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Chapter 2

Exact analytic solutions of nonlinear evolution equations

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Abstract

This chapter presents the main classical methods used to obtain exact explicit solutions of classical nonlinear evolution equations.

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2.1 Introduction

Nonlinear partial differential equations are known to describe a wide variety of phenomena not only in physics, where applications extend over magnetofluid dynamics, water surface gravity waves, electromagnetic radiation reactions, and ion acoustic waves in plasmas, just to name a few, but also in biology and chemistry, and several other fields. One of the important tasks is to seek exact and explicit solutions. In the past several decades both mathematicians and physicists have made many attempts in this direction. Various methods for obtaining exact solutions to NLPDEs had been proposed, as s bilinear method, transformation method, etc... Recently, [5] presented a (HBM) for finding exact solutions of a given NLPDEs. This method provides a convenient analytical technique to construct solitary-wave solutions and has been generalized to obtain (or multiple solitary-wave) solutions [68].

In the following, we present the main classical methods used to obtain exact explicit solutions of Classical nonlinear evolution equations of the form:

$$\mathcal{F}\left(u, u_{t^{s_{i_1}}}, \ldots, u_{t^{s_{i_m}}}, u_{x^{r_{j_1}}}, \ldots, u_{x^{r_{j_n}}}\right) = 0 \ , \ \ r_{i_k} \in \{0, \ldots r_0\} \, , \, s_{j_l} \in \{0, \ldots s_0\} \tag{2.1}$$

where $\mathcal{F}$ denotes a function which depends of the solution u, and its derivatives $u_{t^{s_{i_k}}} = \frac{\partial^{s_{i_k}} su}{\partial t^{s_{i_k}}}$ and $u_{x^{r_{j_l}}} = \frac{\partial^{r_{j_l}} u}{\partial x^{r_{i_l}}}$, r_0, s_0 being natural integers:

1. Hirota's bilinear method;
2. Painlevé's expansion method;
3. the Tanh-Coth function method;
4. the Sine-Cosine function method;
5. the EXP function method;
6. the Jacobi elliptic function method;
7. the $\frac{G}{G'}$ expansion method;
8. Lie symmetry reduction method.

2.2 Hirota's bilinear method

Hirota's bilinear method stands out as a powerful method for finding and solving integrable nonlinear partial differential equations. The idea behind it is to make, first a nonlinear change in the dependent variables, after which soliton solutions of integrable systems can be expressed as polynomials of exponentials, the argument of which are linear in the independent variables.

Assume that equation (4.1) has a traveling wave solution of the form

$$u(x,t) = u(\xi), \quad \xi = x - v\,t \tag{2.2}$$

where v is the wave velocity. The resulting equations may be integrated with respect to ξ to reduce the order. For simplicity, the integration constants can be ignored, assuming that the solution its derivatives vanish at $\xi = \pm\infty$.
Expand then $u(\xi)$ in a power series:

$$u(\xi) = \sum_{i=1}^{+\infty} u_n\, g^n(\xi) \tag{2.3}$$

where $g(\xi) = e^{-K(v)\,\xi}$ solves the linear part of the equation. Hence, the wave number $K(c)$ is related to the velocity c by the dispersion law of (one of) the linearized equation. Substitute the expansion (3.3) into the full nonlinear equation (1), rearrange the sums by using 's rule for multiple series [2] and equate the coefficient of $g^n(\xi)$ leads to a nonlinear system of coupled recursion relations for the coefficients u_n. Quite often, the relation $K(v)$ and appropriate scales on $u(\xi)$ allow to simplify the recursion relations.
Assume then that the u_n are polynomials of degree d_n in n:

$$u_n(\xi) = \sum_{i=1}^{d_n} u_n^i\, n^i \tag{2.4}$$

Substitute them in the recursion relations leads to a now completely algebraic system whose unknowns are the u_n^i. This system can easily be solved using a symbolic calculus tool (*Mathematica* or *Maple*).

2.3 Painlevé's expansion method

The singularity structure analysis (popularly known now as analysis) pioneered by Kovalevskaya, , and their contemporaries provides an effective and algorithmic procedure for predict ing the integrable cases of nonlinear dynamical systems governed *by both ordinary and partial differential equations. The test proposed by Weiss, Tabor, and Carnevale [34] not only provides a valuable first test to know whether a given partial differential equation is completely integrable or not but also yields the integrability properties, including the Bäcklund transformation, Lax pair, 's bilinear representation, etc. [35],[11], [12], [28]. The procedure can be extended to specific non- equations, as the Duffing oscillator, the well-known Lorenz system, ... , and enables one to obtain exact solutions wherever possible.*

The partial differential equation (4.1) is said to possess the property if its solutions are single-valued in the neighbourhood of noncharacteristic, movable singular manifolds [5]. To be precise, if the singularity manifold is determined by

$$\varphi(z_1, z_2, \ldots, z_n) = 0, \tag{2.5}$$

Assume then that in the neighbourhood of the manifold (2.5):

$$u(\xi) = \varphi^{\alpha} \sum_{i=0}^{+\infty} u_i \left(z_1, z_2, \ldots, z_n\right) \varphi^{i} \tag{2.6}$$

is a solution of the partial differential equation (4.1), where $u_0 \neq 0$, the u_i being analytic functions of $z_1, z_2, \ldots, z_n$, and α is a negative integer. When the ansatz (2.6) is correct, the partial differential equation passes the test, and is conjectured to be a possible integrable candidate.
The test (or) essentially consists of the following three steps:

i. determination of the leading order of (2.6) in the neighbourhood of the singularity manifold (2.5);

ii. determination of resonances, that is, the powers at which arbitrary functions enter into (2.6);

iii. verification that a sufficient number of arbitrary functions exist without the introduction of movable critical manifolds. In general, the transformation, Lax pair, and linearization follow by "truncating" the expansion (2.6) at the "constant" level term, i.e.,

$$u(\xi) = u_0\, \varphi^{-N} + u_1\, \varphi^{-N+1} + \ldots + u_N \tag{2.7}$$

which enables one to find the u_i.

2.4 Methods based on the same integration technique

In the following, we present methods that are originally based on the same integration procedure: the Tanh-Coth function method, the Sine-Cosine function method, the Exp function method, the Jacobi elliptic function method, the $\frac{G}{G'}$ *expansion method, etc...*

Assume that equation (4.1) has traveling wave solutions of the form

$$u(x,t) = u(\xi), \quad \xi = x - v\,t \tag{2.8}$$

where v is the wave velocity. Substituting (2.8) into equation (4.1) leads to:

$$\mathcal{F}(u, u^{(r)}, (-v)^s\, u^{(s)}) = 0, \tag{2.9}$$

Performing an integration of (2.9) with respect to ξ leads to an equation of the form:

$$\mathcal{F}_\xi^{\mathcal{P}}(u, u^{(r)}, (-v)^s\, u^{(s)}) = C, \tag{2.10}$$

where $\mathcal{F}_\xi^{\mathcal{P}}(u, u^{(r)}, (-v)^s\, u^{(s)}) = C$, is a function depending on u, $u^{(r)}$, $u^{(s)}$, and C is an arbitrary integration constant, which will be the starting point for the determination of solitary waves solutions.

2.4.1 The EXP-function method

The Exp-function method was first proposed in [8]. As a straightforward and concise method, it has been successfully applied to obtain generalized solitary solutions and periodic solutions of some nonlinear evolution equation arising in mathematical physics and differential-difference equations. The solution procedure of this method, with the aid of a symbolic calculus tool (Mathematica or Maple), is of utter simplicity and this method can be easily extended to other kinds of nonlinear evolution equations.

The method admits the use of the finite series expansion

$$u(\xi) = \frac{\displaystyle\sum_{i=-N_1}^{N_2} u_i\, e^{i\,\xi}}{\displaystyle\sum_{i=-N_3}^{N_4} v_i\, e^{i\,\xi}} \tag{2.11}$$

where N_1, N_2, N_3 and N_4 are positive integer which could be freely chosen, u_i and v_i being unknown constants to be determined.
Substituting the assumption (2.11) into the reduced ODE (2.10) gives a polynomial trigonometric equation in $e^{i\,\xi}$, of degree R. Reduce all fractions to the same denominator, and collect then all coefficients of the same power in $e^{i\,\xi}$, $0 \leqslant i \leqslant R$. Since they have to vanish, one obtains a system of algebraic equations in u_i, v_i. The solutions proposed in (2.11) follows immediately.

2.4.2 The Sine-Cosine function method

The solutions of the reduced ODE equation (2.10) are searched in the form:

$$u(\xi) = \lambda\, \cos^{\beta}(\mu\, \xi) \tag{2.12}$$

or in the form

$$u(\xi) = \lambda\, \sin^{\beta}(\mu\, \xi) \tag{2.13}$$

where λ, μ are parameters that will be determined. μ is the wavenumber.
Substituting the sinecosine assumptions (2.12), (2.13) and and their derivatives into the (2.10) gives a polynomial trigonometric equation in $\cos(\mu\,\xi)$ or $\sin(\mu\,\xi)$, of degree R. The parameters are then determined by first balancing the exponents of each pair of cosine or sine to determine R. Collect then all coefficients of the same power in $\cos^k(\mu\,\xi)$ or $\sin^k(\mu\,\xi)$, $0 \leqslant k \leqslant R$. Since they have to vanish, one obtains a system of algebraic equations in β, λ, μ. The solutions proposed in (2.12), (2.13) follow immediately.

2.4.3 The Tanh-Coth function method

The tanh method has been developed by Malfliet [12], [13], where the tanh is used as a new variable, since all derivatives of a tanh can be represented by a tanh itself. It can easily be extended to coth. It is considered to be one of most straightforward and effective methods to construct solitary wave solutions of nonlinear evolution equations. The success of this method depends on the fact that one circumvents *integration to get explicit solutions based on the fact that soliton solutions are essentially of a localized nature.*

Set:

$$Y(\xi) = \tanh(\mu\,\xi) \tag{2.14}$$

leads to the change of derivatives:

$$\frac{d}{d\,\xi} = \mu\,(1 - Y^2)\,\frac{d}{d\,Y} \tag{2.15}$$

$$\frac{d^2}{d\,\xi^2} = -2\,\mu^2\,Y\,(1 - Y^2)\,\frac{d}{d\,Y} + \mu^2\,(1 - Y^2)^2\,\frac{d^2}{d\,Y^2} \tag{2.16}$$

The method admits the use of the finite series expansion

$$u(\xi) = S(Y) = \sum_{i=0}^{N} u_i\,Y^i \tag{2.17}$$

or:

$$u(\xi) = S(Y) = \sum_{i=0}^{N} u_i\,Y^i + \sum_{i=1}^{N} v_i\,Y^{-i} \tag{2.18}$$

where N is a positive integer, in most cases, that will be determined. Substituting (2.17) or (2.18)), and using (2.15), (2.16) into the ODE (2.10) yields an algebraic equation in powers of Y. To determine the parameter N, one usually balances the linear terms of highest order in the resulting equation with the highest order nonlinear terms. Collect then all coefficients of powers of Y in the resulting equation. since they have to vanish, one obtains a system of algebraic equations involving the parameters u_i, $i = 0\,\ldots, N$, μ and v. Having determined these parameters leads to the analytic solution $u(x,t)$ in a closed form.

2.4.4 The Jacobi elliptic function method

The Jacobi elliptic function methodexpansion method was introduced by Liu et al. [21]. It is more general than the hyperbolic tangent function expansion method, and, contrary to this latter method, as well as the homogeneous balance method, the trial function method [7,8], the nonlinear transformation method [9,10], the sinecosine method, who can only obtain the shock and solitary wave solutions, it enables one to obtain the periodic solutions of nonlinear wave equations

The method admits the use of the finite series expansion

$$u(\xi) = \sum_{i=0}^{N} u_i \, sn^i \, \xi \tag{2.19}$$

where sn denotes the . We recall that, for $0 \leqslant x \leqslant 1$:

$$sn^{-1}(x,k) = \int_0^x \left[(1-t^2)\,(1-k^2\,t^2) \right]^{-\frac{1}{2}} \, dt \tag{2.20}$$

$$cn^{-1}(x,k) = \int_0^x \left[(1-t^2)\,(k'^2 + k^2\,t^2) \right]^{-\frac{1}{2}} \, dt \tag{2.21}$$

k denoting the modulus of the elliptic functions ($0 < k < 1$), and k' the complementary modulus, which satisfyes:

$$k'^2 = 1 - k^2 \tag{2.22}$$

It is interesting to note that:

$$\frac{du}{d\,\xi} = \sum_{i=0}^{N} i\, u_i \, sn^{i-1}(\xi)\, cn\,\xi\, dn\,\xi \tag{2.23}$$

where cn and dn are respectively the and the .
Also:

$$cn^2\,\xi = 1 - sn^2\,\xi \;\;,\;\; dn^2\,\xi = 1 - mk^2\,sn^2\,\xi \tag{2.24}$$

Due to:

$$\frac{d\,sn\,\xi}{d\,\xi} = cn\,\xi\,dn\,\xi \;\;,\;\; \frac{d\,cn\,\xi}{d\,\xi} = -sn\,\xi\,dn\,\xi \;\;,\;\; \frac{d\,dn\,\xi}{d\,\xi} = -m^2\,sn\,\xi\,cn\,\xi \tag{2.25}$$

the highest degree of $\frac{d^p\,u}{d\,\xi^p}$ is taken as $N+p$, $p = 1, 2, 3, \ldots$, and the highest degree of $u^q\,\frac{d^p\,u}{d\,\xi^p}$ is taken as $(q+1)\,N+p$, $q = 0, 1, 2, \ldots$, $p = 1, 2, 3, \ldots$,
Thus, one can select N in (2.19) to balance the derivative term of the highest order and the nonlinear term in (4.1). When $m \to 1$, $sn\xi \to tanh\xi$. (2.19) is then degenerated as the following form:

$$u(\xi) = \sum_{i=0}^{N} u_i \, tanh^i \, \xi \tag{2.26}$$

2.4.5 The $\frac{G}{G'}$ expansion method

The $\frac{G}{G'}$ expansion method was introduced by Wang et al. [16], who demonstrated that it is powerful technique for seeking analytic solutions of nonlinear partial differential equations. Bekir [2] applied this method to obtain traveling wave solutions of various equations. Generalization of the method were given by Zhang et al. [17], [18].

Suppose that:

$$u(\xi) = \sum_{i=1}^{N} u_i \left(\frac{G}{G'}\right)^i \tag{2.27}$$

the u_i being real constants, with $u_N \neq 0$ to be determined, is a positive integer to be determined, and the function G is the general solution of the auxiliary linear ordinary differential equation:

$$G''(\xi) + \lambda\, G' v + \mu\, G(\xi) = 0 \tag{2.28}$$

λ, μ being real constants, to be determined later.
By balancing the highest order nonlinear term(s) with the linear term(s) of highest order in equation (2.10), N can then be determined.
Substituting (2.27) in (2.10), and using (2.28), leads to an algebraic equation involving powers $\frac{G}{G'}$. Equating the coefficients of each power of $\frac{G}{G'}$ to zero, enables one to obtain a system of algebraic equations for the u_i, λ, μ, and v. The constants can then be determined with the aid of a symbolic calculus tool, such as *Mathematica* or *Maple*. Since the solutions of (2.28) have been well known for us depending on the sign of the discriminant $\lambda^2 - 4\,\mu$, the exact solutions of (4.1) can be obtained as a final step.

2.5 Lie symmetry reduction method

Among the methods for studying of evolution equations, one can find group analysis. This analytical approach is based on symmetries of differential equations. It was originally introduced by Sophus Lie at the end of the nineteenth century, and further developed by Ovsynnikov [24], Olver [?] and others. For each system of PDEs there exists a symmetry group, which acts on the space of its independent and dependent variables, leaving the form of the system unchanged. The classical Lie symmetry analysis allows one to construct and study special types of analytical solutions of nonlinear PDEs in terms of solutions of lower dimension equations.

Modern description of the theory can also be found in [4]. The classical Lie method is an algorithmic procedure for which many symbolic manipulation programs were designed [7], [14]. This software has now become imperative in finding symmetries associated with large systems of PDEs.

The main significance of transformations in connection with nonlinear equations is that they have typically associated nonlinear superposition principles whereby infinite sequences of solutions to nonlinear equations may be generated by purely algebraic procedures. solutions of many important nonlinear evolution equations have been thereby constructed.

A G_r is defined as a group of continuous transformations which act on an open subset of the Euclidean space $\mathbb{R}^k$ of variables, which change under the action of G_r. We presently concentrate on a local group, the transformations of which are close to the identity transformation.

A G_r is a group of point transformations, which acts on $X \times U$, the space of the independent variables and the dependent ones:

$$G_r = \{x_i^* = \phi_i(x,u,a);\ u_j^* = \varphi_j(x,u,a),\ i = 1,\ldots,m;\ j = 1,\ldots,n; a = (a_1,\ldots,a_r)\} \quad (2.29)$$

where $x \in X \subset \mathbb{R}^m$ and $u \in U \subset \mathbb{R}^n$.

G_r locally satisfies the group axioms: existence of an identity element, associativity, inversibility, closure under the binary composition operation. The transformation corresponding to a zero parameter is the identity transformation.

Expand the transformations by means of a Taylor series at the zero value of the parameter a:

$$\begin{aligned} x_i^* &= x_i + a_\alpha \frac{\partial \phi_i}{\partial a_\alpha}\Big|_{a=0} + \mathcal{O}(a_\alpha^2),\ \alpha = 1,\ldots,r \\ u_j^* &= u_j + a_\alpha \frac{\partial \varphi_j}{\partial a_\alpha}\Big|_{a=0} + \mathcal{O}(a_\alpha^2),\ \alpha = 1,\ldots,r \end{aligned} \quad (2.30)$$

The derivatives of ϕ_i and φ_j with respect to the parameter a_α are smooth functions, called *infinitesimals of the group G_r*.
Denote by ξ_i^α and η_j^α the infinitesimals of G_r; the point transformation group G_r can be represented by means of the operator $\mathbf{L}_\alpha$:

$$\mathbf{L}_\alpha = \xi_i^\alpha(x,u)\frac{\partial}{\partial x_i} + \eta_j^\alpha(x,u)\frac{\partial}{\partial u_j}, \quad i = 1,\ldots,m;\ j = 1,\ldots,n;\ \alpha = 1,\ldots,r \quad (2.31)$$

The operators $\mathbf{L}_\alpha$, $\alpha = 1,\ldots,r$ are called the *infinitesimal operators* of G_r.

$\{\mathbf{L}_\alpha,\ \alpha = 1,\ldots,r\}$ represents the set of tangent vectors to the manifold G_r, when the zero value is assigned to the parameter a. The set is a basis of the of the infinitesimal operators of G_r, the dimension of which is the same as the one of the G_r.

The knowledge of the $\mathbf{L}_\alpha$ enables us to determine the point transformations of the group G_r by solving the equations:

$$\frac{\partial x_i^*}{\partial a_\alpha} = \xi_i^\alpha(x^*, u^*), \quad \frac{\partial u_j^*}{\partial a_\alpha} = \eta_j^\alpha(x^*, u^*), \quad i = 1,\ldots,m;\ j = 1,\ldots,n;\ \alpha = 1,\ldots,r \tag{2.32}$$

in conjunction with the initial conditions:

$$x_i^*\big|_{a=0} = x_i; \quad u_j^*\big|_{a=0} = u_j \tag{2.33}$$

Denote by $u^{(k_1\ldots k_p)}$ the vector, the components of which are partial derivatives of order p, namely, $u_j^{(k_1\ldots k_p)} = \frac{\partial^p u_j}{\partial x_{k_1}\ldots\partial x_{k_p}}$ $j = 1,\ldots,n$ and $k_1,\ldots,k_p \in \{1,\ldots,m\}$.

Denote by $x = (x_1,\ldots,x_m)$ the independent variables, $u = (u_1,\ldots,u_n)$ the dependent variables, and $(x_{k_1}\ldots x_{k_p})$ a set of elements of the independent variables.

Equation (**??**) is a subset of $X \times U^{(l)}$, a prolongation of the space $X \times U$ to the space of the partial derivatives of u with respect to x up to order l. $X \times U^{(l)}$, which is a smooth manifold, is called the l^th order jet space of $X \times U$. In order to take into account the derivative terms involved in the differential equation, the action of G_r on $X \times U$ needs to be prolonged to the space of the derivatives of the dependent variables.

Denote by $\widetilde{G}_r^{(l)}$ a of point transformations acting on an open subset $M^{(l)}$ of the l^th order jet space $X \times U^{(l)}$ of the independent variables x, dependent variables u and the partial derivatives of u with respect to x, where l denotes the order of equation (4.1).

The l^{th}-prolongation operator of G_r is:

$$\widetilde{\mathbf{L}}_\alpha^{(l)} = \xi_i^\alpha(x,u)\frac{\partial}{\partial x_i} + \eta_j^\alpha(x,u)\frac{\partial}{\partial u_j} + \sigma_j^{\alpha,(k_1)}\frac{\partial}{\partial {u_j}^{(k_1)}} + \cdots + \sigma_j^{\alpha,(k_1\ldots k_l)}\frac{\partial}{\partial {u_j}^{(k_1\ldots k_l)}}, \tag{2.34}$$
$$i = 1,\ldots,m;\ j = 1,\ldots,n;\ \alpha = 1,\ldots,r.$$

The infinitesimal functions ξ_i^α, η_j^α, $\sigma_j^{\alpha,(k_1)}$ and $\sigma_j^{\alpha,(k_1\ldots k_o)}$ are given by:

$$\xi_i^\alpha = \frac{\partial \phi_i}{\partial a_\alpha}\Big|_{a=0}, \quad \eta_j^\alpha = \frac{\partial \varphi_j}{\partial a_\alpha}\Big|_{a=0}, \quad \sigma_j^{\alpha,(k_1)} = \frac{\mathcal{D}\eta_j^\alpha}{\mathcal{D}x_{k_1}} - \sum_{i=1}^{m} \frac{\partial u_j}{\partial x_i}\frac{\mathcal{D}\xi_i^\alpha}{\mathcal{D}x_{k_1}} \tag{2.35}$$

$$\sigma_j^{\alpha,(k_1\ldots k_o)} = \frac{\mathcal{D}\sigma_j^{\alpha,(k_1\ldots k_{o-1})}}{\mathcal{D}x_{k_o}} - \sum_{i=1}^{m} \frac{\partial^o u_j}{\partial x_i \partial x_{k_1}\ldots\partial x_{k_{o-1}}}\frac{\mathcal{D}\xi_i^\alpha}{\mathcal{D}x_{k_o}}, \quad o = 2,\ldots,l$$

where: $\dfrac{\mathcal{D}}{\mathcal{D}x_k} = \dfrac{\partial}{\partial x_k} + \displaystyle\sum_{j=1}^{n} \frac{\partial u_j}{\partial x_k}\frac{\partial}{\partial u_j}$

Theorem:

The system of l^{th}-order differential equations is invariant under the group $\widetilde{G}_r^{(l)}$ if and only if it satisfies the following :

$$\widetilde{\mathbf{L}}_\alpha^{(l)}\mathcal{F}^\lambda\Big|_{\mathcal{F}^\lambda=0} = 0, \quad \alpha = 1,\ldots,r;\ \lambda = 1,\ldots,q \tag{2.36}$$

The knowledge of the infinitesimal generators enables one to obtain new solutions of the studied evolution equation.

2.6 Conclusion

Most of the above methods assume that equation (4.1) has traveling wave solutions of the form $u(x,t) = u(\xi)$, $\quad \xi = x - v\,t$, which enables one to integrate the resulting equation with respect to ξ to reduce the order.

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Chapter 3

The panel of resolution methods

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Abstract

This chapter presents the main classical methods used to obtain solutions of Classical nonlinear evolution equations.

3.1 Introduction

This chapter directly follows the previous one, where analytic methods that enable one to obtain exact solutions of nonlinear evolution equations, were exposed.
We hereafter present the main classical methods used to obtain solutions of Classical nonlinear evolution equations of the form:

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$$\begin{aligned} \mathcal{F}\left(u, u_{t^{s_{i_1}}}, \ldots, u_{t^{s_{i_m}}}, u_{x^{r_{j_1}}}, \ldots, u_{x^{r_{j_n}}}\right) = 0 \\ r_{i_k} \in \{0, \ldots r_0\}, \; s_{j_l} \in \{0, \ldots s_0\} \end{aligned} \tag{3.1}$$

where $\mathcal{F}$ denotes a function which depends of the solution u, and its derivatives $u_{t^{s_{i_k}}} = \frac{\partial^{s_{i_k}} su}{\partial t^{s_{i_k}}}$ and $u_{x^{r_{j_l}}} = \frac{\partial^{r_{j_l}} u}{\partial x^{r_{i_l}}}$, r_0, s_0 being natural integers:

1. the inverse scattering method;
2. the first-integral method;
3. the extended homogeneous balance method.

3.2 The inverse scattering method

Assume that equation (3.1) has a traveling wave solution of the form

$$u(x,t) = u(\xi), \quad \xi = x - v\,t \tag{3.2}$$

where v is the wave velocity. The resulting equations may be integrated with respect to ξ to reduce the order. For simplicity, the integration constants can be ignored, assuming that the solution its derivatives vanish at $\xi = \pm\infty$.
Expand then $u(\xi)$ in a power series:

$$u(\xi) = \sum_{i=1}^{+\infty} u_n\, g^n(\xi) \tag{3.3}$$

where $g(\xi) = e^{-K(v)\,\xi}$ solves the linear part of the equation. Hence, the wave number $K(c)$ is related to the velocity c by the dispersion law of (one of) the linearized equation. Substitute the expansion (3.3) into the full nonlinear equation (1), rearrange the sums by using rule for multiple series [2] and equate the coefficient of $g^n(\xi)$ leads to a nonlinear system of coupled recursion relations for the coefficients u_n. Quite often, the relation $K(v)$ and appropriate scales on $u(\xi)$ allow to simplify the recursion relations.
Assume then that the u_n are polynomials of degree d_n in n:

$$u_n(\xi) = \sum_{i=1}^{d_n} u_n^i\, n^i \tag{3.4}$$

Substitute them in the recursion relations leads to a now completely algebraic system whose unknowns are the u_n^i. This system can easily be solved using a symbolic calculus tool (*Mathematica* or *Maple*).

3.3 The first integral method

The first integral method, which is based on ring theory in commutative algebra, was originally introduced for the BurgersKdV equation in an attempt to seek its traveling wave solutions [7], and generalized in [8], [9].

Assume that equation (3.1) has traveling wave solutions of the form

$$u(x,t) = u(\xi), \quad \xi = x - v\,t \tag{3.5}$$

where v is the wave velocity. Substituting (9.98) into equation (3.1) leads to:

$$\mathcal{F}(u, u^{(r)}, (-v)^s\, u^{(s)}) = 0, \tag{3.6}$$

Performing an integration of (9.99) with respect to ξ leads to an equation of the form:

$$\mathcal{F}_\xi^{\mathcal{P}}(u, u^{(r)}, (-v)^s\, u^{(s)}) = C, \tag{3.7}$$

where $\mathcal{F}_\xi^{\mathcal{P}}(u, u^{(r)}, (-v)^s\, u^{(s)}) = C$, is a function depending on u, $u^{(r)}$, $u^{(s)}$, and C is an arbitrary integration constant, which will be the starting point for the determination of solitary waves solutions.

The first integral method can be applied in the case where $\mathcal{F}_\xi^{\mathcal{P}}(u, u^{(r)}, (-v)^s\, u^{(s)})$ is of the form:

$$\mathcal{F}_\xi^{\mathcal{P}}(u, u^{(r)}, (-v)^s\, u^{(s)}) = G(u, u') + R(u)$$

where u' denotes the derivative of u with respect to ξ, G a two-variable polynomial, and R a one-variable polynomial.

Set $x = u$, $y = u_\xi$. Equation (9.100) is thus equivalent to:

$$\begin{cases} x' &= y \\ y' &= G(x,y) + R(x) \end{cases} \tag{3.8}$$

The for two variables in the complex domain $\mathbb{C}$ is then used to seek one first integral to (3.8) which can reduce Equation (9.100) to a first-order integrable ordinary differential equation. An exact solution to (3.1) is then obtained by solving this first-order ordinary differential equation directly.

Division Theorem.

Suppose that $P(\omega, z)$ and $Q(\omega, z)$ are polynomials in $\mathbb{C}(\omega, z)$, and that $P(\omega, z)$ is irreducible in $\mathbb{C}(\omega, z)$. If $Q(\omega, z)$ vanishes at all zero points of $P(\omega, z)$, then there exists a polynomial $G(\omega, z)$ in $\mathbb{C}(\omega, z)$ such that

$$Q(\omega, z) = P(\omega, z)G(\omega, z)$$

The is a consequence of the :

HilbertNullstellensatz.

Let k be a field and $\mathcal{L}$ an algebraic closure of k. Then:

i. Every ideal of $k[X_1, \ldots, X_n]$ not containing 1 admits at least one zero in $\mathcal{L}^n$;

ii. Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ be two elements of $\mathcal{L}^n$: for the set of polynomials of $k[X_1, \ldots, X_n]$ zero at x to be identical with the set of polynomials of $k[X_1, \ldots, X_n]$ zero at y, it is necessary and sufficient that there exists a k-automorphism s of $\mathcal{L}$ such that $y_i = s(x_i)$ for $1 \leqslant i \leqslant n$;

iii. For an ideal $\mathcal{I}$ of $k[X_1, \ldots, X_n]$ of $k[X_1, \ldots, X_n]$ to be maximal, it is necessary and sufficient that there exists an x in L^n such that a is the set of polynomials of $k[X_1, \ldots, X_n]$ zero at x:

iv. For a polynomial Q of $k[X_1, \ldots, X_n]$ to be zero on the set of zeros in $\mathcal{L}^n$ of an ideal $\mathcal{I}$ of $k[X_1, \ldots, X_n]$, it is necessary and sufficient that there exist an integer m such that $Q^m \in \mathcal{I}$.

The enables us to seek the first integral to (3.8). Suppose that $x = x(\xi)$ and $y = y(\xi)$ are the nontrivial solutions to (3.8), and that $p(x, y) = \sum_{i=0}^{m} a_i(x)\, y^i$ is an irreducible polynomial in $\mathbb{C}[x, y]$ such that:

$$p\left(x(\xi), y(\xi)\right) = \sum_{i=0}^{m} a_i(x)\, y^i = 0 \tag{3.9}$$

where the a_i are polynomials of x and all relatively prime in $\mathbb{C}[x, y]$, $a_m(x) \neq 0$. (3.9) is the first integral to (3.8). Since $p\left(x(\xi), y(\xi)\right) = 0$, one also has $\frac{dp}{d\xi} = 0$. Applying the leads to the existence of a polynomial $H(x, y) = \alpha(x) + \beta(x)\, y$ in $\mathbb{C}[x, y]$ such that:

$$\left[\frac{dp}{d\xi}\right](x(\xi), y(\xi)) = H\,(x(\xi), y(\xi))\; p\,(x(\xi), y(\xi)) \tag{3.10}$$

Since:

$$\left\{\begin{array}{rcl}\left[\frac{\partial p}{\partial x}\,\frac{\partial x}{\partial \xi} + \frac{\partial p}{\partial y}\,\frac{\partial y}{\partial \xi}\right] & = & \left\{\sum_{i=0}^{m} a_i'(x)\, y^i\right\} y + \left\{\sum_{i=1}^{m} i\, a_i(x)\, y^{i-1}\right\} \left(y\,\frac{\partial G}{\partial x_1}(x,y) + \frac{\partial G}{\partial x_2}(x,y) + y\, R'(x)\right) \\ & = & \sum_{i=1}^{m+1} a_{i-1}'(x)\, y^i + \left\{\sum_{i=1}^{m} i\, a_i(x)\, y^i\right\} \left(\frac{\partial G}{\partial x_1}(x,y) + R'(x)\right) + \left\{\sum_{i=1}^{m} i\, a_i(x)\, y^{i-1}\right\} \frac{\partial G}{\partial x_2}(x,y)\end{array}\right. \tag{3.11}$$

and:

$$\left\{\begin{array}{rcl} H(x,y)\, p(x,y) & = & (\alpha(x) + \beta(x)\, y) \left[\sum_{i=0}^{m} a_i(x)\, y^i\right] \\ & = & \alpha(x) \left[\sum_{i=0}^{m} a_i(x)\, y^i\right] + \beta(x) \left[\sum_{i=1}^{m+1} a_{i-}(x)\, y^i\right]\end{array}\right. \tag{3.12}$$

Set:

$$a(x) = \left[\begin{array}{c} a_0(x) \\ \vdots \\ a_m(x)\end{array}\right] \tag{3.13}$$

On equating the coefficients of y^i on both sides of (3.12), one has:

$$a'(x) = A(x).a(x) \tag{3.14}$$

and:

$$a_1\,\frac{\partial G}{\partial x_2} = \alpha(x)\, a_0 \tag{3.15}$$

where:

$$A(x) = \left[\begin{array}{ccccccc} \beta(x) & \alpha(x) - \frac{\partial G}{\partial x_1}(x,y) + R'(x) & -2\,\frac{\partial G}{\partial x_2} & 0 & \dots & \dots & 0 \\ 0 & \beta(x) & \alpha(x) - 2\,\frac{\partial G}{\partial x_1}(x,y) + R'(x) & -3\,\frac{\partial G}{\partial x_2} & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 & \beta(x) & \alpha(x) - (m-1)\,\frac{\partial G}{\partial x_1}(x,y) + R'(x \\ 0 & \dots & \dots & \dots & \dots & 0 & \beta(x)\end{array}\right. \tag{3.16}$$

Remark

The above study can be started by assuming $m = 2$ in (3.9).

The resolution of (3.14) and (3.16) enables one to obtain the values of the a_i, $i = 0, \ldots, m$, and the expressions of α and β, which lead then to the searched solution.

The first integral method appears thus as more general than other ones, in so far as solutions obtained by means of other methods (such, for instance, as the Tanh method) appear as particular cases of the ones obtained using the first integral method. It enables one to seek solutions of (3.1) in more general form, as follows.

A class of traveling wave solutions of the (9.100) equation is established by using a hyperbolic ansatz which is actually a combination of bell-profile waves and kink-profile waves of the form

$$u(x,t) = \sum_{i=1}^{n} \left(B_i \tanh^i \left[C_i(x - vt + x_0) \right] + D_i \operatorname{sech}^i \left[C_i(x - vt + x_0) \right] \right) + B_0, \quad (3.17)$$

where the $B_i's$, $C_i's$ $D_i's$, $(i = 1, \cdots, n)$, v and B_0 are constants to be determined, and x_0 is arbitrary.

Substitution of (3.17) into equation (9.100) leads to an over-determined algebraic system in B_i, C_i, D_i, B_0 and v. Following [3], after balancing the higher-order derivative term and the leading nonlinear term, one can deduce $n = 1$. Then, replace $\operatorname{sech}(C_1 \xi)$ by $\frac{2}{e^{C_1 \xi} + e^{-C_1 \xi}}$, $\sinh(C_1 \xi)$ by $\frac{e^{C_1 \xi} - e^{-C_1 \xi}}{2}$, $\tanh(C_1 \xi)$ by $\frac{e^{C_1 \xi} - e^{-C_1 \xi}}{e^{C_1 \xi} + e^{-C_1 \xi}}$, and reduce all fractions to the same denominator, equation (9.100) can be rewritten in the following form:

$$\sum_{k=0}^{n_0} P_k \left(B_0, \; B_1, \; C_1, \; D_1, \; v \right) e^{k \, C_1 \, \xi} \; = \; 0, \quad (3.18)$$

where the P_k $(k = 0, ..., n_0)$, n_0 being an integer, are polynomials of B_0, B_1, C_1, D_1 and v.

With the aid of mathematical softwares such as *Mathematica*, when $n = 1$, equating the coefficient of each term $e^{k \, C_1 \, \xi}$ $(k = 0, ..., n_0)$ in equation (9) to zero, yields a nonlinear algebraic system that one can solve using a symbolic calculus tool as *Mathematica* or *Maple*.

The first integral method described appears not only efficient but also has the merit of being widely applicable. It can easily be applied to many nonlinear evolution equations, such as the nonlinear Schrödinger equation, the generalized KleinGordon equation, and the high-order KdV-like equation, ...

3.4 The extended homogeneous balance method

The homogeneous balance method and its extensions [15], [16], is a powerful tool to find solitary wave solutions of nonlinear partial differential equations PDEs.

Function φ is called a quasi-solution of equation (3.1), if there exists a function $f = f(\varphi)$ of only one variable so that a suitable linear combination of the constant function equal to 1 and of the following functions:

$$f\left(\varphi_{x^{r_{j_1}},t^{s_{i_1}}}\right) \quad , \quad r_{i_k} \in \{0, \ldots r_0\}\,, \; s_{j_l} \in \{0, \ldots s_0\} \tag{3.19}$$

is actually a solution of equation (3.1). The method for finding $f(\varphi)$, the quasi-solution φ, the suitable linear combination of the functions in (3.19), and then obtaining special exact solutions of equation (3.1) consists of four steps:

i. ***First step:*** choosing a suitable linear combination of the functions in (3.19), where the coefficients have to be determined so that the highest nonlinear terms and the highest-order partial derivatives terms in the given equation are both transformed into polynomials with a highest degree in partial derivatives of φ with respect to φ and its various derivatives.

ii. ***Second step:*** substituting the combination chosen in the first step into equation (3.1), collecting all terms with the highest degree of derivatives of φ and setting its coefficient to zero, one obtains an ordinary differential equation for $f(\varphi)$, and then solve it; in most cases, f is a logarithmic function.

iii. ***Third step:*** starting from the ordinary differential equation for $f(\varphi)$ and its solution obtained in step 2, the nonlinear terms of various derivatives of $f(\varphi)$ in the expression obtained in the second step can be replaced by the corresponding higher-order derivatives of $f(\varphi)$. Then, collecting all terms with the same order derivatives of $f(\varphi)$ and setting the coefficient of each order derivative of $f(\varphi)$ to zero, one obtains a set of equations for φ and the combination coefficients; the left-hand sides of these equations are the k-degree homogeneous functions in various derivatives of φ, where k is the order of $f^{(k)}$. If there exists a solution for these equations of φ and combination coefficients, the combination chosen in the first step can be determined.

iv. ***Fourth step:*** substituting $f(\varphi)$ and φ, as well as some constants obtained in the second and third steps into the combination chosen in the first step, after doing some calculations, enables one to obtain an exact solution of equation (3.1).

New solutions of equation (3.1) can be obtained adding a known solution φ_0 to the combination obtained in the in the first step. The second step is the same as above. In the third step, one obtains a set of equations in φ, some coefficients of which may depend on φ_0. If the set of equations obtained in the third step is solvable, one solves them and thus gets a new quasi-solution φ. Then, proceeding as in the fourth step above, one may obtain new solution(s) of Eq. (3.1).

3.5 Conclusions

The methods described in the above appear efficient and powerful, while having the merit of being widely applicable.

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