Growth, reaction, movement and diffusion from biology

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Chapter 1

Relaxation, perturbation and entropy methods

Solutions to nonlinear parabolic equations and systems can exhibit various and sometimes complex behaviors. This course aims at describing some of them in relation to problems arising from modeling in biology. In which circumstances can such complex behaviors happen? A first answer is given in this chapter by indicating some conditions for relaxation to trivial steady states; then nothing interesting can happen!

We present examples of a first class of equations we have in mind and why they arise in biology. These are 'reaction-diffusion' equations, or in mathematical classification, semilinear equations. Another class, the Fokker-Planck equations will be treated later on. Then, we present relaxation results by perturbation methods (small nonlinearity), and entropy methods, two cases where we cannot hope anything spectacular.

For more biologically oriented textbooks, the reader can also consult [31, 30]; for mathematical tools, he can consult [13, 14].

1.1 Lotka-Volterra systems

A first class arises in the area of population biology, and ecological interactions, and is characterized by birth and death. For \(1 \leq i \leq I\), let \((n_i(t, x))_{1 \leq i \leq I}\) denote the population densities of \(I\) species at the location \(x \in \mathbb{R}^d\) (\(d = 2\) for instance). We assume that these species move randomly according to brownian motions and that they have growth (meaning birth and death rates) \(R_i(t, x)\). Then, the system describing these densities is

\[
\frac{\partial}{\partial t} n_i - D_i \Delta n_i = n_i R_i, \quad t \geq 0, \quad x \in \mathbb{R}^d, \quad i = 1, 2, ..., I. \tag{1.1}
\]
In a first approximation, the diffusion coefficients $D_i > 0$ are constants depending on the species, but the birth and death rates depend on the interactions between species, we express this saying that

$$R_i(t, x) = R_i(n_1(t, x), n_2(t, x), ..., n_I(t, x)).$$

In models of ecology as well as in evolution (in the sense of Darwin) one also finds non-local interactions which means that $R_i(t, x)$ depends upon all the values $n_j(t, y)$, for all $y$’s, not only on $x$. This we denote as

$$R_i(t, x) = R_i(t, x; [n_1(t)], [n_2(t)], ..., [n_I(t)]).$$

Examples are studied in Sections 3.7.2, (3.6) and 6.3.

A classical family is given by the quadratic interactions

$$R_i(n_1, n_2, ..., n_I) = r_i + \sum_{j=1}^I c_{ij} n_j,$$

with $r_i$ the intrinsic growth rate of the species $i$ (it can be positive or negative) and $c_{ij}$ the interaction effect of species $j$ on species $i$. The coefficients $c_{ij}$ are not necessarily symmetric neither non-negative. One can distinguish

- $c_{ij} < 0$, $c_{ji} > 0$: species $i$ is a prey, species $j$ is a predator,
- $c_{ij} > 0$, $c_{ji} > 0$: cooperative interaction (both species help the other and benefit from it),
- $c_{ij} < 0$, $c_{ji} < 0$: direct competition (both species compete for instance for the same food).

The quadratic aspect relates to the necessity of binary encounters for the interaction to occur. Better models include saturation effects. An example is to use

$$R_i(n_1, n_2, ..., n_I) = r_i + \sum_{j=1}^I c_{ij} \frac{n_j}{1 + n_i}.$$

The original prey-predator model of Volterra has two species, $I = 2$ ($i = 1$ the prey, $i = 2$ the predator). The prey (small fishes) can feed on abundant zooplankton and thus $r_1 > 0$, while predators (sharks) will die out without small fishes to eat ($r_2 < 0$). The sharks eat small fishes proportionally to the number of sharks ($c_{12} < 0$), while the shark population grows proportionally to the small fishes they can eat ($c_{21} > 0$). Therefore we find the rule

$$r_1 > 0, \quad r_2 < 0, \quad c_{11} = c_{22} = 0, \quad c_{12} < 0, \quad c_{21} > 0.$$

But it is not always true that the signs of $c_{ij}$ and $c_{ji}$ are opposite and cooperative species benefit from each other, then $c_{ij} > 0$ and $c_{ji} > 0$. Also intraspecific competition is usually
assumed, i.e., $c_{ii} < 0$.

In all generality, solutions to the Lotka-Volterra system (1.1) satisfy very few qualitative properties; there are no a priori conservation laws (because they contain birth and death), neither entropy properties (which does not seem to relevant in ecological systems). As we will see the quadratic aspect may lead to blow-up (solutions exists only for a finite time) Let us only mention that it is consistent with the property that population density be nonnegative.

**Lemma 1.1** Assume that the initial data $n_i^0$ are nonnegative functions of $L^2(\mathbb{R}^d)$ and that there is a locally bounded function $\Gamma(t)$ such that $|R_i(t, x)| \leq \Gamma(t)$. Then, the weak solutions in $C(\mathbb{R}^+; L^2(\mathbb{R}^d))$ to the Lotka-Volterra system (1.1) satisfy $n_i(t, x) \geq 0$.

The definition and usual properties of weak solutions are given in Chapter 2, but we do not need that to show the manipulations leading to this result.

**Proof.** (Formal) We follow the method of Stampacchia. Set $p_i = -n_i$, we still have

$$\frac{\partial}{\partial t} p_i - D_i \Delta p_i = p_i R_i.$$ 

Multiply by $(p_i)_+ := \max(0, p_i)$ and integrate by parts. We have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} (p_i(t, x))_+^2 \, dx + D_i \int_{\mathbb{R}^d} |\nabla (p_i)_+|^2 = \int_{\mathbb{R}^d} (p_i(t, x))_+^2 R_i$$

and thus

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} (p_i(t, x))_+^2 \, dx \leq \Gamma(t) \int_{\mathbb{R}^d} (p_i(t, x))_+^2.$$ 

Therefore, we have

$$\int_{\mathbb{R}^d} (p_i(t, x))_+^2 \, dx \leq e^{2 \int_0^t \Gamma(s) \, ds} \int_{\mathbb{R}^d} (p_0^i(x))_+^2 \, dx.$$ 

But, the assumption $n_i^0 \geq 0$ implies that $\int_{\mathbb{R}^d} (p_0^i(x))_+^2 \, dx = 0$, and thus, for all times we have $\int_{\mathbb{R}^d} (p_i(t))_+^2 \, dx = 0$, which means $n_i(t, x) \geq 0$.

(Rigorous) See Section 2.7.

**Exercise.** In the context of the Lemma 1.1, show that

$$\frac{d}{dt} \int_{\mathbb{R}^d} n_i(t, x) \, dx = \int_{\mathbb{R}^d} R_i(t, x) n_i(t, x) \, dx,$$

$$\int_{\mathbb{R}^d} n_i(t, x)^2 \, dx \leq \int_{\mathbb{R}^d} (n_i^0(x))^2 \, dx \, e^{2 \int_0^t \Gamma(s) \, ds}.$$ 

(Left to the reader; see the proof of the Lemma 1.1).
1.2 Reaction kinetics and entropy

Biochemical reactions lead to a different structure in the right hand sides of semi-linear parabolic equations. They are written

$$\frac{\partial}{\partial t} n_i - D_i \Delta n_i + n_i L_i = G_i, \quad t \geq 0, \quad x \in \mathbb{R}^d, \quad i = 1, 2, ..., I.$$  \hspace{1cm} (1.3)

The quantities $n_i$ are molecular concentrations and the loss terms $L_i$ depend on all the molecules $n_j$ (with which $n_i$ can react) and the gain terms $G_i$, depending on the $n_j$’s also, denote the rates of production of $n_i$ from the other molecules. The main property is the conservation of atoms, which asserts that some quantity should be constant in time. Notice that $D_i$ represents the molecular diffusion of these molecules according to Einstein’s rules. Usually $L_i$ and $G_i$ take the form

$$L_i = r_i \prod n_j^{p_{ij}}, \quad G_i = r'_i \prod n_j^{q_{ij}}.$$ 

Also, in principle such systems are conservative

$$\sum_{i=1}^I [n_i L_i - G_i] = 0, \quad \forall (n_i) \in \mathbb{R}^I.$$  \hspace{1cm} (1.4)

This implies that

$$\frac{\partial}{\partial t} \sum_{i=1}^I n_i = \Delta \sum_{i=1}^I D_i n_i,$$

$$\frac{d}{dt} \int_{\mathbb{R}^d} \sum_{i=1}^I n_i(t, x)dx = 0,$$

and thus

$$\int_{\mathbb{R}^d} \sum_{i=1}^I n_i(t, x)dx = \int_{\mathbb{R}^d} \sum_{i=1}^I n_i^0(x)dx \quad \text{(mass conservation).}$$ \hspace{1cm} (1.5)

As for the Lotka-Volterra systems, nonnegativity holds true

**Lemma 1.2** A weak solution to (1.3) with $G_i \geq 0$ and nonnegative initial data satisfies $n_i \geq 0$, $\forall i = 1, 2, ..., I$.

The rationale behind this lemma is that because we expect the concentrations to be nonnegative, $n_j \geq 0$, then $G_i = r_i \prod n_j^{p_{ij}} \geq 0$. Details can be found in Chapter 2.

We choose the example of the standard degradation reaction of dioxygen in monoxygen. It is usually associated with hyperbolic models for fluid flows because it is a very energetic reaction occurring at very high temperature with reaction rates depending critically on this temperature.
But for our purpose here, we forget these limitations and consider the dissociation rate $k_1$ of $n_1 = [O_2]$ in $n_2 = [O]$, and reversely its recombination with rate $k_2$. This lead to

$$
\begin{cases}
\frac{\partial}{\partial t} n_1 - D_1 \Delta n_1 + k_1 n_1 = k_2(n_2)^2, \\
\frac{\partial}{\partial t} n_2 - D_2 \Delta n_2 + 2k_2(n_2)^2 = 2k_1 n_1,
\end{cases}
$$

(1.6)

with initial data $n_1^0 \geq 0$, $n_2^0 \geq 0$. According to the law of mass action, the term $(n_2)^2$ arises because the encounter of two atoms of monoxygen are needed for the reaction.

We derive the conservation law (number of atoms is constant) by a combination of the equations

$$
\frac{\partial}{\partial t} [2n_1 + n_2] - \Delta [2D_1 n_1 + D_2 n_2] = 0,
$$

which implies that for all $t \geq 0$

$$
\int_{\mathbb{R}^d} [2n_1(t, x) + n_2(t, x)] dx = M := \int_{\mathbb{R}^d} [2n_1^0(x) + n_2^0(x)] dx.
$$

(1.7)

**Exercise.** Deduce from (1.7) and $n_1 \geq 0$, $n_2 \geq 0$ the a priori bound

$$
2k_2 \int_0^T \int_{\Omega} (n_2)^2 dx dt \leq k_1 MT.
$$

[Hint: In (1.6), integrate the equation for $n_2$.]

For the simple case of the reaction (1.6), the entropy is defined by

$$
S(t, x) = n_1 \left[ \ln(k_1 n_1) - 1 \right] + n_2 \left[ \ln(k_2^{1/2} n_2) - 1 \right].
$$

(1.8)

One can readily check that

**Lemma 1.3 (Entropy inequality)**

$$
\frac{d}{dt} \int_{\mathbb{R}^d} S(t, x) dx = -\int_{\mathbb{R}^d} [D_1 \frac{\nabla n_1}{n_1} + D_2 \frac{\nabla n_2}{n_2}] dx
$$

$$
-\int_{\mathbb{R}^d} \left[ \ln(k_2 n_2^2) - \ln(k_1 n_1) \right] [k_2(n_2)^2 - k_1 n_1] dx \leq 0.
$$

From a mathematical point of view, the entropy inequality helps the analysis of this kind of systems because it gives a priori bounds on $S(t, x)$ in $L^1_{\log}$ (thus possibly a better bound than $L^1$).
Exercise. Another example is the reversible reaction $A + B \rightarrow C + D$

\[
\begin{align*}
\frac{\partial}{\partial t} n_1 - D_1 \Delta n_1 + k_1 k_2 n_1 n_2 &= k_3 k_4 n_3 n_4, \\
\frac{\partial}{\partial t} n_2 - D_2 \Delta n_2 + k_1 k_2 n_1 n_2 &= k_3 k_4 n_3 n_4, \\
\frac{\partial}{\partial t} n_3 - D_3 \Delta n_3 + k_3 k_4 n_3 n_4 &= k_1 k_2 n_1 n_2, \\
\frac{\partial}{\partial t} n_4 - D_4 \Delta n_3 + k_3 k_4 n_3 n_4 &= k_1 k_2 n_1 n_2.
\end{align*}
\]

(1.9)

with the $n_i \geq 0$ and the mass conservation law

\[
\int_{\mathbb{R}^d} [n_1(t,x) + n_2(t,x) + n_3(t,x) + n_4(t,x)] dx = M := \int_{\mathbb{R}^d} [n_1^0(x) + n_2^0(x) + n_3^0(x) + n_4^0(x)] dx.
\]

Find the entropy inequality for the chemical reaction (1.9).

For more elaborate chemical reactions, the detailed description of all elementary reactions is unrealistic. Then one simplifies the system by assuming that some reactions are much faster than others, or that some components are in high concentrations. These manipulations may violate the mass conservation and entropy inequality which may be lost. For instance, the famous Belousov-Zhabotinskii reaction is known to produce periodic patterns (discovered in 1951 by Belousov, it remained unpublished because no respectable chemist in that time could accept this idea. Belousov received the Lenin Prize in 1980, a decade after his death). Other examples, more representative of biology are the enzymatic reactions, which are usually associated with the names of Michaelis and Menten\(^1\).

### 1.3 Boundary conditions

When working in a domain (connected open set with smooth enough boundary) $\Omega$ we encounter two natural types of boundary conditions. The reaction-diffusion systems (1.1) or (1.3) are completed either by

- Dirichlet boundary conditions $n_i = 0$ on $\partial \Omega$. This means that individuals or molecules go across the boundary and do not come again in $\Omega$. This interpretation stems from the brownian motion underlying the diffusion equation. But we can see that indeed, if we consider the conservative chemical reactions (1.3) with (1.4), then

\[
\frac{d}{dt} \int_\Omega \sum_{i=1}^{l} n_i(t,x) dx = \sum_{i=1}^{l} D_i \int_{\partial \Omega} \frac{\partial}{\partial \nu} n_i,
\]

\(^1\)Michaelis, L. and Menten, M. I., Die Kinetic der Invertinwirkung. Biochem. Z., 49, 333–369 (1913)
with \( \nu \) the outward normal to the boundary. But with \( n_i(t,x) \geq 0 \) in \( \Omega \) and \( n_i(t,x) = 0 \) in \( \partial \Omega \), we have \( \frac{\partial}{\partial \nu} n_i \leq 0 \), therefore the total mass diminishes

\[
\int_{\Omega} \sum_{i=1}^{I} n_i(t,x) dx \leq \int_{\Omega} \sum_{i=1}^{I} n_0^i(x) dx, \quad \forall t \geq 0.
\]

- Neuman boundary conditions \( \frac{\partial}{\partial \nu} n_i = 0 \) on \( \partial \Omega \), still with \( \nu \) the outward normal to the boundary. This means that individuals or molecules are reflected when they hit the boundary. In the computation above for (1.3) with (1.4), the normal derivative vanishes and we find directly mass conservation

\[
\int_{\Omega} \sum_{i=1}^{I} n_i(t,x) dx = \int_{\Omega} \sum_{i=1}^{I} n_0^i(x) dx.
\]

There is a big difference between the case of the full space \( \mathbb{R}^d \) and the case of a bounded domain. This can be seen by the results of spectral analysis in Section 1.6 which do not hold on \( \mathbb{R}^d \).

### 1.4 Relaxation results by perturbation methods

In this section we describe two types of behaviors in problems when the nonlinearity is 'small' compared to the leading Laplacian term of these parabolic equations. As we will see in the next two subsections, the Dirichlet and Neuman conditions lead to similar relaxation phenomenon but towards different types of elementary solutions.

#### 1.4.1 Asymptotic stability (Dirichlet boundary condition)

The first possible long time behavior for a semilinear parabolic equation is simply the relaxation towards a stable steady state (that we choose to be 0 here). This is possible when the following two features are combined

- the nonlinear part is a small perturbation of a main (linear differential) operator,
- this main linear operator has a kernel reduced to 0.

Of course this simple relaxation behavior is somehow boring and appears as the opposite of pattern formation, as e.g. when Turing instability occurs, that will be described later on, see Chapter 6.

To illustrate this, we consider, on a bounded domain \( \Omega \), the semi-linear heat equation with
Dirichlet boundary condition
\[
\begin{aligned}
\frac{\partial}{\partial t} u_i(t, x) - D_i \Delta u_i(t, x) &= F_i(t, x; u_1, ..., u_I), \quad 1 \leq i \leq I, \quad x \in \Omega, \\
u_i(t, x) &= 0, \quad x \in \partial \Omega, \\
u_i(t = 0, x) &= u_i^0(x) \in L^2(\Omega).
\end{aligned}
\] (1.10)

We assume that \( F(t, x; 0) = 0 \) so that \( u \equiv 0 \) is a steady state solution. Is it stable and attractive?

We will use a technical result. The Laplace operator (with Dirichlet boundary condition) admits a first eigenvalue \( \lambda_1 > 0 \), associated with a positive eigenfunction, \( w_1(x) \), which is unique up to multiplication by a constant,
\[
-\Delta w_1 = \lambda_1 w_1, \quad w_1 \in H^1_0(\Omega).
\] (1.11)

This eigenvalue is characterized as being the best constant in the Poincaré inequality (see Section 1.6 or the book [13])
\[
\lambda_1 \int_{\Omega} |v(x)|^2 \leq \int_{\Omega} |\nabla v|^2, \quad \forall v \in H^1_0(\Omega),
\]
with equality only for \( v = \mu w_1, \mu \in \mathbb{R} \).

**Theorem 1.4 (Asymptotic stability)** Assume \( \min_i D_i = d > 0 \) and that there is a (small) constant \( L > 0 \) such that
\[
|F(t, x; u)| \leq L |u|, \quad \text{or more generally,} \quad F(t, x; u) \cdot u \leq L |u|^2, \quad \forall u \in \mathbb{R}^I, \ t \geq 0, \ x \in \Omega,
\] (1.12)
\[
\delta = d\lambda_1 - L > 0,
\] (1.13)
then, \( u_i(t, x) \) vanishes with exponential rate as \( t \to \infty \), namely,
\[
\int_{\Omega} |u(t, x)|^2 \leq e^{-2\delta t} \int_{\Omega} |u^0(x)|^2.
\] (1.14)

**Proof.** We multiply the parabolic equation (1.10) by \( u_i \) and integrate by parts
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} u_i(t)^2 + D_i \int_{\Omega} |\nabla u_i(t)|^2 = \int_{\Omega} u_i(t) F_i(t, x; u(t)),
\]
and using the characterization (1.11) of \( \lambda_1 \), we conclude
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \sum_{i=1}^I u_i(t)^2 + d\lambda_1 \int_{\Omega} \sum_{i=1}^I u_i(t)^2 \leq L \int_{\Omega} \sum_{i=1}^I u_i(t)^2.
\]
The result follows by the Gronwall lemma. □
1.4.2 Relaxation to homogeneous solution (Neumann boundary condition)

The next easiest possible long time behavior for a parabolic equation is relaxation to an homogeneous (i.e., independent of $x$) solution which is not constant in time. This is possible when two features are combined

- the nonlinear part is a small perturbation of a main (differential) operator,
- this main operator has a non-empty kernel (0 is the first eigenvalue).

Consider again, on a bounded domain $\Omega$ with outward unit normal $\nu$, the semi-linear parabolic equation with Neumann boundary condition

\[
\begin{aligned}
\frac{\partial}{\partial t} u_i(t, x) - D_i \Delta u_i(t, x) &= F_i(t; u_1, ..., u_I), & 1 \leq i \leq I, & x \in \Omega, \\
\frac{\partial}{\partial \nu(x)} u_i(t, x) &= 0, & x \in \partial \Omega, \\
u_i(t = 0, x) &= u_i^0(x) \in L^2(\Omega).
\end{aligned}
\]

(1.15)

The Laplace operator (with Neuman boundary condition) admits $\lambda_1 = 0$ as a first eigenvalue, associated with the constants $w_1(x) = 1/\sqrt{|\Omega|}$ as eigenfunction. We will use its second eigenvalue $\lambda_2$ characterized by the Poincaré-Wirtinger inequality (see [13] and Section 1.6)

\[
\lambda_2 \int_{\Omega} |v(x) - ⟨v⟩|^2 \leq \int_{\Omega} |\nabla v|^2, \quad \forall v \in H^1(\Omega),
\]

(1.16)

with the average of $v(x)$ defined as

\[
⟨v⟩ = \frac{1}{|\Omega|} \int_{\Omega} v.
\]

Notice that this is also the $L^2$ projection on the eigenspace spanned by $w_1$.

**Theorem 1.5** (Relaxation to homogeneous solution) Assume $\min_i D_i = d > 0$ and

\[
(F(u) - F(v)) \cdot (u - v) \leq L |u - v|^2, \quad \forall u, v \in \mathbb{R}^I,
\]

(1.17)

\[
\delta = d\lambda_2 - L > 0,
\]

(1.18)

then, $u_i(t, x)$ tends to become homogeneous with exponential rate, namely,

\[
\int_{\Omega} |u(t, x) - ⟨u(t)⟩|^2 \leq e^{-2\delta t} \int_{\Omega} |u^0(x) - ⟨u^0⟩|^2.
\]

(1.19)

**Proof.** Integrating in $x$ equation (1.15), we find

\[
\frac{d}{dt} ⟨u_i⟩ = ⟨F_i(t; u)⟩,
\]
therefore
\[
\frac{d}{dt}[u_i - \langle u_i \rangle] - D_i \Delta[u_i - \langle u_i \rangle] = F_i(t; u) - \langle F_i(t; u) \rangle.
\]

Thus, using assumption (1.17), we find
\[
\frac{1}{2} \frac{d}{dt} \int \Omega |u - \langle u \rangle|^2 + d \int \Omega |\nabla(u - \langle u \rangle)|^2 = \int \Omega (F(t; u) - \langle F(t; u) \rangle) \cdot (u - \langle u \rangle)
\]
\[
= \int \Omega F(t; u) \cdot (u - \langle u \rangle)
\]
\[
= \int \Omega (F(t; u) - F(t; \langle u \rangle)) \cdot (u - \langle u \rangle)
\]
\[
\leq L \int \Omega |u - \langle u \rangle|^2.
\]

Therefore, with notation (1.18)
\[
\frac{d}{dt} \int \Omega |u - \langle u \rangle|^2 \leq -2\delta \int |u - \langle u \rangle|^2.
\]

The result (1.19) follows directly. \(\Box\)

Exercise. Explain why we cannot allow a dependency on \(x\) in \(F_i(t; u)\).

Exercise. Let \(v \in H^2(\Omega)\) satisfy \(\frac{\partial v}{\partial \nu} = 0\) on \(\partial \Omega\) (Neumann condition).

1. Prove, using the Poincaré-Wirtinger inequality (1.16), that
\[
\lambda_2 \int \Omega |\nabla v|^2 \leq \int \Omega |\Delta v|^2.
\]

[Hint. Integrate by parts the expression \(\int \Omega |\nabla v|^2\)]

2. In the context of Theorem 1.5, assume that
\[
\sum_{i,j=1}^I D_{ij}F_i(t; u)\xi_i\xi_j \leq L|\xi|^2, \forall \xi \in \mathbb{R}^I.
\]

Using the above inequality, prove that
\[
\int \Omega \sum_i |\nabla u_i(t, x)|^2 \leq e^{-2\delta t} \int \Omega \sum_i |\nabla u_i^0(x)|^2.
\]

[Hint. Use the equation on \(\frac{d}{dt} \frac{\partial u_i}{\partial x_i}\)]

3. Deduce a variant of Theorem 1.5.

1.5 Entropy and relaxation

We have seen in Lemma 1.3 that reaction kinetics equation as (1.6) are endowed with an entropy (1.8). It originates from the microscopic \(N\)-particles stochastic systems from which reaction kinetics are derived (at this level it is a Markov jump process which enjoys entropy dissipation as all Markov processes).
This entropy inequality is very useful also because it can be used to show relaxation to the steady state, independently of the size of the constants \( k_1, k_2 \). To do that we consider Neuman boundary conditions in a bounded domain \( \Omega \)

\[
\begin{align*}
\frac{\partial}{\partial t} n_1 - D_1 \Delta n_1 + k_1 n_1 &= k_2 (n_2)^2, \quad t \geq 0, \quad x \in \Omega \\
\frac{\partial}{\partial t} n_2 - D_2 \Delta n_2 + 2k_2 (n_2)^2 &= 2k_1 n_1, \\
\frac{\partial}{\partial \nu} n_1 &= \frac{\partial}{\partial \nu} n_2 = 0 \quad \text{on} \quad \partial \Omega.
\end{align*}
\]

(1.22)

**Theorem 1.6** The solutions to (1.22), with \( n_0^i \geq 0, n_0^i \in L^1(\Omega), n_0^i \ln(n_0^i) \in L^1(\Omega) \), satisfy that

\[ n_i(t, x) \to N_i, \quad \text{as} \quad t \to \infty \]

with \( N_i \) the constants defined uniquely by

\[
2N_1 + N_2 = \int_{\Omega} [2n_0^1(x) + n_0^2(x)] dx, \quad k_2 (N_2)^2 = k_1 N_1.
\]

**Proof.** Then \( S(t, x) = n_1 \ln(k_1 n_1 - 1) + n_2 \ln(k_2^{1/2} n_2 - 1) \) satisfies, following Lemma 1.3,

\[
\frac{d}{dt} \int_{\Omega} S(t, x) dx = \int_{\Omega} [D_1 \frac{|\nabla n_1|^2}{n_1} + D_2 \frac{|\nabla n_2|^2}{n_2}] dx \\
+ \int_{\Omega} [\ln(k_2 n_2^2) - \ln(k_1 n_1)] [k_2 (n_2)^2 - k_1 n_1] dx.
\]

And, because \( S \) is bounded from below, it also gives a bound on the entropy dissipation

\[
\int_0^\infty \int_{\Omega} [D_1 \frac{|\nabla n_1|^2}{n_1} + D_2 \frac{|\nabla n_2|^2}{n_2}] dx dt \\
+ \int_0^\infty \int_{\Omega} [\ln(k_2 n_2^2) - \ln(k_1 n_1)] [k_2 (n_2)^2 - k_1 n_1] dx dt \leq C(n_0^1, n_0^2).
\]

(1.23)

This is again a better estimate in \( x \) than the \( L^1_{\log} \) estimate (derived from mass conservation) because of the quadratic term \( (n_2)^2 \).

From a qualitative point of view, it says that the chemical reaction should lead the system to an equilibrium state which is space homogeneous. Indeed, formally at least, the integral (1.23) can be bounded only if

\[
\nabla n_1 = \nabla n_2 \approx 0 \quad \text{as} \quad t \to \infty,
\]

\[
k_2 (n_2)^2 \approx k_1 n_1, \quad \nabla n_1 = \nabla n_2 \approx 0 \quad \text{as} \quad t \to \infty.
\]

The first conclusion says that the dynamics becomes homogeneous in \( x \) (but may depend on \( t \)). The second conclusion, combined with the mass conservation relation (1.7) shows that there is a unique possible asymptotic homogeneous state because the constant state satisfies

\[
k_2 (N_2)^2 = k_1 N_1 = k_1 (\frac{M}{|\Omega|} - N_2) \]

which has a unique positive root.
Exercise. Consider (1.6) with Neumann boundary conditions and set
\[ S(t, x) = \frac{1}{k_1} \Sigma_1 (k_1 n_1(t, x)) + \frac{1}{2k_2^{1/2}} \Sigma_2 (k_2^{1/2} n_2(t, x)). \]

1. We assume that \( \Sigma_1, \Sigma_2 \) satisfy \( \Sigma_1'(u) = \Sigma_2'(u^{1/2}) \). Show that \( \Sigma_1 \) is convex if and only if \( \Sigma_2 \) is convex.
2. Under the conditions in question 1, show that the equation (1.6) dissipates entropy.
3. Adapt Stampacchia method to prove \( L^\infty \) bounds on \( n_1, n_2 \) (and which is the natural quantity for the maximum principle). What are the natural \( L^p \) bounds.

Exercise. Consider (1.6) with Dirichlet boundary conditions and \( n_i^0 \geq 0 \).
1. Using that formally \( \frac{\partial n_i}{\partial \nu} \leq 0 \) because \( n_j \geq 0 \), show that \( M(t) \) decreases where
\[ M(t) = \int_{\Omega} [2n_1(t, x) + n_2(t, x)] \, dx. \]
2. Consider the entropies of previous exercise with the additional condition \( \Sigma_1'(0) = 0 \). Show that the equation (1.6) dissipates entropy.
(iii) Show that solutions to (1.6) with Dirichlet boundary conditions tend to 0 as \( t \to \infty \).

1.6 The spectral decomposition of Laplace operators

We have used consequences of the spectral decomposition of the Laplace operator with either Dirichlet boundary condition
\[
\begin{cases}
-\Delta u = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]  

or Neuman boundary condition

\[
\begin{cases}
-\Delta u = f & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

1.6.1 Main results

Theorem 1.7 (Dirichlet) Consider a bounded connected open set \( \Omega \), then there is a spectral basis \( (\lambda_k, w_k)_{k \geq 1} \) for (1.24), that is,
(i) \( \lambda_k \) is a nondecreasing sequence with \( 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_k \leq \ldots \) and \( \lambda_k \to \infty \),
(ii) \( (\lambda_k, w_k) \) are eigenelements, i.e., for all \( k \geq 1 \) we have
\[
\begin{cases}
-\Delta w_k = \lambda_k w_k & \text{in } \Omega, \\
w_k = 0 & \text{on } \partial \Omega,
\end{cases}
\]
(iii) \((w_k)_{k \geq 1}\) is an orthonormal basis of \(L^2(\Omega)\).

(iv) we have \(w_1(x) > 0\) in \(\Omega\) and the first eigenvalue \(\lambda_1\) is simple, and for \(k \geq 2\), the eigenfunction \(w_k\) changes sign and maybe multiple.

**Theorem 1.8 (Neuman)** Consider a \(C^1\) bounded connected open set \(\Omega\), then there is a spectral basis \((\lambda_k, w_k)_{k \geq 1}\) for (1.25), i.e.,

(i) \(\lambda_k\) is a nondecreasing sequence with \(0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq ... \leq \lambda_k \leq ...\) and \(\lambda_k \to \infty\) as \(k \to \infty\),

(ii) \((\lambda_k, w_k)\) are eigenelements, i.e., for all \(k \geq 1\) we have

\[
\begin{cases}
-\Delta w_k = \lambda_k w_k & \text{in } \Omega, \\
\frac{\partial w_k}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(iii) \((w_k)_{k \geq 1}\) is an orthonormal basis of \(L^2(\Omega)\)

(iv) \(w_1(x) = \frac{1}{|\Omega|^{1/2}} > 0\), and for \(k \geq 2\), the eigenfunction \(w_k\) changes sign.

**Remark 1.** The hypothesis that \(\Omega\) is connected is just used to guarantee that the first eigenvalue is simple and the corresponding eigenfunction is positive in \(\Omega\). Otherwise we have several non-negative eigenfunctions with first eigenfunction in one component and 0 in the others.

**Remark 2.** The sequences \(w_k\) are also orthogonal in \(H^1_0(\Omega)\) (for Dirichlet conditions) and \(H^1(\Omega)\) for Neuman conditions. Indeed, if \(w_k\) is orthogonal to \(w_j\) in \(L^2(\Omega)\), then from the Laplace equation on \(w_k\) and Stokes formula

\[
\int _\Omega \nabla w_j \cdot \nabla w_k = \lambda_k \int _\Omega w_j w_k = 0.
\]

Therefore orthogonality in \(L^2\) implies orthogonality in \(H^1_0\) or \(H^1\).

Notice that for \(k \geq 2\), the eigenfunction \(w_k\) changes sign because \(\int _\Omega w_1 w_k = 0\) and \(w_1\) has a sign.

**Proof of Theorem 1.7.** We only prove the first Theorem, the second being a variant and we do not give the details; for additional matter see [35] Ch. 7, [18] Ch. 5, [9] p. 96. The result is based on two ingredients. (i) The spectral decomposition of self-adjoint compact linear mappings on Hilbert spaces is a general theory that extends the case of symmetric matrices. (ii) The simplicity of the first eigenvalue with a positive eigenfunction is also usual and is a consequence of the Krein-Rutman theorem (infinite dimension version of the Perron-Froebenius theorem).
**First step. 1st eigenelements.** On the Hilbert space $H = L^2(\Omega)$, we consider the linear subspace $V = H^1_0(\Omega)$. Then we define the minimum on $V$

$$\lambda_1 = \min_{\int_{\Omega}|u|^2=1} \int_{\Omega} |\nabla u|^2 dx.$$ 

It is attained because a minimizing sequence $(u_n)$ will converge strongly in $H$ and weakly in $V$ to $w_1 \in V$ with $\int_{\Omega}|w_1|^2 = 1$, by the Rellich compactness Theorem (see [13]). Therefore $\int_{\Omega} |\nabla w_1|^2 dx \leq \liminf_{n \to \infty} \int_{\Omega} |\nabla u_n|^2 dx = \lambda_1$. This implies the equality and that $\lambda_1 > 0$. The variational principle associated to this minimization problem says that

$$-\Delta w_1 = \lambda_1 w_1,$$

which implies that $w_1$ is smooth in $\Omega$ (by elliptic regularity)

**Second step. Positivity.** Because in $V$, $|\nabla |u|^2 = |\nabla u|^2$ a.e., the construction above tells us that $|w_1|$ is also a first eigenfunction and we may assume that $w_1$ is nonnegative. By the strong maximum principle for the Laplace equation, we obtain that $w_1$ is positive inside $\Omega$ (because it is connected). This also proves that all the eigenfunctions associated with $\lambda_1$ have a sign in $\Omega$ because, on a connected open set, $w_1$ cannot satisfy the three properties (i) be smooth, (ii) change sign and (iii) $|w_1|$ be positive also.

**Third step. Simplicity.** Finally, we can deduce the simplicity of this eigenfunction because if there were two independent, a linear combination would allow to build one which changes sign (by orthogonality to $w_1$ for instance) and this is impossible by the above positivity argument.

**Fourth step. Other eigenelements.** We may iterate the construction. Denote $E_k$ the finite dimensional subspace generated by the $k$-th first eigenspaces. We work on the closed subspace $E_k^\perp$ of $H$, and we may define

$$\lambda_{k+1} = \min_{u \in E_k^\perp \cap V, \int_{\Omega}|u|^2=1} \int_{\Omega} |\nabla u|^2 dx.$$ 

It is attained by the same reason as before. The variational form gives that the minimizers are solutions to the $k + 1$-th eigenproblem. They can form a multidimensional space. But it is finite dimensional; otherwise we would have an infinite dimensional subspace of $L^2(\Omega)$ which unit ball is compact by the Rellich compactness Theorem since $\int_{\Omega} |\nabla u|^2 dx \leq \lambda_{k+1}$ in this ball

Also $\lambda_k \to \infty$ as $k \to \infty$ because one can easily built (with oscillations or sharp gradients) functions satisfying $\int_{\Omega} |u_n|^2 = 1$ and $\int_{\Omega} |\nabla u_n|^2 dx \geq n$.  \Box
1.6.2 Rectangles: explicit solutions

In one dimension, one can compute explicitly the spectral basis because the solutions to \(-u'' = \lambda u\) are all known. On \(\Omega = (0, 1)\) we have

\[w_k = a_k \sin(k\pi x), \quad \lambda_k = (k\pi)^2, \quad \text{(Dirichlet)}\]

\[w_k = b_k \cos((k-1)\pi x), \quad \lambda_k = ((k-1)\pi)^2, \quad \text{(Neuman)}\]

and \(a_k\) and \(b_k\) are normalization constants that ensure \(\int_0^1 |w_k|^2 = 1\).

In two dimensions, on a rectangle \((0, L_1) \times (0, L_2)\) we see that the family is better described by two indices \(k \geq 1\) and \(l \geq 1\) and we have

\[w_{kl} = a_{kl} \sin\left(k\pi \frac{x_1}{L_1}\right) \sin\left(l\pi \frac{x_2}{L_2}\right), \quad \lambda_{kl} = \left((\frac{k}{L_1})^2 + (\frac{l}{L_2})^2\right)\pi^2, \quad \text{(Dirichlet)}\]

\[w_{kl} = b_{kl} \cos\left((k-1)\pi \frac{x_1}{L_1}\right) \cos\left((l-1)\pi \frac{x_2}{L_2}\right), \quad \lambda_{kl} = \left((\frac{k-1}{L_1})^2 + (\frac{l-1}{L_2})^2\right)\pi^2, \quad \text{(Neuman)}\]

These examples indicate that

- The first positive eigenvalue is of order \(1/ \max(L_1, L_2)^2\). On a large domain (even with a large aspect ratio) we can expect that the first eigenvalues is close to zero and that the eigenvalues are close to each other.
- Except the first, eigenvalues can be multiple (take \(L_1/L_2 \in \mathbb{N}\)).
- Large eigenvalues are associated with highly oscillating eigenfunctions.

1.6.3 The numerical artefact

![Figure 1.1: A parasite discrete eigenfunction of Laplace equation in 2 dimensions associated with the eigenvalue \(\frac{2}{\Delta^2}\). This does not approximate a continuous eigenfunction.](image)

The numerical computation of the high eigenvalues is a difficult question. Indeed, the discrete eigenproblem may exhibit parasite eigenvalues that do not converge to a continuous eigenfunction in the limit. Figure 1.1 gives an example of this artefact in 2 dimensions.
1.7 The Lotka-Volterra prey-predator system with diffusion (Problem)

In the case the Lotka-Volterra prey-predator system we can show relaxation towards an homogeneous solution. The coefficients of the model need not be small as it is required in Theorem 1.5. This is because the model comes with a natural quantity (as the entropy) which gives a global control.

Exercise. Consider the prey-predator Lotka-Volterra system without diffusion

\[
\begin{align*}
\frac{\partial}{\partial t} n_1 &= n_1 [ r_1 - an_2], \\
\frac{\partial}{\partial t} n_2 &= n_2 [-r_2 + bn_1],
\end{align*}
\]

where \( r_1, r_2, a \) and \( b \) are positive constants and the initial data \( n_i^0 \) are positive.

1. Show that there are local solutions and that they remain positive.
2. Show that the entropy (lyapunov functional)

\[ E(t) = -r_1 \ln n_2 + an_2 - r_2 \ln n_1 + bn_1, \]

is constant. Show that \( E \) is bounded from below and that \( E \to \infty \) as \( n_1 + n_2 \to \infty \). Conclude that solutions are global.
3. What is the unique steady state solution?
4. Show, using the question 2., that the solutions are periodic (trajectories are closed).

Exercise. Let \( \Omega \) a smooth bounded domain. Consider smooth positive solutions to the Lotka-Volterra equation with diffusion and Neuman boundary condition

\[
\begin{align*}
\frac{\partial}{\partial t} n_1 - d_1 \Delta n_1 &= n_1 [ r_1 - an_2], \\
\frac{\partial}{\partial t} n_2 - d_2 \Delta n_2 &= n_2 [-r_2 + bn_1], \\
\frac{\partial n_i}{\partial \nu} &= 0 \text{ on } \partial \Omega, \quad i = 1, 2,
\end{align*}
\]

where \( d_1, d_2, r_1, r_2, a \) and \( b \) are positive constants and the initial data \( n_i^0 \) are positive.

1. Consider the quantity \( m(t) = \int_{\Omega} [an_1(t, x) + an_2(t, x)] dx \). Show that \( m(t) \leq m(0)e^{rt} \) and give the value \( r \).
2. Show that the convex entropy

\[ E(t) = \int_{\Omega} [-r_1 \ln n_2 + an_2 - r_2 \ln n_1 + bn_1] dx, \]
a) is bounded from below, b) is decreasing.
Conclude that \( m(t) \) is bounded.

3. What finite integral do we obtain from the entropy dissipation?

3. Assume that the quantities \( \nabla \ln n_i(t, x) \) converge, as \( t \to \infty \),
a. What are their limits?
b. What can you conclude on the behavior of \( n_i(t, x) \) as \( t \to \infty \)?
Chapter 2

Weak solutions to parabolic equations in $\mathbb{R}^d$

2.1 Weak solutions in distributions sense

So far, our statements always concern weak solutions $u \in C(\mathbb{R}^+; L^2(\mathbb{R}^d))$ to parabolic equations of the type

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u &= f \quad \text{in} \quad \mathbb{R}^d, \\
u(t = 0, x) &= u^0(x).
\end{aligned}
\] (2.1)

According to the general theory of distributions (due to Laurent Schwartz), these are defined through a formal integration by parts on a test function

**Definition 2.1** Let $f \in L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$, $u^0 \in L^1_{\text{loc}}(\mathbb{R}^d)$. A function $u \in L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ is a weak solution (or a distributional solution) to (2.1) if we have

\[
\int_0^\infty \int_{\mathbb{R}^d} \left( - \frac{\partial \Phi}{\partial t} - \Delta \Phi \right) dx \, dt = \int_0^\infty \int_{\mathbb{R}^d} f(t, x) \Phi(t, x) \, dx \, dt + \int_{\mathbb{R}^d} u^0(x) \Phi(t = 0, x) \, dx,
\]

(2.2)

for all test functions $\Phi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d)$. We can also consider locally bounded measures or derivatives of functions here.

In particular when $u(t, x)$ is a $C^2$ function, this holds true obviously.

Weak solutions to linear equations enjoy many properties that make the interest of this notion.
2.2 Stability of weak solutions

Consider a sequence of weak solutions \( u_n \) of (2.1) corresponding to data \( u_0^n \) and \( f_n \). Assume convergence in some weak topology (for instance \( L^2, L^1, M^1 \) (measures))

\[
u_n^0 \rightharpoonup u^0, \quad f_n \rightharpoonup f, \quad u_n \rightharpoonup u.
\]

Then

**Lemma 2.2** In this situation \( u \) is a weak solution to (2.1).

**Proof.** By definition of weak solution, for a test function \( \Phi \in \mathcal{D}([0, +\infty) \times \mathbb{R}^d) \) we have

\[
\int_0^{\infty} \int_{\mathbb{R}^d} u(t, x) \left[ -\frac{\partial \Phi}{\partial t} - \Delta \Phi \right] dx \, dt = \int_0^{\infty} \int_{\mathbb{R}^d} f_n(t, x) \Phi(t, x) \, dx + \int_{\mathbb{R}^d} u_n^0(x) \Phi(t = 0, x) \, dx,
\]

but \( \Phi \) being fixed, and by the very definition of weak convergence, we can pass to the limit as \( n \to \infty \) and recover the relation (2.2).

2.3 Mass conservation and truncation

Among the desirable properties of solutions is that the mass conservation law holds true. In other we can integrate on the whole space and not bother of the 'boundary terms at infinity'. This is true indeed

**Proposition 2.3** Assume \( f \in L^1([0, T] \times \mathbb{R}^d) \), \( u^0(x) \in L^1(\mathbb{R}^d) \). Let \( u \in L^1([0, T] \times \mathbb{R}^d) \) for all \( T > 0 \) be a weak solution to (2.1), then \( \int_{\mathbb{R}^d} u(t, x) dx \in C(\mathbb{R}^+) \) and

\[
\int_{\mathbb{R}^d} u(t, x) dx = \int_0^t \int_{\mathbb{R}^d} f(s, x) dx \, ds + \int_{\mathbb{R}^d} u^0(x) dx.
\]

In other words, it holds in the weak sense

\[
\frac{d}{dt} \int_{\mathbb{R}^d} u(t, x) dx = \int_{\mathbb{R}^d} f(t, x) dx.
\]

We point out that it is necessary and important that we know a priori that the functions \( u^0, f, u \) are integrable. For \( f \equiv 0, u^0 \equiv 0 \), there are non-zero solutions to the heat equation with a super-exponential growth at infinity.

We leave as an exercise the derivation of the differential equality from the integrated relation.
Proof. In the definition (2.1), we can use a test function \( \chi_R(x) = \chi(\frac{x}{R}) \) with
\[
\begin{cases} 
\chi \in \mathcal{D}(\mathbb{R}^d), & 0 \leq \chi(\cdot) \leq 1, \\
\chi(x) = 1 & \text{for } |x| \leq 1, \\
\chi(x) = 0 & \text{for } |x| \geq 2. 
\end{cases}
\tag{2.3}
\]
We have for any test function \( \phi \in \mathcal{D}(\mathbb{R}^+) \),
\[
\int_0^\infty \int_{\mathbb{R}^d} u(t,x)\chi_R(x)\phi'(t) = \int_0^\infty \int_{\mathbb{R}^d} [u(t,x)\Delta \chi_R(x) + f(t,x)\chi_R(x)]\phi(t) + \int_{\mathbb{R}^d} u^0(x)\chi_R(x)\phi(0)
\]
Because \( \Delta \chi_R(x) = \frac{1}{R^d} \Delta \chi(\frac{x}{R}) \) and \( u, u^0 \) and \( f \) belong to \( L^1([0,T] \times \mathbb{R}^d) \), we can set \( R \to \infty \) and obtain, using the Lebesgue Dominated Convergence Theorem, that
\[
-\int_0^\infty \int_{\mathbb{R}^d} u(t,x)\phi'(t) = \int_0^\infty \int_{\mathbb{R}^d} f(t,x)\phi(t) + \int_{\mathbb{R}^d} u^0(x)\phi(0).
\]
In other words, the functions defined as \( M_u(t) = \int_{\mathbb{R}^d} u(t,x)dx \) and \( M_f(t) = \int_{\mathbb{R}^d} f(t,x)dx \) are related through
\[
\int_0^\infty M_u(t)\phi'(t) = \int_0^\infty M_f(t)\phi(t) + \int_{\mathbb{R}^d} u^0(x)\phi(0).
\]
Because \( M_u(t), M_f(t) \) are simply \( L^1_{\text{loc}}(\mathbb{R}^+) \) functions, we cannot derive (2.3) immediately.

Let \( T > 0 \) and consider the function \( \phi(t) = 1_{\{0 \leq t \leq T\}} \). It is not an admissible test function but we can choose a sequence of functions \( \phi_n \in \mathcal{D}(\mathbb{R}) \) with \( \phi_n \) decreasing, \( \phi_n(t) \leq \phi(t) \) and \( \phi_n(t) = 1 \) on \([0,T-1/n]\) (\( n \) large enough). Then we have from the above equality
\[
\int_0^\infty M_u(t)\phi_n'(t) = \int_0^\infty M_f(t)\phi_n(t) + \int_{\mathbb{R}^d} u^0(x) \xrightarrow{\delta \to 0} \int_0^T \int_0^\infty M_f(t) + \int_{\mathbb{R}^d} u^0(x),
\]
still by the Lebesgue Dominated Convergence Theorem applied to \( M_f \).

From that, we can deduce that \( M_u \) is continuous (or more accurately has a continuous representant in his Lebesgue class). Indeed, choosing \( \phi_n \) nicely, at each Lebesgue point \( T \) of \( M_u \) one can also pass to the limit in the left hand side of the above and obtain
\[
\int_{\mathbb{R}^d} M_u(T) = \int_0^T M_f(t)dt + \int_{\mathbb{R}^d} u^0(x).
\]
This is the rigorous statement for the differential equality stated in Proposition 2.4. \( \square \)

2.4 Regularization of weak solutions (space)

Several general properties of weak solutions can be derived from a regularization argument we give now. It uses a regularizing kernel \( \omega \), this is a function satisfying the properties
\[
\omega \in \mathcal{D}(\mathbb{R}^d), \quad \omega \geq 0, \quad \int_{\mathbb{R}^d} \omega = 1. \tag{2.4}
\]
We regularize functions by convolution and set

$$\omega * u = \int_{\mathbb{R}^d} \omega(x - y)u(y)dy.$$ 

For \(u \in L^1(\mathbb{R}^d)\), we still have \(\omega * u \in L^1(\mathbb{R}^d)\).

**Proposition 2.4** Let \(\omega\) be a regularizing kernel, \(f \in C(\mathbb{R}^+;L^1(\mathbb{R}^d))\) and \(u^0(x) \in L^1(\mathbb{R}^d)\). For a weak solution to \((2.1)\), \(u \in L^1([0,T] \times \mathbb{R}^d) \forall T > 0\), then \(\omega * u\) belongs to \(C^1(\mathbb{R}^+;C^2(\mathbb{R}^d))\) and is a classical solution to \((2.1)\) for a regularized right hand side \(\omega(x) * f\) and initial data \(\omega * u^0\).

Moreover

$$\int_{\mathbb{R}^d} |u(t,x)|dx \leq \int_0^t \int_{\mathbb{R}^d} |f(s,x)|dxds + \int_{\mathbb{R}^d} |u^0(x)|dx, \quad \text{a.e.} \quad (2.5)$$

**Proof.** (Classical solution) We use the test function \(\Phi(t,x) = \phi(t)\omega(y - x)\) with \(y \in \mathbb{R}^d\) a fixed vector and \(\phi \in \mathcal{D}(\mathbb{R}^+)\) a given test function. For this choice, the definition (2.2) gives

$$\int_0^\infty [-u * \omega(t,y) \frac{\partial \phi}{\partial t} - u * \Delta \omega(t,y)\phi(t)]dt = \int_0^\infty f * \omega(t,y)\phi(t)dt + u^0 * \omega(y)\phi(0).$$

We set \(U(t,x) = u * \omega, \ F(t,x) = f * \omega\), these are two smooth functions in \(x\) and the above equality can also be written (changing the name of the variable from \(y\) to \(x\))

$$\int_0^\infty [-U(t,x) \frac{\partial \phi}{\partial t} - \Delta U(t,x)\phi(t)]dt = \int_0^\infty F(t,x)\phi(t)dt + U^0(0)\phi(0).$$

We fix \(x\) and set \(G = F + \Delta U \in L^1_{\text{loc}}(\mathbb{R}^+)\), and we rewrite the above equality as

$$- \int_0^\infty U(t)\phi'(t)dt = \int_0^\infty G(t)\phi(t)dt + U^0\phi(0),$$

for all test functions \(\phi \in \mathcal{D}(\mathbb{R}^+)\). Then we first conclude that \(U \in C(\mathbb{R}^+)\) with the argument at the end of the proof of Proposition 2.3 (choosing \(\phi_n \rightarrow 1_{\{0 \leq t \leq T\}}\)) and

$$U(t,x) = \int_0^t G(s,x)ds + U^0(x).$$

This proves that \(\Delta U \in C(\mathbb{R}^+)\) (because \(\Delta U = \Delta \omega * u\) while \(U = \omega * u\), and the argument also applies with \(\Delta \omega\) in place of \(\omega\)). Therefore \(G \in C(\mathbb{R}^+)\) and thus \(U \in C^1(\mathbb{R}^+)\).

(Integral inequality) The proof is more involved because this inequality is a nonlinear statement. We use the regularization argument of the first part of this proof to obtain, for any regularizing kernel \(\omega_\varepsilon = \frac{1}{\varepsilon^d}\omega(\frac{x}{\varepsilon})\), that the \(C^1\_C^2\) function \(u_\varepsilon = u * \omega_\varepsilon\) satisfies

$$\frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = f * \omega_\varepsilon.$$
We now use a function $S(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ with the properties: it is smooth, it is a sublinear convex function, $S(0) = 0$ (see Figure 2.1 for an example). Then, we also have using the chain rule
\[
\frac{\partial S(u_\varepsilon)}{\partial t} - \Delta S(u_\varepsilon) = -S''(u_\varepsilon)|\nabla u_\varepsilon|^2 + S'(u_\varepsilon)f * \omega_\varepsilon \leq S'(u_\varepsilon)f * \omega_\varepsilon.
\]
Because $S(u_\varepsilon) \in C(\mathbb{R}^+; L^1(\mathbb{R}^d))$, we can use the truncation argument of Section 2.3 to integrate and find
\[
\int_{\mathbb{R}^d} S(u_\varepsilon(t,x)) \, dx \leq \int_0^t \int_{\mathbb{R}^d} S'(u_\varepsilon(s,x)) f(s,x) \, dx \, ds + \int_{\mathbb{R}^d} S(u_\varepsilon^0(x)) \, dx.
\]
As $\varepsilon \to 0$, $u_\varepsilon \to u$ strongly in $L^1[0,T] \times \mathbb{R}^d$, $\forall T > 0$, and we obtain
\[
\int_{\mathbb{R}^d} S(u(t,x)) \, dx \leq \int_0^t \int_{\mathbb{R}^d} S'(u(s,x)) f(s,x) \, dx \, ds + \int_{\mathbb{R}^d} S(u^0(x)) \, dx. \tag{2.6}
\]

![Figure 2.1: The function $S_\delta(u)$ that regularizes $|u|$.](image)

We may finally choose a sequence $S_\delta(\cdot)$ of smooth functions as above that regularizes the absolute value. It can enjoy the properties that (see Figure 2.1 for an example):
\[
\begin{align*}
S_\delta(\cdot) & \text{ is smooth, even and convex,} & \max(0,|u| - \delta) & \leq S_\delta(u) \leq |u|, \\
0 & \leq \text{sgn}(u)S'_\delta(u) \leq 1, & 0 & \leq \frac{\delta}{2} S'_\delta(u) \leq S_\delta(u). \tag{2.7}
\end{align*}
\]

Then, (2.7) gives
\[
\int_{\mathbb{R}^d} S_\delta(u(t,x)) \, dx \leq \int_0^t \int_{\mathbb{R}^d} |f(s,x)| \, dx \, ds + \int_{\mathbb{R}^d} |u^0(x)| \, dx
\]
and passing to the strong limit we find the inequality 2.5. \qed

**Exercise.** Write and prove the same statement as Proposition 2.4 in $L^2(\mathbb{R}^d)$ in place of $L^1(\mathbb{R}^d)$ and in particular
\[
\|u(t)\|_{L^2(\mathbb{R}^d)} \leq \int_0^t \|f(s)\|_{L^2(\mathbb{R}^d)} \, ds + \|u^0\|_{L^2(\mathbb{R}^d)}.
\]
2.5 Regularization of weak solutions (time)

We can relax the assumptions in time using an additional regularization in time. To do so, there is a technical issue because $t \geq 0$ and we have to be careful on convolution. This is the reason we introduce an asymmetric regularizing kernel $\tilde{\omega}$, this is a function satisfying the properties

$$\tilde{\omega} \in D(\mathbb{R}), \quad \tilde{\omega} \leq 0, \quad \tilde{\omega}(s) = 0 \text{ for } s \geq 0 \quad \int_{\mathbb{R}^d} \tilde{\omega} = 1. \quad (2.8)$$

Then, we can regularize a function $u(t) : \mathbb{R}^+ \to \mathbb{R}$ by convolution with the usual formula

$$\omega \ast_t u(t) = \int_{\mathbb{R}} \omega(t-s)u(s)ds,$$

because we have $t-s \leq 0$ and thus for $t \geq 0$ we have $s \geq 0$.

**Theorem 2.5** Let $f \in L^1([0,T] \times \mathbb{R}^d) \forall T > 0$, and $u^0(x) \in L^1(\mathbb{R}^d)$. For a weak solution to (2.1), $u \in L^1([0,T] \times \mathbb{R}^d) \forall T > 0$, then $(\tilde{\omega}(t)\omega(x)) \ast u$ belongs to $C^1(\mathbb{R}^+; C^2(\mathbb{R}^d))$ and is a classical solution to (2.1) for a regularized right hand side $(\tilde{\omega}(t)\omega(x)) \ast f$ and initial data $\omega \ast u^0$. Moreover

$$\int_{\mathbb{R}^d} |u(t,x)|dx \leq \int_0^t \int_{\mathbb{R}^d} |f(s,x)|dxds + \int_{\mathbb{R}^d} |u^0(x)|dx \quad \text{a.e.} \quad (2.9)$$

We just write the main idea of the proof.

**Proof.** We fix the space regularization kernel $\omega$ and use a regularizing kernel in time $\tilde{\omega}_\alpha = \frac{1}{\alpha} \tilde{\omega}(\frac{x}{\alpha})$. We define the smooth functions

$$U_\alpha(t,x) = (\tilde{\omega}_\alpha(t)\omega(x)) \ast u, \quad F_\alpha(t,x) = (\tilde{\omega}_\alpha(t)\omega(x)) \ast f.$$

We use the test function $\Phi(s,y) = \tilde{\omega}_\alpha(t-s)\omega(x-y)$ in the definition of weak solutions to (2.1) written with the variables $(s,y)$. Since $\tilde{\omega}_\alpha(t) = 0$ for $t \geq 0$, we find that the heat equation holds in the classical sense

$$\frac{\partial U_\alpha}{\partial t} - \Delta U_\alpha = F_\alpha, \quad t \geq 0, \quad x \in \mathbb{R}^d.$$

But the initial data is not recovered and we have to use again the argument in Section 2.3 that we may use (formally) the test function in time $\phi(s) = 1_{\{0 \leq s \leq t\}}$ to find

$$\omega \ast u(t,x) = \int_0^t [\Delta(\omega \ast u) + \omega \ast f](s,x)ds + \omega \ast u^0, \quad \text{a.e. } t > 0,$$

and thus integrating $\tilde{\omega}(0-t) dt$, we find

$$U_\alpha(0,x) = \int \tilde{\omega}_\alpha(-t) \int_{s=0}^t [\Delta(\omega \ast u) + \omega \ast f](s,x)ds \, dt + \omega \ast u^0.$$
With $R_\alpha(t) = \int_t^\infty \tilde{\omega}_\alpha(s)\,ds$, this can be written as

$$U_\alpha(0, x) - \omega * u^0(x) = \int_0^\infty R_\alpha(-t) [\Delta(\omega * u) + \omega * f](t, x)\,dt \to 0, \quad \forall x \in \mathbb{R}^d,$$

because $0 \leq R_\alpha(t) \leq 1$, $R_\alpha(t) \to 0$ a.e. And it follows that

$$\|U_\alpha(0, x) - \omega * u^0(x)\|_{L^1(\mathbb{R}^d)} \to 0,$$

because for $\alpha$ small enough the support of $R_\alpha$ is less than a constant $C$ and

$$\left| \int_0^\infty R_\alpha(-t) [\Delta(\omega * u) + \omega * f](t, x)\,dt \right| \leq \int_C |\Delta(\omega * u) + \omega * f|(t, x)|dt,$$

this is a fixed $L^1$ function, and we can apply the Lebesgue Dominated Convergence Theorem.

This allows us to recover the inequality

$$\int_{\mathbb{R}^d} S(U_\alpha(t, x))\,dx \leq \int_s^t \int_{\mathbb{R}^d} S'(U_\alpha(t, x)) F_\alpha(t, x)\,dx\,ds + \int_{\mathbb{R}^d} S(U^0_\alpha(x))\,dx.$$

And in the limit $\alpha \to 0$, we obtain

$$\int_{\mathbb{R}^d} S(\omega * u(t, x))\,dx \leq \int_{s=0}^t \int_{\mathbb{R}^d} S'(\omega * u(t, x)) \omega * f(t, x)\,dx\,ds + \int_{\mathbb{R}^d} S(\omega * u^0(x))\,dx.$$

We are back in the situation of Section 2.4. □

2.6 Uniqueness of weak solutions

A direct consequence of the regularization technique is

**Proposition 2.6** Let $f \in L^1([0, T] \times \mathbb{R}^d)$ $\forall T > 0$, $u^0(x) \in L^1(\mathbb{R}^d)$ then there is at most one weak solution $u \in L^1([0, T] \times \mathbb{R}^d)$ $\forall T > 0$ to (2.1).

**Proof.** Indeed, substracting two possible solutions $u_1$ and $u_2$, we find a solution to (2.1) with $f \equiv 0$, $u^0 \equiv 0$. Applying the inequality (2.9), we find

$$\int_{\mathbb{R}^d} |u_1(t, x) - u_2(t, x)|\,dx \leq 0,$$

which implies that $u_1 = u_2$. □
2.7 Positivity of weak solutions to Lotka-Volterra type equations

The arguments of regularization and truncation are also useful to prove the positivity of weak solutions stated in Lemma 1.1, that is we consider a weak solution $u \in C(\mathbb{R}^+; L^2(\mathbb{R}^d))$ to the parabolic equations

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u &= nR(t, x) \quad \text{in } \mathbb{R}^d, \\
n(t = 0, x) &= n^0(x).
\end{aligned}
$$

Then we have

**Lemma 2.7** Assume that the initial data $n^0$ is a nonnegative function in $L^2(\mathbb{R}^d)$ and that there is a locally bounded function $\Gamma(t)$ such that $|R(t, x)| \leq \Gamma(t)$. Then, the weak solutions in $C(\mathbb{R}^+; L^2(\mathbb{R}^d))$ to the Lotka-Volterra system (1.1) satisfy $n(t, x) \geq 0$.

**Proof.** Again we are going to prove that the negative part vanishes. We set $p = -n$, $p_+ = \max(0, p)$ and we have to justify the equation holds

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} (p(t, x))^2_+ \, dx + \int_{\mathbb{R}^d} |\nabla (p)_+|^2 = \int_{\mathbb{R}^d} (p(t, x))^2_+ R \leq \Gamma(t) \int_{\mathbb{R}^d} (p(t, x))^2_+.
$$

(2.11)

To do so, we can regularize with a smoothing kernel $\omega_\varepsilon(\cdot)$ and write first

$$
\frac{\partial}{\partial t} \omega_\varepsilon * p - \Delta \omega_\varepsilon * p = \omega_\varepsilon * (p R).
$$

Then we handle a smooth function ($C^\infty$ in $x$ and $C^1$ in time if the $R$ are continuous in time) as shown in Section 2.4. We can also truncate using a function $\chi_\rho(\cdot)$. We obtain

$$
\frac{\partial}{\partial t} \chi_\rho \omega_\varepsilon * p - \Delta [\chi_\rho \omega_\varepsilon * p] = \chi_\rho \omega_\varepsilon * (p R) - 2\nabla \chi_\rho \nabla \omega_\varepsilon * p - \omega_\varepsilon * p \Delta \chi_\rho.
$$

Therefore the chain rule indeed gives

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} (\chi_\rho \omega_\varepsilon * p(t, x))^2_+ \, dx + \int_{\mathbb{R}^d} |\nabla (\chi_\rho \omega_\varepsilon * p)_+|^2 = \int_{\mathbb{R}^d} (\chi_\rho \omega_\varepsilon * p(t, x))^2_+ R - 2\int_{\mathbb{R}^d} (\chi_\rho \omega_\varepsilon * p(t, x))_+ [\nabla \chi_\rho \nabla \omega_\varepsilon * p - \omega_\varepsilon * p \Delta \chi_\rho] \, dx
$$

and thus

$$
\frac{1}{2} \int_{\mathbb{R}^d} (\chi_\rho \omega_\varepsilon * p(t, x))^2_+ \, dx \leq \int_0^t \int_{\mathbb{R}^d} (\chi_\rho \omega_\varepsilon * p(s, x))^2_+ \, ds + 2 \int_0^t \int_{\mathbb{R}^d} [\nabla \chi_\rho \nabla \omega_\varepsilon * p|^2 + |\omega_\varepsilon * p \Delta \chi_\rho|^2] \, dx.
$$
We can now precise the choice of the truncation function. We set $\chi_{\rho}(x) = \chi(\frac{x}{\rho})$, with a function $\chi(\cdot)$ that satisfies

\[
\begin{cases}
\chi \in \mathcal{D}(\mathbb{R}^d), & 0 \leq \chi(\cdot) \leq 1, \\
\chi(x) = 1 & \text{for } |x| \leq 1, \\
\chi(x) = 0 & \text{for } |x| \geq 2.
\end{cases}
\]

And we let $\rho \to \infty$, and thus $\chi_{\rho} \to 1$. In the above expression, the last integral vanishes because $|\nabla \chi_{\rho}| \leq C/\rho$ and $|\Delta \chi_{\rho}| \leq C/\rho^2$ while $\nabla \omega_{\varepsilon} * p$ and $\omega_{\varepsilon} * p$ are $L^2$ functions because $p$ is.

Therefore, we obtain using the Lebesgue Dominated Convergence Theorem

\[
\frac{1}{2} \int_{\mathbb{R}^d} \omega_{\varepsilon} * p(t,x) \frac{d}{t} dx \leq \int_0^t \int_{\mathbb{R}^d} \omega_{\varepsilon} * p(s,x) \omega_{\varepsilon} * (p(s) R(s)) ds + \int_0^t \int_{\mathbb{R}^d} \omega_{\varepsilon} * p(s,x) \frac{d}{s}.
\]

Then, we let $\varepsilon \to 0$ and obtain

\[
\frac{1}{2} \int_{\mathbb{R}^d} p(t,x) \frac{d}{t} dx \leq \int_0^t \int_{\mathbb{R}^d} (p(s,x))_+ (p(s) R(s)) ds + \int_0^t \int_{\mathbb{R}^d} (p(s,x))_+ \\
\leq \int_0^t [\Gamma(s) + 1] \int_{\mathbb{R}^d} (p(s,x))_+.
\]

Because $\Gamma(t)$ is locally bounded, the Gronwall lemma implies that $\int_{\mathbb{R}^d} (p(t,x))_+ dx = 0$ and therefore $p(t,x) \leq 0$ almost everywhere.

**Exercise.** Prove the same positivity result in $L^1$ in place of $L^2$. [Hint:] replace $(u)_+$ by a convex function with linear growth at infinity.

### 2.8 Positivity of weak solutions to reaction kinetics equations

In the same way, we consider a weak solution $u \in C(\mathbb{R}^+; L^2(\mathbb{R}^d))$ to the parabolic equations

\[
\begin{cases}
\frac{\partial n}{\partial t} - \Delta n + n R(t,x) = Q(t,x) & \text{in } \mathbb{R}^d, \\
n(t = 0, x) = n^0(x).
\end{cases}
\] (2.12)

**Lemma 2.8** Assume $|R(t,x)| \leq \Gamma(t)$ with $\Gamma \in L^\infty_{\text{loc}}(\mathbb{R}^+)$ and $Q \in C(\mathbb{R}^+; L^2(\mathbb{R}^d))$. If $Q \geq 0$ and $n^0 \geq 0$, then $n \geq 0$.

We leave the proof as an exercise.
2.9 Heat kernel and explicit solutions

Before solving a problem with variable coefficients as (2.12), it is usual (see [13, 34]) to look for the fundamental solution. This is to solve the PDE with the initial data a Dirac mass, for the heat equation this means
\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u &= 0 \quad \text{in } \mathbb{R}^d, \\
u(t = 0, x) &= \delta(x).
\end{align*}
\] (2.13)

**Lemma 2.9** The fundamental solution to the heat equation is given by the explicit form
\[
K(t, x) = \frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/4t}.
\] (2.14)

This means that for all test functions \( \Phi \in D([0, T) \times \mathbb{R}^d) \) there holds
\[
\int_0^T \int_{\mathbb{R}^d} K(t, x) \left[ -\frac{\partial \Phi}{\partial t} - \Delta \Phi \right] dx \, dt = \Phi(t = 0, x = 0).
\] (2.15)

Notice that
\[
\int_{\mathbb{R}^d} K(t, x) dx = 1 \quad \forall t > 0,
\]
\[
K(\varepsilon, x) \rightarrow \delta(x) \quad \text{as } \varepsilon \rightarrow 0^+.
\]

We give two proofs.

**Proof.** (1) For \( \varepsilon > 0 \) small enough, we have, by integration by parts,
\[
\int_0^T \int_{\mathbb{R}^d} K(t, x) \left[ -\frac{\partial \Phi}{\partial t} - \Delta \Phi \right] dx \, dt = \int_{\mathbb{R}^d} \Phi(x) dx + \int_0^T \int_{\mathbb{R}^d} \left[ \frac{\partial K}{\partial t} - \Delta K \right] \Phi dx \, dt
\]
and one readily checks that \( \frac{\partial K}{\partial t} - \Delta K = 0 \) for \( t > 0 \) (left as an exercise). Therefore we arrive at
\[
\int_0^T \int_{\mathbb{R}^d} K(t, x) \left[ -\frac{\partial \Phi}{\partial t} - \Delta \Phi \right] dx \, dt = \int_{\mathbb{R}^d} \Phi(\varepsilon, x) dx + \int_0^T \int_{\mathbb{R}^d} K(t, x) \left[ -\frac{\partial \Phi}{\partial t} - \Delta \Phi \right] dx \, dt.
\]
As \( \varepsilon \rightarrow 0 \), we obtain (2.15) because
\[
\int_0^\varepsilon \int_{\mathbb{R}^d} K(t, x) \left| -\frac{\partial \Phi}{\partial t} - \Delta \Phi \right| dx \, dt \leq \int_0^\varepsilon |\Omega| \sup_{t, x} \left| -\frac{\partial \Phi}{\partial t} - \Delta \Phi \right| \leq C\varepsilon.
\]

**Proof.** (2) This is a variant of the above proof. For \( \varepsilon > 0 \), \( K_\varepsilon(t, x) = K(t + \varepsilon, x) \) is a solution to \( \frac{\partial K_\varepsilon}{\partial t} - \Delta K_\varepsilon = 0 \) for \( t > 0 \) (left as an exercise) with initial data \( K(\varepsilon, x) \). As \( \varepsilon \rightarrow 0 \) we may apply the stability result of section 2.2 and we recover the result because \( K(\varepsilon, x) \rightarrow \delta(x) \) as \( \varepsilon \rightarrow 0^+ \).
Another variant is to consider more generally $K_\varepsilon = K \star \omega_\varepsilon(x)$ with $\omega_\varepsilon$ a family of smooth functions that converge to a Dirac mass as $\varepsilon \to 0$ and apply the following remark.

Once the fundamental solution is known, one can also solve the Cauchy problem
\[
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u &= 0 \quad \text{in } \mathbb{R}^d, \\
u(t = 0, x) &= u^0(x).
\end{aligned}
\]

Its solution is given by the convolution
\[
u(t, x) = u^0 \star K(t) = \int_{\mathbb{R}^d} u^0(y) K(t, x - y) dy.
\]

When $u^0$ is smooth with sub-exponential growth (so as to be able to integrate in the convolution formula), this is the smooth solution to the heat equation.

Even for $u^0 \in L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, this defines a function $u \in C^\infty((0, \infty) \times \mathbb{R}^d)$ which is indeed a smooth solution to the heat equation and also
\[
\|u(t)\|_{L^p(\mathbb{R}^d)} \leq ||u^0||_{L^p(\mathbb{R}^d)}.
\]

Consequently, one can also find the solution to
\[
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u &= Q(t, x) \quad \text{in } \mathbb{R}^d, \\
u(t = 0, x) &= 0.
\end{aligned}
\]

It is given by the Duhamel formula
\[
u(t, x) = \int_0^t \int_{\mathbb{R}^d} K(t - s, x - y) Q(s, y) dy ds.
\]

**Exercise.** Show that
1. for $Q \in L^1([0, T] \times \mathbb{R}^d)$, we have $\int_{\mathbb{R}^d} u(t, x) dx = \int_0^T Q(s, y) dy ds$,
2. for $Q \in L^1([0, T]; L^p(\mathbb{R}^d))$, we have
   2a. $\|u(t)\|_{L^p(\mathbb{R}^d)} \leq \int_0^T \|Q(s)\|_{L^p(\mathbb{R}^d)} ds$, for all $p \in [1, \infty)$.
   2b. $u \in C([0, T]; L^p(\mathbb{R}^d))$ for all $1 \leq p < \infty$.
3. For $Q \in L^1([0, T]; L^\infty(\mathbb{R}^d))$, we have $u \in L^\infty([0, T] \times \mathbb{R}^d)$.

## 2.10 Nonlinear problems

### 2.10.1 A general result for Lipschitz nonlinearities

The most standard existence theory consists in Lipschitz continuous nonlinearities in $L^p(\mathbb{R}^d)$,
\[
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u &= Q(x, u(t))) \quad \text{in } \mathbb{R}^d, \\
u(t = 0, x) &= u^0(x) \in L^p(\mathbb{R}^d).
\end{aligned}
\]
We need two assumptions
\[ \|Q(x,[u])\|_{L^p(\mathbb{R}^d)} \leq M_Q \|u\|_{L^p(\mathbb{R}^d)}, \]  
(2.18)
\[ \|Q(x,[u]) - Q(x,[v])\|_{L^p(\mathbb{R}^d)} \leq L_Q \|u - v\|_{L^p(\mathbb{R}^d)}, \]  
(2.19)

**Theorem 2.10** With the assumptions (2.18)–(2.19) and \( u^0(x) \in L^p(\mathbb{R}^d) \) with \( 1 \leq p < \infty \), then there is a unique solution \( u \in C(\mathbb{R}^+; L^p(\mathbb{R}^d)) \) to (2.17).

**Proof.** We consider a small \( T \) (to be chosen later on), the Banach space \( E = C([0,T]; L^p(\mathbb{R}^d)) \) into itself and the mapping \( \Phi : E \to E \) defined by \( u = \Phi(v) \) is the solution to the equation of the type (2.16)
\[ \begin{cases} \frac{\partial u}{\partial t} - \Delta u = Q(x,[v(t)]) & \text{in } \mathbb{R}^d, \ 0 \leq t \leq T, \\ u(t = 0, x) = u^0(x) \in L^p(\mathbb{R}^d). \end{cases} \]  
(2.20)
Notice that \( u \in E \) because of assumption (2.18). We claim that for \( T \) small enough, \( \Phi \) is a strong contraction because
\[ \|\Phi(v_1) - \Phi(v_2)\|_E \leq L_Q T \|v_1 - v_2\|_E. \]  

Indeed from the properties of the solutions to (2.16) we have, using assumption (2.19),
\[ \|u_1(t) - u_2(t)\|_{L^p(\mathbb{R}^d)} \leq \int_0^T \|Q(s,[v_1(s)]) - Q(s,[v_2(s)])\|_{L^p(\mathbb{R}^d)} ds \leq L_Q \int_0^T \|v_1(s) - v_2(s)\|_{L^p(\mathbb{R}^d)} ds, \]
and thus
\[ \|u_1 - u_2\|_E \leq L_Q T \|v_1 - v_2\|_E. \]

Now choose \( T \) such that \( L_Q T = 1/2 \). The Banach-Picard fixed point theorem asserts there is a unique fixed point \( u \). This is the unique solution to (2.17) on \([0,T]\).

We can iterate the argument to build a solution on \([T,2T], [2T,3T]...\)

\[ \text{2.10.2 Example 1.} \]

Consider the local nonlinear problem
\[ \begin{cases} \frac{\partial u}{\partial t} - \Delta u = Q(u), \\ u(t = 0, x) = u^0(x) \in L^p(\mathbb{R}^d). \end{cases} \]  
(2.21)
with
\[ Q(0) = 0, \quad |Q'(\cdot)| \leq L_Q. \]  
(2.22)

**Corollary 2.11** With assumption (2.22), there is a unique solution to (2.21) in \( C([0,T]; L^p(\mathbb{R}^d)) \).
Proof. Indeed we can apply the Theorem 2.10 because both assumptions (2.18) and (2.19) are satisfied (the details are left to the reader. □

Notice that when \( u^0 \in L^1 \cap L^\infty(\mathbb{R}^d) \) we obtain the same solution in all \( L^p \). This can be seen from the construction of the fixed point in Theorem 2.10. The uniqueness of the weak solution \( u \) for a given \( Q(v) \) shows that the Picard iterations are the same for all \( L^p \).

2.10.3 Example 2.

Consider now the local nonlinear problem

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u &= u(1 - u), \\
\quad u(t = 0, x) &= u^0(x) \quad \text{with } 0 \leq u^0 \leq 1,
\end{aligned}
\]

(2.23)

The Theorem 2.10 cannot be applied directly because \( u \mapsto u(1 - u) \) is not lipschitz continuous on \( \mathbb{R} \). Nevertheless a small variant leads to the

**Corollary 2.12** There is a unique solution to (2.23) \( u \in L^\infty([0, T] \times \mathbb{R}^d) \) such that

\[
0 \leq u(t, x) \leq 1.
\]

Proof. Define \( Q(u) = u_+(1 - u)_+ \), this a Lipschitz continuous function and there is a solution to (2.21). Because \( Q(\cdot) \geq 0 \), the non-negativity result of Lemma 2.7 asserts that \( u \geq 0 \) and the same result for \( 1 - u \) tells us that \( 1 - u \geq 0 \) (in fact we need a variant in \( L^\infty \) in place of \( L^2 \)).

Therefore, for the solution we also have \( Q(u) = u(1 - u) \) and the result follows.

Notice that, still applying the positivity result of Lemma 2.7 set in \( L^\infty \), any solution should satisfy \( 0 \leq u(t, x) \leq 1 \), and thus the bounded solution is unique. □
Chapter 3

Traveling waves

The relaxation results in Chapter 1 show that on bounded domain and with small nonlinearities we cannot expect spectacular behaviors in reaction-diffusion equations. When working on the full space, these conclusions fall down and we can observe a first possible type of interesting behavior: traveling waves.

This chapter gives several examples motivated by models from biology even though combustion waves or phase transitions are among the most noticeable origins of the questioning. In ecology it can describe the progress of an invasive species in a virgin environment. It can also be an epidemic spread as bubonic plague in Europe in the 14th century (see Figure 3.1). In neuroscience it can be calcium pulses propagating along a nerve, and their study motivated J. Evans when he introduced the now-called Evans function\(^1\) for studying their stability.

3.1 Setting the problem

The simplest examples is to consider the equation

\[
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(u), \quad t \geq 0, \ x \in \mathbb{R},
\]

where \(f(u)\) denotes a reaction term. For this kind of equation we define

**Definition 3.1**  A traveling wave solution is a solution of the form \(u(t,x) = \nu(x-ct)\) with \(c \in \mathbb{R}\) a constant called the traveling speed.

It is usual that \(f(u)\) admits two stationary states, say \(f(0) = f(1) = 0\). This is the case for instance for the Fisher/KPP equation where \(f(u) = u(1-u)\) or the bistable case \(f(u) = \)

Figure 3.1: Propagation of the front of plague through Europe in the middle of 14th century.

\[ u(1 - u)(u - 1/2). \] Then, we may complete this definition with the conditions \( v(-\infty) = 1, \) \( v(\infty) = 0. \) When \( c > 0, \) this expresses that the state \( v = 1 \) invades the state \( v = 0 \) and vice-versa. Then we arrive to a simple equation on \( v \)

\[
\begin{aligned}
&v''(x) + cv'(x) + f(v(x)) = 0, \quad x \in \mathbb{R}, \\
v(-\infty) = 1, \quad v(+\infty) = 0.
\end{aligned}
\]

Notice that this problem is translational invariant and \( v(x + a) \) is always a solution if \( v(x) \) is one. Therefore we can normalize it with \( v(0) = \frac{1}{2} \) for instance.

The natural question is to know what properties of the steady state make that one connect them by a traveling wave. We will give examples covering the cases

- 0 is unstable and 1 is stable (Fisher/KPP case),
- 0 and 1 are stable (bistable case),
- the state 0 is connected to itself by a special traveling wave called a pulse in this case (FitzHugh-Nagumo case).
- the state 0 is unstable and 1 is Turing unstable (see Chapter: turing). The example of non-local Fisher/KPP equation is treated in [7].

There are two useful observations. The first consists in integrating on the line and, because \( v'(-\infty) = v'(\infty) = 0, \) we find

\[
c = \int_{-\infty}^{\infty} f(v(x)) \, dx.
\]
The second observation consists in computing the energy of the system
\[ \frac{1}{2} (v'(x)^2)' + c(v'(x)^2) + \frac{d}{dx} W(v(x)) = 0. \]
with
\[ W(u) = \int_0^u f(v)dv. \]
Because \( v'(-\infty) = v'(\infty) = 0 \), we find
\[ c \int_{-\infty}^{\infty} (v'(x)^2)dx = W(1). \] (3.1)
For instance in the Fisher/KPP case, \( f(\cdot) \geq 0 \), therefore \( W(1) > 0 \) and we find that \( c > 0 \). This means that the state \( v = 1 \) is indeed invading.

We first give the simplest results when there is a unique pair \((c^*, v)\) that satisfies this equation (Fisher/KPP equation with ignition temperature, Allen-Cahn). Then we turn to the 'unusual' case of the original Fisher/KPP equation.

### 3.2 The Fisher/KPP equation with ignition temperature

The Fisher/KPP equation with ignition temperature arises from the theory of combustion when a minimum 'temperature' \( 0 < \theta < 1 \) is needed to burn the gas. It gives the model
\[ \frac{\partial}{\partial t} u - \nu \frac{\partial^2}{\partial x^2} u = f(u), \quad t \geq 0, \quad x \in \mathbb{R}. \] (3.2)
with a reaction term given by
\[ f_\theta(u) = \begin{cases} 
0 & \text{for } 0 \leq u \leq \theta, \\
> 0 & \text{for } \theta < u < 1,
\end{cases} \quad f(1) = 0. \] (3.3)

The traveling wave problem is still to find \( c \) and \( v \) such that
\[ \begin{cases} 
\nu v''(x) + cv'(x) + f(v(x)) = 0, & x \in \mathbb{R}, \\
v(-\infty) = 1, & v(+\infty) = 0.
\end{cases} \] (3.4)

We are going to prove the

**Theorem 3.2** For the Fisher/KPP equation with ignition temperature, i.e., (3.2) when \( f(\cdot) \) satisfies (3.3), there is a unique decreasing traveling wave solution \((c^*, v)\) normalized with \( v(0) = \frac{1}{2} \) and it holds that \( c^* > 0 \).
More is known about this problem, see [6, 38]. For instance, we give below explicit bounds on $c^*$.

**Proof.** The easiest proof relies on the *phase space* method for O.D.Es which we follow here. It is however limited to simple problems and more natural PDE methods can be found in [6, 38].

We decompose the proof in three steps: (i) we reduce the problem to a simpler O.D.E. (ii) we prove monotonicity in $c$, (iii) we prove existence.

(i) We reduce the traveling wave problem to an O.D.E. problem. We fix $c$ and set $w = -v'$ (so that $w > 0$ because we look for decreasing $v$). Then, the system (3.4) becomes a system of differential equations

$$
\begin{align*}
&v' = -w, \\
&w' = -\frac{c}{\nu} w + \frac{r}{\nu} f(v), \\
&v(-\infty) = 1, \quad w(-\infty) = 0, \quad v(+\infty) = 0, \quad w(+\infty) = 0.
\end{align*}
$$

(3.5)

It can be further simplified because by monotonicity, we can invert $v(x)$ as a function $X(v)$, $0 \leq v \leq 1$ and define a function $\tilde{w}(v) = w(X(v))$. In place of (3.5), we have to find a solution to

$$
\begin{align*}
&\frac{d\tilde{w}(v)}{dv} = \frac{dw}{dx} \left( \frac{dv}{dx} \right)^{-1} = \frac{c}{\nu} - \frac{r}{\nu} \frac{f(v)}{\tilde{w}}, \quad 0 \leq v \leq 1, \\
&\tilde{w}(0) = \tilde{w}(1) = 0, \quad \tilde{w} \geq 0.
\end{align*}
$$
Therefore, we arrive at the question to know if the solution to the Cauchy problem

\[
\begin{aligned}
\frac{d\tilde{w}_c(v)}{dv} &= \frac{c}{\nu} - \frac{1}{\nu} \frac{f(v)}{\tilde{w}_c(v)}, & 0 \leq v \leq 1, \\
\tilde{w}_c(0) &= 0,
\end{aligned}
\]

(3.6)
can also achieve, for a special value of \(c\), the conditions

\[
\tilde{w}_c(1) = 0, \quad \tilde{w}_c(v) \geq 0, \quad \text{for } 0 \leq v \leq 1.
\]

(3.7)

Figure 3.3: Traveling wave solutions to the Fisher/KPP equation with temperature ignition (3.4) with threshold \(\theta = 0.25\) plotted in the phase space variables (3.6). We have plotted 4 different values of \(c\). The x-axis represents \(v\) and the y-axis represents the function \(\tilde{w}(v)\). Left: two cases where the solution to (3.6) does not vanish (large \(c\)), these are called Type I. Right: the limiting case \(c = c^*\) and a smaller value of \(c\), these are called Type II. Notice that the \(w\) scale is larger on the left than on the right.

Notice that there is a priori a singularity at \(v = 0\) because the numerator and denominator vanish in the right hand side of (3.6). But for \(0 \leq v \leq \theta\), \(f(v) \equiv 0\) and the solution is simply

\[
\tilde{w}_c(v) = \frac{c}{\nu} v, \quad 0 \leq v \leq \theta.
\]

Then it can be continued smoothly as a simple (nonsingular) O.D.E. until either we reach \(v = 1\), either \(\tilde{w}_c\) vanishes and the problem is not defined any longer. Numerics indicate that, depending on \(c\), either we have

- \(\tilde{w}_c(v) > 0\) for \(0 \leq v \leq 1\) (call it Type I), then we set \(v_c = 1\), or
- \(\tilde{w}_c(v_c) = 0\) for some \(0 < v_c < 1\) (call it Type II), then the equation tells us that \(\tilde{w}_c'(v_c) = -\infty\).

In both cases these are not solutions because they cannot fulfill (3.7). In the limiting case \(v_c^* = 1\) there is a solution. These possible behaviors are depicted in Figure 3.3.

(ii) We now prove that this last case can only occur for a single \(v_c\) based on the

**Lemma 3.3** The mapping \(c \mapsto \tilde{w}_c(v)\) is increasing for those \(v\) where it is defined, i.e., for \(0 < v < v_c\). Moreover for \(c' > c\) we have \(v_{c'} > v_c\) (and \(v_{c'} = v_c\) if \(v_c = 1\)) and

\[
\tilde{w}_{c'}(v) \geq \tilde{w}_c(v) + (c' - c) v/\nu, \quad 0 \leq v \leq v_c.
\]

(3.8)
Proof. Set \( z_c(v) = \frac{d\tilde{w}(v)}{dv} \). It satisfies,
\[
\frac{dz_c(v)}{dv} = \frac{1}{\nu} + \frac{1}{\nu} \frac{f(v)}{\tilde{w}(v)^2} z_c(v), \quad z_c(0) = 0.
\]
From this equation we deduce that \( z_c(v) \geq \frac{v}{\nu} \) as long as it is defined, i.e., that \( \tilde{w}_c(v) \) does not vanish which is for \( 0 \leq v < v_c \). The conclusions follow.

After integration in \( c \) we find the inequality on \( \tilde{w} \) which holds up to \( v = v_c \) by continuity. □

Consequently, \( \tilde{w}_c(1) \) is an increasing function of \( c \). Therefore there can indeed be at most one value of \( c \) satisfying the condition \( \tilde{w}_c(1) = 0 \).

(iii) For existence, we choose now \( \nu = 1 \) to simplify. We introduce the two positive real numbers defined (uniquely) by
\[
\bar{c}^2 = \nu \int_0^1 \frac{f(v)}{v} \, dv, \quad \tilde{c}^2 = 4\nu \max_{0 \leq v \leq 1} \frac{f(v)}{v}.
\]
Notice that \( \bar{c} < \bar{c} \). In fact we are going to prove that

Lemma 3.4 For \( c > \bar{c} \), the solution is of Type I. For \( c \leq \bar{c} \), the solution is of Type II. And thus
\[
\frac{c}{2\nu} \geq \frac{d\tilde{w}_c(v)}{dv} = \frac{c}{\nu} - 2\frac{f(v)}{c v} \geq \frac{c}{\nu} + \frac{\tilde{c}^2}{2\nu c},
\]
which contradicts \( c > \bar{c} \). This means that \( v_0 = 1 \) the situation is of Type I.

Proof. Upper estimate. We first show that for \( c > \bar{c} \), the solution is of Type I. We consider the largest interval \([0, v_0] \subset [0, 1]\) on which \( \tilde{w}_c(v) \geq \frac{c}{\nu} v \). Because \( \tilde{w}_c(v) = \frac{c}{\nu} v \) on \([0, \theta]\) clearly \( v_0 > \theta \). If \( v_0 < 1 \) (otherwise we are done), then \( \tilde{w}_c(v_0) \leq \frac{c}{\nu} v_0 \). Then, for \( \theta \leq v \leq v_0 \), we have \( \tilde{w}_c(v) \geq \frac{c}{\nu} \theta \) and thus
\[
\frac{c}{2\nu} \geq \frac{d\tilde{w}_c(v_0)}{dv} = \frac{c}{\nu} - 2\frac{f(v_0)}{c v_0} \geq \frac{c}{\nu} + \frac{\tilde{c}^2}{2\nu c},
\]
which contradicts \( c > \bar{c} \). This means that \( v_0 = 1 \) the situation is of Type I.

Lower estimate. We show that for \( c \leq \bar{c} \) the solution is of Type II. We notice that \( \tilde{w}_c(v) \leq \frac{c}{\nu} v \) as long as it is defined (because \( \frac{f(v)}{\tilde{w}(v)} \geq 0 \)) and thus
\[
\frac{d\tilde{w}_c(v)}{dv} = \frac{c}{\nu} - \frac{f(v)}{\nu \tilde{w}_c(v)} \leq \frac{c}{\nu} - \frac{f(v)}{cv},
\]
and thus (and the inequality is strict for \( v > \theta \))
\[
\tilde{w}_c(v) \leq \frac{c}{\nu} v - \int_0^v \frac{f(s)}{cs} ds.
\]
This implies that, if the solution did not vanish before \( v = 1 \), we would have \( 0 \leq \tilde{w}_c(1) < c - \int_0^1 f(s) \, ds \). This implies that \( c < c \). This proves that the solution is of Type II for \( c \leq c \). □

(iv) We can conclude by a continuity argument on \( v_c \) in the region of Type II. By the monotonicity argument of step (ii) and because of (3.8) as long as \( v_c < 1 \), the point \( v_c \) increases continuously with controlled uniform growth (see exercise below). Therefore, \( \max_c v_c = 1 \), where the \( \max \) is taken on the \( c \) of Type II, and it is achieved for \( c = c^* \), in other words \( v_{c^*}(1) = 0 \). When \( c > c^* \), the solution is of Type I again by the monotonicity argument (we know from lemma 3.3 that \( \tilde{w}_c(1) \) decreases uniformly with \( c \)). □

Exercise Take \( \nu = r = 1 \) to simplify. Find a lower bound on \( \frac{d}{dc} v_c \). Prove it is positive as long as \( v_c < 1 \) and that it is uniformly positive for \( v_c \approx 1 \).

[Hint] \( c \int_0^{v_c} \tilde{w}_c(v) \, dv = \int_0^{v_c} f(v) \, dv \).

\[
\begin{align*}
f(v_c) \frac{dv}{dc} & = \int_0^{v_c} \tilde{w}_c(v) \, dv + c \int_0^{v_c} \tilde{w}_c(v) \, dv \geq \frac{c}{2} \theta^2 + \frac{c}{2} (v_c)^2.
\end{align*}
\]

Exercise For \( \varepsilon > 0 \), study the (regularized) Cauchy problem

\[
\begin{align*}
\begin{cases}
\frac{dw}{dv} = c - \frac{f(v)}{\sqrt{v^2 + w(v)^2}}, & 0 \leq v \leq 1, \\
w(0) = 0.
\end{cases}
\end{align*}
\]

(i) Show that one cannot achieve \( w(1) = 0 \) with \( w(v) \geq 0 \) for \( 0 \leq v \leq 1 \) whatever are \( c \) or \( \varepsilon \).

(ii) Show that, are for all \( v \), the mapping \( c \mapsto w_{c,\varepsilon}(v) \) is increasing and the mapping \( \varepsilon \mapsto w_{c,\varepsilon}(v) \) is non decreasing.

(iii) For \( \varepsilon \) fixed, show that one can find a unique \( c^\varepsilon \) that achieves \( w(v_c) = 0 \), \( w(v) \geq 0 \) on a maximal interval \( 0 \leq v \leq v_c \). What is the value \( w_{c^\varepsilon,\varepsilon}(v_c) \)?

(iv) Draw the solutions for several values of \( c \).

(v) Prove that \( v_c \to 1 \), \( c^\varepsilon \to c^* \) as \( \varepsilon \to 0 \).

Correction (i) Indeed, this implies \( w'(1) \geq 1 \) but the equation implies that \( w'(1) = c > 0 \).

(ii) Same proof as above.

(iii) As in the above proof, for \( c^2 > \|f\|_\infty / \theta \), we have \( w_c'(v) > 0 \) and for \( c < c_\ast \) we have \( w_c(1) < 0 \) with \( c_\ast \) the unique fixed point of \( c_\ast = \frac{1}{\int_0^1 f(v)} / \sqrt{\theta^2 + c_\ast^2} \). So, by monotonicity, there is a larger \( c = c^\varepsilon \) such that \( w \) vanishes at some point, \( w(v_c) = 0 \) and \( w(v) \geq 0 \). Therefore we have \( w'(v_c) = 0 \), which implies \( \varepsilon c^\varepsilon = f(v_c) \).

(iv) As \( \varepsilon \) decreases to 0, one can check (still by monotonicity) that \( c^\varepsilon \) increases to a limit \( c_f > c_\ast \).

On the otherhand, by the previous question, \( f(v_c) \to 0 \). One checks that \( v_c \) remains far from \( [0, \theta] \) and thus, by the assumption on \( f \), we have \( v_c \to 1 \). In the limit we obtain a solution to (3.6) which vanishes at \( v = 1 \).
3.3 Allen-Cahn (bistable) equation

Uniquely defined traveling waves solutions may exist for other nonlinearities. In this section we study the bistable nonlinearity related to the O.D.E.

\[
\frac{d}{dt} u(t) = u(t) \left( 1 - u(t) \right) \left( u(t) - \alpha \right),
\]

for some parameter

\[
0 < \alpha < 1.
\] (3.11)

It has three steady states, \( u \equiv 0 \) and \( u \equiv 1 \) are stable, \( u \equiv \alpha \) is unstable. Of course any solution will converge either to 0, for \( u^0 < \alpha \) or to 1 for \( u^0 > \alpha \). Also the region \( 0 \leq u^0 \leq 1 \) is invariant with time.

This bistable equation uses an improvement of the logistic growth term \( u(1 - u) \); it supposes that too low densities \( u(t) \), less than \( \alpha \), lead to extinction by lack of encounters between individuals. This is called Allee effect, [1]. It however takes his name from the theory of phase transitions\(^2\).

Next, we include motion of individuals and we obtain the Allen-Cahn equation

\[
\frac{\partial}{\partial t} u(t, x) - \Delta u(t, x) = u(t, x) \left( 1 - u(t, x) \right) \left( u(t, x) - \alpha \right).
\] (3.12)

It resembles the Fisher/KPP equation and also admits traveling wave solutions \( u(x, t) = v(x - ct) \), with \( v(\cdot) \) solution to

\[
\begin{aligned}
-c v'(x) - v''(x) &= v(x) \left( 1 - v(x) \right) \left( v(x) - \alpha \right), \\
v(-\infty) &= 1, \quad v(+\infty) = 0, \quad v(0) = \frac{1}{2}.
\end{aligned}
\] (3.13)

We have again imposed the condition $v(0) = \frac{1}{2}$ to avoid the translational invariance.

The following result is similar to the case of Fisher/KPP with ignition temperature

**Theorem 3.5** There exists a unique decreasing solution $(c^*, v)$ to (3.13) and

$$c^* > 0 \quad \text{for } 0 < \alpha < \frac{1}{2}, \quad c^* < 0 \quad \text{for } \frac{1}{2} < \alpha < 1, \quad c^* = 0 \quad \text{for } \alpha = \frac{1}{2}.$$  

The sign follows from the general principle in Section 3.1.

An explicit solution can be found:

**Exercise.** Set $u(x) = \frac{1}{2} - \frac{1}{2} \frac{e^{x/2} - e^{-x/2}}{e^{x/2} + e^{-x/2}}$, for $a \in \mathbb{R}$.

1. Check that it connects the state $u = 1$ to $u = 0$.
2. Check it satisfies the equation (3.13).
3. Write the relation between $\alpha$ and $c$.
4. In which case do we have a Fisher/KPP wave or a Allen-Cahn wave?

However a general proof is available which does not use the specific form of the bistable nonlinearity but only the properties that there is a unique root $\alpha$, $0 < \alpha < 1$ such that

$$f(0) = 0, \quad f'(0) < 0, \quad f(\alpha) = 0, \quad f(1) = 0, \quad f'(1) < 0,$$

$$f(u) < 0 \quad \text{for } 0 < u < \alpha, \quad f(u) > 0 \quad \text{for } \alpha < u < 1.$$

Following again the general principle in Section 3.1, the speed of the wave then depends upon the sign of $W(1)$ with

$$W(u) = \int_0^u f(v) dv.$$

**Proof.** As in Section 3.2, we consider (3.13) as an O.D.E. that we solve as a system of first order equations

$$\begin{cases}
  v'(x) = -w(x), \\
  w'(x) = -c w(x) + v(x)(1 - v(x))(v(x) - \alpha), \\
  v(-\infty) = 1, \quad w(-\infty) = 0, \quad v(+\infty) = 0, \quad w(+\infty) = 0.
\end{cases}$$

And because we look for $v$ decreasing, we can invert $v(x)$ as a function $X(v)$, $0 \leq v \leq 1$ and define a function $\tilde{w}(v) = w(X(v))$. Following the derivation of (3.6) in the case of Fisher/KPP equation with ignition temperature, we arrive here to

$$\begin{cases}
  \frac{d\tilde{w}(v)}{dv} = c - \frac{v(1 - v)(v - \alpha)}{\tilde{w}(v)}, \quad 0 \leq v \leq 1, \\
  \tilde{w}(0) = \tilde{w}(1) = 0, \quad \tilde{w} \geq 0.
\end{cases} \quad (3.14)$$

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Then, we consider the case $\alpha \leq 1/2$ only, because it is enough to begin at $v = 1$ to treat $\alpha > 1/2$ while we begin at $v = 0$ below. We argue in several steps.

(i) Firstly the singularity of the right hand side at $v = 0$ can be handled by expanding $\tilde{w}(v) \approx \tilde{w}'(0)v$ and computing from (3.14)

$$\tilde{w}'(0) - \frac{\alpha}{\tilde{w}'(0)} = c \iff \tilde{w}'(0) = S(c),$$

and the function $c \mapsto S(c) = \frac{1}{2}(c + \sqrt{c^2 + 4\alpha})$ is increasing.

This allows (using a version of Cauchy-Lipschitz theorem with singularities at the origin) to define a unique solution to

$$\begin{cases}
\frac{d\tilde{w}_c(v)}{dv} = c - \frac{v(1-v)(v-\alpha)}{\tilde{w}_c(v)}, \\
\tilde{w}(0) = 0, \quad \tilde{w}_c'(0) = S(c).
\end{cases} \quad (3.15)$$

Because $v(1-v)(v-\alpha) \leq 0$ on $[0, \alpha]$, we have $\frac{d\tilde{w}_c(v)}{dv} \geq c$ and thus

$$\tilde{w}_c(v) \geq cv \quad \text{on} \quad [0, \alpha].$$

Therefore, it is either defined and positive for all $0 \leq v \leq 1$, then we set $v_c = 1$ and call it Type I. Either it is defined on an interval $[0, v_c]$ with $\tilde{w}_c(v) = 0$ and

$$v_c > \alpha, \quad (3.16)$$

when the system reaches another singularity where $\tilde{w}_c'(v_c) = -\infty$, and call it Type II.

(ii) The property holds

$$c \mapsto \tilde{w}_c(v) \quad \text{is} \quad C^1 \quad \text{and increasing for} \quad 0 < v \leq v_c.$$ 

Indeed, following the Fisher/KPP case, we set $z_c(v) = \tilde{w}_c(v)'$ and we have

$$\begin{cases}
\frac{dz_c(v)}{dv} = 1 + z_c(v)\frac{v(1-v)(v-\alpha)}{\tilde{w}_c(v)^2}, \\
z_c(0) = 0, \quad z_c'(0) = S'(c) > \frac{1}{2}.
\end{cases}$$

And the solution to this equation is positive close to $v = 0$, it remains positive for all $v = 0$ because if $z_c(v)$ becomes too small the equation tells us that its derivative is positive.

(iii) It is easy to conclude that for $c$ large enough $\tilde{w}_c(v)$ remains increasing on $[0,1]$. In other words for $c$ large enough the solution is of type I.
We claim that the solution for $c \approx 0$ is of type II. Indeed, we can compute another relation because the equation reads

$$\frac{1}{2} \frac{d}{dv} \tilde{w}_c(v)^2 = c\tilde{w}_c - v(1 - v)(v - \alpha) = c\tilde{w}_c - W'(v),$$

with $W(v) = \int_0^v z(1 - z)(z - \alpha)dz$ depicted in Figure 3.4. We arrive at

$$\frac{1}{2} \tilde{w}_c(v)^2 = c \int_0^v \tilde{w}_c(z)dz - W(v). \tag{3.17}$$

For $c = 0$ this gives

$$\tilde{w}_0(v)^2 = -2W(v).$$

Because $W(v) \leq 0$ on $[0,1]$ only when $\alpha = 1/2$, this shows that $c = 0$ gives the 'standing wave' (traveling wave with speed 0). For $\alpha < 1/2$, $W$ vanishes at a point that we denote by $\beta$

$$W(\beta) = 0, \quad 1/2 < \beta < 1.$$}

In other words $c = 0$, and by continuity $c \approx 0$, give a solution of Type II.

(iv) As $c$ increases from $c = 0$, $v_c$ also increases by point (ii) and we can write from (3.17)

$$0 = c \int_0^{v_c} \tilde{w}_c(z)dz - W(v_c), \quad \tilde{w}_c(v_c) = 0.$$}

Differentiating in $c$, we obtain

$$0 = \int_0^{v_c} \tilde{w}_c(z)dz + c\tilde{w}_c(v_c) \frac{dv_c}{dc} - W'(v_c) \frac{dv_c}{dc} = \int_0^{v_c} \tilde{w}_c(z)dz - W'(v_c) \frac{dv_c}{dc},$$

or also, recalling (3.16),

$$\frac{dv_c}{dc} = \frac{\int_0^{v_c} \tilde{w}_c(z)dz}{v_c(1 - v_c)(v_c - \alpha)} > 0.$$}

So that we can define again $c^*$ as the maximum of the $c$ corresponding to type II. It has to satisfy $\tilde{w}_{c^*}(v_{c^*}) = 0$. By strong monotonicity, or by (3.17), it is also the minimum of the $c$ giving solutions of type I. \hspace{1em} \Box

### 3.4 The Fisher/KPP equation

We can now come to the more basic equation proposed by Fisher\footnote{Fisher, R. A. \textit{The genetical theory of natural selection}. Clarendon Press, 1930. Deuxième éd. : Dover, 1958. Troisième édition, présentée et annotée par Henry Bennett : Oxford Univ. Press, 1999.} for the propagation of a favorable gene in a population. It is to find a solution $u(t, x)$ to

$$\frac{\partial}{\partial t} u - \nu \frac{\partial^2}{\partial x^2} u = r \ u(1 - u), \quad t \geq 0, \ x \in \mathbb{R}, \tag{3.18}$$
with \( \nu > 0, r > 0 \) given parameters. They describe respectively the diffusion ability (due to active motion for instance as in zooplankton) and the growth rate of the population. The same equation is also called the KPP\(^4\) equation and describes a combustion wave in a chemical reaction. It makes sense in any dimension but traveling waves are naturally one dimensional.

A simple observation is as follows: the steady state \( u \equiv 0 \) is unconditionally unstable. This means that any homogeneous small initial perturbation \( \delta u^0 \) will give an exponential growth \( u \approx e^{rt}\delta u^0 \). But the steady state \( u \equiv 1 \) is unconditionally stable; any homogeneous (at least) small initial perturbation \( u^0 = 1 - \delta u^0 \) will relax exponentially to 1. These are the reasons why we expect that the 'colonized' state \( u = 1 \) invades the 'uncolonized' state \( u = 0 \). To describe this invasion process, we again look for solution \( u(t, x) = v(x - ct) \). Inserting this definition in the Fisher/KPP equation (3.18), we obtain

\[
\nu v''(x) + cv'(x) + rv(x)(1 - v(x)) = 0, \quad x \in \mathbb{R},
\]

and because we want it to describe the progression of an invasion front corresponding to \( u = 1 \) into an uncolonized region \( u = 0 \), we complete the definition with the conditions at infinity

\[
v(-\infty) = 1, \quad v(+\infty) = 0, \quad v(0) = 1/2,
\]

and, again, the last condition is to fix the translational invariance.

The situation here very different from the case with ignition temperature and from the Allen-Cahn equation. A famous result is the

**Theorem 3.6** For any \( c \geq c^* := 2\sqrt{\nu r} \), there is a unique (traveling wave) solution \( v, 0 \leq v(x) \leq 1 \), to (3.19)–(3.20). It is monotonically decreasing.

The quantity \( c^* \) is called the minimal propagation speed. There are several ways to motivate that this speed \( c^* \) corresponds to the most stable traveling wave; we mention one later based on perturbation of the nonlinear term by including an ignition temperature \( \theta \) and letting \( \theta \) vanish. It is also the type of wave that appears as the long time limit of the evolution equation with an initial data with compact support. See [6, 38].

The condition \( c \geq c^* \) can be derived in studying the 'tail' of \( v(x) \) for \( x \) close to \(+\infty\). Because \( v = 0 \) is unstable, we can look for exponential decay as \( x \approx \infty \), namely

\[
v(x) \approx e^{-\lambda x}, \quad x \gg 1, \quad \lambda > 0.
\]

Inserting this in (3.19), we find

\[ \nu \lambda^2 - c \lambda + r = 0, \quad \lambda = \frac{c \pm \sqrt{c^2 - 4\nu r}}{2\nu}. \]  
\[ (3.21) \]

Because no oscillation can occur around \( v = 0 \) (due to the condition \( v > 0 \)) we should have \( c^2 \geq c^2 = 4\nu r \). And \( c > 0 \) is needed to have \( \lambda > 0 \), hence we should have \( c \geq c^* \).

The Fisher/KPP equation can be extended to a more general right hand side

\[ \frac{\partial}{\partial t} u - \nu \frac{\partial^2}{\partial x^2} u = f(u), \quad t \geq 0, \quad x \in \mathbb{R}. \]  
\[ (3.22) \]

with \( f : \mathbb{R} \to \mathbb{R} \) a smooth mapping satisfying

\[ f(0) = f(1) = 0, \quad f(u) > 0 \text{ for } 0 < u < 1. \]

When \( f \) is concave on \([0, 1]\) (this is the case of the Fisher/KPP term \( u(1 - u) \)), the linearization method explained above and Theorem 3.6, remain true with the slight modification that the minimal propagation speed is now

\[ c^* = 2\sqrt{f'(0)\nu}. \]

We do not give a complete proof and refer the interested reader to [14, 6, 38]. We just indicate the difficulties and two ways to solve them.

**Proof of Theorem 3.6 (Phase space).** If we try the *phase space* method as before, we set \( w = -v' \). Then, the system (3.19) becomes

\[
\begin{aligned}
v' &= -w, \\
w' &= -\frac{c}{\nu} w + \frac{r}{\nu} v(1 - v), \\
v(-\infty) &= 1, \quad w(-\infty) = 0, \quad v(+\infty) = 0, \quad w(+\infty) = 0.
\end{aligned}
\]
\[ (3.23) \]

It still can be further simplified because by monotonicity, we can still invert \( v(x) \) as a function \( X(v) \) and define a function \( \tilde{w}(v) = w(X(v)) \). In place of (3.23), we have to find a solution to

\[
\begin{aligned}
\frac{d\tilde{w}(v)}{dv} = \frac{dw}{dx} \left( \frac{dv}{dx} \right)^{-1} = \frac{c}{\nu} - \frac{r}{\nu} \frac{v(1 - v)}{\tilde{w}}, \quad 0 \leq v \leq 1, \\
\tilde{w}(0) = \tilde{w}(1) = 0, \quad \tilde{w} \geq 0.
\end{aligned}
\]
\[ (3.24) \]

This differential equation still has singularity at \( v = 0 \) and \( v = 1 \) but it is worse than what we have encountered yet. If we try to guess what is the slope \( \tilde{w}'(0) \), we find

\[ \tilde{w}'(0) = \frac{c}{\nu} - \frac{r}{\nu} \frac{1}{\tilde{w}'(0)}, \quad \tilde{w}'(0) = \frac{1}{2\nu} \left[ c \pm \sqrt{c^2 - 4\nu r} \right]. \]
This only confirms that we can only begin the trajectory when \( c \geq c^* \), but does not tell us which branch to use.

Instead, we can argue by perturbation and consider a family \( \theta \to 0 \) in the model with ignition temperature \eqref{eq:ignition}. We denote the corresponding solution by \( (c^*(\theta), \tilde{w}^\theta(v)) \). For a well-tuned \( f_\theta \) in \eqref{eq:ignition} we have

\[
f_\theta(u) \to ru(1-u) \quad \text{as} \quad \theta \to 0, \quad \text{in} \ C([0,1]),
\]

and the lower and upper bounds \( (c(\theta), \bar{c}(\theta)) \) in Lemma 3.4 are uniformly bounded in \( \theta \). Therefore we can extract a sequence

\[
\theta_n \to 0, \quad c^*(\theta_n) \to c^{**}, \quad 0 < c^{**} < \infty.
\]

On the other hand from \eqref{eq:wn}, we know know that \( \tilde{w}^\theta(v) \leq c^*(\theta) \) and thus

\[
0 \leq \tilde{w}^\theta(v) \leq c^*(\theta)v.
\]

Writing

\[
- \max_{0 \leq v \leq 1} f^\theta(v) \leq \frac{d}{dv} \tilde{w}^\theta(v)^2 = c^*(\theta)\tilde{w}^\theta(v) - f^\theta(v) \leq c^*(\theta)^2v,
\]

we conclude by the Ascoli Theorem that, still after extraction, \( \tilde{w}^\theta(v)^2 \) converges uniformly. Therefore, still for the uniform convergence we have

\[
\tilde{w}^{\theta_n}(v) \xrightarrow{n \to \infty} \tilde{w}^{**}(v), \quad 0 \leq \tilde{w}^{**}(v) \leq c^{**} v.
\]

It is easy to prove that \( \tilde{w}^{\theta_n}(v) \) remains uniformly positive in \( (0,1) \) and, from \eqref{eq:wn}, that \( \frac{d}{dv} \tilde{w}^{\theta_n}(v) \) also converges locally uniformly to a solution to \( \eqref{eq:wte} \).

It is possible to prove that this solution is the traveling wave with minimal speed but we will not do it here. \( \square \)

**Proof of Theorem 3.6 (Physical space).** We consider again the solution \( (c^*(\theta), v^\theta(x)) \) to the model with ignition temperature \eqref{eq:ignition} and prove uniform estimates in \( \theta \) showing that we can extract subsequences which converge. We do that in several steps and drop the dependency upon \( \theta \) in the course of calculations.

1st step. Uniform upper bound on \( c^*(\theta) < c^* = 2 \). This also uses the additional assumption \( f^\theta(v) < f(v) = v(1-v) \) on \( (0,1) \). We argue thy contradiction and assume \( c := c^*(\theta) \geq c^* \). We consider

\[
0 < \lambda = \frac{c - \sqrt{c^2 - 4}}{2} < c, \quad \lambda^2 - c\lambda + 1 = 0.
\]
For a large enough \( A \) we have \( A e^{-\lambda x} > v^{\theta}(x) \), because of the compared behavior at infinity, \( v(-\infty) = 1 \) and \( v(x) \approx e^{-cx} \) at \( +\infty \). Take the largest \( A \) where the two functions touch, that is

\[
A_0 e^{-\lambda x_0} = v^{\theta}(x_0) \quad A e^{-\lambda x} > v^{\theta}(x), \; \forall x \neq x_0.
\]

Then \( v'(x_0) = \lambda A_0 e^{-\lambda x_0}, \; v''(x_0) \leq \lambda^2 A_0 e^{-\lambda x_0} \), and thus, from equation (3.3)

\[
0 = v''(x_0) + v'(x_0) + f^{\theta}(v(x_0)) < \lambda^2 - c \lambda + \frac{v^{\theta}(x_0)(1 - v^{\theta}(x_0))}{v^{\theta}(x_0)} \leq \lambda^2 - c \lambda + 1 = 0,
\]

a contradiction. This proves the inequality.

\textit{2nd step.} Lower bound on \( c^*(\theta) \). We derive it from two equalities. The first is obtained by integrating (3.4) from \( x^- \) to \( x^+ \)

\[
c(v(x^+) - v(x^-)) + (v'(x^+) - v'(x^-)) + \int_{x^-}^{x^+} f^{\theta}(v(x)) \, dx = 0,
\]

and thus, passing to the limits \( x^- \to -\infty, \; x^+ \to \infty \), we find thanks to the conditions at infinity

\[
c^*(\theta) = \int_{-\infty}^{\infty} f^{\theta}(v^{\theta}(x)) \, dx. \tag{3.25}
\]

We can also multiply by \( v \) equation (3.4) and integrate. We find

\[
\frac{c}{2}(v^2(x^+) - v^2(x^-)) + (v v'(x^+) - v v'(x^-)) - \int_{x^-}^{x^+} (v'(x))^2 \, dx + \int_{x^-}^{x^+} v(x) f^{\theta}(v(x)) \, dx = 0,
\]

and in the limit

\[
\frac{c^*(\theta)}{2} = \int_{-\infty}^{\infty} v^{\theta}(x) f^{\theta}(v^{\theta}(x)) \, dx - \int_{-\infty}^{\infty} (v^{\theta'}(x))^2 \, dx. \tag{3.26}
\]

Subtracting (3.26) to (3.25), we obtain

\[
\frac{c^*(\theta)}{2} = \int_{-\infty}^{\infty} (1 - v^{\theta}(x)) f^{\theta}(v^{\theta}(x)) \, dx + \int_{-\infty}^{\infty} (v^{\theta'}(x))^2 \, dx > 0 \quad \text{(uniformly in} \; \theta \text{).} \tag{3.27}
\]

\textit{3rd step.} Uniform bound on \( v' \). We multiply (3.4) by \( v' \) and obtain

\[
c \, (v')^2 + \frac{1}{2} (v')^2' + F^{\theta}(v(x))' = 0,
\]

with, for \( 0 \leq v \leq 1 \),

\[
F^{\theta}(v) = \int_{0}^{v} f^{\theta}(u) \, du \quad \text{(a bounded increasing function)}.
\]
Integrating again this equation as before and passing to the limits, we find
\[ c^* (\theta) \int_{-\infty}^{\infty} \left( v^\theta (x) \right)^2 dx = - \int_{-\infty}^{\infty} F^\theta (v^\theta (x))' dx = F^\theta (1). \] (3.28)

And integrating between \( y \) and \( \infty \), we find
\[ \frac{1}{2} (v^\theta (y))^2 = c^* (\theta) \int_{y}^{\infty} \left( v^\theta (x) \right)^2 dx - F^\theta (y) \leq F^\theta (1) - F^\theta (y). \] (3.29)

4th step. Limit as \( \theta \to 0 \). These bounds combined to the equation (3.4) prove that \( v'' \) is uniformly bounde. Then we can pass to the uniform limits in (3.4) and find a solution to the Fisher/KPP traveling wave problem.

Exercise. Compute the linearized equation of (3.19) around \( u \equiv 1 \) and its exponential solutions. Show that the relations for exponential decay does not bring new conditions on \( c \) compared to (3.21).

Exercise. In (3.3), choose \( f^\theta \) increasing in \( \theta \). Set \( \zeta (v) = \frac{d}{d\theta} \tilde{w}^\theta (v) \)

1. Write differential equation on \( \zeta \)
2. Since \( \zeta (v) = 0 \) on \([0, \theta]\), show that \( \zeta (v) \) is negative.

Therefore solutions of type II will never converge to solutions of Fisher/KPP equations. In practice solutions of type I do not either. Correction: \( \frac{d}{dv} \zeta (v) = - \frac{df^\theta (v)}{d\theta} \frac{1}{\tilde{w}^\theta (v)} + \zeta (v) \frac{f^\theta (v)}{\tilde{w}^\theta (v)^2} \).

3.5 The Diffusive Fisher/KPP system

Another system related to the Fisher/KPP equation arises in modeling both combustion and bacterial colonies\(^5\). The system is
\[
\begin{cases}
\frac{\partial}{\partial t} u - d_u \Delta u = g(u)v, \\
\frac{\partial}{\partial t} v - d_v \Delta v = -g(u)v,
\end{cases}
\] (3.30)

here we have considered again a truncation function \( g(\cdot) \in C^2 ([0, \infty)) \), and because there is a priori no maximum principle for this system, we have to define it on the positive half-line
\[ g(0) = 0, \quad g'(0) = 0, \quad g'(u) > 0 \text{ for } u > 0, \] (3.31)


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Figure 3.5: The traveling wave profile for the Diffusive Fisher equation with $d_u = 1$, $d_v = 10$ and $g(u) = u$. The first unknown $u$ has a decreasing sharp front and $v$ a wide increasing shape.

and typically one takes $g(u) = u^n$ for some $n \geq 1$. See also Section 6.5.3 for related systems.

For combustion $v$ represents the concentration of one reactant and $u$ the temperature. The ratio $Le := d_v/d_u$ is called the Lewis number.

For bacterial colonies, $u$ represents the density of cells (and the colony is growing) and $v$ the nutrient consumed by the cells. See [19].

When $d_u = d_v$, a particular solution of system (3.30) consists in choosing $v = a - u$ ($a > 0$ a given positive number). Then it reduces to the Fisher/KPP equation (with temperature ignition in the case at hand) and thus it admits traveling waves. To avoid the parameter $a$, one can fix it equal to 1 and the traveling problem now reads

$$
\begin{align*}
-cu' - d_u u'' &= g(u)v, & u(-\infty) &= 1, & u(+\infty) &= 0, \\
-cv - d_v v'' &= -g(u)v, & v(-\infty) &= 0, & v(+\infty) &= 1.
\end{align*}
$$

(3.32)

Translational invariance can be normalized by, say $u(0) = 1/2$.

The general study of the solutions, for Lewis numbers $Le \neq 1$, is much harder than for the Fisher/KPP equation.

- Existence of a traveling wave $(c, u, v)$ with $u' < 0$, $v' > 0$, can be found in [8] in the case of ignition temperature,
- Existence for $c$ large enough, and uniqueness, can be found in [25] in the case without ignition temperature and $Le \leq 1$. 

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3.6 Competing species

An example arising in ecology comes from two species in competition for the resources. The model considers two population densities $u_1$ and $u_2$ and reads (after normalization)

$$
\begin{align*}
\frac{\partial}{\partial t} u_1 - d_1 \Delta u_1 &= r_1 u_1 (1 - u_1 - \alpha_2 u_2), \\
\frac{\partial}{\partial t} u_2 - d_2 \Delta u_2 &= r_2 u_2 (1 - \alpha_1 u_1 - u_2).
\end{align*}
$$

(3.33)

Notice that, according to Lemma 1.1 we have $u_1(t, x) \geq 0$ and $u_2(t, x) \geq 0$ when the initial data satisfy $u_1^0 \geq 0$ and $u_2^0 \geq 0$. Also the maximum principle holds: if $u_1^0 \leq 1$ then $u_1(t, x) \leq 1$, and if $u_2^0 \leq 1$ then $u_2(t, x) \leq 1$.

We may assume for instance that the species 1 is more motile than the species 2 that is $d_1 > d_2$. Depending on the predation coefficients $\alpha_1$, $\alpha_2$, and the specific growth rates $r_1$, $r_2$, is this an advantage? Does species 1 invade species 2 or the other way?

It can be noticed that there are several steady states

- the unpopulated steady state $(0, 0)$ is always unstable,
- the one-species (monoculture) steady states are $(0, 1)$ (and $(1, 0)$). They are stable if $\alpha_2 > 1$ (resp. $\alpha_1 > 1$) or unstable (in fact a saddle point) if $\alpha_2 < 1$ (resp. $\alpha_1 < 1$).
- there is another homogeneous steady state defined by
  $$
  \begin{pmatrix}
  1 \\
  \alpha_1 \\
  \end{pmatrix}
  \begin{pmatrix}
  u_1 \\
  u_2 \\
  
  \end{pmatrix}
  =
  \begin{pmatrix}
  1 \\
  1 \\
  \end{pmatrix}.
  $$

We assume that either $\alpha_2 < 1$ and $\alpha_1 < 1$ or $\alpha_2 > 1$ and $\alpha_1 > 1$, so that there is a unique positive solution, the coexistence state,

$$
(U_1, U_2) = \left( \frac{1 - \alpha_2}{1 - \alpha_2 \alpha_1}, \frac{1 - \alpha_1}{1 - \alpha_2 \alpha_1} \right).
$$

The above question is now to know which states can be connected by a traveling wave what is the sign of the speed $c$ of the traveling waves for $v_i(x - ct) = u_i(t, x)$. That is

$$
\begin{align*}
-v_1' - d_1 v_1'' &= v_1(1 - v_1 - \alpha_2 v_2), \\
-v_2' - d_2 v_2'' &= v_2(1 - \alpha_1 v_1 - v_2).
\end{align*}
$$

$v_1(-\infty) = U_1$, $v_1(+\infty) = 0$, $v_2(-\infty) = 0$, $v_2(+\infty) = U_2$.  

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Exercise. Consider the associated O.D.E system. Prove that
1. when $\alpha_1 < 1$ and $\alpha_2 < 1$, the steady states $U_1$ and $U_2$ are less than 1 and are stable,
2. when $\alpha_1 > 1$ and $\alpha_2 > 1$, the steady states $U_1$ and $U_2$ are less than 1 and are unstable.
3. Find $a$, $b$, $c$ such that the quantity $E = au_1 + bu_2 - c\ln(u_1) - \ln(u_2)$ satisfies, for some real numbers $\lambda$, $\mu$, $\nu$

$$\frac{d}{dt}E(t) = -\left(\lambda + \mu u_1(t) + \nu u_2(t)\right)^2.$$ 

4. Derive from this equality the long time behaviour of the system.

Solution. We only solve the stability questions. The linearized matrix around $(U_1, U_2)$ is given by

$$L = \begin{pmatrix} -r_1 U_1 & -\alpha_2 r_1 U_1 \\ -\alpha_1 r_2 U_2 & -r_2 U_2 \end{pmatrix}$$

and $tr(L) = -(r_1 U_1 + r_2 U_2) < 0$, $det(L) = r_1 r_2 U_1 U_2 (1 - \alpha_1 \alpha_2)$. Therefore

1. For $\alpha_1 \alpha_2 < 1$, $det(L) > 0$ and the two eigenvalues have negative real part. The system is stable
2. For $\alpha_1 \alpha_2 > 1$, $det(L) < 0$ and one of the two eigenvalues are real and one is positive (thus unstable) and the other is negative.

3.7 Spike solutions

One of the typical behaviour of solutions to elliptic equations or systems are spikes. These are solutions that vanish at plus and minus infinity while traveling waves take different values at each end. We give several examples from different areas of biology.

3.7.1 A model for chemotaxis

We refer to [32] for more explanations on the subject of chemotaxis. Here we consider a density $u(x)$ of bacteria attracted by a chemoattractant which concentration is denoted by $v(x)$. The variant of the Keller-Segel model we use here takes into account the nonlinear diffusion of cells introducing an exponent $p$. In the modeling literature it is aimed at representing saturation effects in high density regions (volume filling, quorum sensing, signal limiting). It can also be derived from refined models at the mesoscopic (kinetic) or even microscopic (individual centered)
scales. And we also consider one dimension in order to carry out explicit calculations,

\[
\begin{align*}
- (u^{1+p})_{xx} + (\chi w x)_x &= 0, \quad x \in \mathbb{R}, \\
- v_{xx} + \alpha v &= u, \\
u(-\infty) &= v(-\infty) = u(\infty) = v(\infty) = 0.
\end{align*}
\]

(3.34)

We are going to prove the following result

**Proposition 3.7** Assume that \(0 < p < 1\), then the system (3.34) has a unique spike solution which attains its maximum at \(x = 0\) and satisfies \(u \geq 0\), \(v \geq 0\).

**Proof.** 1st step (Reduction to a single equation) Solution \(u\) can be obtained explicitely interms of \(v\), writing \((u^{1+p})_x = \chi w x\) (the constant vanishes because we expect that \(u\), \(v\) and their derivatives vanish at \(\pm \infty\)). This is also, and for the same reason

\[
(u^p)_x = \frac{p\chi}{1 + p} v_x, \quad u = \left( \frac{p\chi}{1 + p} v \right)^{1/p}.
\]

We may insert this expression in the equation on \(v\) which gives

\[
-v_{xx} + \alpha v = \left( \frac{p\chi}{1 + p} v \right)^{1/p}.
\]

(3.35)

One can solve explicitly this type of equation. We multiply by \(v_x\) and find

\[
-\left( \frac{(v_x)^2}{2} \right)_x + \alpha \left( \frac{(v)^2}{2} \right)_x = \frac{1}{2} (g(v))_x, \quad v(-\infty) = v(\infty) = 0,
\]

with \(g(v) = \frac{2p}{1+p} \left( \frac{p\chi}{1+p} \right)^{1/p} v^{1+1/p}\). Therefore we have (the integration constant still vanishes because of the behaviour at infinity)

\[
(v_x)^2 = \alpha v^2 - g(v).
\]

(3.36)

2nd step (Resolution of the reduced equation) We now choose to normalize the translation invariance so that \(v\) attains its maximum at \(x = 0\) and define \(v(0) = v_0 > 0\). Then we should have \(g(v_0) = \alpha v_0^2\) because at a maximum point \(v_x(0) = 0\), and this defines the unique value \(v_0\) because of the assumption on \(p\). For \(v \leq v_0\), we have \(\alpha v^2 - g(v) \geq 0\) and for \(v > v_0\), we have \(\alpha v^2 - g(v) < 0\). Thus we can decide of the square root in equation (3.36) and we find

\[
\begin{align*}
v_x(x) &= - \sqrt{\alpha v(x)^2 - g(v(x))}, \quad v(0) = v_0, \quad x > 0, \\
v_x(x) &= \sqrt{\alpha v(x)^2 - g(v(x))}, \quad v(0) = v_0, \quad x < 0.
\end{align*}
\]

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This defines a unique function $v(x)$ which decreases to 0 as $|x| \to \infty$. \hfill \square

One can prove that the evolution equation associated with (3.34) has the following behaviour. If its initial data satisfies $\int_{\mathbb{R}} u^0(x)dx < \int_{\mathbb{R}} u(x)dx$ then $u(t, x) \to 0$ as $t \to \infty$. If its initial data satisfies $\int_{\mathbb{R}} u^0(x)dx > \int_{\mathbb{R}} u(x)dx$ then $u(t, x)$ blows up (see Chapter 5) in finite time and concentrates as a Dirac mass. See [11].

**Exercise.** What does it happen for $p = 1$ in the above calculation? Are there still spike solutions?

**Exercise.** Compute how the solution $v$ to (3.34) decays to 0 at infinity. Prove that $u$ satisfies $\int_{\mathbb{R}} u(x)dx < \infty$.

**Exercise.** Consider the general chemotaxis model

$$
\begin{align*}
\begin{cases}
-(u^{1+p})_{xx} + (u^q v_x)_x = 0, & x \in \mathbb{R}, \\
-v_{xx} + \alpha v = u^r.
\end{cases}
\end{align*}
$$

For which range of positive parameters $p$, $q$, $r$ are there spike solutions.

### 3.7.2 A non-local model from adaptive evolution

In order to motivate spikes, we may also take an example from the theory of adaptive evolution. A population is structured by a physiological parameter: it could be the size of an organ of the individuals, proportion of resource used for division, or any relevant parameter useful to describe the adaptation of the individuals i.e. their ability to use the nutrient for reproduction. We denote by $u(t, x)$ the density of individuals with trait $x \in \mathbb{R}$ (to make it as simple as possible), and we write the dynamics of the population density as, for instance to take the easiest model,

$$
\begin{align*}
\begin{cases}
\frac{\partial}{\partial t} v(t, x) - \Delta v = v(b(x) - \varrho(t)), & t \geq 0, \ x \in \mathbb{R}, \\
\varrho(t) = \int_{\mathbb{R}} v(t, x)dx, \\
v(t = 0, x) = v^0(x) \geq 0.
\end{cases}
\end{align*}
$$

(3.37)

The term $\Delta v$ takes into account mutations (it should be included in the birth term, but we choose here to simplify the equation as much as possible), $b(x)$ is the birth rate depending of the trait $x$. Finally, $-\varrho$ models the death term, as in the Fisher/KPP equation, with the main difference that he total population is used for this quadratic death term.
We now look for a steady state solution of this model that vanishes at $\pm \infty$ so as to find a spike solution

\[
\begin{cases}
-u''(x) = u(x) \left(b(x) - \varrho\right), & x \in \mathbb{R}, \\
\varrho = \int_{\mathbb{R}} u(x) \, dx, & u(x) > 0, \\
u(\pm \infty) = 0.
\end{cases}
\]  

(3.38)

Notice that the problem (3.38) is simply an eigenvalue problem and is relevant from the Krein-Rutman theory (Perron-Froebenius theory in infinite dimension). The solution exists under quite general assumptions on $b$. We give in Figure 3.6 two examples of solutions for $b(x) = 20 \ast e^{-(x-.25)/.01}$ and $b(x) = 24 \ast e^{-(x-.25)/.01} + 20 \ast e^{-(x-.7)/.03}$.

We give a simple example with an explicit solution that explains the existence of a pulse solution

**Theorem 3.8** Assume $b(x)$ has the form

\[
b(x) = \begin{cases}
 b_- > 0 & \text{for } |x| > a, \\
b_+ > b_- & \text{for } |x| < a,
\end{cases}
\]

then there is a unique spike solution to (3.38), and it is single spiked.

**Theorem 3.9** Assume that there is a constant $C_0$ such that $0 \leq v^0 \leq C_0 u$, then the solution to (3.37) converges as $t \to \infty$ to the steady state solution to (3.38).

**Proof of Theorem 3.8.** We first consider a solution, forgetting the condition $\varrho = \int u$ and find an intrinsic condition on $\varrho$. For $x \leq -a$ the equation is $-u''(x) = u(x)(b_- - \varrho)$ and the conditions $u > 0$ and $u(-\infty) = 0$ impose $\varrho > b_-$ and gives the explicit solution

\[u(x) = \mu_- e^{\lambda_- x}, \quad x \leq -a, \quad \lambda_- = \sqrt{\varrho - b_-}.
\]
Similarly, we have
\[ u(x) = \mu_+ e^{-\lambda_+ x}, \quad x \geq a. \]

In order to connect these two branches, we need that \( \overline{\varrho} < b_+ \) and
\[ u(x) = \mu_1 \cos(\lambda_o x) + \mu_2 \sin(\lambda_o x), \quad -a \leq x \leq a, \quad \lambda_o = \sqrt{b_+ - \overline{\varrho}}. \]

Also the sign condition \( u(x) > 0 \) imposes \( \mu_1 > 0 \) (at \( x = 0 \)) and \( \lambda_o a < \pi/2 \) (otherwise take \( \lambda_o x_o = \pm \pi/2 \)).

Now, we have to check the continuity of \( u \) and \( u' \) at the points \( \pm a \). This gives the conditions
\[
\begin{align*}
\mu_+ e^{-\lambda_+ a} &= \mu_1 \cos(\lambda_o a) + \mu_2 \sin(\lambda_o a), \\
\mu_+ e^{-\lambda_+ a} &= \mu_1 \cos(\lambda_o a) + \mu_2 \sin(\lambda_o a),
\end{align*}
\]
\[
\begin{align*}
\mu_+ \lambda_o e^{-\lambda_+ a} &= \mu_1 \lambda_o \sin(\lambda_o a) + \mu_2 \lambda_o \cos(\lambda_o a), \\
\mu_+ \lambda_o e^{-\lambda_+ a} &= \mu_1 \lambda_o \sin(\lambda_o a) - \mu_2 \lambda_o \cos(\lambda_o a).
\end{align*}
\]

From these equalities we deduce first that the quantity \( \mu = \frac{1}{2}(\mu_- + \mu_+) \) satisfies
\[
\begin{align*}
\mu e^{-\lambda_- a} &= \mu_1 \cos(\lambda_o a), \\
\mu \lambda_o e^{-\lambda_- a} &= \mu_1 \lambda_o \sin(\lambda_o a),
\end{align*}
\]
therefore \( \lambda_- = \lambda_o \tan(\lambda_o a) \), in other words the parameter \( \overline{\varrho} \) should satisfy
\[ b_- \leq \overline{\varrho} \leq b_+, \quad a \sqrt{b_+ - \overline{\varrho}} < \pi/2, \quad \sqrt{\overline{\varrho} - b_-} = \sqrt{b_+ - \overline{\varrho}} \tan(a \sqrt{b_+ - \overline{\varrho}}). \]

By monotonicity we obtain that there is a unique \( \overline{\varrho} \) satisfying these conditions, and that being given \( \mu, \mu_1 \) is proportional to \( \mu \).

Next, we go back to the four conditions and now find
\[
\begin{align*}
\frac{\mu_+ - \mu_-}{2} e^{-\lambda_- a} &= \mu_2 \sin(\lambda_o a), \\
\frac{\mu_+ - \mu_-}{2} \lambda_o e^{-\lambda_- a} &= -\mu_2 \lambda_o \cos(\lambda_o a),
\end{align*}
\]
and straightforward sign considerations show that \( \mu_2 = 0 \) and \( \mu_- = \mu_+ \).

Therefore we have the only free parameter \( \mu \) left. It is needed to realize the mass condition \( \varrho = \int_{\mathbb{R}} u(x)dx \) and thus we have indeed a unique solution.

**Proof of Theorem 3.9.** Consider the solution \( \tilde{v}(t, x) \geq 0 \) to the heat equation
\[
\begin{align*}
&\partial_t \tilde{v}(t, x) - \Delta \tilde{v} = \tilde{v}(b(x) - \overline{\varrho}), \quad t \geq 0, \quad x \in \mathbb{R}, \\
&\tilde{v}(t = 0, x) = v^0(x)
\end{align*}
\]
This is a heat equation with 0 the first eigenvalue of the steady equation. We know from the maximum principle that $0 \leq \tilde{v}(t, x) \leq C_0 u(x)$ (see the proof of Lemma). And from the general theory of dominant eigenvectors of positive operators (Perron-Froebenius),

$$v(t, x) \xrightarrow{t \to \infty} \lambda u(x),$$

for some $\lambda \in \mathbb{R}$.

On the other hand we can look for the solution to (3.37) under the form $v(t, x) = \mu(t)\tilde{v}(t, x)$ with $\mu(t) \geq 0$. We have

$$\frac{\partial}{\partial t} v(t, x) - \Delta v - v(b(x) - \rho(t)) = \dot{\mu}(t)\tilde{v}(t, x) + \mu(t)\tilde{v}(t, x)(\rho(t) - \bar{\rho}),$$

which allows us to find $\mu(t)$ by the equation

$$\dot{\mu}(t) + \mu(t)(\rho(t) - \bar{\rho}) = 0.$$

But we have $\rho(t) = \int_{\mathbb{R}} v(t, x)dx = \mu(t)\int_{\mathbb{R}} \tilde{v}(t, x)dx = \mu(t)\lambda(t)$, with

$$\lambda(t) = \int_{\mathbb{R}} \tilde{v}(t, x)dx \xrightarrow{t \to \infty} \lambda.$$

Finally, we obtain

$$\left\{ \begin{array}{l} \dot{\mu}(t) + \mu(t)(\mu(t)\lambda(t) - \bar{\rho}) = 0, \\ \mu(0) = 1. \end{array} \right.$$

The solution satisfies

$$\mu(t) \xrightarrow{t \to \infty} \bar{\rho}/\lambda.$$

This is exactly the announced result. \Box

**Exercise.** Consider the model

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} u(t, x) - \Delta u = u \left( \frac{b(x)}{1 + \rho(t)} - d(x)\rho(t) \right), \quad t \geq 0, \; x \in \mathbb{R}, \\ \rho(t) = \int_{\mathbb{R}} u(t, x)dx. \end{array} \right. \tag{3.40}$$

Assume that $b, d$ still take two different values (with same discontinuity points) and give a condition for existence of a spike solution.

### 3.8 Traveling pulses

We describe the mathematical mechanism for creating pulses which means spikes that are moving as a traveling wave. The upmost famous example, namely Hodgkin-Huxley system, describes accurately propagation of ionic signal along nerves. It is rather complex and we prefere here to retain only simplified models, which we hope, explains better the mechanism.
3.8.1 Fisher/KPP pulses

A method to create a pulse from the Fisher/KPP equation is to include a component $v(t, x)$ which allows to drive back the state $u = 1$ to zero. This arises with a certain delay because this component $v$ has to increase from zero (a state we impose initially) to a quantity larger than 1 so as to impose that $u$ itself decreases to zero. With these considerations, we arrive to the system

$$
\begin{aligned}
\frac{\partial}{\partial t} u(t, x) - \Delta u &= u[1 - u - v] \\
\frac{\partial}{\partial t} v(t, x) &= u,
\end{aligned}
$$

(3.41)

Figure 3.7: Fisher/KPP pulses, these are traveling wave solutions to the system (3.41). The solution is presented as a function of $x$, at a given time. The pulse propagates from left to right. Left: $u(t, x)$ exhibits a pulse shape, i.e., it vanishes at both endpoints. Right: $v(t, x)$ exhibits a traveling wave as in Fisher/KPP equation but not a pulse shape (by opposition with FitzHugh-Nagumo system).

The solutions at a given time are depicted in Figure 3.7. One can observe that $u$ is indeed a pulse but $v$ does not vanish at $x = -\infty$ by opposition with FitzHugh-Nagumo system.

3.8.2 FitzHugh-Nagumo system

Several parabolic models have been used in neurosciences with a huge impact for the propagation of nerve impulses. Hodgkin-Huxley system, the first model, gave amazing results with comparisons to experiments along the giant squid axon during the 1950’s (see [30]). This model has initiated the theories of electrophysiology, cardiac, neural communication by electrical signaling or neural rhythms.
The FitzHugh-Nagumo system is nowadays the simpler model used to describe pulses propagations in a spatial region. We give two versions, one for electric potential, one for calcium waves.

\begin{equation}
\begin{aligned}
\frac{\partial}{\partial t} u(t, x) - \varepsilon \Delta u &= \frac{1}{\varepsilon} [u(1-u)(u-\alpha) - v] \\
\frac{\partial}{\partial t} v(t, x) &= \gamma u - \beta v, \\
\end{aligned}
\end{equation}

In Figure 3.8 (Left) we depict a pulse traveling with parameters indicated in the figure caption. We choose here \( \alpha < .5 \) in order to propagate a traveling wave in the Allen-Cahn equation on \( u \) (initially \( v \equiv 0 \)). This switches on the equation on \( v \), and with a certain delay it inhibits the pulse and \( u \) goes back to the other stable value \( u = 0 \). For that we need the condition

\[ u(1 - u)(u - \alpha) < \frac{\gamma}{\beta} u \quad \text{for} \quad 0 < u < 1. \]

In the other case we can reach an equilibrium \( \beta v = \gamma u \) which transforms the bistable nonlinearity in a monostable.

The parameter \( \varepsilon \) controls the stiffness of the fronts, \( \beta \) the width of the pulse and \( \frac{\gamma}{\beta} \) the type of wave.

As one can see on Figure 3.8, the solution \( u \) is negative. This is in accordance with electric potential waves. This can be seen as a modeling error for concentration waves. Also the shape of the 'polarization wave' does not always fit with experimental observations (for instance for cardiac electric waves). This is the reason why several variants exist.
A possible way to guarantee that $u$ remains nonnegative is to modify the FitzHugh-Nagumo system into

$$
\begin{align*}
\frac{\partial}{\partial t} u(t, x) - \varepsilon \Delta u &= \frac{1}{\varepsilon} u[(1 - u)(u - \alpha) - v] \\
\frac{\partial}{\partial t} v(t, x) &= v_\infty(u) - v,
\end{align*}
$$

(3.43)

where $v_\infty(\cdot)$ represents the equilibrium on $v$ in the potential (concentration) $u$. The choice of this nonlinear function allows for more generality.

- For $v_\infty(u) = ku$, the system is due to McCulloch$^6$.
- For $v_\infty(u) = ku(u - 1 - a)$, the system is called Aliev-Panfilov$^7$.

Figure 3.9 shows a pulse for equation (3.43) with $\alpha = .2$, $v_\infty(u) = 3 \ast (u - 4)_+$ (here the numbers are adapted to $\alpha = .2$) which propagates from left to right. Two values of $\varepsilon$ are represented.

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Chapter 4

The Fokker-Planck equation

4.1 The Fokker-Planck equation and relative entropy

Not only reaction terms but also drift terms (first order derivatives) appear in parabolic models. The model equation is the (particular) Fokker-Planck equation for a given (smooth) potential $V : \mathbb{R}^d \to \mathbb{R}$:

$$\begin{cases} \frac{\partial}{\partial t} n(t, x) - \Delta n(t, x) - \text{div} (n(t, x) \nabla V(x)) = 0, & t \geq 0, \ x \in \mathbb{R}^d, \\ n(t = 0, x) = n^0(x). \end{cases}$$ (4.1)

The main properties (formal at this stage) are

$$n^0 \geq 0 \implies n \geq 0,$$

and ‘mass’ conservation

$$\int_{\mathbb{R}^d} n(t, x) dx = \int_{\mathbb{R}^d} n^0(x) dx, \ \forall t \geq 0.$$

The other family of noticeable a priori estimates for (4.1) the so-called relative entropy inequality Then, we have the

Proposition 4.1 For any convex function $H : \mathbb{R} \to \mathbb{R}$, we have

$$\frac{d}{dt} \int_{\mathbb{R}^d} e^{-V(x)} H \left( \frac{n(t, x)}{e^{-V(x)}} \right) dx = -D_H(n|e^{-V}) \leq 0,$$

$$D_H(n|e^{-V}) = \int_{\mathbb{R}^d} e^{-V} H'' \left( \frac{n}{e^{-V}} \right) \left| \nabla \frac{n}{e^{-V}} \right|^2 dx.$$

In particular the usual $L^p$ estimates are not true but they come with weights. We choose $H(u) = u^p$ and obtain

$$\frac{d}{dt} \int_{\mathbb{R}^d} e^{(p-1)V(x)} n^p(t, x) dx \leq 0.$$
4.2 The full Fokker-Planck equation

The general Fokker-Planck equation (also called Kolmogorov equation in the theory of diffusion processes) is given by

\[
\begin{cases}
\frac{\partial}{\partial t} n(t, x) - \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} (a_{ij}(t, x)n(t, x)) + \text{div}(n(t, x)U(t, x)) = 0, & t \geq 0, \ x \in \mathbb{R}^d, \\
n(t = 0, x) = n^0(x). & \end{cases}
\]  

(4.2)

Here \( U : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) is the velocity field and \( a_{ij} : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}^+ \) is the diffusion matrix. It is always assumed to satisfy, for some \( \nu > 0 \),

\[ \sum_{i,j=1}^{d} a_{ij}(t, x) \xi_i \xi_j \geq \nu |\xi|^2 \quad \forall \xi \in \mathbb{R}^d. \]

The main properties are still the sign property

\[ n^0 \geq 0 \implies n \geq 0, \]

and 'mass' conservation

\[ \int_{\mathbb{R}^d} n(t, x)dx = \int_{\mathbb{R}^d} n^0(x)dx, \forall t \geq 0. \]

We begin by a priori estimates for (4.1) which are not usual \( L^p \) estimates but come with weights. These are related to the so-called relative entropy inequality which assumes that there is a steady state \( N \) to (4.1), i.e., the coefficients are independent of time \((a_{ij}(x), U(x))\) and

\[ -\sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} (a_{ij}(x)N(x)) + \text{div}(N(x)U(x)) = 0, \quad 0, \ x \in \mathbb{R}^d, \quad N(x) > 0. \]  

(4.3)

The existence of solutions depends heavily on the coefficients but a typical example is as before

\[ a = I \quad (\text{Identity matrix}), \quad U(x) = -\nabla V(x), \]

then one readily checks that, because \( \Delta e^{-V} = \text{div}(\nabla e^{-V}) \), we can choose

\[ N(x) = e^{-V(x)}. \]

Then, we have the

**Proposition 4.2** For any convex function \( H : \mathbb{R} \rightarrow \mathbb{R} \), we have

\[ \frac{d}{dt} \int_{\mathbb{R}^d} N(x)H \left( \frac{n(t, x)}{N(x)} \right) dx = -D_H(n\|N) \leq 0, \]

\[ D_H(n\|N) = \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} N(x)a_{ij}(x)\nabla_i \frac{n}{N} \nabla_j \frac{n}{N} dx. \]
To prove it, just calculate successively the equation on \( u(t, x) = \frac{n(t, x)}{N(x)} \), then on \( H(u(t, x)) \), and then on \( NH(u(t, x)) \). It is tedious but it works.

There are many examples of nonlinear Fokker-Planck equations arising in biology and we give examples later on. They describe the density of a population moving with a deterministic velocity \( U \) added to a 'random noise of intensity' \( a_{ij} \). More generally, the reason why they play a central role is the connection with brownian motion and Stochastic Differential Equations. This material is introduced later on.

### 4.3 Brownian motion

![Simulations of two sample paths of the brownian motion.](image)

We recall that a real brownian motion (also called Wiener process) is a continuous stochastic process \( W(t, \omega) \), defined on a 'big' probability space, such that

(i) (depart from 0) \( W(0) = 0 \) a.s.,
(ii) (gaussian increments) the law of \( W(t) - W(s) \) is \( \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{|x|^2}{2(t-s)}} \) for \( t \geq s \geq 0 \),
(iii) (independent increments) \( W(t) - W(s) \) is independent of \( W(s_2) - W(s_1),...,W(s_n) - W(s_{n-1}) \) for all \( 0 < s_1 < s_2 < ... < s_n < s < t \).

It is useful to notice that, choosing \( s = 0 \) is the second requirement, we have

\[
E(W(t)) = 0, \quad E(|W(t)|^2) = t.
\]

One can build a numerical approximation of the brownian motion being given a time step \( \Delta t \) just as we do for differential equations. A method to generate a sample path at times \( t^k = k\Delta t \) (a given time step), is to
(i) set \( W^{(0)} = 0 \), and at each time step \( k \geq 0 \)
(ii) generate, independently of previous ones, an increment \( A^{(k+1)} \) with a normal law \( \frac{1}{\sqrt{2\pi}} e^{-\frac{|x|^2}{2}} \),
(iii) define \( W_{\Delta t}^{(k+1)} = W_{\Delta t}^{(k)} + \sqrt{\Delta t} A^{(k+1)} \).

One can give an intuitive reason why this algorithm works. Fixing a time \( t = K\Delta t \). We can write \( \sum_{k=1}^{K} A^{(k)} \) and, by the central limit theorem,
\[
P(a' \leq \frac{W_{\Delta t}(t)}{\sqrt{t}} \leq b') \to \frac{1}{\Delta t - 0} \int_{a'}^{b'} e^{-\frac{y^2}{2}} dy ,
\]
or in other words (with \( a = a'\sqrt{t}, b = b'\sqrt{t}, x = y\sqrt{t} \)),
\[
P(a \leq W_{\Delta t}(t) \leq b) \to \frac{1}{\sqrt{2\pi t}} \int_{a}^{b} e^{-\frac{x^2}{2t}} dx .
\]
The same holds for the increments \( W_{\Delta t}(t) - W_{\Delta t}(s) \).

In other words, this numerical construction generates the correct law of the brownian process, which is what we care of. It does not mean convergence pathwise which requires much more involved constructions. Notice also that the random variables \( A^k \) might be chosen with any law with the correct first two moments, and not necessarily a normal law, still according to the central limit theorem.

### 4.4 Stochastic Differential Equations

For a given smooth enough vector field \( U \in \mathbb{R}^d \) and matrix \( \sigma(t, x) \in M_{d,p} \), one can build the solution to the Itô Stochastic Differential Equation
\[
\begin{cases}
    dX(t) = U(t, X(t))dt + \sigma(t, X(t))dW(t), \\
    X(0) = X^0 \in \mathbb{R}^d \quad \text{(a random vector)},
\end{cases}
\]
with \( W(t) = (W^1(t), \ldots, W^p(t)) \), \( p \) independent brownian motions.

The numerical construction which mimicks the one of the brownian motion is to set (here \( d = p = 1 \) but the reader can extend it easily)
\[
X^{(k+1)} = X^{(k)} + \Delta t U(t, X^{(k)}) + \sqrt{\Delta t} \sigma(t, X^{(k)}) A^{k+1}.
\]

Because the random variable \( A^{k+1} \) is independent of \( X^{(k)} \), we have
\[
E\left[ X^{(k+1)} - X^{(k)} \right] = \Delta t E\left[ U(t, X^{(k)}) \right],
\]
\[
E \left[ (X^{(k+1)} - X^{(k)})^2 \right] = E \left[ \Delta t^2 U^2(t, X^{(k)}) + 2\sqrt{\Delta t} U(t, X^{(k)}) \sigma(t, X^{(k)}) A^{k+1} + \Delta t |\sigma(t, X^{(k)}) A^{k+1}|^2 \right] \\
= \Delta t E[\sigma^2(t, X^{(k)})] E[(A^{k+1})^2] + O(\Delta t^2) \\
= \Delta t E[\sigma^2(t, X^{(k)})] + O(\Delta t^2)
\]

The type of approximation used here is very important. The so-called Stratonovich integral corresponds to the semi-implicit approximation

\[
X^{(k+1)} = X^{(k)} + \Delta t U(t, X^{(k)}) + \sqrt{\Delta t} \sigma(t, X^{(k)}) A^{k+1}
\approx X^{(k)} + \Delta t U(t, X^{(k)}) + \sqrt{\Delta t} \left[ \sigma(t, X^{(k)}) + \frac{1}{2} \sigma'(t, X^{(k)}) (X^{(k+1)} - X^{(k)}) \right] A^{k+1}
\approx X^{(k)} + \Delta t U(t, X^{(k)}) + \frac{\Delta t}{2} \sigma'(t, X^{(k)}) \sigma(t, X^{(k)}) (A^{k+1})^2 + \sqrt{\Delta t} \sigma(t, X^{(k)}) A^{k+1} + O(\Delta^{3/2}).
\]

It is remarkable that the semi-implicit scheme does not lead to same leading expectation and thus not to the same laws of \(X(t)\) in the limit.

### 4.5 Ito’s formula

An important tool in the theory of SDEs is Itô’s formula that gives the equation satisfied by the random variable \(u(t, X(t))\), when \(u \in C^2(\mathbb{R}^d; \mathbb{R})\),

\[
du(t, X(t)) = \left[ \frac{\partial u(t, X(t))}{\partial t} + \nabla u(t, X(t)).U(t, X(t)) + a(t, X(t)).D^2 u(t, X(t)) \right] dt \\
+ \nabla u(t, X(t)).\sigma(t, X(t))dW(t)
\]

with

\[
a(t, x) = \frac{1}{2} \sigma(t, x) . \sigma^t(t, x) \quad (\sigma^t \text{ denotes the transposed matrix}).
\]

In other words, the chain rule does not apply to SDEs as it applies to ODEs \((\sigma(t, x) = 0)\) and extra drift (the term \(a(t, X(t)).D^2 u(t, X(t))\) appears).

We recall that

**Definition 4.3** The probability density \(n(x)\) of a random variable \(X\) is defined as

\[
\int_A n(x)dx = E(I_{\{X(\omega) \in A\}}) = P(X(\omega) \in A).
\]

We give the probability density \(n^0(x)\) of \(X^0\) and we denote by \(n(t, x)\) the probability density of the process \(X(t, \omega)\). Then we can deduce from Itô’s formula that \(n\) satisfies the Fokker-Planck
equation (4.1). It turns out that the Fokker-Planck equation comes naturally with the ‘mass conservation’ property because, for a probability density, we have $\int_{\mathbb{R}^d} n(t, x) \, dx = 1$.

This interpretation is closely related to the adjoint to the Fokker-Planck equation

$$-rac{\partial}{\partial t} u(t, x) - \sum_{i,j=1}^{d} a_{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j} u(t, x) - U(t, x).\nabla u(t, x) = 0,$$

an equation that should be understood as a backward Cauchy problem (we give $u(T, x)$ and search for $t \leq T$). When $u$ satisfies this PDE, we have, still from Itô’s formula

$$\frac{d}{dt} E(u(t, X(t))) = 0.$$

**Exercise.** Consider the deterministic differential equation

$$\begin{align*}
\begin{cases}
\frac{dX(t)}{dt} & = U(t, X(t)), \\
X(0) & = X^0 \in \mathbb{R}^d.
\end{cases}
\end{align*}$$

(4.5)

1. Show that the weak solution to the first order PDE

$$\begin{align*}
\begin{cases}
\frac{\partial}{\partial t} n(t, x) + \text{div}(n(t, x)U(t, x)) & = 0, \\
n(t = 0, x) & = n^0(x) := \frac{1}{K} \sum_{k=1}^{K} \delta(x - X^0_k),
\end{cases}
\end{align*}$$

is given by $n(t, x) = \frac{1}{K} \sum_{k=1}^{K} \delta(x - X_k(t))$.

2. Show that the $C^1$ solutions to the equation

$$\begin{align*}
\begin{cases}
-\frac{\partial}{\partial t} u(t, x) - U(t, x).\nabla u & = 0, \\
u(t = 0, x) & = u^0(x),
\end{cases}
\end{align*}$$

are constant along the characteristics, which are defined as the solutions to (4.5).

### 4.6 Active collective motion

The Stochastic Differential Equation (4.4) is well adapted to describe the motion of a large number of cells (or more generally individuals). The Keller-Segel system [21] for chemotaxis is the most famous model in this area and assumes that cells move with a combination of a random (brownian) motion and an oriented drift which is the gradient of a quantity depending on the other individuals. For certain cells, this is a well documented behavior called chemotaxis; each cell emits a chemoattractant, i.e., molecule which diffuses in the medium and attract the other cells. They react by a biased random motion in the direction of higher concentrations of this chemoattractant.
Figure 4.2: A numerical solution to (4.6) at three different times for an attractive interaction kernel $K$. Such models have the property to create high concentrations (aggregations).

This kind of assumptions leads to the nonlinear Fokker-Planck equation

$$\begin{cases}
\frac{\partial}{\partial t} n(t,x) - \Delta n(t,x) + \text{div}(n(t,x)\nabla S(t,x)) = 0, & t \geq 0, \ x \in \mathbb{R}^d, \\
\Phi(t,x) = \int_{\mathbb{R}^d} K(x-y)n(t,y)dy, \\
n(t=0,x) = n^0(x).
\end{cases} \tag{4.6}$$

The convolution kernel $K(\cdot)$ describes the long range effect of an individual located at $y$ on another individual located at $x$, creating the interaction potential (signal) $S(x)$. The gradient $\nabla S(x)$ of this potential defines the preferred direction (and intensity) of the active motion of an individual located at $x$.

Usually, in an homogeneous and isotropic medium, the kernel satisfies $K(x) = \bar{K}(|x|)$. One distinguishes the attractive movement $\bar{K}(\cdot) \geq 0$, $\bar{K}'(\cdot) \leq 0$ and the case $\bar{K}(\cdot) \leq 0$, $\bar{K}'(\cdot) \geq 0$ for repulsive movement (in physics the former corresponds to newtonian gravitational forces, the later to coulombic electric forces). This can be seen from the exercise below. Figure 4.2 depicts the numerical solution to (4.6) in the attractive case. When the total density is high enough the population has tendency to aggregate and form a spike solution (see Section 3.7).

**Exercise.** Assume that $K(x) = \bar{K}(|x|)$.

1. Derive formally the free energy for the solutions to (4.6)

$$\frac{d}{dt} \int_{\mathbb{R}^d} [n(t,x) \ln n(t,x) - \frac{1}{2} n(t,x)S(t,x)] dx := -D(t) \leq$$

and compute $D(t)$. 

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2. Compute $E(t)$ for the dynamics of second moment

$$\frac{d}{dt} \int_{\mathbb{R}^d} \frac{|x|^2}{2} n(t,x) dx = E(t).$$

3. Interpret these relations in terms of attractive or repulsive kernels.

Solution: $D(t) = \int_{\mathbb{R}^d} n|\nabla (S + \ln n)|^2 dx$.

$E(t) = d \int_{\mathbb{R}^d} n^0 + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y| \tilde{K}'(|x - y|) n(t,x)n(t,y) dx dy$

4.7 Passive collective motion

![Graphs showing numerical solution](image)

Figure 4.3: A numerical solution $n(t,x)$ to (4.7) at four different times when $A(n)$ has a decreasing region for $1 \leq n \leq 2$. Top left, the initial data. Bottom right, the last time (the steady state is composed of two constant states separated by two discontinuities).

Movement can also be completely passive, by this we mean just a brownian motion with variable intensity (and thus of zero mean), and nevertheless it can give interesting patterns.
This is the case when \( U \equiv 0 \) in (4.4) but the intensity of the brownian motion depends on the local population density through a smooth function \( \Sigma : \mathbb{R}^+ \to \mathbb{R} \),

\[
\sigma(t, x) = \Sigma(n(t, x)).
\]

When the individuals like (or need) the rest of the population, then \( \Sigma'(\cdot) \leq 0 \) which means that the higher is the population, the lower is the movement. When the individuals do not like high concentrations we take \( \Sigma'(\cdot) \geq 0 \). The Fokker-Planck equation reduces to the parabolic equation

\[
\begin{aligned}
\frac{\partial}{\partial t} n(t, x) - \Delta A(n(t, x)) &= 0, & t \geq 0, & x \in \mathbb{R}^d, \\
A(n) &= n\Sigma(n)^2, \\
n(t = 0, x) &= n^0(x).
\end{aligned}
\]

(4.7)

In Figure 4.3, we present the numerical solutions to a relaxation system close to (4.7):

\[
\begin{aligned}
\frac{\partial}{\partial t} n_\varepsilon(t, x) - \frac{\Delta}{\varepsilon} \left[ \Sigma^2(m_\varepsilon(t, x)) n_\varepsilon(t, x) \right] &= 0, & t \geq 0, & x \in (0, 1), \\
-\varepsilon \Delta m_\varepsilon(t, x) + m_\varepsilon(t, x) &= n_\varepsilon(t, x),
\end{aligned}
\]

(4.8)

together with Neumann boundary conditions on both equations (the total number of individuals remains constant in time). We have computed the case when \( \Sigma(\cdot) \) is decreasing near \( n = 0 \), which means that individuals tend to avoid low densities by moving fast (still with average 0).

In our test case

\[
A(n) = \frac{n^3}{3} - \frac{n^2}{2} + 2n,
\]

and \( A'(n) \leq 0 \) for \( 1 \leq n \leq 2 \). This is an unstable region and that creates discontinuities. The total number constraint makes that a part of the population has to remain at a low level of density.

**Exercise.** Define \( \Phi(n) \) by \( \Phi' = A \). Show that \( \int_{\mathbb{R}^d} \Phi(n(t, x)) \, dx \) is decreasing. When can it be convex?

**Exercise.** Perform a numerical simulation of the system (4.8) and use a decreasing function \( \Sigma \). Compare with the results in Figure 4.3.

The case of two species leads to a similar derivation. Each brownian motion has an intensity which depends on the densities \( n_1 \) and \( n_2 \) of the two species and leads to a Fokker-Planck equation. We arrive to a coupled system

\[
\begin{aligned}
\frac{\partial}{\partial t} n_1(t, x) - \Delta [n_1 a_1(n_1(t, x), n_2(t, x))] &= 0, & t \geq 0, & x \in \mathbb{R}^d, \\
\frac{\partial}{\partial t} n_2(t, x) - \Delta [n_2 a_2(n_1(t, x), n_2(t, x))] &= 0.
\end{aligned}
\]

(4.9)
In such models where the $a_i$ depend on $n_j$, second order derivatives of $n_j$ arise in the equation for $n_1$ and, thus, are called *cross-diffusions*. 
Chapter 5

Breakdown and blow-up of solutions

We know from Chapter 1 that for 'small' nonlinearities, the solutions relax to an elementary state. When the nonlinearity is not lipschitz continuous, it is possible that solutions to nonlinear parabolic equations do not exist for all times. The simplest mechanism is that the solution becomes larger and larger pointwise and eventually becomes infinite in finite time. The scenario follows the case of the differential equations
\[
\dot{z}(t) = z(t)^2, \quad z(0) > 0,
\]
which solution is \(z(t) = z(0)/(1 - tz(0))\) and blows-up in finite time, i.e., solutions tend to infinity at \(t = T^* := 1/z(0)\). It is a typical illustration of the alternative arising in the Cauchy-Lipschitz Theorem, solutions can only tend to infinity in finite time (case \(w(0) > 0\)), or they are globally defined (case \(z(0) < 0\)). The purpose of this Chapter is to describe the extension of this argument to the parabolic equation. The topic of blow-up is very rich and other modalities of blow-up, different blow-up rates and regularizing effects that prevent blow-up are also possible. All these use fundamentally the PDE structure.

We present several methods for proving blow-up in finite time. The first two methods are on semi-linear parabolic equations, the third method is illustrated on the Keller-Segel system.

5.1 Semilinear equations; the method of the eigenfunction

To study the case of nonlinear parabolic equations, we consider the model
\[
\begin{align*}
\frac{\partial}{\partial t} u - \Delta u &= u^2, & t \geq 0, & x \in \Omega, \\
u(x) &= 0, & x \in \partial\Omega, \\
u(t = 0, x) &= u^0(x) \geq 0.
\end{align*}
\tag{5.1}
\]
Here we treat the case when \( \Omega \) is a bounded domain. To define a distributional solution is not completely obvious because the right hand side should be well defined. Later on we call 'solution to (5.1)' a function satisfying for some \( T > 0 \),

\[
u \in L^2([0,T) \times \Omega), \quad u \in C([0,T]; L^1(\Omega)). \tag{5.2}\]

The question is then to know if diffusion is able to win against the quadratic nonlinearity and (5.1) could have global solutions. The answer is given by the next theorems

**Theorem 5.1** Assume that \( \Omega \) is a bounded domain, \( u^0 \geq 0 \), \( u^0 \) is large enough (in a weighted \( L^1 \) space introduced below), then there is a time \( T^* \) for which the solution to (5.1) satisfies

\[
\|u\|_{L^2([0,T) \times \mathbb{R}^d)} \longrightarrow T \to T^* \infty.
\]

Of course, this result means that \( u(t) \) also blows up in all \( L^p \) norms, \( 2 \leq p \leq \infty \) because we are in a bounded domain. It is quite easy to see that the blow-up time is the same for all these norms. The case of Keller-Segel system for chemotaxis is more interesting because it blows-up in all \( L^p \) norms, for all \( p > 1 \), but the \( L^1 \) norm is conserved (it represents the total number of cells in a system where division is ignored).

**Proof of Theorem 5.1.** We are going to obtain a contradiction on the fact that a function \( u \in L^2([0,T) \times \Omega) \) can be a solution to (5.1) when \( T \) overpasses an explicit value we compute below.

Because we are working in a bounded domain, the operator \(-\Delta\) has a smallest eigenvalue associated with a positive eigenfunction (see Section 1.6)

\[
\begin{cases}
-\Delta w_1 = \lambda_1 w_1, & w_1 > 0 \quad \text{in} \quad \Omega, \\
w_1(x) = 0, & x \in \partial \Omega, \quad \int_{\Omega}(w_1)^2 = 1.
\end{cases} \tag{5.3}
\]

For a solution, we can multiply equation (5.1) by \( w_1 \) and integrate by parts. We arrive at

\[
\frac{d}{dt} \int_{\Omega} u(t,x) \, w_1(x) \, dx = \int_{\Omega} \Delta u(t,x) \, w_1(x) \, dx + \int_{\Omega} u(t,x)^2 \, w_1(x) \, dx \\
= \int_{\Omega} u(t,x) \, \Delta w_1(x) \, dx + \int_{\Omega} u(t,x)^2 \, w_1(x) \, dx \\
= -\lambda_1 \int_{\Omega} u(t,x) \, w_1(x) \, dx + \int_{\Omega} u(t,x)^2 \, w_1(x) \, dx \\
\geq -\lambda_1 \int_{\Omega} u(t,x) \, w_1(x) \, dx + \left( \int_{\Omega} u(t,x) \, w_1(x) \, dx \right)^2 \left( \int_{\Omega} w_1(x) \, dx \right)^{-1}
\]

(after using the Cauchy-Schwarz inequality). We set \( z(t) = e^{\lambda_1 t} \int_{\Omega} u(t,x) \, w_1(x) \) and, with \( a = \left( \int_{\Omega} w_1(x) \, dx \right)^{-1} \), the above inequality reads

\[
\frac{d}{dt} z(t) \geq ae^{-\lambda_1 t} z(t)^2,
\]

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and we obtain:
\[
\frac{d}{dt} \frac{1}{z(t)} \leq -ae^{-\lambda_1 t},
\]
\[
\frac{1}{z(t)} \leq \frac{1}{z_0} - a \frac{1 - e^{-\lambda_1 t}}{\lambda_1}.
\]

Assume now the size condition
\[
z_0 > \frac{\lambda_1}{a}.
\] (5.4)

The above inequality contradicts that \(z(t) > 0\) for \(e^{-\lambda_1 t} \leq 1 - \frac{\lambda_1 z_0}{a}\). Therefore the computation, and thus the assumption that \(u \in L^2([0, T] \times \mathbb{R}^d)\), fails before that finite time. \(\Box\)

The size condition is necessary. There are various ways to see it. For \(d \leq 5\) it follows from a general elliptic

**Theorem 5.2** There is a steady state solution \(\bar{u} > 0\) in \(\Omega\) to
\[
\begin{cases}
-\Delta u = u^p, & x \in \Omega, \\
u(x) = 0, & x \in \partial\Omega,
\end{cases}
\]
when \(p\) satisfies
\[
1 < p < \frac{d + 2}{d - 2}.
\]

We refer to [13] for a proof of this theorem and related results (as non-existence for \(p > \frac{d + 2}{d - 2}\)).

One can see more directly that the size condition is needed. We choose \(\mu = \min_\Omega \frac{\lambda_1}{w_1(x)}\) and set \(\tilde{w} = \mu w_1\). We have
\[
\frac{\partial}{\partial t} \tilde{w} - \Delta \tilde{w} = \lambda_1 \tilde{w} \geq \tilde{w}^2.
\]
One concludes that any solution to (5.1) with \(u^0 \leq \tilde{w}\) satisfies \(u(t) \leq \tilde{w}\) for all times \(t\) where the solution exists (and this is enough to prove that the solution is global). Therefore we have the

**Lemma 5.3** Under the smallness condition \(u^0 \leq \min_\Omega \frac{\lambda_1}{w_1(x)} w_1\), there is a global solution to (5.1) and \(u(t) \leq \tilde{w} \\forall t \geq 0\).

**Exercise.** Prove the blow-up in finite time for the case of a general nonlinearity
\[
\begin{cases}
\frac{\partial}{\partial t} u - \Delta u = f(u), & t \geq 0, x \in \Omega, \\
u(x) = 0, & x \in \partial\Omega, \\
u(t = 0, x) = u^0(x) > 0 & \text{large enough},
\end{cases}
\] (5.5)
and $f(u) \geq c|u|^\alpha$ with $\alpha > 1$.

**Exercise.** (Neumann boundary condition) A solution on $[0, T)$ to the equation

$$\begin{cases}
\frac{\partial}{\partial t} u - \Delta u = u^2, & t \geq 0, \ x \in \Omega, \\
\frac{\partial}{\partial \nu} u(x) = 0, & x \in \partial \Omega, \\
u(t = 0, x) = u^0(x) > 0.
\end{cases} \tag{5.6}
$$

is a distributional solution such that $u \in L^2([0, T) \times \Omega)$.

1. Prove that there is no such solution after some time $T^*$ and no size condition is needed here.
2. Prove also that $\|u(T)\|_{L^1(\mathbb{R}^d)} \xrightarrow{T \to T^*} \infty$.

**Exercise.** For $d = 1$ and $\Omega = [-1, 1]$, construct a unique even non-zero solution to

$$-\Delta v = v^2, \quad v(\pm 1) = 0.
$$

*Hint:* Reduce it to $-\frac{1}{2}(v')^2 = \frac{1}{3}v^3 + c_0$ and find a positive real number $c_1$ such that

$$v' = -\sqrt{c_1 - \frac{2}{3}w^3}.
$$

### 5.2 Semilinear equations; energy method

Still for the semilinear parabolic equation (5.1) we consider another method leading to a different size condition and which does not use the sign condition. We give it as an exercise.

**Exercise.** (Energy method) Consider again, for $p > 1$, the problem to analyze non-negative solutions to

$$\begin{cases}
\frac{\partial}{\partial t} u - \Delta u = |u|^{p-1}u, & t \geq 0, \ x \in \Omega, \\
u(x) = 0, & x \in \partial \Omega, \\
u(t = 0, x) = u^0(x) \neq 0.
\end{cases} \tag{5.7}
$$

1. Prove that the energy $E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{p+1} \int_\Omega |u|^{(p+1)}$, decreases with time.
2. Show that (with $\alpha = \frac{1}{2} - \frac{1}{p+1} > 0$)

$$\frac{1}{4} \frac{d}{dt} \int_\Omega u^2 = -E(u) + \alpha \int_\Omega |u|^{(p+1)}.
$$

3. Deduce that for some $\beta > 0$ we have

$$\frac{1}{4} \frac{d}{dt} \int_\Omega u^2 \geq -E(u^0) + \beta \left( \int_\Omega u^2 \right)^{(p+1)/2}.
$$

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4. Prove blow-up for \( E(u^0) \leq 0 \). Find functions \( u^0 \in H^1_0(\Omega) \) that satisfy this condition.

5. Prove blow-up for \( \left( \int_\Omega (u^0)^2 \right)^{(p+1)/2} > \frac{2}{p} E(u^0) \).

Hint: \( 1. \frac{d}{dt} E(t) = - \int_\Omega \left( \frac{\partial}{\partial t} u(t) \right)^2 dx \)

### 5.3 Keller-Segel system; the moment method

We come back to the Keller-Segel model used to describe chemotaxis as mentioned in Section 4.6. We recall that it consists in a system which describes the evolution of the density of cells (bacteria, amoeba,...) \( n(t,x), \ t \geq 0, \ x \in \mathbb{R}^d \) and the concentration \( c(t,x) \) of the chemical attracting substance emitted by the cells themselves,

\[
\begin{align*}
\frac{\partial}{\partial t} n - \Delta n + \text{div}(n \chi \nabla c) &= 0, \quad t \geq 0, \ x \in \mathbb{R}^d, \\
-\Delta c + \tau c &= n, \\
n(t = 0) &= n^0 \in L^\infty \cap L^1_+(\mathbb{R}^d).
\end{align*}
\]

The first equation just expresses the random (brownian) diffusion of the cells with a bias directed by the chemoattractant concentration with a sensitivity \( \chi \). The chemoattractant \( c \) is directly emitted by the cell, diffused on the substrat and \( \tau^{-1/2} \) represents its activation length.

The notation \( L^1_+ \) means nonnegative integrable functions, and the parabolic equation on \( n \) gives nonnegative solutions (as expected for the cell density)

\[
n(t,x) \geq 0, \quad c(t,x) \geq 0.
\]

Another property we will use is the conservation of the total number of cells

\[
m^0 := \int_{\mathbb{R}^d} n^0(x) \, dx = \int_{\mathbb{R}^d} n(t,x) \, dx.
\]

In particular solutions cannot blow-up in \( L^1 \). But we have the

**Theorem 5.4** In \( \mathbb{R}^2 \), take \( \tau = 0 \) and assume \( \int_{\mathbb{R}^2} |x|^2 n^0(x) \, dx < \infty \).

(i) (Blow-up) When the initial mass satisfies

\[
m^0 := \int_{\mathbb{R}^2} n^0(x) \, dx > m_{\text{crit}} := 8\pi/\chi,
\]

then any solution to (5.8) becomes a singular measure in finite time.

(ii) When the initial data satisfies \( \int_{\mathbb{R}^2} n^0(x) |\log(n^0(x))| \, dx < \infty \) and

\[
m^0 := \int_{\mathbb{R}^2} n^0(x) \, dx < m_{\text{crit}} := 8\pi/\chi,
\]
there are weak solutions to (5.8) satisfying the a priori estimates

\[ \int_{\mathbb{R}^2} n[|\ln(n(t))| + |x|^2] \, dx \leq C(t), \]
\[ \|n(t)\|_{L^p(\mathbb{R}^2)} \leq C(p, t, n^0) \quad \text{for} \quad \|n^0\|_{L^p(\mathbb{R}^2)} < \infty, \quad 1 < p < \infty. \]

**Proof.** Formally the blow-up proof is very simple, and the difficulty here is to prove that the solution becomes a singular measure. We follow Nagai’s argument, first assuming enough decay in $x$ at infinity, afterward we state a more precise result. It is based on the formula

\[ \nabla c(t, x) = -\lambda_2 \int_{\mathbb{R}^2} \frac{x - y}{|x - y|^2} n(t, y) \, dy, \quad \lambda_2 = \frac{1}{2\pi}. \]

Then, we consider the second $x$ moment

\[ m_2(t) := \int_{\mathbb{R}^2} \frac{|x|^2}{2} n(t, x) \, dx. \]

We have, from (5.8),

\[
\frac{d}{dt} m_2(t) = \int_{\mathbb{R}^2} \frac{|x|^2}{2} [\Delta n - \text{div}(n\chi \nabla c)] \, dx \\
= \int_{\mathbb{R}^2} [2n + \chi nx \cdot \nabla c] \, dx \\
= 2m^0 - \chi \lambda_2 \int_{\mathbb{R}^2 \times \mathbb{R}^2} n(t, x)n(t, y) \frac{x \cdot (x - y)}{|x - y|^2} \\
= 2m^0 - \frac{\chi \lambda_2}{2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} n(t, x)n(t, y) \frac{(x - y) \cdot (x - y)}{|x - y|^2}
\]

(this last equality just follows by a symmetry argument, interchanging $x$ and $y$ in the integral). This yields finally,

\[ \frac{d}{dt} m_2(t) = 2m^0 \left( 1 - \frac{\chi}{8\pi} m^0 \right). \]

Therefore if we have $m^0 > 8\pi/\chi$, we arrive at the conclusion that $m_2(t)$ should become negative in finite time which is impossible since $n$ is nonnegative. Therefore the solution cannot be smooth until that time.
Chapter 6

Linear instability, Turing instability and pattern formation

In his seminal paper\(^1\) A. Turing ‘suggests that a system of chemical substances reacting together and diffusing through a tissue, is adequate to account for the main phenomena of morphogenesis’. He introduces several concepts as the chemical basis of morphogenesis (and the name ‘morphogen’ itself), spatial chemical patterns and what is now called ‘Diffusion Driven Instability’. The concept of Turing instability has become standard and the aim of this Chapter is to describe what it is (and what it is not!).

The first experimental evidence of a chemical reaction with spatial patterns explained by these principles was obtained 30 years later by P. De Kepper et al\(^2\). Meanwhile, several non-linear parabolic systems exhibiting Turing Patterns have been studied. Some have been aimed at modelling particular examples of morphogenesis as the famous models developed in [26, 27]. Some have been derived as the simplest possible models exhibiting Turing instability conditions. Nowadays, the biological interest for morphogenesis have evolved towards molecular cascades and networks. Biologists doubt that, within cells or tissues, diffusion is adequate to describe molecular spreading. It remains that Turing’s mechanism stays both as one of the simplest explanation for pattern formation, and one of the most counter-intuitive results in PDEs.

This chapter presents this theory and several examples of diffusion driven instabilities. We begin with the historical example of reaction-diffusion systems where the linear theory shows its exceptional originality. Then we present nonlinear examples. The simplest is the nonlocal Fisher/KPP equation, some more standard parabolic systems are also presented.

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6.1 Turing instability in linear reaction-diffusion systems

An amazingly counter-intuitive observation is the instability mechanism proposed by A. Turing [37]. Consider a linear $2 \times 2$ O.D.E. system

$$\begin{cases} \frac{du}{dt} = au + bv, \\ \frac{dv}{dt} = cu + dv, \end{cases} \quad (6.1)$$

with real constant coefficients $a$, $b$, $c$, and $d$. We assume that

$$\mathcal{T} := a + d < 0, \quad \mathcal{D} := ad - bc > 0. \quad (6.2)$$

Consequently, we have

$$(u, v) = (0, 0) \text{ is a stable attractive point for the system } (6.1). \quad (6.3)$$

In other words, the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has two eigenvalues with negative real parts. Indeed its characteristic polynomial is

$$(a - X)(d - X) - bc = X^2 - XT + \mathcal{D},$$

and the two complex roots are

$$X_{\pm} = \frac{1}{2} \left[ T \pm \sqrt{T^2 - 4\mathcal{D}} \right].$$

Now consider a bounded domain $\Omega$ of $\mathbb{R}^d$ and add diffusion to the system (6.1),

$$\begin{cases} \frac{du}{\partial t} - \sigma_u \Delta u = au + bv, \quad x \in \Omega, \\ \frac{dv}{\partial t} - \sigma_v \Delta v = cu + dv, \end{cases} \quad (6.4)$$
with either Neuman or Dirichlet boundary conditions. In both cases the state \((u,v) = (0,0)\) is still a steady solution, to (6.4) now.

In principle adding diffusion in the differential system (6.1) should help for stability. But surprisingly we have the

**Theorem 6.1** We fix both the domain \(\Omega\) and the matrix \(A\), we assume (6.2) with \(a > 0\), \(d < 0\) and consider the system (6.4). Then, for \(\sigma_u\) small enough, \(\sigma_v\) of order one, the steady state \((u,v) = (0,0)\) is linearly unstable. Moreover, only a finite number of eigenmodes are unstable.

The usual interpretation of this result is as follows. Because \(a > 0\) and \(d < 0\), the quantity \(u\) is called an activator and \(v\) an inhibitor. On the other hand fixing a unit of time, the \(\sigma\)'s scale as square of length. The result can be extended as the

**Turing instability alternative.**
- Turing instability \(\iff\) short range activator, long range inhibitor.
- Traveling waves \(\iff\) long range activator, short range inhibitor.

There is no general proof of such a result which can only be applied to nonlinear systems. But it is a general observation that can be checked case by case.

**Proof.** We consider the Laplace operator, with Dirichlet or Neuman conditions according to those considered for the system (6.4). It has an orthonormal basis of eigenfunctions \((w_k)_{k \geq 1}\) associated with positive eigenvalues \(\lambda_k\),

\[- \Delta w_k = \lambda_k w_k.\]

We recall that we know that \(\lambda_k \xrightarrow[k \to \infty]{} \infty\). We use this basis to decompose \(u(t)\) and \(v(t)\), i.e.,

\[ u(t) = \sum_{k=1}^{\infty} \alpha_k(t) \, w_k, \quad v(t) = \sum_{k=1}^{\infty} \beta_k(t) \, w_k. \]

We can project the system (6.4) on these eigenfunctions and arrive to

\[
\begin{cases}
\frac{d\alpha_k(t)}{dt} + \sigma_u \lambda_k \alpha_k(t) = a \alpha_k(t) + b \beta_k(t), \\
\frac{d\beta_k(t)}{dt} + \sigma_v \lambda_k \beta_k(t) = c \alpha_k(t) + d \beta_k(t).
\end{cases}
\]

(6.5)

Now, we look for solutions with exponential growth in time, i.e., \(\alpha_k(t) = e^{\lambda t} \alpha_k, \beta_k(t) = e^{\lambda t} \beta_k\) with \(\lambda > 0\) (in fact a complex number with \(Re(\lambda) > 0\) is enough, but this does not change the
conditions we find below). The system is again reduced to

\[
\begin{align*}
\lambda \alpha_k + \sigma_u \lambda_k \alpha_k &= a \alpha_k + b \beta_k, \\
\lambda \beta_k + \sigma_v \lambda_k \beta_k &= c \alpha_k + d \beta_k.
\end{align*}
\tag{6.6}
\]

This is a 2*2 linear system for \(\alpha_k, \beta_k\) and it has a nonzero solution if and only if its determinant vanishes

\[
0 = \det \begin{pmatrix}
\lambda + \sigma_u \lambda_k - a & -b \\
-c & \lambda + \sigma_v \lambda_k - d
\end{pmatrix}
\]

Hence, there is a solution with exponential growth for those eigenvalues \(\lambda_k\) for which

there is a root \(\lambda > 0\) to \((\lambda + \sigma_u \lambda_k - a)(\lambda + \sigma_v \lambda_k - d) - bc = 0.\) \tag{6.7}

This condition can be further reduced to the dispersion relation

\[
\lambda^2 + \lambda \left[ (\sigma_u + \sigma_v) \lambda_k - T \right] + \sigma_u \sigma_v (\lambda_k)^2 - \lambda_k (d \sigma_u + a \sigma_v) + D = 0.
\]

Because the first order coefficient of this polynomial is positive, it can have a positive root if and only if the zeroth order term is negative

\[
\sigma_u \sigma_v (\lambda_k)^2 - \lambda_k (d \sigma_u + a \sigma_v) + D < 0,
\]

and we arrive to the final condition

\[
(\lambda_k)^2 - \lambda_k (\frac{d}{\sigma_v} + \frac{a}{\sigma_u}) + \frac{D}{\sigma_u \sigma_v} < 0. \quad (6.8)
\]

Because \(\lambda_k > 0\) and \(\frac{D}{\sigma_u \sigma_v} > 0\), this polynomial can take negative values only for \(\frac{d}{\sigma_v} + \frac{a}{\sigma_u} > 0\) and large enough with \(\frac{D}{\sigma_u \sigma_v}\) small enough. It is hardly possible to give an accurate general characterization in terms \(\sigma_u\) and \(\sigma_v\) for \((a, b, c, d)\) fixed because we do not know the repartition of the eigenvalues a priori.

To go further, we set

\[
\theta = \frac{\sigma_u}{\sigma_v},
\]

and we write explicitly the roots of the above polynomial and we need that

\[
\lambda_k \in [\Lambda_-, \Lambda_+], \quad \Lambda_{\pm} = \frac{1}{2\sigma_v \theta} \left[ \frac{d \theta + a \pm \sqrt{(d \theta + a)^2 - 4D \theta}}{2 \sigma_v \theta} \right]. \tag{6.9}
\]

We can restrict ourselves to the regime \(\theta\) small, then the Taylor expansion gives,

\[
\Lambda_{\pm} = \frac{d \theta + a}{2 \sigma_v \theta} \left[ 1 \pm \sqrt{1 - \frac{4D \theta}{(d \theta + a)^2}} \right],
\]

80
\[ \Lambda_{\pm} \approx \frac{d\theta + a}{2\sigma_v \theta} \left[ 1 \pm \left( 1 - \frac{2D\theta}{(d\theta + a)^2} \right) \right] , \]

and thus
\[ \Lambda_- \approx \frac{D}{a \sigma_v} = O(1), \quad \Lambda_+ \approx \frac{a}{\sigma_v \theta} . \]

In the regime \( \sigma_u \) small, \( \sigma_v \) of order 1, the interval \([\Lambda_-, \Lambda_+]\) becomes very large, hence we know that some eigenvalues \( \lambda_k \) will work.

Notice however that, because \( \lim_{k \to \infty} \lambda_k = +\infty \), there are only a finite number of unstable modes \( \lambda_k \).

In principle one will observe the mode \( w_k \) corresponding to the largest possible \( \lambda \) in (6.7) among the \( \lambda_k \) satisfying the condition (6.9). This corresponds to the smallest value of the expression in (6.8). When the large modes are concerned this, in general, does not correspond to the largest \( \lambda_k \).

**Exercise.** Check how the condition (6.8) is generalized if we only impose the more general instability criteria that (6.7) holds with \( \lambda \in \mathbb{C} \) and \( \text{Re}(\lambda) > 0 \).

### 6.2 Spots on the body and stripes on the tail

![Figure 6.2: Examples of animals with spots and stripes.](image)

Turing instability provides us with a possible explanation of why so many animals (especially fishes) have spots on the body and stripes on the tail, see Figures 6.2–6.3. In short words, in a long and narrow domain (a tail) typical eigenfunctions are 'bands' and with a better shaped domain (a mathematical square body), the eigenfunctions are 'spots' or 'chessboards'.

To explain this, we consider Neuman boundary condition and use our computations of eigenvalues in Section 1.6.2. In one dimension, on a domain \([0, L]\), the eigenvalues and eigenfunctions
are
\[ \lambda_k = \left( \frac{\pi k}{L} \right)^2, \quad w_k(x) = \cos \left( \frac{\pi k x}{L} \right), \quad k \in \mathbb{N}. \]

In a rectangle \([0, L_1] \times [0, L_2]\), we obtain the eigenelements, for \(k, l \in \mathbb{N}\)
\[ \lambda_{kl} = \left( \frac{\pi k}{L_1} \right)^2 + \left( \frac{\pi l}{L_2} \right)^2, \quad w_{kl}(x, y) = \cos \left( \frac{\pi k x}{L_1} \right) \cos \left( \frac{\pi l y}{L_2} \right). \]

Consider a narrow stripe, say \(L_2 \approx 0\) and \(L_1 \gg 1\). The condition (6.9), namely \(\lambda_{kl} \in [\Lambda_-, \Lambda_+]\), will impose \(l = 0\) otherwise \(\lambda_{kl}\) will be very large and cannot fit the interval \([\Lambda_-, \Lambda_+]\). The corresponding eigenfunctions are bands parallel to the \(y\) axis.

When \(L_2 \approx L_1\), the repartition of sums of squared integers generically imposes that the \(\lambda_{kl} \in [\Lambda_-, \Lambda_+]\) will be for \(l \approx k\).

We refer to [30] for a detailed analysis of the Turing patterns, and their interpretation in development biology. To conclude this section, we point out that growing domains during the development also influences very strongly the pattern formation.

### 6.3 The simplest nonlinear example: the nonlocal Fisher/KPP equation

As a simple nonlinear example to explain what is Turing instability, we consider the non-local Fisher/KPP equation
\[
\frac{\partial}{\partial t} u - \nu \frac{\partial^2}{\partial x^2} u = r u(1 - K * u), \quad t \geq 0, \quad x \in \mathbb{R},
\]
still with \( \nu > 0, \ r > 0 \) given parameters and the convolution kernel \( K \) is a smooth probability density function

\[
K(\cdot) \geq 0, \quad \int_R K(x)dx = 1, \quad K \in L^\infty(\mathbb{R}) \quad \text{(at least)}.
\]

Compared to the Fisher/KPP equation it takes into account that competition for resources can be of long range (the size of the support of \( K \)) and not just local.

It has been proposed in ecology as an improvement of the Fisher equation that takes into account long range competition for resources [10, 20]. In semi-arid regions the roots of the trees, in competition for water, can cover up ten times the external size of the tree itself (while in temperate regions the ratio is roughly one to one). This leads to the so-called 'tiger bush' landscape [22], see Figure ??.

It has also been proposed as a simple model of adaptive evolution to take account for higher competition between closer trait [17]; \( x \) represents a physiological trait, the Laplace term represents mutations and the right hand side growth and competition. The convolution kernel means that competition between individuals of closer phenotypical traits is higher than between more different traits (see also Section 3.7.2).

The convolution term has a drastic effect on solutions; it can induce that solutions exhibit a behavior quite different from those to the Fisher/KPP equation. The reason is mainly that the maximum principle is lost with the non-local term. Again we notice that the steady state \( u \equiv 0 \) is unstable, that \( u \gg 1 \) is also unstable because it induces a strong decay. In one dimension, for a general reaction function \( f(u) \) the conditions reads \( f(0) = 0, \ f'(0) > 0 \) and \( f(u) < 0 \) for \( u \) large; consequently there is a point \( u_0 \) satisfying (generically) \( f(u_0) = 0, \ f'(u_0) < 0 \), i.e. a stable steady state should be in between the unstable ones. This is the case of the nonlinearities arising in Fisher/KPP equation that we have encountered.

In the infinite dimensional framework at hand, we will see that under certain circumstances, the steady state \( u \equiv 1 \) can be unstable in the sense of the

**Definition 6.2** The steady state \( u \equiv 1 \) is called linearly unstable if there are perturbations such that the linearized system has exponential growth in time.

Then, the following conditions are satisfied

**Definition 6.3** A steady state \( u_0 \) is said nonlinearly Turing unstable if

(i) it is between two unstable states as above (no blow-up, no extinction),

(ii) it is linearly unstable,

(iii) the corresponding growth modes are bounded (no high frequency oscillations).
Obviously when Turing instability occurs, solutions should exhibit strange behaviors because they remain bounded away from the two extreme steady states, cannot converge to the steady state $u^0$ and cannot oscillate rapidly. In other words, they should exhibit Turing patterns. See Figure 6.4 for a numerical solution to (6.10).

In practice, to check linear instability we use a spectral basis. In compact domains the concept can be handled using eigenfunctions of the Laplace operator as we did it in section 6.1. On the full line, we may use the generalized eigenfunctions which are the Fourier modes. We define the Fourier transform as

$$\hat{u}(\xi) = \int_{\mathbb{R}} u(x)e^{-ix\xi}dx.$$  

**Theorem 6.4** *Assume the condition*

$$\exists \xi_0 \text{ such that } \hat{K}(\xi_0) < 0,$$

(6.11)

*and $\nu/r$ small enough (depending on $\xi_0$ and $K(\xi_0)$), then the non-local Fisher/KPP equation (6.10) is nonlinearly Turing unstable.*

A practical consequence of this Theorem is that solutions should create Turing patterns as mentioned above. This can easily be observed on numerical simulations, see Figure 6.4.

![Figure 6.4: Steady state solutions of the nonlocal Fisher/KPP equation (6.10) in 2 dimensions with different diffusion coefficients.](image)

The nonlocal Fisher equation gives also an example of the already mentioned alternative Turing instability alternative.

- Turing instability $\iff K(\cdot)$ is long range.
- Traveling waves $\iff K(\cdot)$ is short range.

Indeed the nonlocal term $K * u$ is the inhibitor (negative) term. The diffusion represents the activator (with a coefficient normalized to 1). The limit of very short range is the case of $K = \delta$, a Dirac mass, and we recover the Fisher/KPP equation. More on this is proved in [7].
Proof.  (i) The state $u \equiv 0$ and $u \equiv \infty$ are indeed formally both unstable (to prove this rigorously is not so easy for $u \equiv \infty$.)

(ii) The linearized equation around $u \equiv 1$ is obtained setting $u = 1 + \tilde{u}$ and keeping the first order terms, we obtain

$$\frac{\partial}{\partial t} \tilde{u} - \nu \frac{\partial^2}{\partial x^2} \tilde{u} = -r K * \tilde{u}.$$ 

And we look for solutions of the form $\tilde{u}(t, x) = e^{\lambda t} v(x)$ with $\lambda > 0$. This means that we should find eigenfunctions associated with the positive eigenvalue $\lambda$ to

$$\lambda v - \nu \frac{\partial^2}{\partial x^2} v = -r K * v.$$ 

We look for a possible Fourier mode $v(x) = e^{ix \xi_1}$ that we insert in the previous equation. Then we obtain the condition

$$\lambda + \nu \xi_1^2 = -r \hat{K}(\xi_1), \quad \text{for some } \lambda > 0. \quad (6.12)$$

And it is indeed possible to such a $\lambda$ and a $\xi_1 = \xi_0$ under the conditions of the Theorem.

(ii) The possible unstable modes $\xi_0$ are obviously bounded because $\hat{K}$ is bounded as the Fourier transform of a $L^1$ function.

Notice however that the mode $\xi_1$ we will observe in practice is that with the highest growth rate $\lambda$.

### 6.4 Phase transition: what is not Turing instability

What happens if the third condition in Definition 6.3 does not hold? The system remains bounded away from zero and infinity by condition (i) and it is unstable by condition (ii). But it may 'blow-up' by high frequency oscillations?

As an example of such an unstable system, which is not Turing stable, we consider the phase transition model also used in Section 4.7,

$$\begin{cases}
\frac{\partial u}{\partial t} - \Delta A(u) = 0, & x \in \Omega, \\
\frac{\partial}{\partial \nu} u = 0 & \text{on } \partial \Omega,
\end{cases} \quad (6.13)$$

with

$$A(u) = u (3 - u)^2. \quad (6.14)$$

Because $A'(u) = 3(3 - u)(1 - u)$, the equation (6.13) is backward-parabolic in the interval $u \in (1, 3)$ because $A'(u) < 0$ there. We expect that linear instability occurs on this interval. We take $\bar{u} = 2$ and set

$$u = 2 + \tilde{u},$$
Inserting this in the above equation we find the linearized equation for $\tilde{u}$

$$
\begin{cases}
\frac{\partial \tilde{u}}{\partial t} - a'(2) \Delta \tilde{u} = 0, & x \in \Omega \\
\frac{\partial \tilde{u}}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
$$

We set $\gamma = -a'(2) > 0$ and we look for solutions $e^{\lambda t}w(x)$ which are unstable, i.e., $\lambda > 0$. These are given by

$$
\lambda w + \gamma \Delta w = 0,
$$

and thus they stem from the Neuman eigenvalue problem in Theorem 1.7. We have

$$
\lambda = \lambda_i \gamma, \quad w = w_i.
$$

We can see that all the eigenvalues of the Laplace operator generate a possible unstable modes and thus they can be of very high frequencies. And we expect to see the mode corresponding to the largest $\lambda$, i.e., to the largest $\lambda_i$ which of course does not exist because $\lambda_i \to \infty$. In the space variable these correspond to highly oscillatory eigenfunctions $w_i$ that we can observe numerically.

Figure 6.5 gives numerical solutions to (6.13)–(6.14) corresponding to two different grids; high frequency solutions are obtained that depend on the grid. This defect explains why we require bounded unstable modes in the definition of Turing instability. It also explains why in Section 4.7 we have introduced a relaxation system.

Figure 6.5: Two numerical solutions to the phase transition system (6.13)–(6.14) for (Right) 80 grid points and (Left) 150 grid points. The oscillation frequencies depend on the grid and these are not Turing patterns.
6.5 Gallery of parabolic systems giving Turing patterns

Many examples of nonlinear parabolic systems exhibiting Turing instabilities (and patterns) have been widely studied. This Section gives very quickly several examples but it is in no way complete, neither for their theory neither for those used in biology and other applications.

6.5.1 The diffusive Fisher/KPP system

We can depart the quest of systems exhibiting Turing patterns with the Fisher/KPP equation (see sections 3.2 and 3.4)

\[ \frac{\partial u}{\partial t} - d_u \Delta u = g(u)(1 - u). \]  

(6.15)

We recall that its main remarkable property is to exhibit traveling waves.

The natural extension to a system leads to the diffusive Fisher/KPP system already mentioned in section 3.5. It reads

\[
\begin{align*}
\frac{\partial u}{\partial t} - d_u \Delta u &= g(u)v, \\
\frac{\partial v}{\partial t} - d_v \Delta v &= -g(u)v.
\end{align*}
\]  

(6.16)

For \(d_u = d_v\), the solution \(v = 1 - u\) reduces (6.16) to (6.15). Therefore this system still exhibits traveling waves. It has been proved that even for \(d_v \neq d_u\), the system admits monotonic traveling waves [8, 24, 25] for a power law nonlinearity \(g(u) = u^n\).

Form the Turing instability alternative in section 6.1 we do not expect Turing patterns at this stage. This is why interesting examples are always a little more elaborated.

**Exercise.** Consider the steady states \((U = \gamma, V = 0), \gamma > 0\) of (6.16) with \(g(u) = u^n\).

1. Compute the linearized equation around this steady state.
2. In the whole space or a bounded domain with Neuman boundary condition, show that this steady state is always stable.

6.5.2 The Brusselator

Prigogine and Lefever\(^3\) proposed the following example. This is certainly the simplest system but with the drawback that there is an input constant \(A > 0\).

Let \(A > 0, B > 0\) be given positive numbers. In a finite domain \(\Omega\) we consider the following

2 \times 2 system with Neuman boundary condition

\begin{align}
\frac{\partial u}{\partial t} - d_u \Delta u &= A - (B + 1)u + u^2v, \\
\frac{\partial v}{\partial t} - d_v \Delta v &= Bu - u^2v.
\end{align}

(6.17)

We check below that there is a single positive steady state that exhibits the linear conditions for Turing instability. Also the solution cannot vanish thanks to the term $A > 0$. But we are not aware of a proof that the solutions remain bounded for $t$ large.

Figure 6.6: The Turing instability region in the $(A, B)$ plane.

Figure 6.7: Two numerical solutions to the brusselator system (6.17) with $A = B = 2$ and 200 grid points. The choice of diffusion coefficients are (Left) $d_v = 1$. and $d_u = 0.005$ (Right) $d_v = 0.1$ and $d_u = 0.001$. The first component $u$ exhibits one or two strong pick(s) upwards while $v$ exhibits one or two minimum(s) and has been magnified by a factor 5.

**Exercise.** 1. Check that the only homogeneous steady state is $(U = A, V = \frac{B}{A})$.
2. Write the linearized system around this steady state.
3. Check that for $B < 1 + A^2$, this steady state is attractive for the associated differential equation ($d_u = d_v = 0$).

4. Let $d_u > 0$, $d_v > 0$ and consider an eigenpair of Laplace equation with Neuman boundary condition ($\lambda_i, w_i$). Write the condition on $A$, $B$, $d_u$, $d_v$, $\lambda_i$ which degenerates an unstable mode $\lambda > 0$.

5. Show that for $\theta := \frac{d_u}{d_v} < 1$ the interval $[\Lambda_-, \Lambda_+]$ for $\lambda_i$ is not empty.

6. Show that for $\theta$ small we have $\Lambda_- \approx \frac{A^2}{d_v}$, $\Lambda_+ \approx \frac{B-1}{2d_v\theta}$. Conclude that for $d_v$ fixed and $d_u$ small, the steady state becomes unstable.

Solution. 2.

\[
\begin{align*}
\frac{\partial u}{\partial t} - d_u \Delta \tilde{u} &= (B - 1)\tilde{u} + A^2\tilde{v}, \\
\frac{\partial v}{\partial t} - d_v \Delta \tilde{v} &= -B\tilde{u} - A^2\tilde{v}.
\end{align*}
\]

3. $det = A^2$, $tr = B - 1 - A^2$ and see Section 6.1 for the condition $tr < 0$ which leads to $B < 1 + A^2$.

4. As in the general theory we arrive to a second order polynomial for $\lambda$ which implies that the constant term should be negative leading to the condition

\[
d_u d_v \lambda_i^2 + \lambda_i (A^2 d_u - (B - 1) d_v) + A^2 < 0.
\]

5. To have two positive roots we need a negative slope at origin, i.e., $B > 1 + \frac{d_u}{d_v} A^2$ and also

\[(B - 1) d_v - A^2 d_u > 2 A \sqrt{d_u d_v} \iff B > 2 \sqrt{\theta} A + \theta A^2.
\]

This is compatible with the condition of question 3. if and only if $\theta < 1$.

6. We have

\[
\Lambda_{\pm} = \frac{1}{2d_v \theta} [B - 1 - A^2 \theta \pm \sqrt{(B - 1 - A^2 \theta)^2 - 4A^2 \theta}].
\]

The Taylor expansion for $\theta$ small reads

\[
\Lambda_{\pm} \approx \frac{1}{2d_v \theta} (B - 1 - A^2 \theta) \left[ 1 \pm \sqrt{1 - \frac{4A^2 \theta}{(B - 1 - A^2 \theta)^2}} \right]
\]

and thus

\[
\Lambda_- \approx \frac{A^2}{d_v}, \quad \Lambda_+ \approx \frac{B - 1}{d_v \theta}.
\]

For $d_v$ fixed and $\theta$ small this interval will contain eigenvalues $\lambda_i$.

The two parabolic regions in $(A, B)$ are drawn in Figure 6.6.
Figure 6.8: Two dimensional simulations of the Gray-Scott system (6.19) on a square grid (the quantities $u$, $v$ and $\int_0^1 u(x,s)ds$ are depicted).

### 6.5.3 Gray-Scott system

Gray and Scott\(^4\) introduced this system as a model of chemical reaction between two constituents. It differs from the Brusselator (6.17) only from the reaction terms

\[
\begin{align*}
\frac{\partial u}{\partial t} - d_u \Delta u &= u^n v - Au, \\
\frac{\partial v}{\partial t} - d_v \Delta v &= -u^n v + B(1 - v).
\end{align*}
\]  

(6.18)

Again $A > 0$, $B > 0$ are constants (input, degradation of constituents) and $n$ is an integer, the number of molecules $u$ react with a single $v$.

One sometimes refers to the ‘Gray-Scott system’ for the particular case \( n = 2, B = 0 \)

\[
\begin{align*}
\frac{\partial u}{\partial t} - d_u \Delta u &= u^2 v - Au, \\
\frac{\partial v}{\partial t} - d_v \Delta v &= -u^2 v.
\end{align*}
\] (6.19)

It is already very rich and exhibits beautiful dynamical patterns shown in Figures 6.8 and 6.9. These are not Turing patterns because there is no unstable steady state in this particular case. In fact the only steady state is obviously \((0,0)\) which is stable.

The system (6.18) has the advantage over the Brusselator (6.17) that it satisfies the Turing Instability Principle in section 6.1. To explain this, we only consider the case

\[ n = 2, \quad B > 4A^2. \] (6.20)
We first notice that there are three steady states; the trivial one \((U_0 = 0, V_0 = 1)\) and non-vanishing ones \((U_\pm, V_\pm)\) to the Gray-Scott system, given by

\[
UV = A, \quad AU = B(1 - V).
\]

We can eliminate \(V\) or \(U\) and find 
\[
AU^2 - BU + AB = 0 \quad \text{and} \quad BV^2 - BV + A^2 = 0,
\]
this means that
\[
U_\pm = \frac{B \pm \sqrt{B^2 - 4A^2B}}{2A}, \quad V_\pm = \frac{B + \sqrt{B^2 - 4A^2B}}{2B}.
\] (6.21)

It is easy (but tedious) to see that the following results hold:

**Lemma 6.5** With the assumption (6.20), the steady state \((U_-, V_-)\) is linearly unstable, the state \((U_0 = 0, V_0 = 1)\) is linearly stable and \((U_+, V_+)\) is linearly stable under the additional condition (6.22) below.

**Proof.** The linearized systems read

\[
\begin{align*}
\frac{\partial \tilde{u}}{\partial t} - d_u \Delta \tilde{u} &= (2UV - A)\tilde{u} + U^2\tilde{v}, \\
\frac{\partial \tilde{v}}{\partial t} - d_v \Delta \tilde{v} &= -2UV\tilde{u} - (U^2 + B)\tilde{v}.
\end{align*}
\]

We begin with the trivial steady state \((U_0, V_0)\). Along with the general analysis of section 6.1, we compute, for the trivial steady state, the quantities

\[
D_0 = AB > 0, \quad T = -A - B < 0.
\]

This means that for the steady state \((U_0, V_0)\), the corresponding O.D.E. system is linearly attractive.

For the non-vanishing ones, using the above rule \(UV = A\), we also compute

\[
\begin{align*}
\frac{\partial \tilde{u}}{\partial t} - d_u \Delta \tilde{u} &= A\tilde{u} + U^2\tilde{v}, \\
\frac{\partial \tilde{v}}{\partial t} - d_v \Delta \tilde{v} &= -2A\tilde{u} - (U^2 + B)\tilde{v},
\end{align*}
\]

\[
D = A(U^2 - B) > 0, \quad T = A - B - U^2.
\]

We deduce from (6.21)

\[
U_\pm^2 - B = \frac{B^2}{2A^2} - 2B \pm \frac{B}{2A^2} \sqrt{B^2 - 4A^2B}
\]

\[
= \frac{B}{2A^2} \left[ \frac{B}{2A^2} \pm \sqrt{B^2 - 4A^2B} \right].
\]
Therefore obviously $D_+ > 0$ and

$$D_- = \frac{B}{2A^2} \sqrt{B^2 - 4A^2B} \left[ \sqrt{B - 4A^2} - \sqrt{B} \right] < 0.$$ 

Therefore, for the steady state $(U_-, V_-)$, the corresponding O.D.E. system is unstable.

It remains to check the trace condition for $(U_+, V_+)$. We have

$$T = A - \frac{B^2}{2A^2} - \frac{B}{2A^2} \sqrt{B^2 - 4A^2B} < 0,$$

which is an additional condition to be checked. Notice that for the limiting value $B = 4A^2$ it means that $A > 1/8$. Therefore it is clearly compatible with (6.20). 

We can now come back to the traveling wave solutions. We consider the particular case $d_u = d_v$, $A = B$ and choosing $v(t, x) = 1 - u(t, x)$. Then the two equations of (6.18) reduce to the single equation

$$\frac{\partial u}{\partial t} - d_u \Delta u = u^2(1 - u) - Au = u(u - u^2 - A).$$

This is the situation of the Allen-Cahn (bistable) equation where, for $A$ small enough, we have three steady states $U_0 = 0$ is stable, $U_-$ is unstable and $U_+ > U_- > 0$ is stable. Therefore we have indeed a unique traveling wave solution (see section 3.3). For an extended study of traveling waves in Gray-Scott system, we refer to [29]

### 6.5.4 Schnakenberg system

The Schnakenberg system\(^5\) is still another variant written as

$$\begin{cases}
\frac{\partial u}{\partial t} - d_u \Delta u = C + u^2v - Au, \\
\frac{\partial v}{\partial t} - d_v \Delta v = B - u^2v.
\end{cases}$$

(6.23)

With $C = 0$, it has been advocated by M. Ward as a simple model for spot-patterns formation and spot-splitting in the following asymptotic regime

$$\begin{cases}
\frac{\partial u}{\partial t} - \varepsilon \Delta u = u^2v - Au, \\
\varepsilon \frac{\partial v}{\partial t} - d_v \Delta v = B - \frac{u^2v}{\varepsilon}.
\end{cases}$$

(6.24)

6.5.5 The diffusive FitzHugh-Nagumo system

Consider again the FitzHugh-Nagumo system as already studied in Section 3.8.2, but with diffusion on both components,

\[
\begin{aligned}
\frac{\partial u}{\partial t} - d_u \Delta u &= u(1-u)(u - \frac{1}{2}) - v, \\
\frac{\partial v}{\partial t} - d_v \Delta v &= \mu u - v,
\end{aligned}
\]  

(6.25)

with Neuman boundary conditions.

Assume that

\[ \mu > (1-u)(u - \frac{1}{2}) \quad \forall u \in \mathbb{R}. \]

Then the only homogeneous steady state is \((0,0)\) and it is stable for the associated differential equation.

**Exercise.** We fix \(d_v > 0\). Show that for \(d_u\) small enough the steady state becomes unstable.

6.5.6 Gierer-Meinhardt system

Gierer-Meinhardt\(^6\) system is one of the most famous exhibiting Turing instability and Turing patterns. It can be considered as model for chemical reactions with only two reactants denoted by \(u(t,x)\) and \(v(t,x)\).

\[
\begin{aligned}
\frac{\partial u}{\partial t}(t,x) - d_1 \Delta u(t,x) + Au &= u^p/v^q, \quad t \geq 0, \ x \in \Omega, \\
\frac{\partial v}{\partial t}(t,x) - d_2 \Delta v(t,x) + Bv &= u^r/v^s,
\end{aligned}
\]  

(6.26)

with Neuman boundary conditions.

Among this class, several authors have used the particular case with a single parameter

\[
\begin{aligned}
\frac{\partial u}{\partial t}(t,x) - d_1 \Delta u(t,x) + Au &= u^2/v, \quad t \geq 0, \ x \in \Omega, \\
\frac{\partial v}{\partial t}(t,x) - d_2 \Delta v(t,x) + v &= u^2,
\end{aligned}
\]  

(6.27)

The diffusion coefficients satisfy

\[ d_1 \ll 1 \ll d_2. \]

A possible limit is \(d_2 \to \infty, v \to \text{constant}\) and thus we arrive to the reduced system for steady state

\[
\begin{aligned}
-\varepsilon^2 \Delta u + u &= u^p, \quad x \in \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]  

(6.28)

Following Berestycki and Lions, there is a unique radial spike like solution \( u = u_0(\frac{x}{\varepsilon}) \) with
\[
-\Delta u_0 + u_0 = u_0^p, \quad x \in \mathbb{R}^d, \quad u_0 > 0,
\]
when \( p < \frac{d+2}{d-2} \).

But (see Andrea Malchiodi and Montenegro) there are solutions concentrating on the boundary \( \partial \Omega \) without limitation on \( p \).

Also for systems there is a large literature\(^7\) on the numerous types of solutions, and their concentration properties in the case
\[
\begin{align*}
-\varepsilon^2 \Delta u + u &= \frac{u^2}{v}, \quad x \in \mathbb{R}, \\
-\Delta v + v &= u^2.
\end{align*}
\]

### 6.6 Models from ecology

There are many standard models from ecology. The simplest versions never satisfy the conditions for Turing instability and we review some such examples first. Then we conclude with a more elaborate model which satisfies the Turing conditions for instability.

#### 6.6.1 Competing species and Turing instability

Let us come back to models of competing species as already mentioned in section 3.6. Let the coefficients \( r_1, r_2, \alpha_1 \) and \( \alpha_2 \) be positive in the system
\[
\begin{align*}
\frac{\partial}{\partial t} u_1 - d_1 \Delta u_1 &= r_1 u_1 (1 - u_1 - \alpha_2 u_2), \\
\frac{\partial}{\partial t} u_2 - d_2 \Delta u_2 &= r_2 u_2 (1 - \alpha_1 u_1 - u_2).
\end{align*}
\]

We have seen in section 3.6 that the positive steady state \((U_1, U_2)\) is stable iff \( \alpha_1 < 1, \alpha_2 < 1 \). As stated in Theorem 6.1, to be Turing unstable, we need that one of the diagonal coefficients is positive in the linearized matrix
\[
L = \begin{pmatrix}
-r_1 U_1 & -\alpha_2 r_1 U_1 \\
-\alpha_1 r_2 U_2 & -r_2 U_2
\end{pmatrix}
\]
We see it is not the case. There is no activator in such systems.

---

6.6.2 Prey-predator system

In system (6.31), we can also consider a prey-predator situation where \( \alpha_1 < 0 \) (\( u_1 \) is the prey) and \( 0 < \alpha_2 < 1 \) (\( u_2 \) is the predator). With these conditions, the positive steady state is given by

\[
(U_1, U_2) = \left( \frac{1 - \alpha_2}{1 - \alpha_2 \alpha_1}, \frac{1 - \alpha_1}{1 - \alpha_2 \alpha_1} \right).
\]

Because

\[
tr(L) = -(r_1 U_1 + r_2 U_2) < 0, \quad det(L) = r_1 r_2 U_1 U_2 (1 - \alpha_1 \alpha_2) > 0,
\]

this steady state is stable.

Again, according to Theorem 6.1, it cannot be Turing unstable because both diagonal coefficients are negative.

6.6.3 Prey-predator system with Turing instability (problem)

Consider the prey-predator system

\[
\begin{aligned}
\frac{\partial}{\partial t} u - d_u \Delta u &= u(1 + u - \gamma \frac{u^2}{2} - \beta v), \\
\frac{\partial}{\partial t} v - d_v \Delta v &= v(1 - v + \alpha u),
\end{aligned}
\]

(6.32)

The purpose of the problem is to show there are parameters \( \alpha > 0, \beta > 0, \gamma > 0 \) for which Turing instability occurs. We assume

\[
\beta < 1, \quad \alpha \beta < 1.
\]
1. Show there is a unique homogeneous stationary state \((\bar{u} > 0, \bar{v} > 0)\). Show that
\[
\gamma \bar{u} > 2(1 - \alpha \beta).
\]

2. Compute the linearized matrix \(A\) of the differential system around this stationary state.

3. Compute the trace of \(A\) and show that \(\text{Tr}(A) < 0\) if and only if the following condition is satisfied: \(\bar{u}[1 - \alpha - \gamma \bar{u}] < 1\).

4. Compute the determinant of \(A\) and show that \(\text{Det}(A) > 0\).

5. Which sign condition is also needed on one of the coefficients of this matrix? How is it written in terms of \(\gamma \bar{u}\)?

6. Suppose also that \(\alpha \beta > \frac{1}{2}, \alpha > 1\). Show that the above conditions are satisfied for \(\gamma\) small enough.

7. State a Turing instability result on \(d_u, d_v\) for coefficients as in 6.

Figure 6.10 shows a numerical simulation of system (6.32) in the Turing instability regime. Notice also that the system has a priori bounds that follow from the maximum principle. If initially true, we have
\[
\gamma \frac{u^2}{2} \leq 1 + u \implies u \leq u_M := \frac{1 + \sqrt{1 - 2\gamma}}{\gamma},
\]
\[
v \leq v_M := 1 + u_M.
\]

**Solution**

1. The non-zero homogeneous steady state is given by \(\ddot{v} = \alpha \dot{u} + 1\) and
\[
0 = \gamma \frac{\dot{u}^2}{2} - \ddot{u} + \beta \ddot{v} - 1 = \gamma \frac{\dot{u}^2}{2} + \ddot{u} (\alpha \beta - 1) + \beta - 1.
\]
Since \(\beta < 1\), its solution positive is given by
\[
\gamma \ddot{u} = 1 - \alpha \beta + \sqrt{(1 - \alpha \beta)^2 + 2\gamma(1 - \beta)} > 2(1 - \alpha \beta).
\]

2. The linearized matrix about this steady state is
\[
A = \begin{pmatrix}
\ddot{u}(1 - \gamma \ddot{u}) & -\beta \ddot{u} \\
\alpha \ddot{v} & -\ddot{v}
\end{pmatrix}
\]

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3. We have $Tr(A) = \bar{u}(1 - \gamma \bar{u} - \alpha) - 1$ and $Tr(A) < 0$ if and only if $\bar{u}[1 - \alpha - \gamma \bar{u}] < 1$.

4. We have

$$Det(A) = -\bar{u}(1 - \gamma \bar{u})\bar{v} + \alpha \beta \bar{u} \bar{v} = \bar{u} \bar{v} (\alpha \beta + \gamma \bar{u} - 1)$$

and using 1.

$$Det(A) > \bar{u} \bar{v} (1 - \alpha \beta) > 0.$$ 

5. The last condition to have Turing instability is that one of the diagonal coefficients of $A$ is positive. It can only be the upper left coefficient and this is satisfied if $\gamma \bar{u} < 1$.

6. As $\gamma \to 0$, we have $\gamma \bar{u} \to 2(1 - \alpha \beta)$ and the condition $\alpha \beta < 1/2$ is enough to ensure $\gamma \bar{u} < 1$ for $\gamma$ small enough. We also have to ensure $\bar{u}[1 - \alpha - \gamma \bar{u}] < 1$ (from 3) and this converges to $\bar{u}[-1 - \alpha + 2\alpha \beta]$ and with $\alpha > 1$ we have in fact $[-1 - \alpha + 2\alpha \beta] < 0$ which guarantees the desired condition.

7. With these conditions we know from Theorem 6.1 that in a bounded domain, for $d_u$ small enough and $d_v$ of order one, the steady state is linearly unstable.

6.7 Keller-Segel with growth

Exercise. Consider the one dimensional Keller-Segel system with growth

$$\begin{cases} u_t - u_{xx} + \chi(uv)_x = u(1 - u), & x \in \mathbb{R}, \\ -dv_{xx} + v = u. \end{cases} \tag{6.33}$$

1. Show that $u = 1, v = 1$ is a steady state.

2. Linearize the system around this steady state $(1,1)$.

3. In the Fourier variable, reduce the system to a single equation on $\hat{U}(t,k)$,

$$\hat{U}_t + \hat{U} \Lambda(k) = 0,$$

and compute $\Lambda(k)$.

4. Show that it is linearly stable under the condition $\chi \leq (1 + \sqrt{d})^2$.

Solution 1. is easy.

2. $U_t - U_{xx} + \chi V_{xx} = -U, \quad -dV_{xx} + V = U$.

3. $\hat{U}_t + k^2 \hat{U} + k^2 \chi \hat{V} = -\hat{U}, \quad -dk^2 \hat{V} + \hat{V} = \hat{U}$.

We can eliminate $\hat{V}$ and find $\Lambda(k) = [k^2 + 1 - \chi \frac{k^2}{1+dk^2}]$. 

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4. The linear stability condition is that all solutions have the time decay $e^{-\lambda t}$ with $\lambda > 0$, in other words $\Lambda(k) > 0$. Using the shorter notation $X = k^2 \geq 0$, it reads $1 + X(d + 1 - \chi) + dX^2 \geq 0$. The analysis of the roots of this polynomial leads to 4.
Chapter 7

Strong reactions and free boundary problems

Departing from reaction-diffusion systems one can rescale the problem and consider the global length of the (experimental, computational, observation) domain rather the natural scale which, e.g. in population biology, is the individual size scale. Doing so we rescale the space variable and this leads to also rescale time so as to use the propagation time scale rather the generation scale.

There are many different ways to change scale and we will discuss several of them in the next chapters.

Here we give the example of the parabolic scale where the old time is defined as $\tilde{t} = \frac{t}{\varepsilon}$, with $t$ the new time. Then we also replace the old space variable $\tilde{x}$ by $x$ with the relation $\tilde{x} = \frac{x}{\sqrt{\varepsilon}}$.

7.1 Derivation of the Stefan problem (no latent heat)

As a first and simple example, consider the reaction-diffusion equation

$$
\left\{
\begin{array}{ll}
\frac{\partial}{\partial t} u_\varepsilon - d_1 \Delta u_\varepsilon = -\frac{1}{\varepsilon} u_\varepsilon v_\varepsilon, & t \geq 0, \ x \in \mathbb{R}^d, \\
\frac{\partial}{\partial t} v_\varepsilon - d_2 \Delta v_\varepsilon = -\frac{1}{\varepsilon} u_\varepsilon v_\varepsilon, \\
u_\varepsilon(t = 0, x) = u^0(x) \geq 0, & v_\varepsilon(t = 0, x) = v^0(x) \geq 0, \quad u^0, v^0 \in L^\infty \cap L^1(\mathbb{R}^d).
\end{array}
\right.
$$

(7.1)

Because the right hand sides are nonpositive, it is easy to see that there is a unique solution and that for all $t \geq 0$

$$
0 \leq u_\varepsilon(t, x) \leq \|u^0\|_\infty, \quad 0 \leq v_\varepsilon(t, x) \leq \|v^0\|_\infty,
$$

$$
\int_{\mathbb{R}^d} u_\varepsilon(t, x) dx \leq \int_{\mathbb{R}^d} u^0(x) dx, \quad \int_{\mathbb{R}^d} v_\varepsilon(t, x) dx \leq \int_{\mathbb{R}^d} v^0(x) dx.
$$
Stronger a priori estimates are given in the

**Theorem 7.1** Assume that in (7.1), \( \nabla u^0, \nabla v^0 \in L^1(\mathbb{R}^d) \), then we have, additionally to the above a priori bounds,
\[
\|\nabla u_\varepsilon(t)\| + \|\nabla v_\varepsilon(t)\| \leq \|\nabla u^0\| + \|\nabla v^0\|,
\]
\( u_\varepsilon \to u, \quad v_\varepsilon \to v \quad \text{strongly in } L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d), \)
\( u, v \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\mathbb{R}^d)) , \)
and \( w = u - v \) satisfies the Stefan problem (7.2) below.

**Proof.** We first show why the singular limit of (7.1) is described by the Stefan problem without latent heat
\[
\frac{\partial}{\partial t} w - \Delta A(w) = 0, \quad t \geq 0, \quad x \in \mathbb{R}^d, \quad (7.2)
\]
with
\[
A(w) = \begin{cases} 
  d_1 w & \text{for } w \geq 0, \\
  d_2 w & \text{for } w \leq 0.
\end{cases}
\]

The derivation is as follows: since
\[
u_\varepsilon v_\varepsilon = \varepsilon \left[ \frac{\partial}{\partial t} u_\varepsilon - d_1 \Delta u_\varepsilon \right],
\]
we may pass to the limit in distribution sense and the limit vanishes. Therefore, with the strong convergence stated in Theorem 7.1, we deduce
\[
u_\varepsilon v_\varepsilon = \lim_{\varepsilon \to 0} u_\varepsilon v_\varepsilon = 0. \quad (7.3)
\]

Next, we define
\[
w_\varepsilon = u_\varepsilon - v_\varepsilon \quad \text{\( \varepsilon \to 0 \)} \quad w = u - v.
\]

We write, subtracting the equations on \( u_\varepsilon \) and \( v_\varepsilon \),
\[
\frac{\partial}{\partial t} w_\varepsilon - \Delta [d_1 u_\varepsilon - d_2 v_\varepsilon] = 0.
\]
Passing to the limit in the distribution sense, we find, with \( A = \lim_{\varepsilon \to 0} (d_1 u_\varepsilon - d_2 v_\varepsilon) \),
\[
\frac{\partial}{\partial t} w - \Delta A = 0.
\]

It remains to identify \( A(t, x) \). For that we argue as follows

- For \( w(t, x) > 0, \) then \( u(t, x) > 0 \), therefore \( v_\varepsilon(t, x) \to v(t, x) = 0 \) and \( u_\varepsilon(t, x) \to u > 0 \), and thus
\[
A(t, x) = d_1 u(t, x) = d_1 w(t, x).
\]
• For \( w(t, x) < 0 \), then \( v(t, x) < 0 \), therefore \( u_\varepsilon(t, x) \to u(t, x) = 0 \) and \( v_\varepsilon(t, x) \to v < 0 \), and thus

\[
A(t, x) = d_2v(t, x) = d_2w(t, x).
\]

It remains to show the strong convergence of \( u_\varepsilon \) and \( v_\varepsilon \). This follows from the a priori estimates which imply local compactness and that the full sequence (and not only a subsequence) converges, follows from the uniqueness of the solution to (7.2), and thus of the limit (a fact that we do not prove here).

Next, we prove the \( BV \) estimates. We work on the equations of (7.1) and differentiate them with respect to \( x_i \). We multiply by the sign and obtain

\[
\frac{d}{dt} \int_{\mathbb{R}^d} \left[ |\frac{\partial}{\partial x_i} u_\varepsilon(t, x)| + |\frac{\partial}{\partial x_i} v_\varepsilon(t, x)| \right] dx \\
\leq - \int_{\mathbb{R}^d} \left[ \frac{\partial}{\partial x_i} u_\varepsilon(t, x)v_\varepsilon \right] \left[ \operatorname{sgn} \left( \frac{\partial}{\partial x_i} u_\varepsilon(t, x) \right) + \operatorname{sgn} \left( \frac{\partial}{\partial x_i} v_\varepsilon(t, x) \right) \right] dx \\
= - \int_{\mathbb{R}^d} \left[ \frac{\partial}{\partial x_i} u_\varepsilon(t, x)v_\varepsilon + \frac{\partial}{\partial x_i} v_\varepsilon(t, x)u_\varepsilon + \frac{\partial}{\partial x_i} u_\varepsilon(t, x) \operatorname{sgn} \left( \frac{\partial}{\partial x_i} u_\varepsilon(t, x) \right) v_\varepsilon + \frac{\partial}{\partial x_i} v_\varepsilon(t, x) \operatorname{sgn} \left( \frac{\partial}{\partial x_i} v_\varepsilon(t, x) \right) u_\varepsilon \right] dx \\
\leq 0.
\]

This concludes the proof of the Theorem.

\[ \square \]

### 7.2 Derivation of the full Stefan problem

In order to include latent heat in the Stefan problem, the reaction-diffusion system should be extended. Following [?]? we consider

\[
\begin{aligned}
\frac{\partial}{\partial t} u_\varepsilon - d_1\Delta u_\varepsilon &= -\frac{1}{\varepsilon} u_\varepsilon(v_\varepsilon + \lambda p_\varepsilon), \quad t \geq 0, \ x \in \mathbb{R}^d, \\
\frac{\partial}{\partial t} v_\varepsilon - d_2\Delta v_\varepsilon &= -\frac{1}{\varepsilon} v_\varepsilon(u_\varepsilon + \lambda w_\varepsilon), \\
\frac{\partial}{\partial t} w_\varepsilon &= \frac{1}{\varepsilon} [p_\varepsilon u_\varepsilon - v_\varepsilon w_\varepsilon], \\
\frac{\partial}{\partial t} p_\varepsilon &= \frac{1}{\varepsilon} [v_\varepsilon w_\varepsilon - p_\varepsilon u_\varepsilon],
\end{aligned}
\tag{7.4}
\]

and we give initial data satisfying

\[
u_\varepsilon^0 \geq 0, \quad v_\varepsilon^0 \geq 0, \quad w_\varepsilon^0 \geq 0, \quad p_\varepsilon^0 \geq 0,
\]

and also

\[
w_\varepsilon^0 + p_\varepsilon^0 = 1.
\]

The quantity \( \lambda > 0 \) is called the latent heat.
Chapter 8

Vanishing diffusion and the Hamilton-Jacobi equation

8.1 A linear example; the exponential variable

The simplest example of small diffusion limit is when \( \varepsilon \) vanishes in the parabolic Lotka-Volterra equation

\[
\begin{cases}
\frac{\partial}{\partial t} u - \varepsilon^2 \Delta u = u R(x), & t \geq 0, \ x \in \mathbb{R}, \\
u(t = 0, x) = u^0(x) > 0,
\end{cases}
\]

where the growth rate \( R(x) \) accounts for birth and death and has apriori no definite sign.

The limit \( \varepsilon \to 0 \) is borrying because we can expect that the solution converges to that of the ordinary differential system obtained with \( \varepsilon = 0 \). Therefore, one also re-scale time according to the diffusion scale \( X = \sqrt{T} \) to obtain

\[
\begin{cases}
\varepsilon \frac{\partial}{\partial t} u_\varepsilon - \varepsilon^2 \Delta u_\varepsilon = u_\varepsilon R(x), & t \geq 0, \ x \in \mathbb{R}, \\
u_\varepsilon(t = 0, x) = u^0_\varepsilon(x) > 0,
\end{cases}
\]

In the zones where \( R \) is positive we expect exponential growth, in the zones where \( R \) is negative we expect exponential decay. Therefore the limit \( u_\varepsilon \) is not well defined. But the formal expansion \( u_\varepsilon(t, x) \approx u^0_\varepsilon e^{R(x)/\varepsilon} \) suggests to change variable and set

\[
u_\varepsilon(t, x) = e^{\varphi_\varepsilon(t,x)/\varepsilon}, \quad \varphi_\varepsilon(t, x) = \varepsilon \ln \left( u_\varepsilon(t, x) \right).
\]

The chain rule gives

\[
\begin{align*}
\frac{\partial}{\partial t} u_\varepsilon &= \frac{1}{\varepsilon} e^{\varphi_\varepsilon(t,x)/\varepsilon} \frac{\partial}{\partial t} \varphi_\varepsilon(t, x), \\
\Delta u_\varepsilon &= \frac{1}{\varepsilon} e^{\varphi_\varepsilon(t,x)/\varepsilon} \Delta \varphi_\varepsilon(t, x) + \frac{1}{\varepsilon^2} e^{\varphi_\varepsilon(t,x)/\varepsilon} |\nabla \varphi_\varepsilon(t, x)|^2.
\end{align*}
\]
Inserting these in the parabolic equation (8.1) gives
\[
\begin{align*}
\frac{\partial}{\partial t}\varphi_\varepsilon(t, x) - \varepsilon\Delta \varphi_\varepsilon(t, x) - |\nabla \varphi_\varepsilon(t, x)|^2 &= R(x), \quad t \geq 0, \quad x \in \mathbb{R}, \\
\varphi_\varepsilon(t = 0, x) &= \varphi^0_\varepsilon(x).
\end{align*}
\] (8.3)

The theory of viscosity solutions has been developed to handle this limit. We can pass to the limit as a regular perturbation and under suitable assumptions, it can be proved that \( \varphi_\varepsilon \to \varphi \) locally uniformly and this limit satisfies the Hamilton-Jacobi equation
\[
\begin{align*}
\frac{\partial}{\partial t}\varphi(t, x) &= |\nabla \varphi(t, x)|^2 + R(x), \quad t \geq 0, \quad x \in \mathbb{R}, \\
\varphi(t = 0, x) &= \varphi^0(x).\nonumber 
\end{align*}
\] (8.4)

This dynamics defines the function \( \varphi(t, x) \) which tells us the important, but rough, information on the behavior of \( u_\varepsilon(t, x) \). When \( \varphi(t, x) > 0 \) then \( u_\varepsilon(t, x) \) grows exponentially, when \( \varphi(t, x) < 0 \) then \( u_\varepsilon(t, x) \) decays exponentially.

The weakness of this approach is that when \( \varphi(t, x) = 0 \), we have no information on \( u_\varepsilon(t, x) \). More generally when \( u_\varepsilon(t, x) = O(1) \) then \( \varphi_\varepsilon(t, x) = O(\varepsilon) \) and we do not obtain any relevant information.

8.2 A priori Lipschitz bounds

8.3 Dirac concentrations

Figure 8.1: Numerical steady state solution of the Brusselator system (6.17) with \( A = B = 2 \), \( d_v = 1. \) and, from left to right \( d_u = 0.005 \), \( d_u = 0.0005 \) and \( d_u = 0.0001 \). The first component \( u(x) \) concentrates more and more while \( v \) (magnified for plotting) remains flat.
Chapter 9

Parabolic equations from biology

This chapter aims at showing how the simple behaviors described previously can be used, combined or generalized in order to give really intriguing patterns related to experimental observations. It is impossible to give an exhaustive presentation of these issues because the parabolic formalism covers too many subjects where an enormous amount of models have been used.

9.1 Tumour spheroids growth (nutrient limitation)

Figure 9.1: Spheroid structure of a early stage tumour: outer rim of proliferating cells and inner necrotic core. From Sutherland et al., Cancer Res. 46 (1986), 5320–5329.

One of the simplest model of tumor growth goes back to H. Greenspan. It is a free boundary problem based on the idea that the tumor has a constant density of cells \( N \) (incompressibility of tissue) and that nutrient is the limiting quantity for growth of the tumor. The nutrients are provided by blood vessels which do not access to the tumor; this is true at the early stage of tumor development (radius of order 1mm), afterwards the necrotic cells in the center of the tumor

\(^1\)H. Greenspan, On the growth and stability of cell cultures and solid tumors, J. Th. Biology, 56(1) 229–242, 1976
emits Vascular Endothelial Growth Factors that attract new blood vessels. This phenomena is called angiogenesis and makes that, very early, the nutrient is no longer a limitant factor.

The tumor is considered as a ball $B_{R(t)}$ of $R^3$ centered at the origin and of radius $R(t)$, the nutrient available is $c(r, t)$ for $0 \leq r \leq R(t)$ and the bulk velocity of cells is $v(r, t)$ with $r = |x|$. We write the system

$$
\begin{align*}
-\Delta c + \lambda c &= 0, \\ c &= c_b, \\
N \text{div } v &= N [B(c(x, t)) - D(c(x, t))] , \\
\dot{R}(t) &= v(R(t), t).
\end{align*}
$$

The equation on $c$ just means that the nutrient is diffused and consumed (with rate $\lambda$) from the (vascularized) boundary where the nutrient is available at a high level $c_b$. The functions $B, D : \mathbb{R}^+ \to \mathbb{R}^+$ are respectively the birth and death rates of cells (depending upon the available nutrient). One can think of the equation on $v$ as an usual density advection

$$
\frac{\partial}{\partial t} n + \text{div}(nv) = n(t, x) \left[ B(c(x, t)) - D(c(x, t)) \right],
$$

of fluid mechanics with the constant density assumption $n(t, x) \equiv N$. This assumption is based on experimental observations and direct individual based simulations (see Chapter ???).

The assumptions are as follows

$$
\begin{align*}
B'(\cdot) &> 0, & D'(\cdot) &< 0, \\
B(0) - D(0) < 0, & B(c_b) - D(c_b) > 0.
\end{align*}
$$
Theorem 9.1 In dimension three, and with assumptions (9.4), (9.5) and an intial radius $R_0 > 0$ given, there is a unique solution to (9.1)–(9.3) with radial symmetry satisfying $v(0, t) = 0$ and $R(t)$ is increasing and satisfies

$$R(t) \to R_\infty < \infty \text{ as } t \to \infty.$$ 

The model is irrealistic in supposing that the dead cells disappear immediately and thus are replaced by other cells moving back from a proliferating rim to the center. This creates an equilibrium between proliferation and death, leading to the existence of a limiting radius (sometimes refered to as the Gompertz law, and which is no longer accepted, neither in vivo nor in vitro).

**Proof.** The problem can be rewritten, taking into account the radial symmetry, and rescalling, as

$$c(t, x) = C(R(t), |x|/R(t)),$$

where $C(\rho, |x|)$ is the solution to

$$\begin{cases} 
- \Delta C + \lambda \rho^2 C = 0, & |x| \leq 1, \\
C = c_b, & \text{on } \{|x| = 1\}.
\end{cases}$$

Notice that the monotonicity property of Laplace equation implies that

$$C(\rho, \cdot) \text{ is decreasing in } \rho, \quad C(0, r) = c_b, \quad c(\infty, r) = 0.$$ 

The equation on $v$ can also be explicitely solved, thanks to the radial symmetry, writing

$$\frac{1}{r^2} \frac{\partial}{\partial r} [r^2 v(t, r)] = \Phi \left( C(R(t), \frac{r}{R(t)}) \right) := \left[ B(c(r, t)) - D(c(r, t)) \right].$$

We obtain

$$v(t, r) = \frac{1}{r^2} \int_0^r \sigma^2 \Phi \left( C(R(t), \frac{\sigma}{R(t)}) \right) d\sigma$$

$$= \frac{R(t)^3}{r^2} \int_0^{r/R(t)} s^2 \Phi \left( C(R(t), s) \right) ds$$

after changing variable $s = \sigma/R(t)$.

All together, we can obtain the equation on $R(t)$; we define

$$v(t, R(t)) = R(t) \int_0^1 s^2 \Phi \left( C(R(t), s) \right) ds =: R(t) W(R(t))$$
and arrive to
\[ \dot{R}(t) = R(t) W(R(t)), \tag{9.8} \]
with
\[ W(R) = \int_0^1 s^2 \Phi(C(R, s)) \, ds. \]

Recall the elementary properties: \( C(0, r) = c_0, \) \( C(\rho, r) \to 0 \) as \( \rho \to \infty, \) and thus, thanks to (9.7),
\[ \frac{d}{dr} W(r) < 0, \quad W(0) = \frac{1}{3} \Phi(c_0), \quad W(\infty) < 0. \]
Therefore, using the assumptions (9.4), (9.5), there is a unique radius \( R_\infty \) such that
\[ W(R_\infty) = 0. \]

The result follows directly from the sign property of \( W \) (inherited from its monotonicity) in the differential equation (9.8).

**Remark 9.2** In a general domain, the setting is as follows. The equation for the nutrient is unchanged but solved on the domain \( \Omega(t) \). The velocity is supposed to be a gradient \( v = \nabla p(t, x) \), where the pressure satisfies some kind of Darcy's law
\[ \begin{cases} -\Delta p = \frac{1}{\eta} \left[ B(c(x, t)) - D(c(x, t)) \right], & x \in \Omega(t), \\ p = \eta \kappa(t, x) \quad \text{on} \quad \partial \Omega(t). \end{cases} \tag{9.9} \]
The boundary condition represents surface tension, with \( \kappa \) the mean curvature of the domain. Finally, the domain moves with the velocity \( \nu(t, x) \),\( \nu(t, x) \) with \( \nu \) the outward normal to the boundary of \( \Omega(t) \). See Cristini, Lowengrub and Nie\(^2\), Sominet-Walker.

In case of a spheroid, the curvature is a constant and \( p \) just depends additively on this constant. Because the model only uses the velocity which is the pressure gradient, this boundary condition is not useful as the above resolution.

Nonradial solutions and stability questions are surveyed by A. Friedman\(^3\).

### 9.2 Tumour spheroids growth (necrotic core)

M. Ward and D. King proposed a more realistic model which is able to represent the dead cells, the so-called necrotic core that is shown in Figures 9.1 and 9.2. The modeling idea is that
\[^3\text{Mathematical analysis and challenges arising from models of tumor growth, M^3\text{AS}, Vol. 17, Supplement (2007), 1751–1772.}\]
nutrients are too low in the center of the tumor and under a certain threshold cells die (necrosis).

The model reads

\[
\begin{cases}
-\Delta c + \lambda c = f(c, N), & x \in \Omega(t), \\
c = c_b, & \text{on } \partial B_{R(t)}, \\
v = \nabla c,
\end{cases}
\]

\[\text{(9.10)}\]

\[
\begin{cases}
\text{div}(N_a v) = N \left[ G(c) - D(c) \right], & x \in \Omega(t), \\
\text{div}(N_d v) = N D(c), & x \in \Omega(t),
\end{cases}
\]

\[\text{(9.11)}\]

where \(c(t, x)\) still represents the available nutrient, \(N_a\) the density of living (active) tumor cells and \(N_d\) the density of dead cells (in principle they are in the center where nutrients are not enough, this is called the necrotic core). Around the necrotic core, a proliferating rim may subsist that creates unlimited growth of the tumor.

The equations on the active and dead cells can be seen as 'constant density' cases of the evolution equations

\[
\begin{cases}
\frac{\partial}{\partial t} n + \text{div}(n v) = g_1(c, n), & x \in \Omega(t), \\
\frac{\partial}{\partial t} d + \text{div}(d v) = g_2(c, n, d), & x \in \Omega(t),
\end{cases}
\]

More recent and realistic models are due to Byrne, Chaplain, Preziosi. A recent survey paper is by T. Roose, S. J. Chapman and P. K. Maini\textsuperscript{4}.

9.3 Tumour spheroids growth (mechanical model)

Another simple model of tumour growth is based on increase of pressure by cell division without reference to a nutrient which is supposed sufficient (e.g. by angiogenesis effects).

The model we present now is taken from H. Byrne and D. Drasdo (Individual based and continuum models of growing cell populations: a comparison, preprint 2007).

To present the model we recall the Heavyside function

\[H(x) = \begin{cases} 
1 & \text{for } x > 0, \\
0 & \text{for } x \leq 0.
\end{cases}\]

We still suppose the density of cells is constant and denoted by \(N\) (this assumption is still based on experimental observations and direct individual based simulations, see Chapter ???)\textsuperscript{4}.

Then, we postulate Darcy’s law for the pressure and velocity

\[
\begin{aligned}
\text{div } v &= -\Delta p = \frac{1}{N}H(p_0 - p) \quad x \in B_{R(t)}, \\
p &= \frac{\eta}{R(t)} \quad \text{on } \partial B_{R(t)}.
\end{aligned}
\tag{9.12}
\]

Here again the boundary condition takes into account for surface tension, along with Remark 9.2 since in a spheroid the curvature is just given by \( \kappa(t,x) = 1/R(t) \).

As before Darcy’s assumptions allows us to elated the growth speed and the pressure with

\[ v(t, x) = -\nabla p(t, x). \]

The tumour still grows with the velocity on its boundary

\[ \dot{R}(t) = v(R(t), t). \]  \tag{9.13}

**Exercise.** Compute the radius \( R(t) \) in this model following the calculation in Section 9.1. Show that it depends on the curvature coefficient \( \eta \) by opposition to the nutrient model in Section 9.1.

### 9.4 Mimura’s system for dentritic bacterial colonies

![Bacterial colonies of salmonella Bacillus Subtilis. Experiments by S. Seror et al, Institut de Génétique et Microbiologie, Université Paris-Sud.](image)

Bacterial colonies often exhibit remarkable patterns, see Figure 9.3 for instance. They follow from complex collective interactions between cells due to several effects: random motion of the individual cells, cell division and colony growth, release of various molecules in their environment resulting e.g. in chemoattraction or surface tension (see [19]). The patterns depend heavily on the type of bacteria and on the support used for the experiment (solid, semi-solid, liquid). A good account on this issue can be found in Murray’s book [30].
A simple model for dentritic colonies growth is due to Mimura [28]. It takes into account only three simple effects: (i) a brownian motion of actives cells which density is denoted by \( n(t, x) \) below, (ii) a nutrient of concentration \( c(t, x) \) which is diffused in the medium an consumed by the active cells for growth, (iii) the active cells become frozen proportionally to \( n \) and then they do not move. The specific form of the equations proposed by Mimura are given in system (9.14) and numerical results are presented in Figures 9.4 and 9.5. It is well established that the biophysical ingredients of this model are not those in the real experiments leading to Figure 9.3, nevertheless they share several common patterns.

\[
\begin{aligned}
\frac{\partial}{\partial t} n(t, x) - d_1 \Delta n(t, x) &= n \left( c - \frac{1}{(1+n)(1+c)} \right), & t \geq 0, \ x \in \mathbb{R}^2, \\
\frac{\partial}{\partial t} c(t, x) - d_2 \Delta c(t, x) &= -nc, \\
\frac{\partial}{\partial t} f(t, x) &= n \frac{1}{(1+n)(1+c)}.
\end{aligned}
\]  

\[(9.14)\]

Figure 9.4: Solutions of Mimura’s system (9.14) at a fixed time. Left: frozen bacteria \( f(t, x) \). Right: active bacteria \( n(t, x) \). Computations by A. Marrocco.

Figure 9.5: Solutions of Mimura’s system (9.14) with a higher nutrient level \( S \) than in Figure 9.4. Computations by A. Marrocco.

The initial data represents an inoculum, i.e., a large amount of bacteria located at the center of the ball (the computational domain). The reason why the dentritic pattern occurs in this model is that the solution \( n(t, x) \) to the first equation has a tendency to create Dirac concentrations. These concentrations move towards the larger values of the nutrient \( c \), and this is the boundary
of the computational domain. Indeed, the nutrient is consumed progressively inside the domain and this creates a nutrient gradient towards the exterior.

9.5 Fisher/KPP system with competition

\[
\begin{align*}
\frac{\partial}{\partial t} u(t, x) - d_1 \Delta u(t, x) &= au(1 - u) - kuv, \quad t \geq 0, \ x \in \Omega \subset \mathbb{R}^d, \\
\frac{\partial}{\partial t} v(t, x) - d_2 \Delta v(t, x) &= bv(1 - v) - \alpha kuv,
\end{align*}
\]  

(9.15)

The first interesting question is \( k \to \infty \) and certainly the second is \( d_1 = d_2 = \varepsilon \) and \( a = b = 1/\varepsilon, \ k = 1/\varepsilon^2 \).

As \( k \to \infty \) we obtain the Stefan problem with zero latent heat, with \( w = \alpha u - v \)

\[
\begin{align*}
uv &= 0, \\
\frac{\partial}{\partial t} u(t, x) - d_1 \Delta u(t, x) &\leq au(1 - u), \\
\frac{\partial}{\partial t} v(t, x) - d_2 \Delta v(t, x) &\leq bv(1 - v), \\
\frac{\partial}{\partial t} w(t, x) - \Delta [d_1 w_+ + d_2 w_-] &= G(w),
\end{align*}
\]  

(9.16)

with \( G(w) = \frac{aw}{\alpha} (1 - \frac{w}{\alpha}) + bw_-(1 - w_-). \)
Bibliography


[34] Schwartz, Laurent, Méthodes mathématiques pour les sciences physiques. Hermann, 1983.


