Overview of domain decomposition strategies applied to the simulation of electromagnetic testing by the boundary element method

Practical examples & focus on the local multi-trace formalism

Edouard DEMALDENT, CEA LIST, DISC (Department of Imaging and Simulation for the Control)

In collaboration with Marc BONNET, POems, and Xavier CLAEYS, LJLL
Design & Evaluation of **non destructive testing** (NDT) processes

- Both **modelling** and **instrumentation** skills
- 70% Ultrasonic, 15% X-ray, 15% Electromagnetic (Eddy Current Testing)
- 100 people including 20 PhD/post-doc, 30 working on software issues, 3 on simulation by FEM/BEM
- 200 **CIVA** licenses worldwide (NDT simulation platform)
Scientific collaboration around the NDT simulation platform CIVA

- New trends such as statistical tools & meta-models
- It still motivates the search for fast and accurate direct models
- Favorable location for collaborating on BEM
CONTEXT & MOTIVATION
PRACTICAL EXAMPLES

NOTATIONS & BIE FORMALISM

EXTENSION OF THE LOCAL MULTI-TRACE BIE TO THE EDDY CURRENT REGIME
X. Claeys, E. Demaldent

ASYMPTOTIC EXPANSION OF THE SINGLE-TRACE BIE OF THE FIRST KIND (PMCHWT)
M. Bonnet, E. Demaldent, A. Vigneron

OPTIMISATION ATTEMPT OF THE EDDY CURRENT MULTI-TRACE BIE
## CONTEXT & MOTIVATIONS
### EDDY CURRENT TESTING

**Eddy Current Testing**
- Detection of perturbations (crack, distortion)
- Input: Loop sinusoidal current in vacuum (coil)
- Compute: Eddy currents in the conductive medium
- Output: Variation of impedance through a Reciprocity Th

\[
\Delta Z = \frac{1}{I^2} \int \mathbf{n} \cdot (\mathbf{E}_s \times \mathbf{H} + \mathbf{H}_s \times \mathbf{E}) \ d\sigma
\]

Working at **low frequency** to penetrate the **conductive** medium
- Conductivity \( \sigma = 1e6 \) S/m
- Frequency \( f = 100 \) kHz

\[
\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} \approx 1.6 \text{ mm}
\]

skin depth (half space)

\[
\kappa_0^2 = \omega^2 \varepsilon_0 \mu_0 \approx 4 \times 10^{-6}
\]

\[
\kappa_1^2 = \kappa_0^2 - s i \omega \mu \sigma \approx 4 \times 10^{-6} - s i 8 \times 10^{+5} \quad (s = \mp 1)
\]

**EC approximation:** \( \kappa_1^2 \sim -s i 8 \times 10^{+5}, \quad \kappa_0^2 \sim 0. \)
Major application: Inspection of steam generators in pressurized water reactors

- Thousands of tubes checked every year
  - Diameter \( \approx 20 \text{ mm} \)
  - Thickness \( \approx 1 \text{ mm} \)
  - Conductivity \( \approx 1 \text{ MS/m} \)
  - Frequency of the testing \( \approx 100 \text{ kHz} \)

6000 U-bend tubes
\( (L_{\text{tot}} = 140 \text{ km, } S_{\text{tot}} = 8000 \text{ m}^2) \)
CONTEXT & MOTIVATIONS
ECT OF STEAM GENERATOR TUBES

- Various kinds of **flaws**

Looking for the variation of impedance
* Phase
* Shape
* Amplitude

- Various kinds of **sensors**

Axial probe  |  French rotating probes  |  US rotating probe  |  Multi-element probe

Numerical methods for wave propagation and applications | E. Demaldent | 7
Various kinds of **distortions**

- **Outer wall** (radius variation)
- **Inner wall** (radius variation)
- **Dent**
- **Ovalization, thickness variation…**
- **Anti-vibration bar**
- **Friction wear**
- **Tube supports**

- **(naïve) pilger noise**

- **Meta-parameters** to be fixed, **simplified modelling** to be evaluated
**CONTEXT & MOTIVATIONS**

**SIMULATION TOOLS**

- **CIVA-ECT**: Fast solution on canonical geometries (stratified media, pipes…)

- Various alternative solutions to handle 3D components…

- **Boundary Elements**
  - From a modified Maxwell integral form
  - That exploits suitable domain decompositions
  - With the use of high-order approximation tools

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Iterative coupling between a ferrite core (BEM) and a canonical conductive workpiece (modal)

Iterations of the coupling (Rototest probe)

Real part of the impedance variation with a notch

CIVA (+Point probe)
Iterative coupling between a ferrite core (BEM) and a canonical conductive workpiece (modal)

CIVA
(no numerical parameter)

2D grid with respect to the ferrite’s border
Iterative coupling between a ferrite core (BEM) and a canonical conductive workpiece (modal)

CIVA
(no numerical parameter)

Extraction & smoothing of the boundary
Iterative coupling between a ferrite core (BEM) and a canonical conductive workpiece (modal)
Iterative coupling between a ferrite core (BEM) and a canonical conductive workpiece (modal)

- Here the coupling between the ferrite and the defect is simplified (ok with low-signal perturbations)
- Few min for pre-computation time (LU BEM) + few sec (iterative process & modal response)

```
\begin{align*}
\begin{bmatrix}
FF & FP \\
PF & PP
\end{bmatrix}
\begin{bmatrix}
X_F \\
X_P
\end{bmatrix} &=
\begin{bmatrix}
Y_F \\
Y_P
\end{bmatrix} & \quad [PP]X_P^0 = (Y_P - [PF][FF]^{-1}Y_F)

[PP](X_P^{i+1} - X_P^i) = [PF][FF]^{-1}[FP](X_P^i - X_P^{i-1})
\end{align*}
```

Hyp. Stable relative position of the probe (constant signal vs scan)

```
\begin{align*}
\begin{bmatrix}
FF & F\tilde{P} \\
\tilde{P}F & \tilde{P}\tilde{P}
\end{bmatrix}
\begin{bmatrix}
\bar{X}_F \\
\bar{X}_{\tilde{P}}
\end{bmatrix} &=
\begin{bmatrix}
Y_F \\
Y_{\tilde{P}}
\end{bmatrix} & \quad \text{VIM}

[\tilde{P}\tilde{P}]\bar{X}_{\tilde{P}} = Y_{\tilde{P}} - [\tilde{P}F]X_F
\end{align*}
```

Hyp. $[\tilde{P}F][FF]^{-1} ([F\tilde{P}]\bar{X}_F - [FP]X_P) \approx 0$

Simplified formalism
Raising order reduces digital noise

- It allows movement of the probe (shift, tilt)

Truncation of the workpiece by scanning subzones

- Meshing of a straight tube
curved quadrilaterals, centered probe

- Distortion on the U-bend tube
w.r.t. the targeted axial position of the probe

- A single calculation for all shift/tilt of the probe
Simulation tools to help determine the path of the probe

- Envelope of the mechanically possible positions of the probe
- Computation of the EC signal on a representative basis
- Fast comparison of several parametric trajectories

Hyp. Smooth $\Delta x \implies$ Smooth $\Delta Z$
The simulation of a defect’s response for the chosen trajectory requires the full scan of the defect’s zone (increasing number of unknowns).
Magnetic testing of pipes
- Simplified model: linear regime, slow motion (no induced current)

High contrast of scales: $L_{tot} \sim 1 \times 1 \text{ m}^2$, $L_{def} \sim 10 \times 0.1 \text{ mm}^2$
- Domain decomposition to isolate the defect

Comparison with experimental data
(Vallourec Research Center France)
Magnetic testing of pipes

- Simplified model: linear regime, slow motion (no induced current)

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- Domain decomposition to isolate the defect

Solution in the defect's area with the equivalent source
Magnetic testing of pipes

- Simplified model: linear regime, slow motion (no induced current)

High contrast of scales: $L_{tot} \sim 1 \times 1 \text{ m}^2$, $L_{def} \sim 10 \times 0.1 \text{ mm}^2$

- Domain decomposition to isolate the defect

**PRACTICAL EXAMPLES**

**MAGNETIC FLUX LEAKAGE**

- Healthy workpiece
- Defect & near area

Hyp. $B$ large enough to neglect $[AC]\{W_C\}$

Simplified formalism
Examples of domain decompositions for ECT: intuitive assumption, empirical setting
- Iterative coupling between a sensor component and the healthy workpiece
- Truncation of the workpiece by scanning subzones
- Truncation of the workpiece in the neighbourhood of the defect

Domain decompositions + Boundary Integral Equations + high-order approximation
- It allows the use of a direct LU-based solver
- It simplifies the construction of databases

In what follows:
Study of the local multi-trace BIE approach for the consideration of adjacent objects
CONTEXT & MOTIVATION
PRACTICAL EXAMPLES

NOTATIONS & BIE FORMALISM

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OPTIMISATION ATTEMPT OF THE EDDY CURRENT MULTI-TRACE BIE
\[ \mathbb{R}^3 = \bigcup_{i=0}^{n} \Omega_i, \; \Omega_i \cap \Omega_j = \emptyset \; \forall i \neq j, \quad \Gamma = \bigcup_{i=0}^{n} \Gamma_i, \; \Gamma_i = \partial \Omega_i. \]

- **Trace Operators**
  - \( u_i = u|_{\Omega_i} \)
  - \( \tau^i_N(u) = n . u|_{\Gamma_i} \)
  - \( \gamma^i(u) = \begin{pmatrix} \gamma^i_D(u) \\ \gamma^i_N(u) \end{pmatrix} = \begin{pmatrix} u|_{\Gamma_i} \times n_i \\ (\nabla \times u)|_{\Gamma_i} \times n_i \end{pmatrix} \)
  - \( \gamma^i_c(u) = -\gamma^j(u) \) (if \( \Gamma_i \cap \Gamma_j \))
  - \( [\gamma^i] = \gamma^i - \gamma^i_c, \quad \{\gamma^i\} = \frac{1}{2}(\gamma^i + \gamma^i_c) \)

- **Maxwell Problem**
  \[
  \begin{cases}
  \nabla \times \nabla \times u_i - \kappa_i^2 u_i = 0, & \text{div}(u_i) = 0 \quad (\Omega_i) \quad (\text{outgoing in } \Omega_0) \\
  [\gamma^i_D](u) = -[\gamma^i_D](u_s) \quad (\Gamma_i) \\
  [\gamma^i_N](\xi u) = -[\gamma^i_N](\xi u_s) \quad (\Gamma_i) \\
  \end{cases}
  \]

- \( (M_\xi) \): \( u = \xi - \xi_s, \quad \xi = \mu^{-1} \)
Integral Representation theorem

\[ u \in H_{\text{loc}}(\text{curl}, \Omega), \nabla \times \nabla \times u - \kappa^2 u = 0, \ \text{div}(u) = 0 \]

\[ \kappa^2 \in \mathbb{C}_+ := \{ \lambda \in \mathbb{C} \mid \text{Re}(\lambda) \geq 0, \text{Im}(s\lambda) \geq 0 \} \]

\[ \nabla \times \Psi_k(y_D(u)) + \Psi_k(y_N(u)) + \nabla \Psi_k(\tau_N(u)) = u_{1\Omega} \]

The Maxwell case

• When \( \kappa^2 \in \mathbb{C}_+^* \) we have \( \tau_N(u) = \frac{1}{\kappa^2} \text{div}_\Gamma(y_N(u)) \) and so

\[ G_k(y(u)) = u_{1\Omega} \]

\[ G_k(v) = \nabla \times \Psi_k(v) + \Psi_k(q) + \kappa^{-2} \nabla \Psi_k(\text{div}_\Gamma q) \]
Cauchy Data local to $\Omega$:
\[
C_\kappa := \{ \nu | \nabla \times \nabla \times \nu - \kappa^2 \nu = 0, \text{div} (\nu) = 0 \} \subset \mathbb{H} = H \times H, \quad H := H_{||}^{-1/2} (\text{div}, \Gamma)
\]

- Calderón’s Projector
  \[
  \gamma \cdot G_\kappa (U) = U \iff U \in C_\kappa
  \]

- Jump Conditions
  \[
  [\nu] \cdot G_\kappa = I
  \]

- Characterization of Cauchy data
  \[
  A_\kappa = \{ \gamma \} \cdot G_\kappa = \left( \gamma - \frac{1}{2} [\nu] \right) \cdot G_\kappa
  \]
  \[
  \left( A_\kappa - \frac{I}{2} \right) U = 0 \quad \forall U \in C_\kappa
  \]
Boundary Integral Equation

- Linear combinations of the local blocks
- taking into account the transmission conditions

\[
\left( A_{ki}^i - \frac{I_i}{2} \right) U^i = 0 \quad (\Gamma_i)
\]

\[
U^i + U_s^i = - \left[ \begin{array}{cc} 1 & 0 \\ 0 & \mu_i/\mu_j \end{array} \right] (U^j + U_s^j) |_{\Gamma_i} \quad (\Gamma_i \cap \Gamma_j)
\]

with \( U^i = \gamma^i (\mathcal{E}_t - \mathcal{E}_s) \)
Boundary Integral Equation

- Linear combinations of the local blocks
- taking into account the transmission conditions

\[
\left( \mathbb{A}_{\kappa_i}^i - \frac{\mathbb{I}^i}{2} \right) U^i = -U^i_s \quad (\Gamma_i)
\]

\[
U^i = -\begin{bmatrix} 1 & 0 \\ 0 & \mu_i/\mu_j \end{bmatrix} U^j |_{\Gamma_i} \quad (\Gamma_i \cap \Gamma_j)
\]

with \( U^i = \gamma^i (\mathcal{E}_t) \)
Boundary Integral Equation

- Linear combinations of the local blocks
- Taking into account the transmission conditions

\[
\begin{align*}
\left( \tilde{A}^i_{\kappa_i} - \frac{I^i}{2} \right) U^i &= -U^i_s \quad (\Gamma_i) \\
U^i &= -U^j_{|\Gamma_i} \quad (\Gamma_i \cap \Gamma_j) \quad (P_i)
\end{align*}
\]

with

\[
U^i = (s\omega\mu_0)^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \mu_1/\mu_0 \end{bmatrix} \gamma^i (\mathcal{E}_t) \sim \left( \mathcal{M}^i \right)
\]

\[
\tilde{A}^i_{\kappa_i} = \begin{bmatrix} 1 & 0 \\ 0 & \mu_0/\mu_i \end{bmatrix} A^i_{\kappa_i} \begin{bmatrix} 1 & 0 \\ 0 & \mu_i/\mu_0 \end{bmatrix}
\]

Magnetic & Electric current densities
NOTATIONS & BIE FORMALISM

BOUNDARY INTEGRAL EQUATION

Boundary Integral Equation
- Linear combinations of the local blocks
- taking into account the transmission conditions

Single-Trace BIEs
- $\mathbb{R}^3 = \overline{\Omega}_0 \cup \overline{\Omega}_1$, $n = n_1 = - n_0$
- $(\mathcal{M}, J) = (n.n_i)(\mathcal{M}^i, J^i)$
- 1st kind BIE PMCHWT: $(\mathcal{P}_0) + (\mathcal{P}_1)$ [Poggio, Miller 73] [Chang, Harrington 77] [Wu, Tsai 77]
- 2nd kind BIE: $(\mathcal{P}_0) - (\mathcal{P}_1)$

$$(\mathcal{A}_{\kappa_i}^i - \frac{I}{2}) U^i = -U_s^i \quad (\Gamma_i)$$

$$U^i = -U^j|_{\Gamma_i} (\Gamma_i \cap \Gamma_j) \quad (P_i)$$

with $U^i = (s\omega \mu_0)^{-1} \begin{bmatrix} 0 \\ \mu_1/\mu_0 \end{bmatrix} \gamma^i(\mathcal{E}_t) \sim (\mathcal{M}^i, J^i)$ Magnetic & Electric current densities

$\mathcal{A}_{\kappa_i}^i = \begin{bmatrix} 1 & 0 \\ 0 & \mu_i/\mu_0 \end{bmatrix} \mathcal{A}_{\kappa_i}^i \begin{bmatrix} 1 & 0 \\ 0 & \mu_i/\mu_0 \end{bmatrix}$
BOUNDARY INTEGRAL EQUATION

Boundary Integral Equation
- Linear combinations of the local blocks
- taking into account the transmission conditions

Local Multi-Trace BIE
- \( \mathbb{R}^3 = \overline{\Omega}_0 \cup \overline{\Omega}_1 \cup \overline{\Omega}_2 \) ...


\[ \begin{pmatrix} \tilde{A}^i_{K_i} - \frac{\Pi^i_i}{2} \end{pmatrix} U^i = -U^i_s \quad (\Gamma_i) \]
\[ U^i = -U^j|_{\Gamma_i} \quad (\Gamma_i \cap \Gamma_j) \quad (P_i) \]

with \( U^i = (s\omega\mu_0)^{-1} \begin{bmatrix} 1 & 0 \\ \mu_1/\mu_0 & 0 \end{bmatrix} \gamma^i(\mathcal{E}_t) \sim (\mathcal{M}^i_j) \) Magnetic & Electric current densities

\[ \tilde{A}^i_{K_i} = \begin{bmatrix} 1 & 0 \\ 0 & \mu_0/\mu_i \end{bmatrix} A^i_{K_i} \begin{bmatrix} 1 & 0 \\ 0 & \mu_i/\mu_0 \end{bmatrix} \]

\[ (A - \frac{\Pi}{2}) U = -U_s \]

\[ \begin{pmatrix} A^0_{K_0} \\ A^1_{K_1} \\ A^2_{K_2} \end{pmatrix} \]
\[ \begin{pmatrix} U^0 \\ U^1 \\ U^2 \end{pmatrix} \]
Bilinear form inspired from the second green formula to provide duality pairing

\[
\begin{bmatrix}
(u, p) \\
p, q
\end{bmatrix}
^i = \langle u, q \rangle^i_x + \langle p, v \rangle^i_x
\]

\[
\langle u, v \rangle^i_x = \int_{\Gamma_i} (u \times v) \cdot n \, d\sigma = \int_{\Gamma_i} (v \times n) \cdot u \, d\sigma
\]

- \( \mathbb{H}(\Gamma) = (H, H), H = H_{||}^{-1/2}(\text{div}_\Gamma, \Gamma) \)
- Single-trace BIE: \( \mathbb{H} = \prod \mathbb{H}(\Gamma_i \cap \Gamma_j) \) & specific treatment with multiple adjacent domains
- Multi-trace BIE: \( \mathbb{H} = \prod \mathbb{H}(\Gamma_i) \)
NOTATIONS & BIE FORMALISM
DISCRETIZATION

\[ \Gamma_h \sim \Gamma, \ \mathbb{H}_h \sim \mathbb{H} \]

- **Div-conforming (edge-)elements**: RWG / Rooftop / Raviart-Thomas functions
  - \( H = H_{||}^{-1/2}(\text{div}_\Gamma, \Gamma) \)

- **Quasi-Helmholtz decomposition**: loop & Star or Tree combinations
  - \( H = H_L \oplus H_S \) where \( H_L = H_{||}^{-1/2}(\text{div}_\Gamma = 0, \Gamma) \)
  - \( H_L = n \times \nabla_\Gamma H^{1/2}(\Gamma) \) (local loop functions) on simply connected geom
  - **Global loop** functions complete local ones on non-simply connected geom (harmonic solutions)
CONTEXT & MOTIVATION

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OPTIMISATION ATTEMPT OF THE EDDY CURRENT
MULTI-TRACE BIE
This contribution concerns the modeling of \textit{adjacent homogeneous media}

- With a minimal effort for describing the adjacency
- Towards* an independent discretization of each component
  *at some point singular functions should be added to handle the singular behavior of the field
- that opens the way to new subdomain resolution strategies and optimization (not discussed here)
**LOCAL MULTI-TRACE BIE**

**MAXWELL MATRIX FORM**

- Single object (2 media)

<table>
<thead>
<tr>
<th>$\kappa_0^2 A_0 - S_0$</th>
<th>$B_0$</th>
<th>$\frac{1}{2} I_{10}^\times$</th>
<th>$M^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_0$</td>
<td>$\left( A_0 - \frac{1}{\kappa_0^2} S_0 \right)$</td>
<td>$\frac{1}{2} I_{10}^\times$</td>
<td>$J^0$</td>
</tr>
<tr>
<td>$\frac{1}{2} I_{10}^\times$</td>
<td>$\frac{\mu_0}{\mu_1} (\kappa_1^2 A_1 - S_1)$</td>
<td>$B_1$</td>
<td>$M^1$</td>
</tr>
<tr>
<td>$\frac{1}{2} I_{10}^\times$</td>
<td>$B_1$</td>
<td>$\frac{\mu_1}{\mu_0} \left( A_1 - \frac{1}{\kappa_1^2} S_1 \right)$</td>
<td>$J^1$</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\Omega_0 & \quad \mu_0, \kappa_0 \\
\Omega_1 & \quad \mu_1, \kappa_1
\end{align*}
\]

\[
[\mathbf{1}]_{tb} = \int_{\Gamma_i \times \Gamma_i} (n_i \times t) \cdot b | n_i \, d\sigma \\
[\mathbf{A}]_{tb} = \int_{\Gamma_i \times \Gamma_i} g_{\kappa_i} t \cdot b | n_i \, d\sigma^2 \\
[\mathbf{S}]_{tb} = \int_{\Gamma_i \times \Gamma_i} g_{\kappa_i} \text{div}_T(t) \cdot \text{div}_T(b) | n_i \, d\sigma^2 \\
[\mathbf{B}]_{tb} = \int_{\Gamma_i \times \Gamma_i} \nabla g_{\kappa_i} \cdot (t \times b) | n_i \, d\sigma^2
\]

The local multi-trace BEM doubles the number of unknowns!

(However it can be reduced to the single-trace form by a simple combination with non-adjacent domains)
### Local Multi-Trace BIE

**Maxwell Matrix Form**

- **Two (or more) adjacent objects**

<table>
<thead>
<tr>
<th>$k_0 A_0 - S_0$</th>
<th>$B_0$</th>
<th>$\frac{1}{2} U_0$</th>
<th>$\frac{1}{2} U_0$</th>
<th>$\mathcal{M}^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_0$</td>
<td>$(A_0 - \frac{1}{k_0^2} S_0)$</td>
<td>$\frac{1}{2} U_0$</td>
<td>$\frac{1}{2} U_0$</td>
<td>$\mathcal{M}^0$</td>
</tr>
<tr>
<td>$\frac{1}{2} U_0$</td>
<td>$\frac{1}{2} U_0$</td>
<td>$B_1$</td>
<td>$\frac{1}{2} U_0$</td>
<td>$\mathcal{M}^1$</td>
</tr>
<tr>
<td>$\frac{1}{2} U_0$</td>
<td>$B_1$</td>
<td>$\frac{1}{2} U_0$</td>
<td>$\frac{1}{2} U_0$</td>
<td>$\mathcal{M}^1$</td>
</tr>
<tr>
<td>$\frac{1}{2} U_0$</td>
<td>$\frac{1}{2} U_0$</td>
<td>$B_2$</td>
<td>$\frac{1}{2} U_0$</td>
<td>$\mathcal{M}^2$</td>
</tr>
</tbody>
</table>

The transmission conditions are handled by the differential terms. This allows for the adjacency of distinct components with a minimal effort.

- $\varepsilon_r = 2$, $\mu_r = 2$

- $2^{nd}$ ($f_0$, $f_1$) and $3^{rd}$ ($f_1$) order interp. functions
- $96 (f_0) + 96 (f_1) + 96 (f_2)$ quad. with curved edges
Quadrature rules must be defined on the intersection of non-conformal meshes to compute the twisted identity terms.

- Conforming meshes
  
- Nonconforming meshes
  Compatible quadrature rules are required (non-conforming & poor quality sub-meshes are ok)

- Nonconformal objects
  Additional unknowns are required (singular bf or local refinement)
LOCAL MULTI-TRACE BIE
THE EDDY CURRENT PROBLEM

- **Eddy current testing**
  - $\Omega_s \in \Omega_0$, $\nabla \cdot J_s = 0$, $n \cdot J_s = 0$ ($\partial \Omega_s$)
  - $\Rightarrow E_s = -s\omega \mu_0 \Phi J_s$ where $\Phi J_s = \frac{1}{4\pi} \int_{\Omega_s} \frac{J_s(y)}{|x-y|} \, dy$
  - $\exists i > 0 \text{ s.t. } \sigma_i \gg 1$

- **Eddy current approximation**
  - $\varepsilon_{\text{dielect}} \sim 0 \Rightarrow k \sim k^{EC} := (1 - st) \sqrt{\frac{\omega \mu \sigma}{2}}$
  - $\Rightarrow k_0 \sim 0$

> When $k^2 \in \mathbb{C}_+^*$ we have $\tau_N(u) = \frac{1}{k^2} \text{div}_I(\gamma_N(u))$ and so $\mathcal{G}_k(\gamma(u)) = u \ 1_\Omega$

\[
\nabla \times \Psi_k(\gamma_D(u)) + \Psi_k(\gamma_N(u)) + \nabla \Psi_k(\tau_N(u)) = u \ 1_\Omega
\]
LOCAL MULTI-TRACE BIE
THE EDDY CURRENT PROBLEM

\[ \nabla \times \Psi_\kappa (\gamma_D(u)) + \Psi_\kappa (\gamma_N(u)) + \nabla \Psi_\kappa (\tau_N(u)) = u \quad 1_\Omega \]

- **Jump conditions**
  - \[
    \begin{pmatrix}
      \gamma_D \\
      \gamma_N
    \end{pmatrix}
    \begin{pmatrix}
      \nabla \times \Psi_\kappa \\
      \Psi_\kappa \\
      \nabla \Psi_\kappa
    \end{pmatrix} = \begin{pmatrix}
      I & 0 & 0 \\
      0 & I & 0
    \end{pmatrix} \quad \text{in } \mathbb{H} \times H^{-1/2}(\Gamma).
  - The gradient term does not play any role in the jump formula.

  Hence a similar identity holds for the reduced potential operator
  \[
  g_\kappa^* (\nu) = \nabla \times \Psi_\kappa (\nu) + \Psi_\kappa (q)
  \]

- **Alternative space**
  - New characteristic constraint: \( \kappa_i^2 = 0 \Rightarrow \text{div}_\Gamma \left( \gamma_N^i (u) \right) = \tau_N (\nabla \times \nabla \times u) = 0. \)
  - Local space: \( \mathbb{H}^* = (H_L \times H_S) \times H_L \)
  - Global space: \( \mathbb{H}(U \Gamma_i) = \prod_{\kappa_i=0} \mathbb{H}^*(\Gamma_i) \times \prod_{\kappa_i \neq 0} \mathbb{H}(\Gamma_i). \)
The cross identity term has to be properly discretized.
A mix of Primal & Dual functions is now required
- (Primal) edge functions on $\mathcal{J}$
- Dual basis functions on $\mathcal{M}$ (Buffa/Christiansen-like).
- The cross identity terms between $\mathcal{J}$ and $\mathcal{M}$ are now well-posed.
- Otherwise the Eddy Current linear system is not invertible.

Two adjacent objects (or more)
No available description of high-order dual basis functions

The Helmholtz decomposition can be applied to the full Maxwell form (no EC approximation)

The cross identity term still affects conditioning but the linear system remains invertible (until what point?)

We will try to overcome the need for dual functions
CONTEXT & MOTIVATION
PRACTICAL EXAMPLES

NOTATIONS & BIE FORMALISM

EXTENSION OF THE LOCAL MULTI-TRACE BIE TO THE EDDY CURRENT REGIME
X. Claeys, E. Demaldent

ASYMPTOTIC EXPANSION OF THE SINGLE-TRACE BIE OF THE FIRST KIND (PMCHWT)
M. Bonnet, E. Demaldent, A. Vigneron

OPTIMISATION ATTEMPT OF THE EDDY CURRENT MULTI-TRACE BIE
SINGLE-TRACE BIE MOTIVATIONS

<table>
<thead>
<tr>
<th><strong>Maxwell Model</strong></th>
<th><strong>Eddy Current Model</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nabla \times \mathbf{E} = -s i \omega \mu \mathbf{H} )</td>
<td></td>
</tr>
<tr>
<td>( \nabla \times \mathbf{H} = \mathbf{J}_s + s i \omega \varepsilon \mathbf{E} )</td>
<td></td>
</tr>
<tr>
<td>where ( \varepsilon = \varepsilon_{\text{diel}} - s i \frac{\sigma}{\omega} )</td>
<td></td>
</tr>
<tr>
<td>Heuristic form by considering ( \varepsilon_{\text{diel}} \sim 0 )</td>
<td></td>
</tr>
<tr>
<td>Additional hyp. on the source</td>
<td></td>
</tr>
<tr>
<td>( \Omega_s \subseteq \Omega_0, \nabla \cdot \mathbf{J}_s = 0, \mathbf{n} \cdot \mathbf{J}_s = 0 (\partial \Omega_s) )</td>
<td></td>
</tr>
<tr>
<td>( \Rightarrow \varepsilon_s = -s i \omega \mu_0 \Phi \mathbf{J}_s )</td>
<td></td>
</tr>
<tr>
<td>where ( \Phi \mathbf{J}<em>s = \frac{1}{4 \pi} \int</em>{\Omega_s} \frac{J_s(y)}{</td>
<td>x-y</td>
</tr>
</tbody>
</table>

- Low frequency Maxwell transmission (single-trace) BIE **PMCHWT**
  See e.g. [Zhao, Chew 00], [Andriulli 12] …

- EC model as an asymptotic case of the Maxwell Integral form
  Related work [Ammari, Buffa, Nédélec 00], [Hiptmair 07], [Schmidt, Sterz, Hiptmair, 08]
SINGLE-TRACE BIE MOTIVATIONS

Maxwell Model
\begin{align*}
\nabla \times \mathcal{E} &= -s \omega \mu \mathcal{H} \\
\nabla \times \mathcal{H} &= \mathcal{J}_s + s \omega \varepsilon \mathcal{E}
\end{align*}
where \( \varepsilon = \varepsilon_{\text{diel}} - s \frac{\sigma}{\omega} \)

Eddy Current Model
Heuristic form by considering \( \varepsilon_{\text{diel}} \sim 0 \)
Additional hyp. on the source
\( \Omega_s \Subset \Omega_0, \nabla \cdot \mathcal{J}_s = 0, n \cdot \mathcal{J}_s = 0 \ (\partial \Omega_s) \)
\[ \Rightarrow \mathcal{E}_s = -s \omega \mu_0 \Phi \mathcal{J}_s \]
where \( \Phi \mathcal{J}_s = \frac{1}{4\pi} \int_{\Omega_s} \frac{\mathcal{J}_s(y)}{|x-y|} \, dy \)

- **Maxwell** Single-trace BIE of the 1st kind \textit{PMCHWT} [Poggio, Miller 73]
  [Chang, Harrington 77]
  [Wu, Tsai 77]
  \( \mathcal{E}_0^s = -s \omega \mu_0 \Phi \mathcal{J}_s^s \)
  \((\mathcal{P}_0) + (\mathcal{P}_1)\)
  \( (\mathcal{M}^i) = (n \cdot n_i) \left( \begin{array}{c} \mathcal{M} \\ \mathcal{J} \end{array} \right) \)
  \( (\mathcal{A} \kappa_0 + \mathcal{A} \kappa_1) \left( \begin{array}{c} \mathcal{M} \\ \mathcal{J} \end{array} \right) = - \left( \nabla \times \Phi \mathcal{J}_s \right) \)
Quasi Helmholtz decomposition to isolate the solenoidal term

Find $X \in \mathbb{H}$ such that $\langle \nabla, ZX \rangle_x = -\langle \nabla, Y \rangle_x \quad \forall Y \in \mathbb{H} = (H_L, H_S, H_L, H_S)$

<table>
<thead>
<tr>
<th>$A_M$</th>
<th>$A_M$</th>
<th>$B$</th>
<th>$B$</th>
<th>$M_L$</th>
<th>$\nabla \times \Phi J_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_M$</td>
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<tr>
<td>$B$</td>
<td>$B$</td>
<td>$A_J$</td>
<td>$A_J$</td>
<td>$J_L$</td>
<td>$\Phi J_s$</td>
</tr>
<tr>
<td>$B$</td>
<td>$B$</td>
<td>$A_J$</td>
<td>$Z_J$</td>
<td>$J_S$</td>
<td>$\Phi J_s$</td>
</tr>
</tbody>
</table>

$A_{J,l}[u] = \frac{\mu_l}{\mu_0} \{y_D \psi_{\kappa_1}\}(u)$,

$Z_{J,l}[u] = A_{J,l}[u] + \frac{1}{\kappa_0^2 \varepsilon_l} \{y_D \nabla \psi_{\kappa_1}\}(\nabla \Gamma \cdot u)$,

$A_{M,l}[u] = \kappa_0^2 \varepsilon_l \{y_D \psi_{\kappa_1}\}(u)$,

$Z_{M,l}[u] = A_{M,l}[u] + \frac{\mu_l}{\mu_0} \{y_D \nabla \psi_{\kappa_1}\}(\nabla \Gamma \cdot u)$,

$B_l[u] = \{y_N \psi_{\kappa_1}\}(u)$

$H_L := H^{-1/2}_\parallel (\text{div}_\Gamma = 0, \Gamma)$

$H_L \oplus H_S := H^{-1/2}_\parallel (\text{div}_\Gamma, \Gamma)$

$Z_J = Z_{J,0} + Z_{J,1}$

$A_J = A_{J,0} + A_{J,1}$

$A_M = A_{M,0} + A_{M,1}$

$Z_M = Z_{M,0} + Z_{M,1}$

$B = B_0 + B_1$
## SINGLE-TRACE BIE

### LOW-FREQUENCY MAXWELL BIE

- **Quasi Helmholtz decomposition** to isolate the solenoïdal term

Find $\mathbf{x} \in \mathbb{H}$ such that $\langle \psi, \zeta \mathbf{x} \rangle_\times = -\langle \psi, \mathbf{y} \rangle_\times \forall \psi \in \mathbb{H} = (H_L, H_S, H_L, H_S)$

<table>
<thead>
<tr>
<th>$\mathcal{A}_M$</th>
<th>$\mathcal{A}_M^*$</th>
<th>$\mathcal{B}$</th>
<th>$\mathcal{B}$</th>
<th>$\mathcal{M}_L$</th>
<th>$\nabla \times \Phi \mathcal{J}_S$</th>
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<tr>
<td>$\mathcal{A}_M$</td>
<td>$\mathcal{Z}_M$</td>
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<td>$\nabla \times \Phi \mathcal{J}_S$</td>
</tr>
<tr>
<td>$\mathcal{B}$</td>
<td>$\mathcal{B}$</td>
<td>$\mathcal{A}_J$</td>
<td>$\mathcal{A}_J$</td>
<td>$\mathcal{J}_L$</td>
<td>$\Phi \mathcal{J}_S$</td>
</tr>
<tr>
<td>$\mathcal{B}$</td>
<td>$\mathcal{B}$</td>
<td>$\mathcal{A}_J$</td>
<td>$\mathcal{Z}_J$</td>
<td>$\mathcal{J}_S$</td>
<td>$\Phi \mathcal{J}_S$</td>
</tr>
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**With combination**

**Without (LF breakdown)**

### Loop & Star combination

**Div-conforming functions**

**Numerical methods for wave propagation and applications | E. Demaldent | 48**
**SINGLE-TRACE BIE ASYMPTOTIC EXPANSION**

- **Investing the limiting case of the Maxwell BIE**

  \[ \gamma := \sqrt{\omega \varepsilon_0 / \sigma} = o(1) \quad \text{for} \quad \tau := D \sqrt{\omega \mu_0 \sigma} = O(1) \quad \text{where} \quad D := \text{diam}(\Omega) \]

  - Rescaled low-freq. Maxwell BIE: \( \langle \mathcal{V}, Z_{\gamma} X_{\gamma} \rangle_x = -\langle \mathcal{V}, Y \rangle_x \) with \( \mathbb{H} = (H_L, H_S, H_L, H_S) \)

  - Asymptotic expansion: \( Z_{\gamma} = Z_{\gamma}^{(0)} + \gamma Z_{\gamma}^{(1)} + O(\gamma^2), \quad Y = Y^{(0)} + \gamma Y^{(1)} + O(\gamma^2) \)

  - Ansatz: \( X_{\gamma} = X_{\gamma}^{(0)} + \gamma X_{\gamma}^{(1)} + O(\gamma^2) \)

  Zeroth-order pb: \( Z_{\gamma}^{(0)} X_{\gamma}^{(0)} = Y^{(0)} \)

  First-order pb: \( (Z_{\gamma}^{(1)} X_{\gamma}^{(0)} + Z_{\gamma}^{(0)} X_{\gamma}^{(1)}) = Y^{(1)} \)
Asymptotic expansion: \( a = a^{(0)} + \gamma a^{(1)} + \mathcal{O}(\gamma^2) \)

- \( \kappa_0 = \gamma \tau \implies \kappa_0^2 = \mathcal{O}(\gamma^2) \)

- \( g_{\kappa_0}(z) = \frac{1}{4\pi|z|} - \gamma \frac{\sin\tau}{4\pi D} + \mathcal{O}(\gamma^2) \) \( \Rightarrow \nabla g_{\kappa_0}^{(1)} = 0, \int g_{\kappa_0}^{(1)} u_L = 0, \Phi J_s = \Phi^{(0)} J_s + \mathcal{O}(\gamma^2) \)

- \( D\kappa_1^2 = -s\mu_r\tau^2 + \mathcal{O}(\gamma^2) \), \( g_{\kappa_1}(z) = \frac{\exp(-s\mu_r\tau^2\mu_r\sqrt{2|z|/D})}{4\pi|z|} + \mathcal{O}(\gamma^2) \)
Asymptotic expansion: $a = a^{(0)} + \gamma a^{(1)} + O(\gamma^2)$

- $\kappa_0 = \gamma \tau \implies \kappa_0^2 = O(\gamma^2)$
- $g_{\kappa_0}(z) = \frac{1}{4\pi|z|} - \gamma \frac{s\tau}{4\pi D} + O(\gamma^2) \implies \nabla g_{\kappa_0}^{(1)} = 0$, $\int g_{\kappa_0}^{(1)} u_L = 0$, $\Phi J_S = \Phi^{(0)} J_S + O(\gamma^2)$
- $D\kappa_1^2 = -s\mu_r \tau^2 + O(\gamma^2)$, $g_{\kappa_1}(z) = \frac{\exp(-s(1-si)\tau \sqrt{\mu_r/2|z|/D})}{4\pi|z|} + O(\gamma^2)$

\[
A_J^{(0)}[u](x) = \left( \int_{\Gamma} \left( g_{\kappa_0}^{(0)} + \mu_r g_{\kappa_1}^{(0)} \right)(x,y) u(y) \, dy \right) \times n(x) = 0
\]

\[
A_J^{(1)}[u](x) = g_{\kappa_0}^{(1)} \left( \int_{\Gamma} u(y) \, dy \right) \times n(x) = A_J^{(1)}[u_L] \equiv 0
\]

\[
Z_{J,y}^{(1)}[u] \equiv 0 \quad A_M^{(1)}[u] \equiv 0 \quad Z_M^{(1)}[u] \equiv 0 \quad B^{(1)}[u] \equiv 0
\]

\[
A_{M}^{(0)}[u](x) = -\frac{s}{D^2} \left( \int_{\Gamma} g_{\kappa_1}^{(0)}(x,y) u(y) \, dy \right) \times n(x)
\]

\[
Z_{M}^{(0)}[u](x) = \left( \nabla \int_{\Gamma} \left( g_{\kappa_0}^{(0)} + \frac{1}{\mu_r} g_{\kappa_1}^{(0)} \right)(x,y)(\nabla \cdot u)(y) \, dy \right) \times n(x) + A_{M}^{(0)}[u](x)
\]

\[
B^{(0)}[u](x) = \left( \nabla \int_{\Gamma} \left( g_{\kappa_0}^{(0)} + g_{\kappa_1}^{(0)} \right)(x,y) u(y) \, dy \right) \times n(x)
\]

\[
g_{\kappa_0/1} = g_{\kappa_{EC}}^{0}
\]

\[
Z_{\gamma}^{(1)}[\mathbf{x}] \equiv 0 \quad \forall \mathbf{x} \in \mathbb{H}
\]
- Zeroth-order BIE \( \langle \nu, \mathcal{Z}_y^{(0)} \mathcal{X}_y^{(0)} \rangle_\times = -\langle \nu, \mathcal{Y}^{(0)} \rangle_\times \)

\[
\begin{array}{c|c|c}
\mathcal{A}_M^{(0)} & \mathcal{A}_M & B^{(0)} \\
\hline
\mathcal{A}_M^{(0)} & \mathcal{Z}_M^{(0)} & B^{(0)} \\
\hline
B^{(0)} & B^{(0)} & \mathcal{A}_j^{(0)} \\
\hline
B^{(0)} & B^{(0)} & \mathcal{Z}_{j,y}^{(0)} \\
\end{array}
\begin{array}{c}
\mathcal{M}_L^{(0)} \\
\hline
\mathcal{M}_S^{(0)} \\
\hline
\mathcal{J}_L^{(0)} \\
\hline
\mathcal{J}_S^{(0)} \\
\end{array}
\begin{array}{c}
\nabla \times \Phi^{(0)} J_S \\
\hline
\nabla \times \Phi^{(0)} J_S \\
\hline
\Phi^{(0)} J_S \\
\hline
\Phi^{(0)} J_S \\
\end{array}
\]

- It coïncides with PMCHWT-type BIE for EC model [Hiptmair 07]: \( \mathcal{X}_y^{(0)} = \mathcal{X}_{EC} \)

- \( \mathcal{J}_S^{(0)} \) (the charge) is computed in a second step

- First-order BIE \( \left( \mathcal{Z}_y^{(1)} \mathcal{X}_y^{(0)} + \mathcal{Z}_y^{(0)} \mathcal{X}_y^{(1)} \right) = \mathcal{Y}^{(1)} \)

  - \( \mathcal{Z}_y^{(1)} \mathcal{X} = 0 \forall \mathcal{X} \in \mathbb{H} \), \( \mathcal{Y}^{(1)} = 0 \) \( \Rightarrow \mathcal{X}_y = \mathcal{X}_{EC} + \mathcal{O}(\gamma^2) \)

- Improvement of [Ammari, Buffa, Nédélec 00] \( \lim_{\omega \searrow 0} (\mathcal{X} - \mathcal{X}_{EC}^\omega) = 0 \)
SINGLE-TRACE BIE
ASYMPTOTIC EXPANSION

Integral Representation

\( \mathcal{E}_0 = \gamma \left( s \eta_0 \frac{\tau}{D} \right) \left( -\Psi_0^{(0)} [\mathcal{J}_L^{(0)}] - \frac{D^2}{\tau^2} \nabla \Psi_0^{(0)} [\nabla \cdot \mathcal{J}_S^{(0)}] + \nabla \times \Psi_0^{(0)} [\mathcal{M}_L^{(0)} + \mathcal{M}_S^{(0)}] \right) + \mathcal{O}(\gamma^3) \)

\( \mathcal{E}_1 = \gamma \left( s \eta_0 \frac{\tau}{D} \right) \left( \mu_r \Psi_1^{(0)} [\mathcal{J}_L^{(0)}] - \nabla \times \Psi_1^{(0)} [\mathcal{M}_L^{(0)} + \mathcal{M}_S^{(0)}] \right) + \mathcal{O}(\gamma^3) \)

\( \mathcal{H}_0 = \nabla \times \Psi_0^{(0)} [\mathcal{J}_L^{(0)}] - \nabla \Psi_0^{(0)} [\nabla \cdot \mathcal{M}_S^{(0)}] + \mathcal{O}(\gamma^2) \)

\( \mathcal{H}_1 = -\nabla \times \Psi_1^{(0)} [\mathcal{J}_L^{(0)}] - s \frac{\tau^2}{D^2} \Psi_1^{(0)} [\mathcal{M}_L^{(0)} + \mathcal{M}_S^{(0)}] + \frac{1}{\mu_r} \nabla \Psi_1^{(0)} [\nabla \cdot \mathcal{M}_S^{(0)}] + \mathcal{O}(\gamma^2) \)

• Consistent with estimates from [Schmidt, Sterz, Hiptmair 08]

Impedance (reciprocity theorem)

\( \Delta Z = \frac{1}{i^2} \gamma \left( s \eta_0 \frac{\tau}{D} \right) \left( \left[ \mathcal{J}_L^{(0)}, \gamma_D (\Phi^{(0)} \mathcal{J}_S) \right] \right)_x - \left[ \mathcal{M}_S^{(0)}, \gamma_D (\nabla \times \Phi^{(0)} \mathcal{J}_S) \right]_x \right) + \mathcal{O}(\gamma^3) \)

\( \mathcal{J}_S^{\text{EC}} \) is only needed in the calculation of \( \mathcal{E}_0^{\text{EC}} \)

\( \mathcal{M}_L^{\text{EC}} \) is not used in the calculation of \( \mathcal{H}_0^{\text{EC}} \) and \( \Delta Z^{\text{EC}} \)
(natural outputs in eddy current testing)
Numerical methods for wave propagation and applications | E. Demaldent | 54
\[ r = 10^{-1} \]
\[(f, \sigma) = (10^4, 10^3)\]
\[(\gamma, \tau) \simeq (2 \times)\]

Maxwell, EC

EC without \(J_T\)

Incorrect \(E\)-field in air
SINGLE-TRACE BIE SUMMARY

- $\chi^{EC}$ can be derived from the Maxwell solution by asymptotic expansion w.r.t. $\gamma := \sqrt{\omega \varepsilon_0 / \sigma}$

- $J_S^{EC}$ is needed only in the calculation of $E_0^{EC}$ and can be computed in post-processing

- $M_L^{EC}$ is not used in the calculation of $H_0^{EC}$ and $\Delta Z^{EC}$ (natural outputs in eddy current testing)
SINGLE-TRACE BIE
SUMMARY

- $\chi^{EC}$ can be derived from the Maxwell solution by asymptotic expansion w.r.t. $\gamma := \sqrt{\omega \varepsilon_0 / \sigma}$
- $J_S^{EC}$ is needed only in the calculation of $E_0^{EC}$ and can be computed in post-processing
- $M_L^{EC}$ is not used in the calculation of $H_0^{EC}$ and $\Delta Z^{EC}$ (natural outputs in eddy current testing)
- $M_L^{EC}$ is not needed at the interface between non-conductive objects ($A_{M,0}^{(0)} = 0$)

We will try to take advantage of these results to improve the ECT multi-trace BIE
CONTEXT & MOTIVATION
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OPTIMISATION ATTEMPT OF THE EDDY CURRENT MULTI-TRACE BIE

Work in progress…
**LOCAL MULTI-TRACE – ECT OPTIM**

**LOW-FREQUENCY MAXWELL PB**

- **Low-frequency Maxwell pb**
  - Quasi-Helmholtz decomposition (*loop-star combinations*)
  - Rescaling ($\gamma := \sqrt{\omega \varepsilon_0 / \sigma}$)

<table>
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<tr>
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<td>$\kappa_0^2 A_0$</td>
<td>$B_0$</td>
<td>$\gamma^2 B_0$</td>
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<td>$\gamma^2 \frac{1}{2} I_{01}^\times$</td>
<td>$\mathcal{M}_L^0$</td>
<td>$-{ L t, \gamma^0_\Phi J_s }_0^0$</td>
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<tr>
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<td>$\kappa_0^2 A_0 - S_0$</td>
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<td>$\frac{1}{2} I_{01}^\times$</td>
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Numerical methods for wave propagation and applications | E. Demaldent | 59
Zeroth-order pb (w.r.t. $\gamma \equiv \sqrt{\omega \varepsilon_0 / \sigma}$)

- We consider $\kappa \sim \kappa^{\text{EC}}$
- We get rid of the charge terms $J_S^0$ (previous ECT-form) as well as $J_S^1$
**LOCAL MULTI-TRACE – ECT OPTIMIZED EDDY CURRENT APPROXIMATION**

- **Zeroth-order pb (w.r.t. $\gamma := \sqrt{\omega \varepsilon_0 / \sigma}$)**
  - We consider $\kappa \sim \kappa^{EC}$
  - We get rid of the charge terms $J_S^0$ (previous ECT-form) as well as $J_S^1$
  - Null terms are considered (*local loop* functions on a **simply connected geom**.)

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<td>$\frac{1}{2} I_{10}^x$</td>
<td>$\frac{1}{2} I_{10}^x$</td>
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<tr>
<td>L</td>
<td>B_0</td>
<td>B_0</td>
<td>A_0</td>
<td>$\frac{1}{2} I_{10}^x$</td>
<td>$\frac{1}{2} I_{10}^x$</td>
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<tr>
<td>L</td>
<td></td>
<td>$\frac{1}{2} I_{10}^y$</td>
<td>$\mu_0 \frac{\kappa_1^2}{\mu_1} A_1$</td>
<td>$\mu_0 \frac{\kappa_1^2}{\mu_1} A_1$</td>
<td>B_1</td>
<td></td>
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<tr>
<td>S</td>
<td></td>
<td>$\frac{1}{2} I_{10}^y$</td>
<td>$\frac{\mu_0}{\mu_1} A_1$</td>
<td>$\mu_0 \frac{\kappa_1^2}{\mu_1} (A_1 - S_1)$</td>
<td>B_1</td>
<td></td>
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<tr>
<td>L</td>
<td></td>
<td>$\frac{1}{2} I_{10}^y$</td>
<td>$\frac{1}{2} I_{10}^x$</td>
<td>B_1</td>
<td>B_1</td>
<td>$\frac{\mu_1}{\mu_0} A_1$</td>
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</tbody>
</table>

$$
\begin{align*}
\Omega_0 & \\
\mu_0, \kappa_0 & \\
\Omega_1 & \\
\mu_1, \kappa_1 & \\
\end{align*}
$$

**Numerical methods for wave propagation and applications | E. Demaldent | 61**
LOCAL MULTI-TRACE – ECT OPTIM
EDDY CURRENT APPROXIMATION

- Zeroth-order pb (w.r.t. $\gamma := \sqrt{\omega \varepsilon_0 / \sigma}$)
  - We consider $\kappa \sim \kappa^{EC}$
  - We get rid of the charge terms $\mathbf{J}_S^0$ (previous ECT-form) as well as $\mathbf{J}_S^1$
  - Null terms are considered (local loop functions on a simply connected geom.)
  - We suppress the solenoïdal magnetic current $\mathbf{M}_L^0$ in the dielectric medium

$$\nabla \times \mathbf{A} = \frac{1}{\mu_0} \mathbf{J}_S^0 + \frac{1}{\mu_0} \mathbf{J}_S^1 \times \mathbf{B}_0 + \frac{1}{\mu_0} \mathbf{J}_S^1 \times \mathbf{B}_1$$

$$-\frac{1}{\mu_0} \mathbf{J}_S^0 \times \mathbf{B}_0 - \frac{1}{\mu_0} \mathbf{J}_S^1 \times \mathbf{B}_1 = \left( \mathbf{J}_L^0 - \mathbf{J}_L^1 \right)$$

- Usual (primal-primal) discretization can be now used (performance & simplicity)

**But there is a trick!**
Non-simply connected geometry

- \( LB_0 L' \not\equiv 0 \implies M_L^0 \) cannot be killed
- \( \Omega_0 \) is split into simply connected sub-domains (\( \Omega_0 = \Omega_{0,a} \cup \Omega_{0,b} \))
- It increases the nb of unknowns but \( M_L^{0,a} \) and \( M_L^{0,b} \) should be killed so that usual functions could be used

\[ \Omega_0 \quad \Omega_0 \]

Work in progress…
Non-simply connected geometry

- $LB_0 L' \neq 0 \implies M_L^0$ cannot be killed
- $\Omega_0$ is split into simply connected sub-domains ($\Omega_0 = \Omega_{0,a} \cup \Omega_{0,b}$)
- It increases the nb of unknowns but $M_L^{0,a}$ and $M_L^{0,b}$ should be killed so that usual functions could be used

![Diagram](image)

It failed!

$$L 1^i_X L' \neq 0 \text{ as } H_L^i \cap H_L^j \notin H_L^{ij} \text{ in general}$$

Same pb on simply connected geom. with adjacent domains
CONCLUSION

Introduction of the local multi-trace BIE formalism for the Maxwell problem
- Assets and limitations of the method in the presence of a non-conforming discretization

Extension of the multi-trace formalism to the eddy current problem
- Requires working with a dual-primal discretization on each boundary
- Alternative: Maxwell low frequency problem with usual (primal) functions
- Until what point? Conditioning gets worse as frequency decreases

Optimization attempt inspired by the asymptotic form of the single-trace BIE
- Seems to require a splitting of the loop space: $H_L^i = \prod_{j \neq i} H_L^{ij} + H_L^{ic}$
- In contradiction with the spirit of multi-trace formalism

Work in progress…

Another topic of interest:
Weighted BIE of the 2nd kind with non-conforming full Helmholtz decomposition (div-free x curl-free)
THANK YOU FOR YOUR ATENTION!
edouard.demaldent@cea.fr