From Gross-Pitaevskii equation to Euler Korteweg system, existence of global strong solutions with small irrotational initial data

Corentin Audiard [*,†] and Boris Haspot [‡]

Abstract

We consider the Euler equations with a capillary tensor in the case of the so-called quantum hydrodynamics, which is formally equivalent to the Gross-Pitaevskii equation. Our main result is the existence and uniqueness of global solutions without vortices for small data in dimension at least 3. The absence of vortices means that the density remains bounded away from 0. Previous results include existence of global solutions without uniqueness (Antonelli-Marcati) and lower bounds on the first occurrence of vortices (Béthuel-Danchin-Smets) Our proof uses in a crucial way some deep results on the scattering of the Gross-Pitaevskii equation due to Gustafson, Nakanishi and Tsai. The optimality of our assumptions is also discussed, in particular we show that for less regular initial data the density is not bounded. For the convenience of the reader, we also sketch in the appendix the key arguments for the scattering of solutions of the Gross-Pitaevskii equation.

Contents

[*]Sorbonne Universités, UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France

[†]CNRS, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France

[‡]Ceremade, Umr Cnrs 7534, Université Paris Dauphine, place du Maréchal De Lattre De Tassigny 75775 Paris cedex 16 (France), haspot@ceremade.dauphine.fr

1 Introduction

1.1 The Euler-Korteweg equations and the Madelung transform

The motion of an Euler-Korteweg compressible fluid is described by the following system:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla P(\rho) = \mathrm{div} K, \\ (\rho, u)_{/t=0} = (\rho_0, u_0). \end{cases} \tag{1.1}$$

Here $u = u(t, x) \in \mathbb{R}^N$ stands for the velocity field, $\rho = \rho(t, x) \in \mathbb{R}^+$ is the density and P the pressure. We restrict ourselves to the case $N \geq 3$. The general Korteweg tensor reads as follows:

$$\mathrm{div} K = \mathrm{div}\Big(\big(\rho\kappa(\rho)\Delta\rho + \frac{1}{2}(\kappa(\rho) + \rho\kappa'(\rho))|\nabla\rho|^2\big) Id - \kappa(\rho)\nabla\rho \otimes \nabla\rho \Big). \tag{1.2}$$

The capillary coefficient κ is a smooth function $\mathbb{R}^{+*} \to \mathbb{R}^{+*}$. The Euler-Korteweg system has been studied by Benzoni, Danchin and Descombes in [5] where they prove for a general capillary coefficient the local existence of strong solutions for large data such that $(\rho_0 - 1, u_0)$ belong to $H^{s+1}(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$ with $s > \frac{N}{2} + 1$. The proof relies on tricky energy inequalities, but the lack of uniform bounds does not allow to obtain global solutions. On the other hand, the equation has some dispersive structure so that global well-posedness is expectable at least if the dimension is large enough and the initial data small enough. Throughout the paper, we denote the space variable $x \in \mathbb{R}^N$ and we shall deal with the specific case:

$$\kappa(\rho) = \frac{\kappa_1}{\rho} \quad \text{so that} \quad \mathrm{div} K = 2\kappa_1 \rho \nabla\Big(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\Big), \ \kappa_1 \in \mathbb{R}^{+*}, P(\rho) = \rho^2/2.$$

This capillary coefficient corresponds to the so called quantum pressure. This case is of special interest because it corresponds to the fluid formulation of the Gross-Pitaevskii equation. More precisely, when the velocity $u = \nabla\theta$ is irrotational, $\lim_{|x| \to +\infty} \rho = 1$, the Madelung transform $\psi = \sqrt{\rho} e^{i\frac{\theta}{2\sqrt{\kappa_1}}}$ allows formally to rewrite the Euler-Korteweg system as the Gross-Pitaevski equation [1] (GP):

$$\begin{cases} 2i\sqrt{\kappa_1}\partial_t\psi + 2\kappa_1\Delta\psi = (|\psi|^2-1)\psi, \\ \psi(0,\cdot) = \psi_0. \end{cases} \tag{1.3}$$

with the boundary condition $\lim_{|x|\to+\infty}|\psi| = 1$. The Gross-Pitaevskii equation is the Hamiltonian evolution associated to the Ginzburg-Landau energy:

$$\begin{aligned} E(\psi) &= \int_{\mathbb{R}^N} \big(\kappa_1|\nabla\psi(t,x)|^2 + \frac{1}{4}(|\psi|^2-1)^2\big)dx \\ &= \int_{\mathbb{R}^N} \big(\kappa_1|\nabla\varphi(t,x)|^2 + \frac{1}{4}(2Re\varphi + |\varphi|^2)^2\big)dx. \end{aligned} \tag{1.4}$$

with $\psi = 1+\varphi$. Note that the fluid counterpart of this energy can be obtained by multiplying the momentum equation by $2u$ and reads

$$\begin{aligned} &\int_{\mathbb{R}^N} \big(4\kappa_1|\nabla\sqrt{\rho}|^2(t,x) + (\rho|u|^2)(t,x) + (\rho-1)^2(t,x)\big)dx \\ &\qquad = \int_{\mathbb{R}^N} \big(4\kappa_1|\nabla\sqrt{\rho_0}|^2(x) + (\rho_0|u_0|^2)(x) + (\rho_0-1)^2(x)\big)dx. \end{aligned} \tag{1.5}$$

Taking advantage of this correspondence, Antonelli and Marcati proved in [1, 2] the existence of global weak solution for the system (1.1) for irrotational initial data when $N = 2,3$ and for pressures that correspond to defocusing nonlinear Schrödinger equations (NLS) (see also [11] for a simpler argument). It is important to mention that in [1, 2] the authors deal with initial density which are close to the vacuum, indeed ρ_0 belongs to $L^2(\mathbb{R}^N)$. The proofs consist in constructing a sequence of global smooth solutions of the system (1.1) (for regularized initial data). The main difficulty to pass from a solution of NLS to a solution of (1.1) is that ψ can vanish, so that $u = \mathrm{Im}(\overline{\psi}\nabla\psi/|\psi|^2)$ is not clearly defined, even as a distribution. Next they prove the convergence to a global weak solution solution of the system (1.1). The key point of the proof is the strong L^2_{loc} convergence of the nonlinear terms $\sqrt{\rho_n}u_n \otimes \sqrt{\rho_n}u_n$ and $|\nabla\sqrt{\rho_n}|^2$. This terms are in fact intertwined and converge using classical regularizing effects of Kato type for the Schrödinger equation. Uniqueness was left open as no control of the vacuum was provided.

On the other hand, Béthuel, Danchin and Smets studied in [6] the Gross-Pitaevskii equation in the long wave regime (small data and slow oscillations), and proved the well-posedness of 1.1 for large times. More precisely, they prove that for such times the density remains bounded away from zero, which in this context corresponds to the absence of vortices. It relied in a crucial way on dispersive properties of the Schrödinger equation. However the question of global well-posedness was left open.

The main novelty of our results is that we construct solutions that are both unique, without vortices, and global. The price to pay is that we need to take small initial data and the dimension $N \geq 3$ in order to fully benefit of dispersive effects. Let us mention that we obtained very recently in [3] the global well-posedness of (1.1) for small initial data and general capillarity and pressure by a direct approach. In particular the Madelung transform is not used. The drawback is that much stronger restrictions on the regularity of the initial data are required (basically $\rho_0 \in 1 + H^{50}$).

[1] It should be pointed out that for a general capillarity the Euler-Korteweg system can also be rewritten as some degenerate quasi linear Schrödinger equation, see [5]. The change of variable does not involve the Madelung transform but is still singular near vacuum.

In this work we loosen the assumptions on the initial data : first we build upon the scattering results for (GP) (see [27, 28, 29]) to construct a global solution ψ that remains bounded away from 0 but is merely in $L^\infty \cap H^s$ with $s \simeq N/2$, second we use the Madelung transform (which is well-defined since ψ is bounded away from 0) to construct a global strong solution of the system (1.1). Uniqueness requires $(\rho_0, u_0) \in (1 + H^{s+1}) \times H^s,\ s > N/2 + 1$.
Before stating our main results we give a (incomplete) review on scattering for NLS and Gross-Pitaevskii.

1.2 On the Gross Pitaevskii equation

Global well-posedness and solitons : Due to the unusual boundary condition at infinity, the analysis of the Cauchy problem for the Gross-Pitaevskii equation is more involved than for a defocusing NLS. Up to a change of variable and for simplicity we will take in (1.3) $2\sqrt{\kappa_1} = 2\kappa_1 = 1$. The natural energy space is not $H^1(\mathbb{R}^N)$, and the $L^2(\mathbb{R}^N)$ norm is not conserved (we will see that it is related to the low frequencies behavior of the linearized equation near $\psi = 1$). The natural energy space associated to the Gross-Pitaevskii equation is

$$E_1 = \{\psi \in H^1_{loc}(\mathbb{R}^N),\ \nabla\psi \in L^2(\mathbb{R}^N),\ |\psi|^2 - 1 \in L^2(\mathbb{R}^N)\}.$$

Global well-posedness with large initial data in E_1 has been proved by C. Gallo and P. Gérard in [17, 18] in dimension $N \leq 3$ and by Kilipp et al in [35] in the critical case $N = 4$. It was also proved that for $s \geq 1$ the $H^s(\mathbb{R}^N)$ regularity is also propagated but without uniform bounds in time.
A striking difference between (GP) and defocusing Schrödinger equations is the existence of traveling waves, namely solutions of the form (up to symmetry):

$$\psi(t, x) = u_c(x_1 - ct, x_2, \cdots, x_N),$$

where u_c satisfies:

$$ic\partial_1 u_c - \Delta u_c - u_c(1 - |u_c|^2) = 0. \tag{1.6}$$

For $N \geq 2$, due to the correspondence with the Euler equations, it was conjectured more than thirty years ago that non constant solutions do not exists for $|c| > \sqrt{2}$, this was rigorously proved in [24]). Solutions of finite energy were constructed for small c in the pioneering paper [8], and the full range $0 < |c| < \sqrt{2}$ was obtained by Maris in [36] in dimension $N \geq 3$. Béthuel and al proved in [7] that there is a lower bound on the energy of non trivial traveling waves for (1.8) in dimension $N = 3$, the result was then extended in any dimension ≥ 4 by de Laire [16]

$$0 < \mathcal{E}_0 = \inf\{E(\psi), \psi(t, x) = u_c(x_1 - ct, x_2, \cdots, x_N) \text{ solves (1.8) for } c > 0\}. \tag{1.7}$$

On the other hand if $N = 2$ there exists non trivial traveling waves of arbitrary small energy (this it was conjectured by Jones et al [33], see [7] for a proof). This is a clear obstruction to scattering.

The scattering problem : We rewrite (1.3) for $\varphi = \psi - 1$:

$$i\partial_t\varphi + \Delta\varphi - 2Re\varphi = F(\varphi) = (\varphi + 2\bar{\varphi} + |\varphi|^2)\varphi. \tag{1.8}$$

The strongest nonlinearity $|\varphi|^2\varphi$ corresponds to the defocusing cubic nonlinear Schrödinger equation, but the dynamic is actually very different. The linearized system reads

$$i\partial_t\varphi + \Delta\varphi - 2Re\varphi = 0. \tag{1.9}$$