# Master 2 Internships
# Algorithms in real algebraic geometry under composition

We are seeking top candidates willing to prepare a PhD at the interface of computational mathematics, algebra, geometry and their application areas.

**Scientific context and positioning.** Real algebraic geometry is the area of mathematics which studies properties of real solution sets to polynomial systems of equations and inequalities with real coefficients and properties of maps between such sets. Algorithms in real algebraic geometry are those algorithms which allow one to *compute* with and over such sets, e.g. to *decide* if a given polynomial system is consistent over the reals, to compute sample points per connected components and count the number of those components, answer connectivity queries, etc.

Such algorithms are developed at least since Fourier and tremendous progress has been brought to the state-of-the-art since the last 30 years, both from the viewpoint of complexity theory and from the practical side with the raise of some efficient software. Such algorithms are now used in areas of engineering sciences such as robotics [6] and biology [5], to cite a few: polynomial system solving over the reals appears in many applications.

Still, most algorithmic problems which arise in this area have complexity exponential in the dimension of the ambient space (but polynomial in the maximum degree of the input polynomial constraints) [1]. Hence, taking advantage of any structural property (of geometric or algebraic nature) to accelerate algorithms of real algebraic geometry is of first importance. This internship project aims at paving the way to faster algorithms solving over the reals polynomial systems whose entries are explicit compositions of polynomials.

**Objectives of the internship.** We will study first systems of polynomial equations

$$f_1 = \ldots = f_p = 0, \quad f_i \in \mathbb{R}[x_1, \ldots, x_n]$$

where $f_i = g_i \circ h_i$ with $g_i \in \mathbb{R}[y_1, \ldots, y_t]$ and $h_i \in \mathbb{R}[x_1, \ldots, x_n]$. We assume that the $g_i$'s and the $h_i$'s are known to the user. We let $V_{\mathbb{R}}$ be the set of solutions of this system in $\mathbb{R}^n$.

Such polynomial systems arise typically in robotics where systems encoding the position of the end-effector of a serial robot are obtained thanks to composition formulæ encoding the position of the end-effector of each arm of the robot.

We will study algorithms for computing at least one point per connected component in $V_{\mathbb{R}}$ with the goal to take advantage of the knowledge of the $g_i$'s and the $h_i$'s. Of course, the key parameters are $t$, $n$, $\gamma = \max_i(\deg(g_i))$ and $\eta = \max_i(\deg(h_i))$

and $\delta = \max_i(\deg(f_i))$. The state-of-the-art *general* algorithms are running in cubic time in $\delta^n$. The goal of this internship is to identify for which situations, one could obtain algorithms running in time subexponential in $n$.

A key algorithmic ingredient for algorithms computing sample points per connected component of a real algebraic set is the so-called critical point method (see e.g. [7]). It consists in identifying a polynomial map which reaches its extrema on each connected component at finitely many points (which are then critical points of the considered map). Then, it remains to compute them using computer algebra techniques, such as Gröbner bases [3, 4], to solve polynomial systems defining these critical points. Such systems contain the input polynomial $f_i$'s and encode some rank deficiency of some Jacobian matrices.

Hence, in our context, the problem is reduced first to the analysis of the critical point computation which we should perform with a better complexity, an extreme case being the situation where exponential algorithms can be replaced with ones running in polynomial time.

Such a breakthrough has already been obtained recently in some highly structured contexts [2]. For instance, when all entries $f_i$ are invariant under the action of the symmetric group, they can be written $g_i \circ h_i$ where $h_i$ is the $i$-th elementary symmetric polynomial. Such a composition formula then leads to a natural matrix factorization of the Jacobian matrices we consider.

In our more general context, a similar phenomenon arises but of course, controlling the complexity in this more general context is more challenging.

**Scientific environment.** This internship will be co-supervised by Vincent Neiger[1] and Mohab Safey El Din[2]. It will take place at Sorbonne University, in the computer science lab LIP6, which is located on the Pierre and Marie Curie campus, at the heart of Paris. The intern will be welcome in the PolSys team which develops and implements fast computer algebra algorithms for polynomial system solving and their applications. The intern will work in a kind and international environment gathering PhD students and Post-Docs representing 6 nationalities and animated with several working groups and a monthly seminar. All computing facilities will be provided.

**How to apply.** All interested applicants should send a full CV, a letter of motivation and the grades obtained during the last two years to

vincent.neiger@lip6.fr and mohab.safey@lip6.fr .

# References


[1] S. Basu, R. Pollack, and M.-F. Roy. *Algorithms in real algebraic geometry*, volume 10 of *Algorithms and Computation in Mathematics*. Springer, 2006. 2nd ed.

[2] J.-C. Faugère, G. Labahn, M. Safey El Din, É. Schost, and T. X. Vu. Computing critical points for invariant algebraic systems. working paper or preprint, Sept. 2020.

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[1] Sorbonne Université, vincent.neiger@lip6.fr

[2] Sorbonne Université, mohab.safey@lip6.fr

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[7] M. Safey El Din and E. Schost. Polar varieties and computation of one point in each connected component of a smooth real algebraic set. In *Proceedings of the 2003 international symposium on Symbolic and algebraic computation*, pages 224–231, 2003.