Un principe unificateur discret pour les schémas numériques Lagrangiens en CFD

Lagrange

MÉMOIRE

SUR LA

THÉORIE DU MOUVEMENT DES FLUIDES (*).

(*Nouveaux Mémoires de l'Académie royale des Sciences et Belles-Lettres de Berlin*, année 1781.)

(1781)

MÉCHANIQUE

ANALITIQUE;

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M. DCC. LXXXVIII.

AVEC APPROBATION ET PRIVILEGE DU ROI.

(1788)

-20/06/2014 : 1910 Google-scholar answers to Lagrangian CFD.
-18/10/2015 : 2830 Google-scholar answers to Lagrangian CFD.

-18/10/2015 : 2650 Google-scholar answers to Eulerian CFD.

Introduction

Lagrangian schemes

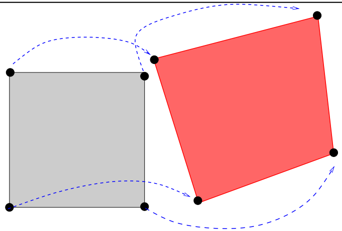
2. Maintenant, à cause de la continuité du fluide, on peut imaginer que chaque particule dm ait la figure d'un parallélépipède rectangle, et que son volume soit par conséquent exprimé par $\delta x\, \delta y\, \delta z$, en supposant que δx, δy, δz soient les côtés du parallélépipède et représentent les variations des coordonnées x, y, z, pour les particules adjacentes, dans la direction de ces coordonnées.

Si donc on nomme Δ la densité de chaque particule dm, on aura

$$dm = \Delta\, \delta x\, \delta y\, \delta z,$$

et la quantité Δ devra être pareillement une fonction de x, y, z, t.

3. Dans l'instant suivant, le parallélépipède changera à la fois de place et de forme, mais la masse dm demeurera la même.



$$\frac{1}{V}\frac{d}{dt}V = \nabla \cdot \mathbf{u}$$

Compressible non viscous fluid

Lagrangian derivative is $D_t = \partial_t + \mathbf{u} \cdot \nabla$, specific vol. is $\tau = \frac{1}{\rho}$.

$\left\{ \begin{array}{l} \rho D_t \tau - \nabla \cdot \mathbf{u} = 0, \\ \rho D_t \mathbf{u} + \nabla p = 0, \\ \rho D_t e + \nabla \cdot (p\mathbf{u}) = 0. \end{array} \right.$	$\left\{ \begin{array}{l} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u}^2 + p\mathbf{I}) = 0, \\ \partial_t (\rho e) + \nabla \cdot (p\mathbf{u}) = 0. \end{array} \right.$
Lagrange	Euler

Pressure may be function of mass and temperature : $p = (\gamma - 1)\rho\varepsilon$, $\varepsilon = C_v T = e - \frac{1}{2}|\vec{u}|^2$.

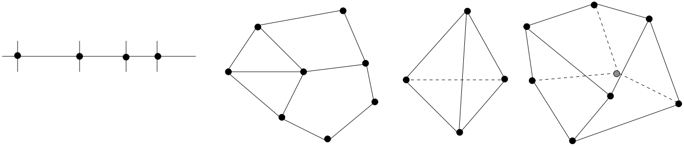
Follow Lagrange :

consider that foundation is ODEs for small volumes ,

PDEs will be derived at the end.

Lagrangians meshes

General cells and general meshes



Notations : j for cells, n for time steps, r for nodes.

For example V_j^n denotes the volume of cell j at time step n, and vertices (nodes) at time t_n are denoted as $\mathbf{x}_r^n$.

Moving Lagrangian cells have a constant mass

$$M_j^n = M_j \quad \forall \text{ time steps } n$$

Discrete analogue of Lagrange idea is moving Lagrangian cells, with volume $V_j(t)$ function the vertices (nodes)

$$\mathbf{x}(t) = (\mathbf{x}_1(t), \cdots, \mathbf{x}_r(t), \cdots) \in \left(\mathbb{R}^d\right)^{N_v}.$$

The volume is a function $V_j : \left(\mathbb{R}^d\right)^{N_v} \to \mathbb{R}^+$

$$(\mathbf{x}_1, \cdots, \mathbf{x}_r, \cdots) \mapsto V_j(\mathbf{x}_1, \cdots, \mathbf{x}_r, \cdots).$$

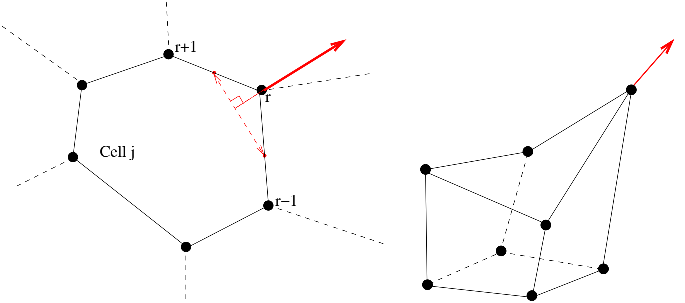
The chain rule writes

$$\frac{d}{dt} V_j = \sum_r (\mathbf{C}_{jr}, \mathbf{u}_r)$$

where a fundamental discrete object is

$$\boxed{\mathbf{C}_{jr} = \nabla_{\mathbf{x}_r} V_j \in \mathbb{R}^d}$$

Geometrical interpretation



$\mathbf{C}_{jr} = \nabla_{\mathbf{x}_r} V_j = O(h^{d-1})$ is a kind of corner vector.

Some properties of the corner vectors

- Chain rule : $\frac{d}{dt} V_j = \left(\nabla_{\mathbf{x}} V_j, \mathbf{x}'\right) = \sum_r \left(\mathbf{C}_{jr}, \mathbf{u}_r\right)$.
- Homogeneity : $V_j = \frac{1}{d} \left(\nabla_{\mathbf{x}} V_j, \mathbf{x}\right) = \frac{1}{d} \sum_r \left(\mathbf{C}_{jr}, \mathbf{x}_r\right)$.
- Translation invariance for all cell j : $\sum_r \mathbf{C}_{jr} = \mathbf{0}$.
- For all interior node $\mathbf{x}_r$: $\sum_j \mathbf{C}_{jr} = \mathbf{0}$.
- A even more general formula (related to rotational invariance)

$$\boxed{\sum_j \mathbf{C}_{jr} \otimes \mathbf{x}_r = V_j \, \mathbf{I}_d}$$

- Similar criterion on edges fundamental for FV diffusion schemes.
- A unified approach to mimetic FD, hybrid FV and MFVM, Droniou-Eymard-Gallouet-Herbin, M3AS, 2010.

FV formulation on moving meshes

$$\begin{cases} \frac{d}{dt}\int_{V_j(t)} \rho \, dV = 0, & \text{mass cons.,} \\ \frac{d}{dt}\int_{V_j(t)} dV - \int_{S_j(t)} \mathbf{u}.\mathbf{n} \, dS = 0, & \text{volume cons.,} \\ \frac{d}{dt}\int_{V_j(t)} \rho\mathbf{u} \, dV + \int_{S_j(t)} p\mathbf{n} \, dS = \mathbf{0}, & \text{momentum cons.,} \\ \frac{d}{dt}\int_{V_j(t)} \rho e \, dV + \int_{S_j(t)} p\mathbf{u}.\mathbf{n} \, dS = 0, & \text{total energy cons.,} \end{cases}$$

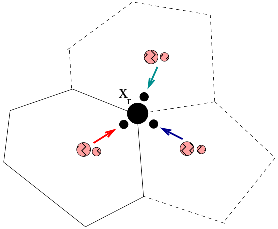
with $S_j(t) = \partial V_j(t)$ the boundary, and $\mathbf{n}$ the outgoing normal.

One gets

$$\begin{cases} \mathbf{x}_r'(t) & = \mathbf{u}_r \, , \\ M_j \tau_j'(t) & = \sum_r \left(\mathbf{C}_{jr} , \mathbf{u}_r \right) , \\ M_j \mathbf{u}_j'(t) & = - \sum_r \mathbf{C}_{jr} p_{jr} \, , \\ M_j e_j'(t) & = - \sum_r \left(\mathbf{C}_{jr} , \mathbf{u}_r \right) p_{jr} \, . \end{cases}$$

The definition of the corner based fluxes has to be completed through a convenient definition of $\mathbf{u}_r$ and p_{jr}.

Corner solvers (unknowns are $\mathbf{u}_r$ and $(p_{jr})_{j\in\mathcal{N}(r)}$)



(big bullets are velocities)

One assumes 1D acoustic Riemann relation in corner direction
$p_{jr} - p_j + \rho_j c_j \left(\frac{\mathbf{C}_{jr}}{|\mathbf{C}_{jr}|}, \mathbf{u}_r - \mathbf{u}_j \right) = 0.$
This formula is all what is retained from the theory of Riemann solvers, which is therefore is very little output for Lagrangian CFD.

Around node r the sum of forces vanishes : $\sum_j \mathbf{C}_{jr} p_{jr} = 0$ (in $\mathbb{R}^d$).
Solution under the form

$$\mathbf{A}_r \mathbf{u}_r - \mathbf{b}_r = \sum_j \mathbf{C}_{jr} p_{jr} = 0.$$

Determine the node velocity as the unique solution of

$$\boxed{\mathbf{A}_r \mathbf{u}_r = \mathbf{b}_r}$$

where the nodal matrix is positive

$$\mathbf{A}_r = \mathbf{A}_r^t = \sum_j \rho_j c_j \frac{\mathbf{C}_{jr} \otimes \mathbf{C}_{jr}}{|\mathbf{C}_{jr}|} \in \mathbb{R}^{d \times d} > 0,$$

and the nodal RHS is

$$\mathbf{b}_r = \sum_j \mathbf{C}_{jr} p_j + \sum_j \rho_j c_j \frac{\mathbf{C}_{jr} \otimes \mathbf{C}_{jr}}{|\mathbf{C}_{jr}|} \mathbf{u}_j \in \mathbb{R}^d.$$

The solution is $\mathbf{u}_r = \mathbf{A}_r^{-1} \mathbf{b}_r$.
Finally the nodal pressures are computed as follows $p_{jr} = p_j + \rho_j c_j \left(\mathbf{u}_r - \mathbf{u}_j, \mathbf{n}_{jr} \right)$.

Remark

One notices the remarkable property that $\mathbf{u}_r$ is also the minimum of a minimization problem. Set

$$J_r(\mathbf{u}_r) = \frac{1}{2}(\mathbf{A}_r \mathbf{u}_r, \mathbf{u}_r) - (\mathbf{b}_r, \mathbf{u}_r) \,.$$

Then

$$\boxed{J_r(\mathbf{u}_r) \leq J_r(\mathbf{v}) \qquad \forall \mathbf{v} \in \mathbb{R}^d}$$

is equivalent to $\mathbf{A}_r \mathbf{u}_r = \mathbf{b}_r$.

Constraints can be introduced for contact problems for example (with Clair and Labourasse).

Example of a first order time loop

1) Compute all geometrical vectors $\mathbf{C}_{jr}^n$
2) Compute all nodal quantities $\mathbf{u}_r^n$ and p_{jr}^n using the nodal solver.
3) For all cell, update velocity

$$M_j \frac{\mathbf{u}_j^{n+1} - \mathbf{u}_j^n}{\Delta t} = - \sum_{r \in nodes} \mathbf{C}_{jr}^n p_{jr}^n.$$

and total energy

$$M_j \frac{e_j^{n+1} - e_j^n}{\Delta t} = - \sum_{r \in nodes} \left(\mathbf{C}_{jr}^n, \mathbf{u}_r^n \right) p_{jr}^n.$$

4) Move nodes $\mathbf{x}_r^{n+1} = \mathbf{x}_r^n + \Delta t \ \mathbf{u}_r^n$.
5) Update Lagrangian density $\rho_j^{n+1} = \dfrac{M_j}{V_j^{n+1}}$.

Properties

- Since FV (P^0), the algorithm is simple to implement
- Conservation of (local) mass, volume, impulse, energy.
- The method is entropy increasing $S_j^{n+1} \geq S_j^n$ under CFL.
- Under usual Lax stability assumptions :

weak consistency (in $\mathcal{D}'$) of the whole Lagrangian algorithm with the usual Eulerian equations of fluid dynamics is demonstrated.

$$\left\{ \begin{array}{ll} \int_{t\in\mathbb{R}} \int_{\mathbf{x}\in\mathbb{R}^d} (\rho\partial_t\varphi + \rho\mathbf{u}\cdot\nabla\varphi)\, dtdx = 0, & \forall\varphi\in\mathcal{D}_0, \\ \int_{t\in\mathbb{R}} \int_{\mathbf{x}\in\mathbb{R}^d} (\rho\mathbf{u}\cdot\partial_t\varphi + \rho\mathbf{u}\otimes\mathbf{u} + p\mathbf{I}\cdot\nabla\varphi)\, dtdx = 0, & \forall\varphi\in\mathcal{D}_0^d, \\ \int_{t\in\mathbb{R}} \int_{\mathbf{x}\in\mathbb{R}^d} (\rho e\partial_t\varphi + \rho e\mathbf{u}\cdot\nabla\varphi)\, dtdx = 0, & \forall\varphi\in\mathcal{D}_0, \\ \int_{t\in\mathbb{R}} \int_{\mathbf{x}\in\mathbb{R}^d} (\rho S\partial_t\varphi + \rho S\mathbf{u}\cdot\nabla\varphi)\, dtdx \leq 0, & \forall\varphi\in\mathcal{D}_0^+, \end{array} \right. \tag{1}$$

where $\mathcal{D}_0$ is the set of smooth test functions with compact support,
and $\mathcal{D}_0^+ \subset \mathcal{D}_0$ is the set of smooth non negative test functions with compact support.

Transport part

For simplification, consider the semi-discrete scheme and simplexes (triangles, tets).
Set

$$\rho_j(t) = \frac{M_j}{V_j(t)}, \quad \rho_h(t,\mathbf{x}) = \sum_j \mathbf{1}_{\mathbf{x}\in\Omega_j(t)}\rho_j(t).$$

and the P^1 velocity reconstruction in $\mathbb{R}^d$

$$\overline{\mathbf{u}}_h(t,\mathbf{x}) = \sum_1^{d+1} \lambda_r^q(\mathbf{X})\mathbf{u}_{r_j(q)}(t), \quad \overline{\mathbf{u}}_h(t,\mathbf{x}) = \sum_j \mathbf{1}_{\mathbf{x}\in\Omega_j(t)}\overline{\mathbf{u}}_j(t,\mathbf{x}).$$

One has the identity on simplexes

$$\partial_t \rho_h + \nabla\cdot(\rho_h \overline{\mathbf{u}}_h) = 0 \text{ in } \mathcal{D}'.$$

On general meshes (with an additional stability assumption, and control of the discrete BV norm)

$$\partial_t \rho_h + \nabla\cdot(\rho_h \overline{\mathbf{u}}_h) = O(h) \text{ in } \mathcal{D}'.$$

The crux of the proof : the pressure gradient

Set

$$p_h(\mathbf{x}) = \sum_j p_j \mathbf{1}_{\mathbf{x}\in\Omega_j} \in \mathbb{R}$$

and

$$B_h(\mathbf{x}) = \sum_j \frac{\sum_r \mathbf{C}_{jr} p_{jr}}{V_j} \mathbf{1}_{\mathbf{x}\in\Omega_j} \in \mathbb{R}^d.$$

- Assume that the speed of sound is within the bounds $0 < \alpha_1 \leq \rho_j c_j < \alpha_2$ everywhere.
- Assume that the mesh is regular in the sense $\alpha_3 h^d \leq V_j$ and $\operatorname{diam}(\Omega_j) \leq \alpha_4 h$ for all cells, with uniform constants $\alpha_3, \alpha_4 > 0$.
- Assume that (p_j) and $(\mathbf{u}_j)$ are bounded in L^∞ and are bounded in BV in the sense

$$\sum_j \sum_{k\in V(j)} h^{d-1} |p_j - p_k| \leq \alpha_5 \text{ and } \sum_j \sum_{k\in V(j)} h^{d-1} |\mathbf{u}_j - \mathbf{u}_k| \leq \alpha_6.$$

Then $B_h - \nabla p_h = O(h)$ in the weak sense.

Some references

The discrete unifying principle is the seminal Lagrange idea on a mesh (ODEs) : Lax strategy is invoked to obtain the PDEs.

- 2003 : Symmetrization of Lagrangian gas dynamic in dimension two and multidimensional solvers, CRAS (Mazeran+D.) : **First evidence that nodal solver are compatible with GCL** while edge-solvers are not.
- 2005 : Lagrangian gas dynamics in 2D, ARMA (M.+D.),
- 2007 : Cell-centered Lagrangian scheme for two-dimensional compressible flow problems (Maire and al).
- 2009 : Cell-centered Lagrangian hydro. on general unstructured meshes in arbitrary dimension, (Carre+D. and al, JCP) ; the **GLACE** scheme.

- Dumbser and al
- Burton, Shashkov and al, C.W. Shu and al, . . ., Loubere, . . .
- Many variants, can be seen as different quadratures.
- Muscl second order extension.
- ALE (Arbitrary Lagrange Euler).
- Various techniques can be used to enhance the stability of the mesh which is already excellent for many problems.