=\frac{\tau}{2}\,u^2-h^2\,\Omega \tag{37}$$

For sake of simplicity, only the first derivative of the space dependant variable is taken into account, which yields:

$$C=10^{-4}\,t\,(t\,u+x)^2\,u_x \tag{38}$$

Two analogous methods provide a direct symmetry analysis of finite difference schemes and can lead to the definition of adapted evolutionary meshes, whose geometrical structure is preserved by the entire group. The first direct method has been introduced by Dorodnitsyn [**?**] and is based on Lie algebra techniques, using the infinitesimal operators. The second method has been introduced by Olver [**?**] and is based on the theory of the Cartan moving frame. The method proposed by Yanenko [**?**] and Shokin [**?**] consists of a symmetry study of the differential approximation. Although the last method is not fully exact, the numerical results in [**?**] and [**?**] has proved its effectiveness.

The scheme proposed below is associated to an uniform orthogonal mesh.
We hereafter propose to approximate the Burgers equation by the finite difference scheme:

$$\frac{u_i^{n+1}-u_i^n}{\tau}+\frac{1}{h}(\mu\delta-\frac{\mu\delta^3}{6})(\frac{u^2}{2})_i^n-\nu\frac{1}{h^2}(\delta^2-\frac{\delta^4}{12})(u_i^n)-h\Big(\Omega^n_{i+\frac{1}{2}}\delta^+-\Omega^n_{i-\frac{1}{2}}\delta^-\Big)u_i^n=0 \tag{39}$$

where $\Omega_i^n=\Omega(x_i,t_n,u_i^n)$ is defined next so that the related differential representation is preserved by the symmetries of the Burgers equation.
The scheme has second-order accuracy in space and first-order accuracy in time. The derivatives $(u^2)_x$ and u_{xx} are approximated by fourth order accuracy difference expressions:

$$(\frac{\mu\delta}{h}-\frac{\mu\delta^3}{6h})(u_i^n)=\big(u_x-\frac{h^4}{30}u_{5x}\big)_i^n+O(h^6),\ (\frac{\delta^2}{h^2}-\frac{\delta^4}{12h^2})(u_i^n)=\big(u_{xx}-\frac{h^4}{90}u_{6x}\big)_i^n+O(h^6) \tag{40}$$

The truncation error of the difference scheme (**??**) can be written as:

$$\varepsilon \;=\; \frac{\tau}{2}u_{tt}-h^2\Big(\Omega u_x\Big)_x+O(\tau^2)+O(h^4)$$

u_{tt} is replaced by an expression involving partial derivatives with respect to x, by using the Burgers equation.
Replacing the obtained expression in the truncation error leads to:

$$\varepsilon \;=\; \Big(Cu_x\Big)_x-\frac{\nu\tau}{2}\Big(uu_{xx}\Big)_x-\frac{\nu\tau}{2}(\frac{u^2}{2})_{xxx}+\frac{\nu^2\tau}{2}u_{xxxx}+O(\tau^2)+O(h^4)$$

where $C=\frac{\tau}{2}u^2-h^2\Omega$.
It is convenient for the calculation of C that the truncation error is reduced to:

$$\varepsilon \;=\; \Big(Cu_x\Big)_x+O(\tau^2)+O(h^4)$$

The related finite difference scheme is the following first order accuracy in time and second order accuracy in space:

$$\begin{aligned}&\frac{u_i^{n+1}-u_i^n}{\tau}+\frac{1}{h}(\mu\delta-\frac{\mu\delta^3}{6})(\frac{u^2}{2})_i^n-\nu\frac{1}{h^2}(\delta^2-\frac{\delta^4}{12})(u_i^n)-h\Big(\Omega^n_{i+\frac{1}{2}}\delta^+-\Omega^n_{i-\frac{1}{2}}\delta^-\Big)u_i^n\\&\quad+\frac{\nu\tau}{2}\Big(u^n_{i+\frac{1}{2}}\frac{\mu\delta^2}{h^2}(u^n_{i+\frac{1}{2}})-u^n_{i-\frac{1}{2}}\frac{\mu\delta^2}{h^2}(u^n_{i-\frac{1}{2}})\Big)-\frac{\nu^2\tau}{2}\frac{\delta^4}{h^4}u_i^n+\frac{\nu\tau}{2}\frac{\mu\delta^3}{h^3}(\frac{u^2}{2})_i^n=0\end{aligned} \tag{41}$$

and the differential approximation can be written as:

$$\mathcal{P}(x,t,u,\nu,u_x,u_t,u_{xx})=u_t+u\,u_x-\nu\,u_{xx}+(Cu_x)_x=0 \tag{42}$$

The von Neumann stability analysis of scheme (**??**) under a linearized form provides the following necessary conditions for S, CFL and $\Omega_\tau=\Omega\tau$:

$$CFL^2-2S-2\Omega_\tau\le 0,\quad 0\le(4S)/3-2S^2+\Omega_\tau\le 1/2 \tag{43}$$

If Ω is sufficiently close to zero, these conditions become then sufficient for the linear formulation.

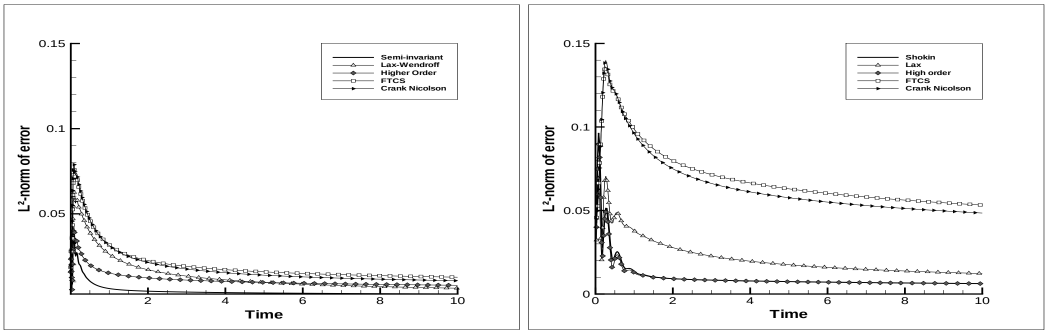

Figure 1. Evolution in time of the L^2 norm of the error in the reference frame, and in the moving one

5.2. Numerical application

i. We hereafter consider the damping of a sinusoidal wave.
Figure (**??**) displays the L^2 norms of the error, obtained by classical methods, and by the semi-invariant one, in the reference frame, and in the moving one.

In the moving frame, as it is the case for classical methods, the semi-invariant one appears to be consistent with the new equation:

$$u_t + \nu_G\, u_x + \left(\frac{u^2}{2}\right)_x - \nu\, u_{xx} = 0 \tag{44}$$

Moreover, the semi-invariant method gives more accurate results in the moving frame.

ii. Consider now the case where the exact solution is given by:

$$u_3(x,t) = -2\nu\, \frac{\partial}{\partial x}\left[\ln F_3(x,t)\right] \ , \forall t > 0\, , x \in\ [0,1] \tag{45}$$

where:

$$F_3(x,t) = \frac{1}{\sqrt{4\nu\pi t}} \int_0^1 exp\left[-\frac{(x-\xi)^2}{4\nu t} - \frac{1}{2\nu}\int_0^{\xi} \sin\left[2\pi(\zeta-1)\right] d\zeta\right] d\xi \tag{46}$$

$$u_3(x,t=0) = \sin\left[2\pi(x-1)\right] \tag{47}$$

Based on the reference value $U = 2$, numerical values of the parameters are:

$$\tau = 5.10^{-3}\ ,\ 10^{-2}\ ,\ \nu = 10^{-3}\ ,\ Reh = 20\ ,\ CFL = 1 \tag{48}$$

Because of the artificial viscosity term $C = 10^{-4}\, t\, (t\, u + x)^2\, u_x$, numerical instabilities lead to the explosion of the numerical solution.
In order to fix these numerical problems, an additionnal viscosity term is taken into account:

$$C = 10^{-4}\, h\, u_x^2 \tag{49}$$

Figures (**??**), (**??**) respectively display the L^2 norms of the error, obtained by the Crank Nicolson scheme, and by the semi-invariant one, in the reference frame, and in the moving one.

The equivalent equation remains invariant under the symmetries of the Burgers equation.

As a primary conclusion, one can note that the solution obtained by means of the semi-invariant method is better, since oscillations decrease.
However, it is delayed, as it is the case for the solution obtained by classical methods.
Also, the invariance condition based on the equivalent equation is weaker.
Moreover, the geometrical structure of the mesh is not fitted enough. The mesh is orthogonal in the reference frame. According to the invariance condition introduced by Dorodnitsyn [**?**], one has to preserve the geometrical structure of the mesh in the moving frame, which is not the case here, since it does not remain orthogonal. Hence, the mesh is not invariant under the Galilean transformation.

6. Conclusion

In the above, we have given a weak invariance condition for the semi-invariant method. It results in a consistancy problem in the moving frame, which leads to a delay with respect to the expected solution. As concerns classical schemes, the additionnal error which appears in the moving frame results in a growth of the amplitude of the oscillations around discontinuity points. We have shown that the semi-invariant method enables one to obtain a damping of the oscillations, and, thus, to reduce this error.
Also, the semi-invariant method enables one to carry out a local analysis of the symmetries, which is not the case with the sole use of the equivalent equation, which cannot take into account geometric properties, and, thus, does not allow an invariance of the stencil under the Galilean transformation.
In [**?**], applying the Dorodnytsin method, we propose an alternative resolution, which takes into account all the stencil variables, and which will we be generalized in other papers.

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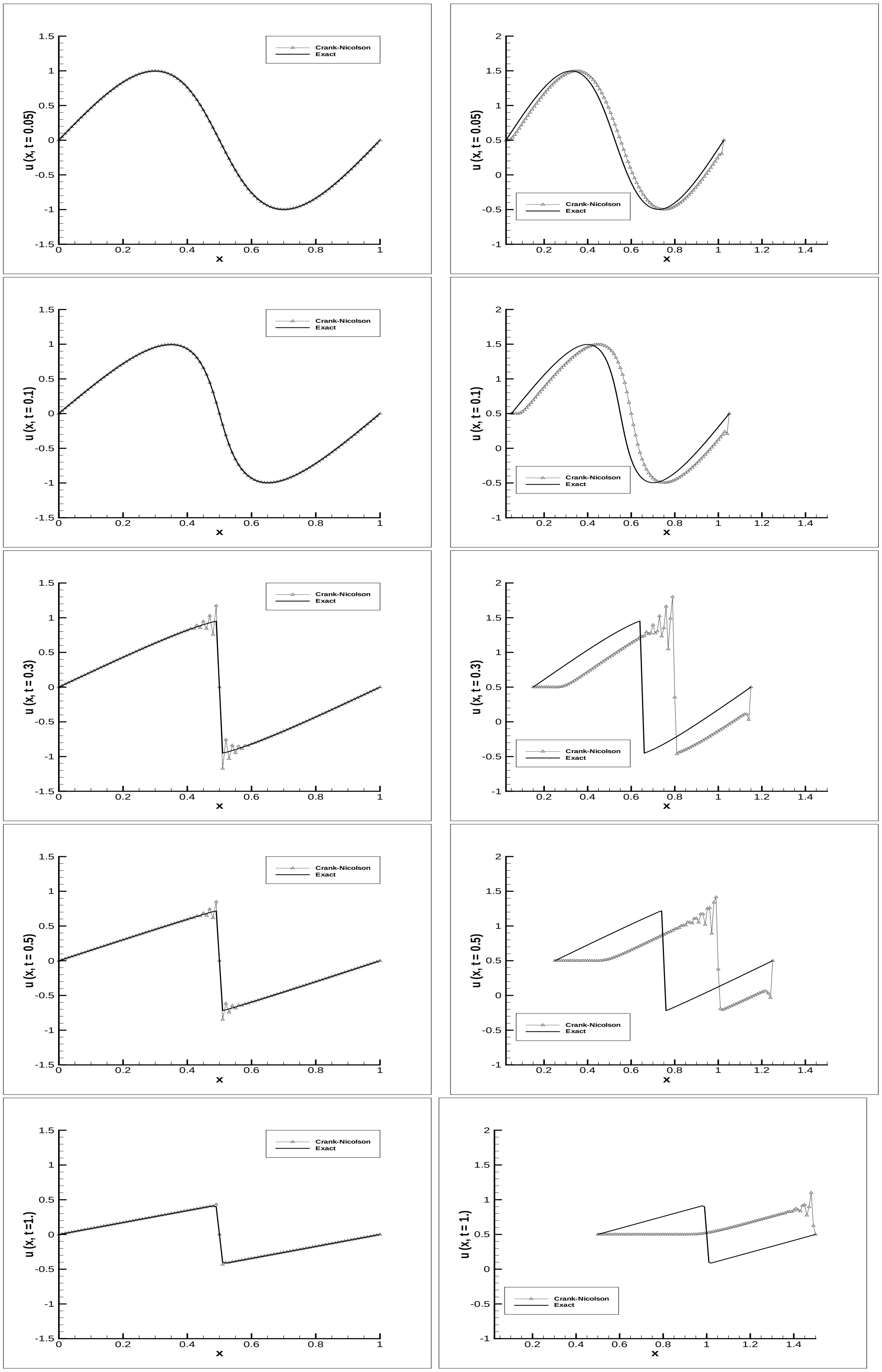

Figure 2. Evolution in time of the L^2 norm of the error for the Crank Nicolson scheme, in the reference frame, and in the moving one

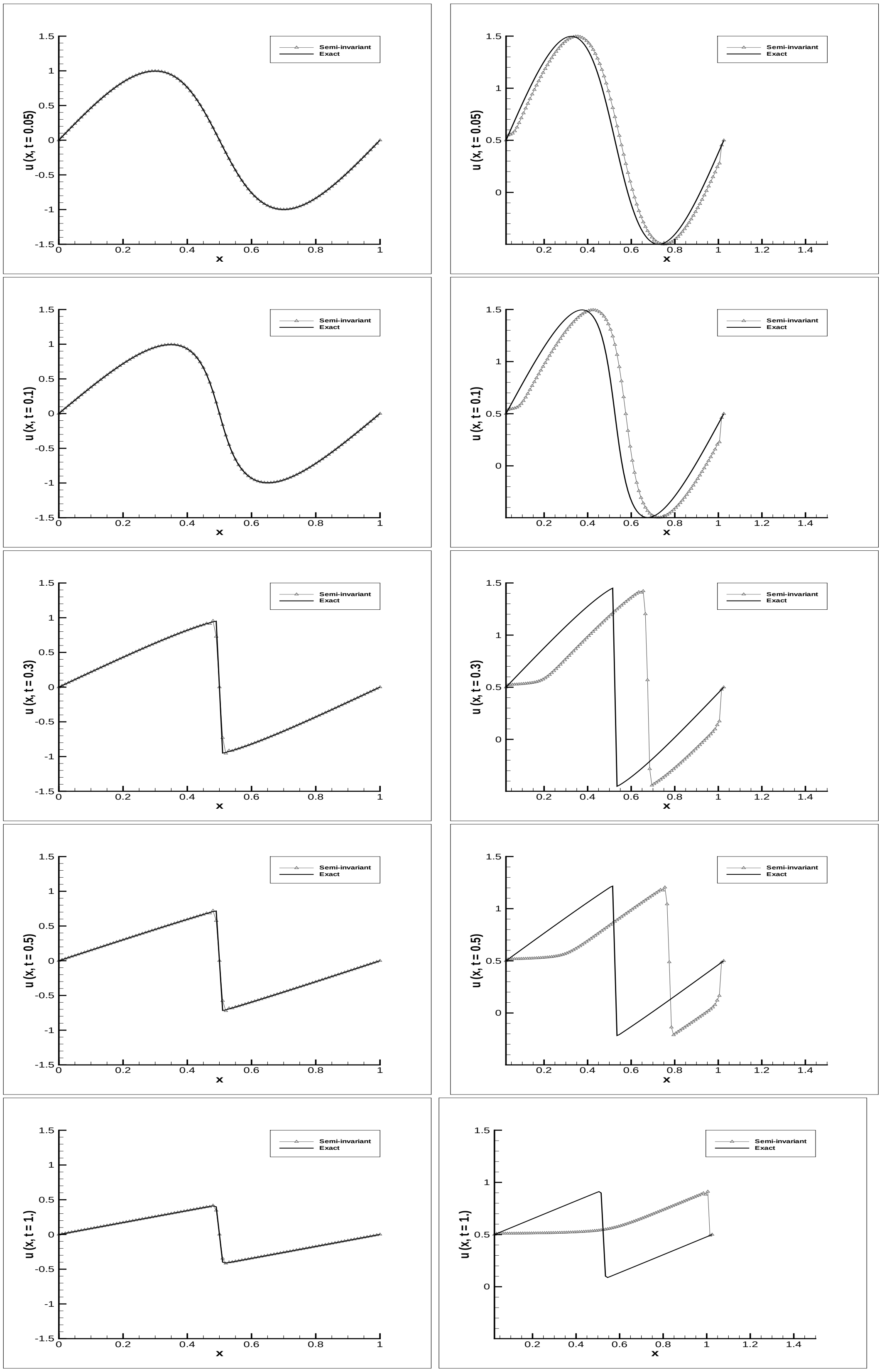

Figure 3. Evolution in time of the L^2 norm of the error for the semi-invariant scheme, in the reference frame, and in the moving one