CHAPTER 4

Asymptotic behavior of solitary waves

Claire David [1], **Qingguo Meng** [2]

[1] *Université Pierre et Marie Curie-Paris 6*
Institut Jean Le Rond d'Alembert, UMR CNRS 7190,
Boîte courrier n^{0}162, *4 place Jussieu, 75252 Paris, cedex 05, France*
[2] *Department of Mechanics Tianjin University of Technology and Education, Tianjin 300222, China*

Abstract

This chapter presents a study of the asymptotic behavior of solitary waves.

[1]Claire.David@upmc.fr

4.1 Introduction

In the following, we study the asymptotic stability of solutions of classical nonlinear evolution equations of the form:

$$\begin{array}{c} \mathcal{F}\left(u, u_{t^{s_{i_1}}}, \ldots, u_{t^{s_{i_m}}}, u_{x^{r_{j_1}}}, \ldots, u_{x^{r_{j_n}}}\right) = 0 \\ r_{i_k} \in \{0, \ldots r_0\}\,,\, s_{j_l} \in \{0, \ldots s_0\} \end{array} \tag{4.1}$$

where $\mathcal{F}$ denotes a function which depends of the solution u, and its derivatives $u_{t^{s_{i_k}}} = \frac{\partial^{s_{i_k}} su}{\partial t^{s_{i_k}}}$ and $u_{x^{r_{j_l}}} = \frac{\partial^{r_{j_l}} u}{\partial x^{r_{i_l}}}$, r_0, s_0 being natural integers, in the specific case where:

$$\mathcal{F}_\xi^{\mathcal{P}}(u, u^{(r)}, (-v)^s\, u^{(s)}) = G(u, u') + R(u) \tag{4.2}$$

G denoting a two-variable polynomial, and R a one-variable polynomial.

4.2 Qualitative analysis

We hereafter follow the traditional definitions.
Suppose that $u(x,\ t) = u(x - vt) = u(\xi)$ is a continuous solution of equation (4.1) for $\xi \in (-\infty,\ +\infty)$ and $\lim\limits_{\xi \to +\infty} u(\xi) = l^+$, $\lim\limits_{\xi \to -\infty} u(\xi) = l^-$, where both l^+ and l^- are constants.

We follow that:

i. $u(x,\ t)$ is called a *bell-profile* solitary wave solution if $l^+ = l^-$;

ii. $u(x,\ t)$ is called a *kink-profile* solitary wave solution if $l^+ \neq l^-$. In the $(x,\ t,\ u)$-space, the graph of the solitary wave solution is called the solitary wave.

Assume that the solution of equation (4.1) satisfies the condition

$$u'(\xi), \quad u''(\xi) \to 0, \quad \text{as } |\xi| \to \infty. \tag{4.3}$$

Multiplying both sides of equation (4.2) by $u'(\xi)$, we have

$$\int_{-\infty}^{\xi} G(u, u')\, u'(\xi)\, d\xi + S(u) = d, \tag{4.4}$$

where S denotes a primitive of R, and d an arbitrary integration constant. From (7) and (8), when ξ goes to negative infinity, we get

$$S(l^-) = d \tag{4.5}$$

Substituting (4.5) into (4.4), when ξ goes to positive infinity, we have

$$\int_{-\infty}^{+\infty} G(u,u')\,u'(\xi)\,d\xi = S(l^-) - S(l^+) \tag{4.6}$$

The analysis of this latter relation will enable one to determine the nature of solitary wave solutions of (4.1).

It can be illustrated in the case of the *CBKDV* equation:

$$u''(\xi) - ru'(\xi) - au(\xi)^3 - bu(\xi)^2 - cu(\xi) = 0, \tag{4.7}$$

where $r = \frac{\mu}{s}$, $a = \frac{\beta}{3s}$, $b = \frac{\alpha}{2s}$ and $c = -\frac{v}{s}$.

Thus:

$$r\int_{-\infty}^{+\infty} [u'(\xi)]^2 d\xi = \frac{c}{2}(l^{-2} - l^{+2}) + \frac{b}{3}(l^{-3} - l^{+3}) + \frac{a}{4}(l^{-4} - l^{+4}). \tag{4.8}$$

When ξ goes to negative infinity, (4.7) yields:

$$al^{+3} + bl^{+2} + cl^{+} = 0 \tag{4.9}$$

In the same way, when ξ goes to positive infinity, (4.7) yields:

$$al^{-3} + bl^{-2} + cl^{-} = 0 \tag{4.10}$$

Thus:

$$\int_{-\infty}^{+\infty} [u'(\xi)]^2 d\xi = \frac{1}{6\,r}\big[c(l^{-2} - l^{+2}) - \frac{a}{3}(l^{-4} - l^{+4})\big]. \tag{4.11}$$

Substituting (4.11) into (4.8), we obtain

$$\int_{-\infty}^{+\infty} [u'(\xi)]^2 d\xi = \frac{1}{6\,r}\big[c(l^{-2} - l^{+2}) - \frac{a}{3}(l^{-4} - l^{+4})\big]. \tag{4.12}$$

Consequently:

i. If $u(x,\ t)$ is a traveling wave solution to equation (4.7) with condition (4.3), then the necessary and sufficient condition is

$$r \quad \text{and} \quad c(l^{-2} - l^{+2}) - \frac{a}{3}(l^{-4} - l^{+4})$$

must have the same sign.

ii. Under condition (4.3), equation (4.7) does not have any bell-profile traveling solitary waves.

iii. If $u(x,\ t)$ is a traveling wave solution to equation (4.7) with condition (4.3), then for fixed values l^- and l^+, the smaller $|r|$ is, the smaller dissipative effect is and the steeper the wave shape of $u(\xi)$ is.

The above conclusions agree with phase plane analysis.

4.3 Approximate Solution of the Burgers-Korteweg-de Vries Equation

The standard form of the Burgers-KDV equation is

$$u_t + \alpha u u_x + \beta u_{xx} + s u_{xxx} = 0, \tag{4.13}$$

where α, β and s are real constants with $\alpha\beta s \neq 0$.
All explicit traveling wave solutions of the Burgers-KdV equation presented in the literature are algebraically equivalent to each other. That is, essentially only one bounded traveling solitary wave solution to the Burgers-KdV equation is known which can be expressed as a composition of a bell-profile solitary wave and a kink-profile solitary wave with the velocity $v = \pm\frac{6\beta^2}{25s}$:

$$\begin{aligned}(i)\ \ u(x,\ t) \ &= \ \frac{3\beta^2}{25\alpha s}\text{sech}^2\Big[\frac{1}{2}\big(-\frac{\beta}{5s}x + \frac{6\beta^3}{125s^2}t\big)\Big] \\ &\quad -\frac{6\beta^2}{25\alpha s}\tanh\Big[\frac{1}{2}\big(-\frac{\beta}{5s}x + \frac{6\beta^3}{125s^2}t\big)\Big] + \frac{6\beta^2}{25\alpha s},\end{aligned} \tag{4.14}$$

$$
\begin{aligned}
(ii)\quad u(x,\ t) \quad = \quad & \frac{3\beta^2}{25\alpha s}\text{sech}^2\Big[\frac{1}{2}\big(-\frac{\beta}{5s}x-\frac{6\beta^3}{125s^2}t\big)\Big] \\
& -\frac{6\beta^2}{25\alpha s}\tanh\Big[\frac{1}{2}\big(-\frac{\beta}{5s}x-\frac{6\beta^3}{125s^2}t\big)\Big]-\frac{6\beta^2}{25\alpha s}.
\end{aligned} \tag{4.15}
$$

Therefore, qualitative analysis as well as approximate techniques to tackle traveling waves of the Burgers-KdV equation appears to be of much interest in this case.
In this Section, our purpose is to present a qualitative study and an approximate solution of the Burgers-KdV equation (4.13). Next, we will show that the Burgers-KdV equation is integrable in the sense of Liouville with certain parametric condition. After that, we make a series of transformations and convert the Burgers-KdV equation (4.13) to the Emden-Fowler equation. An approximate solution is obtained by applying the Adomian decomposition method.

4.3.1 Liouville Integrability

Suppose that equation (4.13) has a traveling wave solution of the form

$$u(x,\ t)=u(x-vt)=u(\xi). \tag{4.16}$$

Substituting (4.16) into equation (4.13) and performing one integration yields a second-order ordinary differential equation

$$u''(\xi)-ru'(\xi)-au^2(\xi)-bu(\xi)-d=0, \tag{4.17}$$

where $r=-\frac{\beta}{s}$, $a=-\frac{\alpha}{2s}$, $b=\frac{v}{s}$ and d is an arbitrary integration constant. If we let $u_\xi(\xi)=z(\xi)$, this latter equation is equivalent to an autonomous system

$$\begin{cases} \dot{u}=z=P(u,\ z), \\ \dot{z}=rz+au^2+bu+d=Q(u,\ z), \end{cases} \tag{4.18}$$

where an overdot denotes differentiation with respect to ξ, Since $\frac{\partial P(u,\ z)}{\partial u}+\frac{\partial Q(u,\ z)}{\partial z}=r$, when $r\neq 0$, by virtue of the Poincaré–Bendixson Theorem, system (4.18) has no closed orbit in the Poincaré phase plane. This implies that equation (**??**) has neither the bell-profile solitary wave solution, nor the periodic traveling wave solution. Moreover, when $\sqrt{b^2-4ad}>0$, equation (4.17) has two homogeneous stationary states, $u=-\frac{b}{2a}-\frac{\sqrt{b^2-4ad}}{2a}$ and $u=-\frac{b}{2a}+\frac{\sqrt{b^2-4ad}}{2a}$. A kink-profile traveling wave solution of (**??**) describes a constant-velocity front of transition from one homogeneous state to another.
In this subsection, we show that under a given parametric condition, the Burgers-KdV equation is integrable in the sense of Liouville. Our result is summarized as follows

Theorem 1 *When the velocity v satisfies $v^2 = \frac{36\beta^4 - 1250\alpha s^3 d}{625 s^2}$, then system (4.18) is Liouville integrable. When $d = 0$, the associated traveling wave solutions are given as (2) and (3), respectively.*

proof 1 (Proof of Theorem 1) *Consider the differential operator*

$$X = P(u,\ z)\frac{\partial}{\partial u} + Q(u,\ z)\frac{\partial}{\partial z}.$$

*Here we say that the polynomial system (**??**) is Liouville integrable, if there is a nontrivial function Ω of u and z with the order greater than or equal to 2, such that*

$$X\Omega = P(u,\ z)\frac{\partial \Omega}{\partial u} + Q(u,\ z)\frac{\partial \Omega}{\partial z} = 0, \quad \textit{for any} \quad (u,\ z).$$

*We know that the second order polynomial system (**??**) is Liouville integrable if and only if there exists an integrating factor $\mu(u,\ z)$ for the associated first order differential equation*

$$Q(u,\ z)du - P(u,\ z)dz = 0, \tag{4.19}$$

and such that both

$$\phi_1(u,\ z) = \frac{1}{\mu(u,\ z)}\frac{\partial \mu}{\partial u} \quad \textit{and} \quad \phi_2(u,\ z) = \frac{1}{\mu(u,\ z)}\frac{\partial \mu}{\partial z},$$

are rational functions in u and z. Using this fact and denoting $B = -\left(\frac{\partial P}{\partial u} + \frac{\partial Q}{\partial z}\right)$, we have

$$\begin{aligned} &\frac{\partial \phi_1}{\partial z} = \frac{\partial \phi_2}{\partial u}, \\ &P(u,\ z)\phi_1 + Q(u,\ z)\phi_2 - B = 0, \end{aligned}$$

from which we have

$$\begin{aligned} PX\phi_2 &= P^2(u,\ z)\left(\frac{\partial}{\partial z}\frac{B}{P}\right) - P^2(u,\ z)\left(\frac{\partial}{\partial z}\frac{Q}{P}\right)\phi_2, \\ &= \left(Q\frac{\partial P}{\partial z} - P\frac{\partial Q}{\partial z}\right)\phi_2 + \left(P\frac{\partial B}{\partial z} - B\frac{\partial P}{\partial z}\right). \end{aligned} \tag{4.20}$$

Usually, for a given autonomous system, it is difficult for us to find $\mu(u,\ z)$ in a straightforward manner. Conversely, if ϕ_1 and ϕ_2 are known, we can derive μ and Ω directly. So we suppose that

$$\phi_2 = \frac{\psi_1}{\psi_2},$$

where ψ_1, ψ_2 are polynomials and relatively prime to each other. Then, from (4.20), we have

$$\psi_2\left[PX\psi_1 - \left(Q\frac{\partial P}{\partial z} - P\frac{\partial Q}{\partial z}\right)\psi_1 - \left(P\frac{\partial B}{\partial z} - B\frac{\partial P}{\partial z}\right)\psi_2\right] = \psi_1(PX\psi_2). \quad (4.21)$$

Separating (4.21) into coupled equations, we find that if there is a polynomial $G(u,\ z)$ such that

$$\begin{aligned} PX\psi_1 &= \left(Q\frac{\partial P}{\partial z} - P\frac{\partial Q}{\partial z} + G\right)\psi_1 + \left(P\frac{\partial B}{\partial z} - B\frac{\partial P}{\partial z}\right)\psi_2, \\ PX\psi_2 &= G\psi_2, \end{aligned} \quad (4.22)$$

then system (4.22) is Liouville integrable too. In other words, the existence of non-trivial polynomials $G(u,\ z)$, ψ_1 and ψ_2 satisfying system (4.22) is another sufficient and necessary condition for Liouville integrability of system (4.18).

Solving system (4.22) consistently, we find that only when $v^2 = \frac{36\beta^4 - 1250\alpha s^3 d}{625 s^2}$ and $G(u,\ z)$ is linear with respect to z, we can obtain the nontrivial polynomials for $G(u,\ z)$, ψ_1 and ψ_2 as follows

$$\begin{aligned} G(u,\ z) &= -\frac{6\beta}{5s}z, \\ \psi_1(u,\ z) &= -\frac{5}{3}z - \frac{2\beta}{3s}u + \frac{4\beta^3}{25\alpha s^2} + \frac{2\beta v}{3\alpha s}, \\ \psi_2(u,\ z) &= z^2 - \left[\frac{4r}{5}u + \frac{2br}{5a}\left(\frac{6r^2}{25b}+1\right)\right]z - \frac{2a}{3}u^3 - bu^2 - \frac{2r^2}{25}u^2 \\ &\quad -2du - \frac{2br^2}{25a}\left(\frac{6r^2}{25b}+1\right)u + \frac{b^3}{12a^2}\left(\frac{6r^2}{25b}+1\right)\left(\frac{36r^4}{625b^2}-1\right). \end{aligned}$$

Therefore, system (4.18) is Liouville integrable. Furthermore, since $\frac{\partial\psi_2}{\partial u} = \frac{\partial[(B/P)-\psi_2(Q/P)]}{\partial z}$, we may find an integrating factor for equation (4.19) according to

$$\frac{1}{\mu(u,\ z)}\frac{\partial\mu}{\partial z} = \frac{\psi_1}{\psi_2}, \quad \frac{1}{\mu(u,\ z)}\frac{\partial\mu}{\partial u} = \frac{B}{P} - \frac{Q}{P}\frac{\psi_1}{\psi_2},$$

which brings us the following first integral of system (4.18):

$$z^2 - \left(\frac{4r}{5}u + \frac{2br}{5a}(k+1)\right)z - \frac{2a}{3}u^3 - bu^2 - \frac{2r^2}{25}u^2 - 2du - \frac{2br^2}{25a}(k+1)u + D = 0.$$

where $k = \frac{6r^2}{25b}$, $k \in \mathbb{R}$ and $D = \frac{b^3}{12a^2}(k+1)(k^2-1)$. Note that the above first integral is the same as that obtained in the literature. Using this first integral, we reduce *equation (4.17) to a first order ordinary differential equation*

$$\frac{du}{d\xi} = \frac{2r}{5a}\left(au + \frac{(k+1)b}{2}\right) \pm \left(au + \frac{(k+1)b}{2}\right)\sqrt{\frac{2}{3a^2}\left(au + \frac{(k+1)b}{2}\right)}. \quad (4.23)$$

Solving equation (4.23) and changing to the original variables, we obtain

$$u(x,\ t) = -\frac{12\beta^2}{25\alpha s}\left(\frac{e^{-\frac{\beta}{5s}(x-vt+\xi_0)}}{e^{-\frac{\beta}{5s}(x-vt+\xi_0)} + c_0}\right)^2 + \frac{(k+1)v}{\alpha}, \quad (4.24)$$

where c_0 *and* ξ_0 *are arbitrary constants. It is notable that making use of the identity* $4A[\frac{e^{2t}}{1+e^{2t}}]^2 = -Asech^2 t + 2A\tanh t + 2A$, *one can see that solutions (2) and (3) are only particular cases of (4.24) where* $k = \pm 1$ *and* $c_0 = 1$.

4.3.2 Approximate Solution

Although many methods have been proposed for seeking explicit solitary wave solutions of equation (4.13), as far as our knowledge goes, all bounded traveling solitary wave solutions of equation (4.13) expressed in the explicit functional forms in the literature are essentially equivalent to each other. That is, a feature of this solution is that it is a linear combination of particular solutions of the KdV equation and the Burgers equation. No explicit approximate solution to the Burgers-Korteweg-de Vries equation with any order derivatives has been presented previously.

In this section, applying an elegant and powerful treatment—the Adomian decomposition method, we are concerned with the approximate solution of the Burgers-Korteweg-de Vries equation. Our work stems mainly from the Adomian decomposition method that provides the solution in the form of a convergent series. The main advantage of the Adomian decomposition method is that it can be applied directly to many types of differential and integral equations, linear or nonlinear, homogeneous or nonhomogeneous equations. In most cases, six or seven components of the series solution may give an insight through the behavior of the solution. Moreover, the accuracy level can be dramatically enhanced by evaluating more terms of the series solution.

We summarize our new result as

Theorem 2 *When the velocity* v *satisfies* $v^2 = \frac{4\beta^4 - 162\alpha s^3 d}{81 s^2}$, *then equation (4.13)* *has an approximate solution*

$$\begin{aligned} u(x,\ t) \ &= \ \frac{s}{\alpha}\sqrt{\frac{v^2}{s^2}+\frac{2\alpha d}{s}}+\frac{v}{\alpha}+\frac{2\beta^2}{9\alpha s}\Big[e^{-\frac{2\beta}{3s}(x-vt+\xi_0)}-\frac{1}{6}e^{-\frac{4\beta}{3s}(x-vt+\xi_0)} \\ &+\frac{1}{60}e^{-\frac{2\beta}{s}(x-vt+\xi_0)}-\frac{22}{3\cdot 7!}e^{-\frac{8\beta}{3s}(x-vt+\xi_0)}+\frac{384}{9\cdot 9!}e^{-\frac{10\beta}{3s}(x-vt+\xi_0)} \\ &+\frac{11302}{45\cdot 11!}e^{-\frac{4\beta}{s}(x-vt+\xi_0)}+\cdots\Big], \end{aligned} \tag{4.25}$$

where ξ_0 is an arbitrary constant.

proof 2 (Proof of Theorem 2) *To remove the constant term in equation (4.17), we make transformation*

$$u(\xi) = -\frac{1}{a}u^*(\xi) - \frac{b}{2a} - \frac{\sqrt{b^2-4ad}}{2a}. \tag{4.26}$$

Substituting (4.26) into (4.17) and omitting the asterisks for notational simplicity gives

$$u''(\xi) - ru'(\xi) + u^2(\xi) - \mu u(\xi) = 0, \tag{4.27}$$

where $\mu = \sqrt{b^2-4ad}/a$. Make the natural logarithmic transformation

$$\xi = \frac{1}{r}\ln\tau,$$

then equation (4.27) becomes

$$r^2\tau^2\frac{d^2u}{d\tau^2} + u^2 - \mu u = 0. \tag{4.28}$$

Take the variable transformation as

$$q = \tau^l, \quad u = r^2 l^2 \tau^{-\frac{1}{2}(l-1)}\cdot\rho(q),$$

then equation (4.28) reduces to

$$\frac{d^2\rho}{dq^2} = -q^{\frac{1-5l}{2l}}\rho^2, \tag{4.29}$$

where l is given by

$$l^2 = 1 - \frac{4\sqrt{b^2-4ad}}{r^2}. \tag{4.30}$$

We continue our attention by considering the Lane-Emden equation of index m *with boundary conditions*

$$x^{-2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) = -\nu y^m = -f(y), \qquad (4.31)$$
$$y(0) = 1, \quad y'(0) = 0,$$

where y *and* x *are dimensionless variables,* ν *is an parameter proportional to the gravitational constant, and* m *is an index related to the ratio of specific heats of the gas comprising the star. Equation (4.31) is one of the basic equations in the theory of stellar structure, which describes the temperature variation of a spherical gas cloud under the mutual attraction of its molecules and subject to the laws of classical thermodynamics. It also describes the variation of density as a function of the radial distance for a polytrope. Physically interesting values of* m *lie in the interval [0, 5]. Exact solutions for equation (4.31) are known only for the values* $m = 0,\ 1$ *and 5. For other values of* m *the study of equation (4.31) is dominated qualitatively and numerically.*
Our motivation for studying this system with index m *comes actually from a transformation*

$$\rho(x) = xy(x), \qquad (4.32)$$

from which we have

$$\rho'_x = y(x) + xy'_x, \quad \rho''_{xx} = xy''_{xx} + 2y'_x. \qquad (4.33)$$

Inserting (4.32) and (4.33) into (4.31) gives

$$\frac{d^2\rho}{dx^2} = -\nu x^{1-m}\rho^m, \qquad (4.34)$$
$$\rho(0) = 0, \quad \rho'(0) = 1.$$

Note that when $\nu = 1$*,* $m = 2$ *and* $l = 1/3$*, (4.34) is of the same form as equation (4.29).*
To apply the Adomian decomposition method, we define the differential operator

$$T = \frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d}{dx}\right),$$

and the corresponding inverse operator T^{-1} *is defined as*

$$T^{-1} = \int_0^x \frac{1}{x^2}\int_0^x x^2(\cdot)dxdx.$$

Equation (4.31) may be re-expressed in an operator form

$$Ty = -f(y). \tag{4.35}$$

Operating with T^{-1} carries (4.35) into

$$y = 1 - T^{-1}[f(y)]. \tag{4.36}$$

The Adomian decomposition method introduces the infinite decomposition series

$$y(x) = \sum_{i=0}^{\infty} y_n(x), \tag{4.37}$$

and the infinite series of polynomials

$$f(y) = \sum_{i=0}^{\infty} A_n(y_0,\ y_1,\ \cdots,\ y_n), \tag{4.38}$$

where the components $y_n(x)$ of the solution $y(x)$ will be determined recurrently, and A_n are Adomian polynomials.
Substituting (4.37) and (4.38) into (4.36) yields

$$\sum_{i=0}^{\infty} y_n(x) = 1 - T^{-1}\sum_{i=0}^{\infty} A_n(y_0,\ y_1,\ \cdots,\ y_n). \tag{4.39}$$

Using the recursive relation

$$\begin{aligned} &y_0(x) = 1, \\ &y_{j+1}(x) = -\int_0^x \frac{1}{x^2}\int_0^x x^2(A_j)dxdx, \quad j \geqslant 0, \end{aligned} \tag{4.40}$$

and when $\nu = 1$, we list the first several Adomian polynomials defined as

$$\begin{aligned} &A_0 = f(y_0) = y_0^m, \\ &A_1 = y_1 f'(y_0) = m y_1 y_0^{m-1}, \\ &A_2 = y_2 f'(y_0) + y_1^2 f''(y_0)/(2!) = m y_2 y_0^{m-1} + m(m-1) y_1^2 y_0^{m-2}/(2!), \\ &A_3 = y_3 f'(y_0) + y_1 y_2 f''(y_0) + f'''(y_0) y_1^3/(3!), \\ &\quad = m y_3 y_0^{m-1} + m(m-1) y_1 y_2 y_0^{m-2} + m(m-1)(m-2) y_0^{m-3} y_1^3/(3!), \\ &A_4 = y_4 f'(y_0) + (y_1 y_3 + y_2^2/2) f''(y_0) + y_1^2 y_2 f'''(y_0)/(2!) + f^{(4)}(y_0) y_1^4/(4!), \\ &\quad = m y_4 y_0^{m-1} + m(m-1)(y_1 y_3 + y_2^2/2) y_0^{m-2} \\ &\qquad + m(m-1)(m-2) y_1^2 y_2 y_0^{m-3}/(2!) \\ &\qquad + m(m-1)(m-2)(m-3) y_1^4 y_0^{m-4}/(4!), \\ &\cdots\cdots \end{aligned} \tag{4.41}$$

Substituting (4.41) into (4.40), we get

$$\begin{aligned}
&y_0 = 1,\\
&y_1 = -T_1^{-1}(A_0) = -\int_0^x \frac{1}{x^2}\int_0^x x^2 dxdx = -\frac{1}{6}x^2,\\
&y_2 = -T_1^{-1}(A_1) = -\int_0^x \frac{1}{x^2}\int_0^x x^2 A_1 dxdx = \frac{m}{5\cdot 4!}x^4,\\
&y_3 = -T_1^{-1}(A_2) = -\int_0^x \frac{1}{x^2}\int_0^x x^2 A_2 dxdx = -\frac{m(8m-5)}{3\cdot 7!}x^6,\\
&y_4 = -T_1^{-1}(A_3) = -\int_0^x \frac{1}{x^2}\int_0^x x^2 A_3 dxdx\\
&\quad = \frac{m(70-183m+122m^2)}{9\cdot 9!}x^8,\\
&y_5 = -T_1^{-1}(A_4) = -\int_0^x \frac{1}{x^2}\int_0^x x^2 A_4 dxdx\\
&\quad = \frac{m(3150-1080m+12642m^2-5032m^3)}{45\cdot 11!}x^{10},\\
&\cdots\cdots
\end{aligned} \tag{4.42}$$

Inserting (4.42) into (4.39), we derive that a solution of equation (4.31) for arbitrary positive m in a series form is expressed as follows

$$\begin{aligned}
y(x) \;=\; & 1-\frac{1}{6}x^2+\frac{m}{5\cdot 4!}x^4-\frac{m(8m-5)}{3\cdot 7!}x^6+\frac{m(70-183m+122m^2)}{9\cdot 9!}x^8\\
&+\frac{m(3150-1080m+12642m^2-5032m^3)}{45\cdot 11!}x^{10}+\cdots .
\end{aligned} \tag{4.43}$$

Consequently, when $m = 2$ and $l = 1/3$, from (4.30), namely, $v^2 = \frac{4\beta^4-162\alpha s^3 d}{81s^2}$, we obtain an approximate solution of equation (17):

$$\rho(q) = q - \frac{1}{6}q^3 + \frac{1}{60}q^5 - \frac{22}{3\cdot 7!}q^7 + \frac{384}{9\cdot 9!}q^9 + \frac{11302}{45\cdot 11!}q^{11} + \cdots .$$

Using the inverses of the transformations introduced at the beginning of this section and changing to the original variables, we obtain an approximate solution as (13) for equation (4.13) immediately.

Note that one can compute higher terms as far as he wishes without any difficulty, but truncating the power series up to six terms will be sufficient to give highly accurate results. For instance, when $m = 1$ and $\nu = 1$, equation (4.34) has an exact solution $\rho = \sin x$. Transformation (4.32) and expression (4.43) gives $\rho = \sin x$

Table 4.1: Comparison of the power series solution of equation (4.34) with six terms and the exact solution for $m = 1$ and $\nu = 1$.

Points x	Power Series Solution $\rho = xy(x)$	Exact Solution $\rho = \sin x$
0.00	0.000 000	0.000 000
0.50	0.479 425	0.479 425
1.00	0.841 471	0.841 470 9
1.50	0.997 495	0.997 494 9
2.00	0.909 296	0.909 297
2.50	0.598 470	0.598 472

exactly. Comparison of the power series solution up to six terms and the exact solution in the case of $m = 1$ and $\nu = 1$ is illustrated in Table (4.3.2).

It is worthwhile to mention that formula (4.43) may also be derived by using the Taylor series approach. However, as clearly illustrated in the literature, the decomposition method has the advantage of easy integrals to compute, whereas the product of series provides cumbersome work in the Taylor series approach. As observed from the above discussion, the components $y_0,\ y_1,\ y_2,\ \cdots\cdots$ are determined simply in a recurrent manner. The method minimizes the computational difficulties of the Taylor series in that the components are determined elegantly by using simple integrals. Furthermore, the accuracy level of the approximation can be increased by evaluating further components.

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