ON THE STABILITY OF STATIONARY STATES FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH AN EXTERNAL MAGNETIC FIELD

THIERRY CAZENAVE
MARIA J. ESTEBAN

Analyse Numérique
Université Pierre et Marie Curie
4, Place Jussieu
75252 Paris - Cedex 05, France

ABSTRACT: We study the stability properties of the standing waves for nonlinear Schrödinger equations in $\mathbb{R}^3$, in presence of an external, constant magnetic field. For that purpose, we first establish well-posedness of the local Cauchy problem for a wide class of nonlinearities. Then, under some growth conditions on the nonlinearity, we show that the ground state is orbitally stable. We also show that a class of symmetric excited states is stable, for certain perturbations.

Key words: Stability • standing waves • nonlinear Schrödinger equations

RESUMO: SOBRE A ESTABILIDADE DE ONDAS ESTACIONÁRIAS PARA EQUAÇÕES DE SCHRÖDINGER NÃO-LINEARES COM CAMPO MAGNÉTICO EXTERIOR. Neste artigo estuda-se a estabilidade das ondas estacionárias de equações de Schrödinger não-lineares em $\mathbb{R}^3$ com um campo magnético exterior constante. Para isso, mostramos primeiramente que o problema de Cauchy correspondente está bem posto para uma grande classe de não-linearidades. Depois, mostramos que quando as não-linearidades satisfazem certas condições de crescimento, a solução de “ground state” é orbitalmente estável. Mostra-se também que uma classe de estados excitados é estável para certas perturbações.

Palavras-chave: Estabilidade • ondas estacionárias • equações de Schrödinger não-lineares

Received 01/XI/88.
1. INTRODUCTION

We consider nonlinear Schrödinger equations in $\mathbb{R}^3$, of the form

$$i u_t - \Delta u - 2i A \cdot \nabla u + |A|^2 u - i \text{div}(A)u + Vu = g(x,u). \quad (1)$$

Here $u$ is a complex-valued function defined on $[0,T) \times \mathbb{R}^3$ for some $T > 0$. $A$ is a vector-valued potential (with real-valued components) modelling the effect of an external magnetic field $B = \text{curl}(A)$. $V$ is a real-valued potential and $g$ is some nonlinear mapping. Such equations as (1) were considered by Avron, Herbst and Simon [1, 2, 3], Combes, Schrader and Seiler [7], Eboli and Marques [8], Kato [13], Reed and Simon [15], Simon [16].

The existence of standing waves for (1), that is solutions of the form $u(t,x) = e^{-i\omega t} \phi(x)$, was established in Esteban and Lions [9] under natural hypotheses on $A$, $V$ and $g$. We study here the stability properties of the standing waves of (1). The results that we obtain are quite similar to those available in the case $A = 0$. However, in the presence of an external magnetic field certain excited states are stable (see Theorem 3), while in the case $A = 0$ such a property seems to be unknown. In order to establish the stability results, we need first to study the Cauchy problem (initial value problem) for (1).

Before stating our main results, let us introduce some notation. In all what follows, we assume that the magnetic field $B$ is a constant $B \in \mathbb{R}^3$ (and so $\text{div}(A) = 0$). Without loss of generality, we can assume that

$$B = (0,0,b),$$

for some $b \in \mathbb{R} \setminus \{0\}$ (see [9]). In this case, and up to a gauge transform, the potential $A$ can be chosen in the following way (see [9])

$$A(x) = \frac{b}{2} (-x_2, x_1, 0), \quad \text{for every } x = (x_1, x_2, x_3) \in \mathbb{R}^3. \quad (2)$$

We consider the Hilbert space $L^2(\mathbb{R}^3) = (u: \mathbb{R}^3 \rightarrow \mathbb{C}, \int |u|^2 dx < \infty)$, equipped with the norm

$$\|u\|_{L^2} = \left( \int_{\mathbb{R}^3} |u(x)|^2 dx \right)^{1/2}.$$

We also consider the Hilbert space $H^1_A$ defined by

$$H^1_A = \{ u \in L^2(\mathbb{R}^3), \quad Vu + iAu \in L^2(\mathbb{R}^3) \};$$
equipped with the norm (see [9])

$$\|u\|_{1,A} = \left(\int_{\mathbb{R}^3} |\nabla u + iAu|^2 \, dx\right)^{1/2}.$$ 

We define the self-adjoint operator $L_A$ on $L^2(\mathbb{R}^3)$ by

$$D(L_A) = \{u \in H^1_A, \quad -\Delta u - 2iA \cdot \nabla u + |A|^2 u \in L^2(\mathbb{R}^3)\};$$

$$L_A u = -\Delta u - 2iA \cdot \nabla u + |A|^2 u, \quad \text{for } u \in D(L_A).$$

Note that

$$\|u\|_{1,A} = \langle L_A u, u \rangle^{1/2} \quad \text{for every } u \in D(L_A),$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(\mathbb{R}^3)$. On the other hand, $H^1_A \subseteq L^2(\mathbb{R}^3)$ with continuous and dense embedding (see [9]); and so, $L_A$ defines a bounded operator from $H^1_A$ onto $H^{-1}_A$, where

$$H^{-1}_A = (H^1_A)'.$$

Also note that

$$H^1_A \subseteq L^p(\mathbb{R}^3), \quad \text{with continuous and dense embedding (see [9]) for every } p \in [2,6]; \quad \text{and so}$$

$$L^q(\mathbb{R}^3) \subseteq H^{-1}_A, \quad \text{for every } q \in [6/5,2].$$

With the above notation, equation (1) becomes (recall that $\text{div}(A)=0$)

$$iu_t + L_A u + Vu = g(x,u). \quad (5)$$

We now describe our assumptions on the potential $V$ and the nonlinearity $g$. We assume that $V: \mathbb{R}^3 \to \mathbb{R}$ satisfies

$$V \in C^\infty(\mathbb{R}^3) + L^\infty(\mathbb{R}^3), \quad (6)$$

for some $\alpha > 3/2$. We assume that the complex-valued function $g(x,u)$ is measurable in $x$ for all $u \in \mathbb{C}$ and continuous in $u$ for almost all $x \in \mathbb{R}^3$, and that there exist $\alpha \in [0,4)$ and a constant $C$ such that for almost all $x \in \mathbb{R}^3$,

$$|g(x,v) - g(x,u)| \leq C(1 + |v|^\alpha + |u|^\alpha)|v-u|, \quad \text{for all } u, v \in \mathbb{C}. \quad (7)$$
Furthermore, we assume that
\[ g(x,0) = 0, \quad \text{for almost all } x \in \mathbb{R}^3; \]
\[ g(x,u) = g(x,|u|) \frac{u}{|u|}, \quad \text{for almost all } x \in \mathbb{R}^3 \text{ and all } u \neq 0. \]  

We define the real-valued function \( G(x,u) \) by
\[ G(x,u) = \int_{0}^{u} g(x,s) \, ds, \quad \text{for almost } x \in \mathbb{R}^3, \]
and the energy \( E_{V,G} \) by
\[ E_{V,G}(u) = \int_{\mathbb{R}^3} \left( \frac{1}{2} |\nabla u + iAu|^2 + \frac{1}{2} |u|^2 - G(x,u(x)) \right) \, dx, \quad \text{for all } u \in H_A. \]

Our main result concerning the initial-value problem for (5) is stated as follows.

**Theorem 1.** Let \( A \) be given by (2), let \( V \) satisfy (6) and let \( g \) satisfy (7)-(9). For every \( \phi \in H_A \), there exists a unique maximal solution \( u \in C([0,T^*],H_A^1) \cap C^1([0,T^*],H_A^{-1}) \) of (5) such that \( u(0) = \phi \). If \( T^* < \infty \), then \( ||u(t)||_{1,A} \rightarrow \infty \) as \( t \uparrow T^* \). In addition, we have both conservation of charge and conservation of energy, that is
\[ \int_{\mathbb{R}^3} |u(t,x)|^2 \, dx = \int_{\mathbb{R}^3} |\phi(x)|^2 \, dx, \quad \text{for all } t \in [0,T^*); \]
\[ E_{V,G}(u(t)) = E_{V,G}(\phi), \quad \text{for all } t \in [0,T^*). \]

Notice that for the model case \( g(x,u)=\lambda |u|^\alpha u \), Theorem 1 applies to \( \alpha \in [0,4) \).

For the case of \( A=0 \), the initial value problem for the nonlinear Schrödinger equation in \( H^1 \) has been studied in the past few years; in particular by Ginibre and Velo [10, 11, 12], Kato [14], Cazenave and Weissler [5, 6]. The methods are of a perturbative nature and rely basically on sharp dispersive properties of the linear equation. Our method for proving Theorem 1 is an obvious adaptation of the method of Cazenave and Weissler [5], combined with an estimate of \( \exp(itL_A) \). We also establish a regularity result (see Proposition 1).

Standing waves of equation (5) are solutions of the form \( u(t,x) = e^{-i\omega t}\phi(x) \). The equation for \( \phi \) is
\[ L_A \phi + \omega \phi + V\phi = g(x,\phi). \]
Of course, we are interested in solutions $\phi \in H^1_A$, $\phi \neq 0$ of (14). Nontrivial solutions of (14) are obtained in Esteban and Lions [9] under natural hypotheses on $V$ and $g$, by solving certain variational problems. In particular, in some cases there exists a ground state $\phi$ solving the following variational problem.

$$I_{V,G} = E_{V,G}(\phi) = \min \{ E_{V,G}(u), \quad u \in H^1_A \text{ and } \| u \|_{L^2} = 1, \| \phi \|_{L^2} = 1 \}, \quad (15)$$

and all minimizing sequences of (15) are relatively compact in $H^1_A$ up to a translation and a change of gauge. This allows us to apply the method described in Cazenave and Lions [4], in order to obtain the following result of orbital stability. For simplicity we assume

$$V = 0, \quad g(x,u) = \lambda |u|^\alpha u. \quad (16)$$

In such a simple case the results in [9] hold, and there exist nontrivial solutions of (14), for some $w$, provided that $0 < \alpha < 4/3$. Moreover, it is also shown in [9] that, under these assumptions, all minimizing sequences for $I_{V,G}$ are relatively compact in $H^1_A$ up to a translation and a change of gauge. Therefore, if we denote by $\Sigma$ the set of solutions of (14), we can prove the following theorem.

**Theorem 2.** Assume that $V$ and $g$ satisfy (16) with $0 < \alpha < 4/3$. Then $\Sigma$ is stable in the following sense. For every $\delta > 0$, there exists $\epsilon > 0$ such that, if $\phi \in H^1_A$ verifies

$$\inf_{w \in \Sigma} \| \phi - w \|_{L^1} \leq \epsilon,$$

then the corresponding solution $u$ of (5) is global and

$$\inf_{w \in \Sigma} \inf_{y \in \mathbb{R}^3} \| e^{iA(y)}x'_{u(t,x+y)} - w \|_{L^1} \leq \delta, \quad \text{for all } t \geq 0.$$

Note that the type of stability described above is the same as in the case $A = 0$ (compare with Cazenave and Lions [4]).

Still in the case where $V$ and $g$ satisfy (16) with $0 < \alpha < 4/3$, there exist symmetric excited states of (14). More precisely, let us set

$$S = \{ v \in L^6(\mathbb{R}^3), \quad v \text{ is spherically symmetric in } x' = (x_1, x_2) \}.$$

It is proved in [9] that, for every $k \in \mathbb{Z}$, (14) has nontrivial solutions of the form

$$u_k(t,x) = e^{-it} \phi_{k}(x),$$

where $\phi_k$ is a solution of (15).
\[ \phi(x) = \left( \frac{x_2 + ix_1}{\sqrt{x_1^2 + x_2^2}} \right)^k \chi(x), \quad (17) \]

for some \( \chi \in \mathcal{S} \). \( \chi \) is a solution of the following minimization problem

\[ E_{\psi G}(\chi^*) = \min \{ E_{\psi G}(v^*), \quad \psi \in \mathcal{S} \text{ and } \| v \|_{L^2} = 1, \| \chi \|_{L^2} = 1 \}, \quad (18) \]

where

\[ u^*(x) = \left( \frac{x_2 + ix_1}{\sqrt{x_1^2 + x_2^2}} \right)^k u(x), \quad \text{for } u \in \mathcal{S}, \]

and all minimizing sequences of (18) are relatively compact in \( L^q(\mathbb{R}^3), \quad 2 \leq q < 6 \), up to a translation and a change of gauge. Thus we can also apply the method of [4], in order to obtain the following result of orbital stability for the excited states of the form (17)-(18). We consider \( k \in \mathbb{Z} \) and we denote by \( \Sigma_k \) the set of solutions of (14) of the form (17), for some \( \chi \) solution of the minimization problem (18), and by \( H_{A,k}^1 \) the set of functions of the form (17) with \( \chi \in \mathcal{S} \). Note that \( H_{A,k}^1 \) is a closed subset of \( H_A^1 \).

**Theorem 3.** If \( V=0 \) and \( g \) is as above, then \( \Sigma_k \) is stable in the following sense. For every \( \delta > 0 \), there exists \( \varepsilon > 0 \) such that, if \( \phi \in H_{A,k}^1 \) verifies

\[ \inf_{w \in \Sigma_k} \| \phi - w \|_{L^1} \leq \varepsilon, \]

then the corresponding solution \( u \) of (5) is global and

\[ \inf_{w \in \Sigma_k} \inf_{y \in \mathbb{R}^3} \| e^{iA(y)} x(t,x+y) - w \|_{L^1} \leq \delta, \quad \text{for all } t \geq 0. \]

Theorem 3 asserts that the standing waves of the form (17)-(18) are stable with respect to perturbations of a certain form. We do not know whether or not they are stable with respect to arbitrary perturbations in \( H_A^1 \).

The paper is organized as follows. Section 2 is devoted to the proof of Theorem 1 and to some further comments on the Cauchy problem for (5) and Section 3 is devoted to the proof of Theorems 2 and 3.
2. THE CAUCHY PROBLEM

A basic step in the proof of Theorem 1 is a collection of estimates of the solutions of the linear equation, which we describe below. Let us set

\[ S(t) = e^{-itL_A}, \quad \text{for } t \in \mathbb{R}. \]

Since \( L_A \) is self-adjoint in \( L^2(\mathbb{R}^3) \), \( iL_A \) is skew-adjoint; and so \( S(t) \) is a group of isometries in \( L^2(\mathbb{R}^3) \), and we have

\[ S(t)^* = S(-t), \quad \text{for } t \in \mathbb{R}. \]  

Furthermore, we have the following formula (see [1]).

\[ S(t)\phi(x) = \frac{b}{4\pi \sin(bt)(4\pi it)^{1/2}} \int_{\mathbb{R}^3} e^{-iF(x,y,t)}\phi(y)dy, \quad \text{for } x \in \mathbb{R}^3, \]

for every \( t \neq 0 \) and every \( \phi \in D(\mathbb{R}^3) \), where the function \( F \) is given by

\[ F(x,y,t) = \frac{(x_1^2 + y_1^2)^2}{4t} + \frac{b}{4} \cotg(bt) \left( (x_1 - y_1)^2 + (x_2 - y_2)^2 \right) - \frac{b}{2} \left( x_1y_2 - x_2y_1 \right), \]

for \( x, y \in \mathbb{R}^3, \ t \neq 0. \)

In particular, for every \( t \neq 0 \), \( S(t) \) is bounded from \( L^1(\mathbb{R}^3) \) to \( L^\infty(\mathbb{R}^3) \), with a norm

\[ \| S(t) \|_{L^1(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)} \leq \frac{|b|}{\sqrt{|t|} |\sin(bt)|}. \]

By interpolation, it follows that for every \( t \neq 0 \) and for every \( p \in [2, \infty] \), \( S(t) \) is bounded from \( L^p(\mathbb{R}^3) \) to \( L^p(\mathbb{R}^3) \), with the estimate

\[ \| S(t)\phi \|_{L^p} \leq \left( \frac{|b|}{\sqrt{|t|} |\sin(bt)|} \right)^{1 - \frac{2}{p}} \| \phi \|_{L^p}. \]

Lemma 1. Let \( r \in [2, 6) \) and \( q \in (2, \infty] \) with \( 2/q = 3(1/2 - 1/r) \). Let \( T > 0 \), and \( \phi \in L^2(\mathbb{R}^3) \). Then \( S(\cdot)\phi \in C([0,T], L^r(\mathbb{R}^3)) \cap L^q(0,T, L^r(\mathbb{R}^3)) \). Furthermore, there exists \( C \) depending only on \( r \) and \( T \) such that

\[ \| S(\cdot)\phi \|_{L^q(0,T, L^r)} \leq C \| \phi \|_{L^2}, \quad \text{for every } \phi \in L^2. \]
Proof. Lemma 1 is proved in Ginibre and Velo [12] for the case of $A=0$. However, the proof given in [12] applies as well to the present case, and it relies only on (19) and (20). The dependence of the constant on $T$ comes from the fact that $\| S(t) \|_{L^r(L^r',L^r)}$ has periodic singularities due to the presence of the term $\sin(bt)$.

For every $T>0$ and every $f \in L^1(0,T,H^{r-1})$, we consider $F_f \in C([0,T],H^{r-1})$ defined by

$$F_f(t) = \int_0^t S(t-s)f(s)ds,$$

for every $t \in [0,T]$.

The basic estimates for the inhomogeneous equation are described in the following lemma.

Lemma 2. Let $r,p \in [2,6)$ and $q, \gamma \in (2,\infty]$ with $2/q=3(1/2-1/r)$ and $2/\gamma=3(1/2-1/p)$. Let $T>0$, and let $f \in L^q(0,T,L^{r'}(R^3))$. Then $F_f \in C([0,T],L^r(R^3)) \cap L^q(0,T,L^{r'}(R^3))$. Furthermore, there exists $C$ depending only on $r,p$ and $T$ such that

$$\| F_f \|_{L^q(0,T,L^{r'})} \leq C \| f \|_{L^q(0,T,L^{r'})},$$

for every $f \in L^q(0,T,L^{r'}(R^3))$.

Proof. For the case of $A=0$, Lemma 2 is proved in Yajima [18] for the special cases of $r=p, r=2, p=2$, and in Cazenave and Weissler [5] for the general case. As for Lemma 2, the proofs given in [18] and [5] apply as well to the present case, and it relies only on (19) and (20). For the explanation of the dependence of the constant on $T$, see the proof of Lemma 1.

Proof of Theorem 1. This is an obvious adaptation of the argument given in Cazenave and Weissler [5] (apply (3)-(4) in the place of usual Sobolev's inequalities), and we only give the sketch of the proof. Let $M>0$, and let $\phi \in H^{r-1}$ be such that $\| \phi \|_{1,A} \leq M$. Approximate $V$ and $g$ by the sequence of potentials $V_m \in L^\infty(R^3)$ and of globally Lipschitz-continuous nonlinearities $g_m$ given by (see Examples 1 and 3 in [5])

$$V_m(x) = \begin{cases} m, & \text{if } V(x)>m; \\ V(x), & \text{if } |V(x)| \leq m; \\ -m, & \text{if } V(x)<-m; \end{cases}$$

and consider $u_m \in C([0,\infty),H^{r-1}) \cap C^1([0,\infty),H^{r-1})$, the corresponding solution of the regularized version of (5), with initial datum $\phi$. It follows from the conserva-
tions laws of the approximate problem that there exists $T_M$ (depending on $M$), such that $u_m$ is bounded in $L^\infty(0,T_M,H^1_A)$. It follows from Lemma 2 that the sequence $(u_m)_{m \in \mathbb{N}}$ converges as $m \to \infty$ to a solution $u$ of (5) such that $u(0) = \phi$. Uniqueness is an immediate consequence of Lemma 2, and the conservation laws are obtained from the passage to the limit, combined with uniqueness. The theorem follows from considering the maximal solution corresponding to the initial datum.

Remark 1. Note that the method of Kato [14] does not seem to be applicable, even when $g$ does not depend on $x$ and $V$ is smooth. Indeed, a basic step in the proof is an estimate obtained by considering the equation satisfied by $Vu$. Here, $V$ does not commute with $L_A$ and, on the other hand, $(V-iA)$ commutes with $L_A$ but is not compatible with the nonlinearity.

Remark 2. More general nonlinearities can be considered, such as Hartree-type nonlinearities. In fact, the local Cauchy problem can be solved for the class of nonlinearities considered in [5].

Remark 3. In the case where $g$ satisfies (7) for some $\alpha < 4/3$, the initial value problem can be solved in $L^2(R^3)$, by using again Lemma 2, and the method of Tsutsumi [17] (see also Cazenave and Weissler [6]). In this case, all solutions are global. Note that the assumptions on $V$ are the same as in the $H^1_A$ case.

Remark 4. It follows immediately from the conservation laws and Sobolev's inequalities (3) that, if $g$ is such that

$$G(x,s) \leq C + Cs^\delta,$$

for all $s \geq 0$ and almost all $x \in R^3$, where $C$ is some constant and $\delta \in [0,10/3)$, then all the solutions of (5) are global and bounded in $H^1_A$.

Remark 5. It is rather easy to show that the solutions of (5) depend continuously on the initial data in the following sense. The mapping $\phi \mapsto T^*(\phi)$ is lower semi-continuous and, if $T \in [0,T^*(\phi))$ and $\phi_n \to \phi$ in $H^1_A$ as $n \to \infty$, then the corresponding solutions $u_n$ of (5) verify $u_n \to u$ as $n \to \infty$, in $C([0,T],H^1_A)$.

In order to state our regularity result, we make the following definition. Let $H^2_A$ be the Hilbert space $D(L_A)$, equipped with the graph norm

$$\| u \|_{2,A} = \left( \| u \|^2_{L^2} + \| L_A u \|^2_{L^2} \right)^{1/2}.$$
Some properties of $H_A^2$ are summarized in the following lemma.

**Lemma 3.** For every $u \in H_A^2$ we have $|\nabla u| \in L^2(R^3) \cap L^6(R^3)$, and there exists a constant $C$ such that

$$
\| (|\nabla u|) \|_{L^2} + \| (|\nabla u|) \|_{L^6} \leq C \| u \|_{H^2},
$$

for every $u \in H_A^2$. (23)

In particular,

$$
H_A^2 = L^\infty(R^3).
$$

(24)

**Proof.** We recall that (see [9]) $\| (|\nabla u|) \|_{L^2} \leq \| u \|_{H^1}$. Since $\| u \|_{H^1} = \| u \|_{L^2}$, this proves the first part of (23). Now let $u \in H_A^2$, and let $f = L_A u \in L^2(R^3)$. An elementary calculation shows that, for every $j \in \{1, 2, 3\}$, we have

$$
\mathcal{L}_A((\partial_j + iA_j)u) = (\partial_j - iA_j)f + 2i(\nabla u + iAu) \cdot (\partial_j A - VA_j), \quad \text{in } D(R^3).
$$

(25)

Next, observe that $|((\partial_j A - VA_j)| \leq b$, and that $(\nabla + iA)$ is by definition a bounded operator from $H_A^1$ to $L^2(R^3)$. Therefore, by duality, $(\nabla + iA)$ is a bounded operator from $L^2(R^3)$ to $H_A^{-1}$. Thus $(\partial_j - iA_j)f + 2i(\nabla u + iAu) \cdot (\partial_j A - VA_j) \in H_A^{-1}$; and so it follows from (25) that $(\partial_j + iA_j)u \in H_A^1$. Multiplying (25) in the duality $H_A^{-1} - H_A^1$ by $(\partial_j + iA_j)u$, we get

$$
\| (\partial_j + iA_j)u \|_{H^1} \leq \| (\nabla + iA)f \|_{-1, A} + 2b \| (\nabla u + iAu) \|_{-1, A}
$$

$$
\leq C(\| f \|_{L^2} + \| u \|_{H^1}) \leq C \| u \|_{H^2}.
$$

Taking successively $j = 1, 2, 3$, we obtain $(\nabla + iA)u \in H_A^1$, and

$$
\| (\nabla + iA)u \|_{H^1} \leq C \| u \|_{H^2}.
$$

(26)

Since $|\nabla u|^2 \leq \| \nabla u \|$ almost everywhere (see [9]), (23) follows from (26) and (3). Finally, (24) is a consequence of (3), (23) and Sobolev's inequalities.

Notice that, according to (8), (7), (24) and the condition $\alpha < 4$, it follows that, for every $u \in H_A^2$, $g(\cdot, u(\cdot)) \in L^2(R^3)$ and

$$
\| g(\cdot, u(\cdot)) \|_{L^2} \leq (1/2) \| u \|_{H^2} + K(\| u \|_{H^1}), \quad \text{for every } u \in H_A^2,
$$

(27)

where $K$ is a locally bounded function $R_+ \rightarrow R$. Following the idea of Kato [14] of estimating $u_t$, we obtain the following regularity result.
**Proposition 1.** Let $A$ be given by (2), let $V$ satisfy (6) and let $g$ satisfy (7)-(9). Consider $\phi \in H^1_A$, and let $u \in C([0,T^*),H^2_A) \cap C^1([0,T^*),H^{-1}_A)$ be the unique maximal solution of (5) such that $u(0)=\phi$. Then if $\phi \in H^2_A$, we have $u \in C([0,T^*),H^2_A) \cap C^1([0,T^*),L^2(R^3))$.

**Proof.** Coming back to the beginning of the proof of Theorem 1, and assuming $\phi \in H^2_A$, we have $u \in C([0,\infty),H^2_A) \cap C'([0,\infty),L^2(R^3))$. Note that, by Lemma 3, we have

$$
(u_m)_t(0) = iL_A \phi + iV_m \phi - ig_m(x,\phi) + iL_A \phi + iV \phi - ig(x,\phi) \quad \text{in} \ L^2(R^3), \quad \text{as} \ m \to \infty.
$$

Considering the equation satisfied by $u_m$, applying Lemmas 1 and 2, and using the fact that $(u_m)_{m \in \mathbb{N}}$ is bounded in $L^\infty(0,T_M,H^1_A)$, it follows that $(u_m)_t$ is bounded in $L^\infty(0,T_M,L^2)$ and in $L^q(0,T_M,L^r)$, where $(q,r)$ is any pair as in Lemma 1, by possibly choosing $T_M$ smaller (but still depending only on $M$). It follows, in particular, that $u \in W^{2,q}(0,T_M,L^2) \cap W^{1,q}(0,T_M,L^r)$. Applying again Lemma 2 with $f=(g(x,u))$, we obtain $u \in C([0,T_M],L^2)$. Therefore, it follows from (1) and (27) that $u \in L^\infty(0,T_M,H^2_A)$. Hence, $g(x,u) \in C([0,T_M],L^2)$; and so $u \in C([0,T_M],H^2_A)$. Since $T_M$ depends only on $M$, it follows that, as long as $u$ remains bounded in $H^2_A$, it remains bounded in $H^2_A$. Hence the result.

### 3. Stability of the Stationary States

Theorems 2 and 3 are proved by the method of Cazenave and Lions [4], by using the compactness properties of the minimizing sequences of problems (15) and (18) established in Esteban and Lions [9].

**Proof of Theorem 2.** First note that Remark 4 applies, and so all solutions of (5) are global and bounded in $H^1_A$. Arguing by contradiction, if the conclusion of the theorem does not hold, there exist $\varepsilon>0$, a sequence $(\phi_n)_{n \in \mathbb{N}}$ such that the distance of $\phi_n$ to $\Sigma$ goes to $0$ as $n \to \infty$, and a sequence $(t_n)_{n \in \mathbb{N}}$ such that

$$
\inf_{w \in \Sigma} \inf_{y \in \mathbb{R}^3} \| e^{iA(y) \cdot x} u_n(t_n,x+y)-w \|_{1, A} \geq \varepsilon, \quad (29)
$$

where $u_n$ denotes the solution of (5) with initial datum $\phi_n$. In particular, we have

$$
\| \phi_n \| + 1, \quad (30)
$$

$$
E(\phi_n) \to M = \min \{ E(u), \ u \in H^1_A \text{ and } \| u \|_{L^2} = 1 \}. \quad (31)
$$
Therefore, if we set \( w_n = u_n(t_n) \), it follows from (30), (31) and the conservation laws (12) and (13) that \((w_n)_{n \in \mathbb{N}}\) is a minimizing sequence for the problem (15). Applying Theorem III.3 of [9] and the remarks that follow it, we get a contradiction with (29). This proves Theorem 2.

Proof of Theorem 3. The proof is the same as that of Theorem 2, by applying Theorem III.6 of [9] instead of Theorem III.3. Indeed, it follows easily from uniqueness that, for initial data of the form (17), the corresponding solution of (5) has the same form.

Remark 6. Theorems 2 and 3 still hold for functions \( V \) and \( g \) much more general than those satisfying (16). Actually, they hold as soon as \( V \) and \( g \) are such that \( \Sigma = 0 \) and all minimizing sequences for \( I_{V,G} \) (respectively (18)) are relatively compact in \( \mathcal{H}_{1}^{1} \) (respectively \( L^{q}(\mathbb{R}^{3}), 2 \leq q < \infty \)). We do not intend to give here the most general assumptions under which this happens. Let us only give a hint of how to proceed in order to determine whether or not the conclusions of Theorem 2 and 3 hold for some given \( V \) and \( g \). More complete results can be found in [9]. Let us consider the non-symmetric case, that is Theorem 2.

Assume that \( V = a + V_{o} \), where \( a \in \mathbb{R}, a > |b| \) and \( V_{o} \in L^{r}(\mathbb{R}^{3}) + (L^{s}(\mathbb{R}^{3}) \cap L^{1}(\mathbb{R}^{3})) \), \( 3/2 < r, s < \infty \) and also assume that \( g \) satisfies (8)-(9) and

\[
\lim \sup_{t \to 0} t^{-1/3} G(x,t) = 0; \\
\lim \sup_{t \to 0} t^{-2} G(x,t) = 0; \\
G(\cdot, t) \in C(\mathbb{R}^{3}, \mathbb{R}) \quad \text{for all } t \in \mathbb{R}; \\
G(t) = \lim_{|x| \to \infty} G(x,t) \text{ exists}; \\
\lim_{|t| \to 0} t^{-1/3} G(t) = +\infty.
\]

Then, as it is proved in [9], the strict inequality

\[
I_{V,G} < I_{a,G}
\]

is a sufficient condition for the relative compactness \( \mathcal{H}_{1}^{1} \) of all minimizing sequences of \( I_{V,G} \) and, in particular, there is a minimum for \( I_{V,G} \), i.e. \( \Sigma = 0 \). Moreover, in some cases we can easily decide whether or not (32) holds. For instance, if we assume that \( V_{o} \leq 0 \) a.e., \( V_{o} \neq 0 \) and \( G(s,t) \geq G(t) \) for all \( t \), then
(32) holds. On the other hand, let us remark that (32) is only a sufficient condition. So, for the case \( V_0 = 0 \) a.e. and \( g \) independent of \( x \), (32) does not hold, since then \( I_{V_0} G = I_{A} G \). Nevertheless, in such a case, all minimizing sequences are still relatively compact in \( H^1_A \) and \( \Sigma \neq 0 \).

Remark 7. There are examples which show that, as in the case \( A = 0 \), the stability property that we have proved in Theorems 2 and 3 can not be improved. More precisely, in order to have stability, one has to take into account the changes of gauge and the translations. For instance, one can easily see that, if \( u \) is a solution of (15), and if we consider the functions

\[
\phi_n(t, x) = \exp(i\lambda_n(t, x) + i\omega t) u(x + b_n(t)),
\]

with

\[
\lambda_n(t, x) = (1/n)(x_1 \sin(bt) + x_2 \cos(bt) + x_3) + 2t/n^2,
\]

\[
b_n(t) = (2/nb)(-\cos(bt), \sin(bt), bt),
\]

then they are solutions of equation (1) relative to the initial conditions

\[
\phi_n(x) = \exp((i/n)(x_2 + x_3)) u(x_1 - 2b/n, x_2, x_3).
\]

Moreover, one can see that \( \phi_n \rightarrow u \) in \( H^1_A \) as \( n \rightarrow \infty \). However the functions \( \phi_n(t, x) \) are not globally in time as close as we want from the set \( \{e^{iu(t)}u | u \text{ solution of (14)}\} \). Finally let us remark that, as the above example shows, the presence of a nonnull external magnetic field induces, as \( t \) increases, phenomena of rotation and translation which involve some kind of oscillation.

REFERENCES

Stability of nonlinear Schrödinger equations