Shape optimization for the heat propagation and other engineering problems

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March 10, 2021

1 Framework

The mathematical theory of shape optimization is a developing area of applied mathematics. Its progress allows to solve or give some ideas on how to solve different problems of a real engineer application connected with an object’s shape. For instance, what is an optimal shape to dissipate the acoustical ways [1], what is an optimal shape of a most resistant and stable roof or a bridge [2 3]. For the introduction to shape optimization, we advise to read these two books [4 5]. Generally, the shape optimization problems are ill-posed: if it is possible to prove the existence of an optimal shape, there are no unicity of it, and sometimes there are infinitely many optimal shapes. This non-unicity question is mainly related to the possibility for different shapes to have the same eigenvalues of the corresponding operators as the Laplacian operator [6] or the Dirichlet-to-Neumann operator [7 8 9]. In addition, it is crucial to find the admissible class of domains (or boundary shapes) on which we minimize the energy or other objective functional depending on the solution of a considered boundary valued PDE problem. Partially the main constraints to impose are related to the engineer restrictions of the construction or the physical meaning of the problem. The main goal is to work on a compact set and to have uniform a priori estimates of solutions on this set of domains independing on their shapes.

Recent and current results of our team [10 3 11 1 12 13] in the area of shape optimization allow considering a lot of new problems, for instance, the problem to find an optimal shape which allows having a maximal speed of heat exchange between two medium. Typically, there are two different media separated by a high conductive boundary layer. In one of the mediums, there is a fixed heat source with compact support. The question is which geometry is optimal to maximize the heat propagation speed inside the initially cold medium during the shortest time. Understanding this problem is relevant to improve heat exchanges, e.g., cooling of metallic radiators or thermal
isolation of pipes and buildings. Depending on the application, the cooling rate has to be either enhanced (e.g., in the case of microprocessors or nuclear reactors), or slowed down (e.g., in the case of pipes and buildings).

Partially this question is related to the length of the perimeter of the boundary or even on its dimension following [14, 15, 16]. As a fractal boundary has a bigger dimension than the regular boundary, any fractal boundary is expected to be more efficient than the regular one.

The recent advances in the theory of PDEs on fractals [17, 18, 19] allow to treat not only the shape optimization in the regular class of boundary with a finite uniform length but also in a class of fractals with any dimension $d$ ($n - 1 \leq d < n$ for $n$ the dimension of the domain).

2 Ph.D. Thesis topic

This work is in collaboration with Prof. M. R. Lancia, University Roma 1 SAPIENZA, Italy. Thus, this Ph.D is planned to be joint with University Sapienza (thèse en cotutelle), where the candidate will spend 12 months. This will allow the candidate to obtain two Ph.D. diplomas from University Paris-Saclay and Sapienza.

We are mainly interested in considering a shape optimization problem in the framework of the heat propagation described by the Venttsel problem, for which there exists the well-posedness theory in the fractal framework too [20, 21]. The physical derivation of the model is given in [22].

We start by considering the shape optimization for the stationary problem. Let us introduce it to fix the ideas.

We consider a bidimensional rectangular domain $\Omega$, divided in two subsets $\Omega_1$ and $\Omega_2$ by an interface $K^0$. The shape of $K^0$ can change in a compact subset $G$ of $\Omega$, but $K^0$ any time has endpoints $A = (0, 0)$ and $B = (1, 0)$.

Actually, we have

$$\Omega = \Omega_1 \cup \Omega_2 \cup K^0.$$  

\[
(P) \begin{cases}
-\Delta u_i = f & \text{in } \Omega_i^0, \\
-\Delta_{K^0} u = [\partial u / \partial n] & \text{on } K^0, \\
u_1 = u_2 & \text{on } K^0, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]  

(1)

where $u_i$ denotes the restriction of $u$ to $\Omega_i^0$ and $[\partial u / \partial n]$ denotes the jump of the normal derivative across $K^0$.

The first task is to pose the shape optimization problem correctly for a fixed $f$ such that it would be possible to prove the existence of an optimal shape $K_{opt}$ of a finite length (by the usual Lebesgue measure), which minimize on a suitable set of shapes the energy in a subdomain (the volume of
which is kept constant for all class of admissible shapes)

\[ J(\Omega_1, u(\Omega)) = A \int_{\Omega_1} |\nabla u|^2 \, dx + B \int_{K^1} |\nabla_t u|^2 \, ds. \]  \hspace{1cm} (2)

Here \( \nabla_t u \) is the tangential gradient on the boundary \( K^1 \). The same question is open in the class of fractals: does exist an optimal fractal?

The next questions are how to derive the energy functional (2) over a regular shape, is it possible to do the same for the Lipschitz boundaries, what happens in the irregular fractal case.

There is a numerical part to be developed to visualize the optimal shapes and perform the numerical analysis of the problem. This part of the work will be done taking into account the level of computer skills of the candidate under the supervision of Prof. F. Magoulès.

Once the steady model is understood, there are the same questions for the heat propagation in time. Further developments could also be related to other types of boundary conditions, possibly involving non-local operators.

As the area of shape optimization problems is huge on developments of different applications, further studies could also consider other applications (fluid mechanics, architecture problems for elastic materials, ultrasounds, obstacle problems, or others).
3 Skills and Profile

A strong background in partial differential equations is required. Skills in programming would be a plus.

4 Contact

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5 Place of work

CentraleSupélec, Université Paris Saclay/ École Doctorale Jaques Hadamard (2 years) and Spaienza, Roma, Italy (1 year).

References


