MÉTHODES MATHÉMATIQUES EN ÉLASTICITÉ[1,2]

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Resumé. Ces notes présentent de façon concise les principales méthodes mathématiques dans la théorie statique de corps élastiques. Pour un exposé plus complet de la théorie de l'élasticité, nous renvoyons le lecteur à Ciarlet [6].

Mathematical methods in elasticity

Abstract. These notes present in a concise form the principal mathematical methods in the static theory of elastic bodies. For a more comprehensive exposition of theory of elasticity, we refer the reader to Ciarlet [6].

Contents

(1) Version 0.1.

(2) Ces notes de cours sont très largement tires de [9].

Preliminaries

All spaces, matrices, etc., are real. The Kronecker symbol is denoted δ_i^j. Lower case letters such as $f, g, ...$ denote real-valued *functions*, boldfaced lower case letters such as $\boldsymbol{u}, \boldsymbol{v}, ...$ denote *vector fields* (i.e., vector-valued functions), and boldfaced upper case letters such as $\boldsymbol{A}, \boldsymbol{B}, ...$ denote *matrix fields* (i.e., matrix-valued functions).

The physical space is identified with the three-dimensional vector space $\mathbb{R}^3$ by fixing an origin and a cartesian basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. In this way, a point x in space is defined by its cartesian coordinates x_1, x_2, x_3 or by the vector $\boldsymbol{x} := \sum_i x_i \mathbf{e}_i$. The space $\mathbb{R}^3$ is equipped with the Euclidean inner product $\boldsymbol{u} \cdot \boldsymbol{v}$ and with the Euclidean norm $|\boldsymbol{u}|$, where $\boldsymbol{u}, \boldsymbol{v}$ denote vectors in $\mathbb{R}^3$. The exterior product of two vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$ is denoted $\boldsymbol{u} \wedge \boldsymbol{v}$.

For any integer $n \geq 2$, we define the following spaces or sets of real square matrices of order n:

$\mathbb{M}^n$: the space of all square matrices,
$\mathbb{A}^n$: the space of all anti-symmetric matrices,
$\mathbb{S}^n$: the space of all symmetric matrices,
$\mathbb{M}^n_+$: the set of all matrices $\boldsymbol{A} \in \mathbb{M}^n$ with $\det \boldsymbol{A} > 0$,
$\mathbb{S}^n_>$: the set of all positive-definite symmetric matrices,
$\mathbb{O}^n$: the set of all orthogonal matrices,
$\mathbb{O}^n_+$: the set of all orthogonal matrices $\boldsymbol{R} \in \mathbb{O}^n$ with $\det \boldsymbol{R} = 1$.

The notation (a_{ij}) designates the matrix in $\mathbb{M}^n$ with a_{ij} as its element at the i-th row and j-th column. The identity matrix in $\mathbb{M}^n$ is denoted $\boldsymbol{I} := (\delta_j^i)$. The space $\mathbb{M}^n$, and its subspaces $\mathbb{A}^n$ and $\mathbb{S}^n$ are equipped with the inner product $\boldsymbol{A} : \boldsymbol{B}$ and with the spectral norm $|\boldsymbol{A}|$ defined by

$$\boldsymbol{A} : \boldsymbol{B} := \sum_{i,j} a_{ij} b_{ij},$$

$$|\boldsymbol{A}| := \sup\{|\boldsymbol{A}\boldsymbol{v}|;\ \boldsymbol{v} \in \mathbb{R}^n,\ |\boldsymbol{v}| \leq 1\},$$

where $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$ denote matrices in $\mathbb{M}^n$. The determinant and the trace of a matrix $\boldsymbol{A} = (a_{ij})$ are denoted $\det \boldsymbol{A}$ and $\operatorname{tr} \boldsymbol{A}$. The cofactor matrix associated with an invertible matrix $\boldsymbol{A} \in \mathbb{M}^n$ is defined by $\mathbf{Cof}\,\boldsymbol{A} := (\det \boldsymbol{A})\boldsymbol{A}^{-T}$.

Let Ω be an open subset of $\mathbb{R}^3$. Partial derivative operators of order $m \geq 1$ acting on functions or distributions defined over Ω are denoted

$$\partial^{\boldsymbol{k}} := \frac{\partial^{|\boldsymbol{k}|}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3}}$$

where $\boldsymbol{k} = (k_i) \in \mathbb{N}^3$ is a multi-index satisfying $|\boldsymbol{k}| := k_1 + k_2 + k_3 = m$. Partial derivative operators of the first, second, and third order are also denoted $\partial_i := \partial / \partial x_i$, $\partial_{ij} := \partial^2 / \partial x_i \partial x_j$, and $\partial_{ijk} := \partial^3 / \partial x_1 \partial x_2 \partial x_3$.

The gradient of a function $f : \Omega \to \mathbb{R}$ is the vector field $\boldsymbol{\nabla} f := (\partial_i f)$, where i is the row index. The gradient of a vector field $\boldsymbol{v} = (v_i) : \Omega \to \mathbb{R}^n$ is the matrix field $\boldsymbol{\nabla} \boldsymbol{v} := (\partial_j v_i)$, where i is the row index, and the divergence of the same vector field is the function $\operatorname{div} \boldsymbol{v} := \sum_i \partial_i v_i$. Finally, the divergence of a matrix field $\boldsymbol{T} = (t_{ij}) : \Omega \to \mathbb{M}^n$ is the vector field $\mathbf{div}\, \boldsymbol{T}$ with components $(\sum_{j=1}^n \partial_j t_{ij})_i$.

The space of all continuous functions from a topological space X into a normed space Y is denoted $\mathcal{C}^0(X;Y)$, or simply $\mathcal{C}^0(X)$ if $Y = \mathbb{R}$.

For any integer $m \geq 1$ and any open set $\Omega \subset \mathbb{R}^3$, the space of all real-valued functions that are m times continuously differentiable over Ω is denoted $\mathcal{C}^m(\Omega)$. The space $\mathcal{C}^m(\overline{\Omega})$, $m \geq 1$, is defined as that consisting of all vector-valued functions $f \in \mathcal{C}^m(\Omega)$ that, together with all their partial derivatives of order $\leq m$, possess continuous extentions to the closure $\overline{\Omega}$ of Ω. If Ω is bounded, the space $\mathcal{C}^m(\overline{\Omega})$ equipped with the norm

$$\|f\|_{\mathcal{C}^m(\overline{\Omega})} := \max_{|\boldsymbol{\alpha}| \leq m} \left(\sup_{x \in \Omega} |\partial^{\boldsymbol{\alpha}} f(x)| \right)$$

is a Banach space.

The space of all indefinitely derivable functions $\varphi : \Omega \to \mathbb{R}$ with compact support contained in Ω is denoted $\mathcal{D}(\Omega)$ and the space of all distributions over Ω is denoted $\mathcal{D}'(\Omega)$. The duality bracket between a distribution T and a test function $\varphi \in \mathcal{D}(\Omega)$ is denoted $\langle T, \varphi \rangle$.

The usual Lebesgue and Sobolev spaces are respectively denoted $L^p(\Omega)$, and $W^{m,p}(\Omega)$ for any integer $m \geq 1$ and any $p \geq 1$. If $p = 2$, we use the notation $H^m(\Omega) = W^{m,2}(\Omega)$. The space $W^{m,p}_{\mathrm{loc}}(\Omega)$ is the space of all mesurable functions such that $f|_U \in W^{m,p}(U)$ for all $U \Subset \Omega$, where the notation $f|_U$ designates the restriction to the set U of a function f and the notation $U \Subset \Omega$ means that $\overline{U}$ is a compact set that satisfies $\overline{U} \subset \Omega$.

The space $W^{m,p}_0(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$ and the dual of the space $W^{m,p}_0(\Omega)$ is denoted $W^{-m,p'}(\Omega)$, where $p' = \frac{p}{p-1}$. If the boundary of Ω is Lipschitz-continuous and if $\Gamma_0 \subset \partial\Omega$ is a relatively open subset of the boundary of Ω, we let

$$W^{1,p}_{\Gamma_0}(\Omega) := \{f \in W^{1,p}(\Omega);\ f = 0 \text{ on } \Gamma_0\},$$
$$W^{2,p}_{\Gamma_0}(\Omega) := \{f \in W^{2,p}(\Omega);\ f = \partial_\nu f = 0 \text{ on } \Gamma_0\},$$

where ∂_ν denote the outer normal derivative operator along $\partial\Omega$ (since Ω is Lipschitz-continuous, a unit outer normal vector (ν_i) exists $\partial\Omega$-almost everywhere along $\partial\Omega$, and thus $\partial_\nu = \nu_i \partial_i$).

If Y is a finite dimensional vectorial space (such as $\mathbb{R}^n$, $\mathbb{M}^n$, etc.), the notation $\mathcal{C}^m(\Omega;Y)$, $\mathcal{C}^m(\overline{\Omega};Y)$, $L^p(\Omega;Y)$ and $W^{m,p}(\Omega;Y)$ designates the spaces of all mappings from Ω into Y whose components in Y are respectively in $\mathcal{C}^m(\Omega)$, $\mathcal{C}^m(\overline{\Omega})$, $L^p(\Omega)$ and $W^{m,p}(\Omega)$. If Y is equipped with the norm $|\cdot|$, then the spaces $L^p(\Omega;Y)$ and $W^{m,p}(\Omega;Y)$ are respectively equipped with the norms

$$\|f\|_{L^p(\Omega;Y)} := \left\{ \int_\Omega |f(x)|^p dx \right\}^{1/p}$$

and

$$\|f\|_{W^{m,p}(\Omega;Y)} := \Big\{ \int_\Omega \big(|f(x)|^p + \sum_{|\boldsymbol{k}|\le m} |\partial^{\boldsymbol{k}} f(x)|^p \big) dx \Big\}^{1/p}.$$

Throughout this article, a *domain* in $\mathbb{R}^n$ is a bounded and connected open set with a Lipschitz-continuous boundary, the set Ω being locally on the same side of its boundary (see, e.g., Adams [1], Grisvard [13], or Nečas [17]). Note that this definition differs from the usual one, according to which a domain is a connected and open set. If $\Omega \subset \mathbb{R}^n$ is a domain, then the following formula of integration by parts is satisfied

$$\int_\Omega \mathbf{div}\, \boldsymbol{F} \cdot \boldsymbol{v}\, dx = -\int_\Omega \boldsymbol{F} : \boldsymbol{\nabla} \boldsymbol{v}\, dx + \int_{\partial\Omega} (\boldsymbol{F}\boldsymbol{n}) \cdot \boldsymbol{v}\, da$$

for all smooth enough matrix field $\boldsymbol{F} : \Omega \to \mathbb{M}^k$ and vector field $\boldsymbol{v} : \Omega \to \mathbb{R}^k$, $k \ge 1$ (smooth enough means that the regularity of the fields $\boldsymbol{F}$ and $\boldsymbol{v}$ is such that the above integrals are well defined; for such instances, see, e.g., Evans & Gariepy [11]). The notation da designates the area element induced on the surface $\partial\Omega$ by the volume element dx. We also record the Stokes formula:

$$\int_\Omega \mathbf{div}\, \boldsymbol{F}\, dx = \int_{\partial\Omega} \boldsymbol{F}\boldsymbol{n}\, da.$$

1. Modelisation

The displacement and the stress arising in an elastic body in response to given loads are predicted by means of a system of partial differential equations in three variables (the coordinates of the physical space). This system is derived from physical laws in the form of two basic sets of equations, the *equations of equilibrium* (Section 1.1) and the *constitutive equations* (Section 1.2). The equations of *nonlinear elasticity* are then obtained by adjoining appropriate boundary conditions to these equations (Section 1.3). The equations of *linear elasticity* are obtained from the nonlinear ones by linearization with respect to the displacement field (Section 1.4).

1.1. **Equations of equilibrium.** In this section, we begin our study of the stress and deformation arising in an elastic body in response to given forces. We consider that the body occupies the closure of a domain $\Omega \subset \mathbb{R}^3$ in the absence of applied forces, henceforth called the *reference configuration* of the body. Any other configuration that the body might occupy when subjected to applied forces will be defined by means of a *deformation*, that is, a mapping

$$\boldsymbol{\Phi} : \overline{\Omega} \to \mathbb{R}^3$$

that is *orientation preserving* (i.e., $\det \boldsymbol{\nabla\Phi}(x) > 0$ for all $x \in \overline{\Omega}$) and *injective* on the open set Ω (i.e., no interpenetration of matter occurs). The image $\boldsymbol{\Phi}(\overline{\Omega})$ is called the *deformed configuration* of the body defined by the deformation $\mathbf{\Phi}$. The "difference" between a deformed configuration and the reference configuration is given by the *displacement*, which is the vector field defined by

$$\boldsymbol{u} := \mathbf{\Phi} - \mathbf{id},$$

where $\mathbf{id} : \overline{\Omega} \to \overline{\Omega}$ is the identity map. It is sometimes more convenient to describe the deformed configuration of a body in terms of the displacement $\boldsymbol{u}$ instead of the deformation $\mathbf{\Phi}$, notably when the body is expected to undergo small deformations (as typically occurs in linear elasticity).

Our objective in this section is to determine, among all possible deformed configurations of the body, the ones that are in "static equilibrium" in the presence of *applied forces*. More specifically, let the applied forces acting on a specific deformed configuration $\tilde{\Omega} := \mathbf{\Phi}(\Omega)$ be represented by the *densities*

$$\tilde{\boldsymbol{f}} : \tilde{\Omega} \to \mathbb{R}^3 \text{ and } \tilde{\boldsymbol{g}} : \tilde{\Gamma}_1 \to \mathbb{R}^3,$$

where $\tilde{\Gamma}_1 \subset \partial\tilde{\Omega}$ is a relatively open subset of the boundary of $\tilde{\Omega}$.

Remark. If the body is subjected for instance to the gravity and to a uniform pressure on $\tilde{\Gamma}_1$, then the densities $\tilde{\boldsymbol{f}}$ and $\tilde{\boldsymbol{g}}$ are given by $\tilde{\boldsymbol{f}}(\tilde{x}) = -g\tilde{\rho}(\tilde{x})\mathbf{e}_3$ and $\tilde{\boldsymbol{g}}(\tilde{x}) = -\pi\tilde{\boldsymbol{n}}(\tilde{x})$, where g is the gravitational constant, $\tilde{\rho} : \tilde{\Omega} \to \mathbb{R}$ is the mass density in the deformed configuration, $\tilde{x}$ denotes a generic point in $\{\tilde{\Omega}\}^-$, $\tilde{\boldsymbol{n}}(\tilde{x})$ is the unit outer normal to $\partial\tilde{\Omega}$, and π is a constant, called pressure.

These examples illustrate that an applied force density may, or may not, depend on the unknown deformation. $\square$

Our aim is thus to determine equations that a deformation $\mathbf{\Phi}$ corresponding to the static equilibrium of the loaded body should satisfy. To this end, we first derive the "equations of equilibrium" from a fundamental axiom due to Euler and Cauchy. The *three-dimensional equations of elasticity* will then be obtained by combining these equations with a "constitutive equation" (Section 1.2).

Let

$$S_2 := \{\boldsymbol{v} \in \mathbb{R}^3;\ |\boldsymbol{v}| = 1\}$$

denote the set of all unit vectors in $\mathbb{R}^3$. Then, according to the **stress principle of Euler and Cauchy**, a body $\tilde{\Omega} \subset \mathbb{R}^3$ subjected to applied forces of densities $\tilde{\boldsymbol{f}} : \tilde{\Omega} \to \mathbb{R}^3$ and $\tilde{\boldsymbol{g}} : \tilde{\Gamma}_1 \to \mathbb{R}^3$ is in equilibrium if there *exists* a vector field

$$\tilde{\boldsymbol{t}} : \{\tilde{\Omega}\}^- \times S_2 \to \mathbb{R}^3$$

such that, for all domains $\tilde{A} \subset \tilde{\Omega}$,

$$\int_{\tilde{A}} \tilde{\boldsymbol{f}}\, d\tilde{x} + \int_{\partial \tilde{A}} \tilde{\boldsymbol{t}}(\tilde{x}, \tilde{\boldsymbol{n}}(\tilde{x}))\, d\tilde{a} = \mathbf{0},$$
$$\int_{\tilde{A}} \tilde{\boldsymbol{x}} \wedge \tilde{\boldsymbol{f}}\, d\tilde{x} + \int_{\partial \tilde{A}} \tilde{\boldsymbol{x}} \wedge \tilde{\boldsymbol{t}}(\tilde{x}, \tilde{\boldsymbol{n}}(\tilde{x}))\, d\tilde{a} = \mathbf{0},$$
$$\tilde{\boldsymbol{t}}(\tilde{x}, \tilde{\boldsymbol{n}}(\tilde{x})) = \tilde{\boldsymbol{g}}(\tilde{x}) \text{ for } \partial\Omega\text{-almost all } \tilde{x} \in \partial\tilde{A} \cap \tilde{\Gamma}_1,$$

where $\tilde{\boldsymbol{n}}(\tilde{x})$ denotes the exterior unit normal vector at $\tilde{x}$ to the surface $\partial\tilde{A}$ (because $\tilde{A}$ is a domain, $\tilde{\boldsymbol{n}}(\tilde{x})$ exists for $d\tilde{x}$-almost all $\tilde{x} \in \partial\tilde{A}$).

This axiom postulates in effect that the "equilibrium" of the body to the applied forces is reflected by the existence of a force of density $\tilde{\boldsymbol{t}}$ that acts on the boundary of any domain $\tilde{A} \subset \tilde{\Omega}$ and depends only on the two variables $\tilde{x}$ and $\tilde{\boldsymbol{n}}(\tilde{x})$.

The following theorem, which is due to Cauchy, shows that the dependence of $\tilde{\boldsymbol{t}}$ on the second variable is necessarily linear:

Theorem 1.1. *If $\tilde{\boldsymbol{t}}(\cdot, \tilde{\boldsymbol{n}}) : \{\tilde{\Omega}\}^- \to \mathbb{R}^3$ is of class $\mathcal{C}^1$ for all $\tilde{\boldsymbol{n}} \in S_2$, $\tilde{\boldsymbol{t}}(\tilde{x}, \cdot) : S_2 \to \mathbb{R}^3$ is continuous for all $\tilde{x} \in \{\tilde{\Omega}\}^-$, and $\tilde{\boldsymbol{f}} : \{\tilde{\Omega}\}^- \to \mathbb{R}^3$ is continuous, then $\tilde{\boldsymbol{t}} : \{\tilde{\Omega}\}^- \times S_2 \to \mathbb{R}^3$ is linear with respect to the second variable.*

Proof. The proof consists in applying the stress principle to particular subdomains in $\{\tilde{\Omega}\}^-$. For details, see, e.g., Ciarlet [6] or Gurtin & Martins [14]. □

In other words, there exists a matrix field $\tilde{\boldsymbol{T}} : \{\tilde{\Omega}\}^- \to \mathbb{M}^3$ of class $\mathcal{C}^1$ such that

$$\tilde{\boldsymbol{t}}(\tilde{x}, \tilde{\boldsymbol{n}}) = \tilde{\boldsymbol{T}}(\tilde{x})\tilde{\boldsymbol{n}} \text{ for all } \tilde{x} \in \{\tilde{\Omega}\}^- \text{ and all } \tilde{\boldsymbol{n}} \in S_2.$$

Combining Cauchy's theorem with the stress principle of Euler and Cauchy yields, by means of Stokes' formula (see Section "Preliminaries"), the following **equations of equilibrium in the deformed configuration**:

Theorem 1.2. *The matrix field $\tilde{\boldsymbol{T}} : \{\tilde{\Omega}\}^- \to \mathbb{M}^3$ satisfies*

$$\begin{aligned} -\mathbf{div}\, \tilde{\boldsymbol{T}}(\tilde{x}) &= \tilde{\boldsymbol{f}}(\tilde{x}) \text{ for all } \tilde{x} \in \tilde{\Omega}, \\ \tilde{\boldsymbol{T}}(\tilde{x})\tilde{\boldsymbol{n}}(\tilde{x}) &= \tilde{\boldsymbol{g}}(\tilde{x}) \text{ for all } \tilde{x} \in \tilde{\Gamma}_1, \\ \tilde{\boldsymbol{T}}(\tilde{x}) &\in \mathbb{S}^3 \text{ for all } \tilde{x} \in \tilde{\Omega}. \end{aligned} \tag{1.1}$$

The system (1.1) is defined over the deformed configuration $\tilde{\Omega}$, which is unknown. Fortunately, it can be conveniently reformulated in terms of functions defined over the *reference configurence* Ω of the body, which is known. To this end, we use the change of variables $\tilde{x} = \boldsymbol{\Phi}(x)$ defined by the unknown deformation $\boldsymbol{\Phi} : \overline{\Omega} \to \{\tilde{\Omega}\}^-$, assumed to be *injective*, and the following formulas between the volume and area elements in $\{\tilde{\Omega}\}^-$ and $\overline{\Omega}$

(with self-explanatory notations)

$$d\tilde{x} = |\det \boldsymbol{\nabla\Phi}(x)|\, dx,$$
$$\tilde{\boldsymbol{n}}(\tilde{x})\, d\tilde{a} = \mathbf{Cof}\,\boldsymbol{\nabla\Phi}(x)\boldsymbol{n}(x)\, da.$$

We also define the vector fields $\boldsymbol{f} : \Omega \to \mathbb{R}^3$ and $\boldsymbol{g} : \Gamma_1 \to \mathbb{R}^3$ by

$$\tilde{\boldsymbol{f}}(\tilde{x})\, d\tilde{x} = \boldsymbol{f}(x)\, dx,$$
$$\tilde{\boldsymbol{g}}(\tilde{x})\, d\tilde{a} = \boldsymbol{g}(x)\, da.$$

Note that, like the fields $\tilde{\boldsymbol{f}}$ and $\tilde{\boldsymbol{g}}$, the fields $\boldsymbol{f}$ and $\boldsymbol{g}$ may, or may not, depend on the unknown deformation $\boldsymbol{\Phi}$.

First of all, assuming that $\boldsymbol{\Phi}$ is smooth enough and using the change of variables $\boldsymbol{\Phi} : \overline{\Omega} \to \{\tilde{\Omega}\}^-$ in the first equation of (1.1), we deduce that, for all domains $A \subset \Omega$,

$$\int_A \boldsymbol{f}(x)\, dx + \int_{\partial A} \tilde{\boldsymbol{T}}(\boldsymbol{\Phi}(x))\mathbf{Cof}\,\boldsymbol{\nabla\Phi}(x)\boldsymbol{n}(x)\, da = \mathbf{0}.$$

The matrix field $\boldsymbol{T} : \overline{\Omega} \to \mathbb{M}^3$ appearing in the second integral, viz., that defined by

$$\boldsymbol{T}(x) := \tilde{\boldsymbol{T}}(\boldsymbol{\Phi}(x))\mathbf{Cof}\,\boldsymbol{\nabla\Phi}(x) \ \text{ for all } x \in \overline{\Omega},$$

is called the **first Piola-Kirchhoff stress tensor field**. In terms of this tensor, the above relation read

$$\int_A \boldsymbol{f}(x)\, dx + \int_{\partial A} \boldsymbol{T}(x)\boldsymbol{n}(x)\, da = \mathbf{0},$$

which implies that the matrix field $\boldsymbol{T}$ satisfies the following partial differential equation:

$$-\mathbf{div}\,\boldsymbol{T}(x) = \boldsymbol{f}(x) \ \text{ for all } \ x \in \Omega.$$

Using the identity

$$\boldsymbol{\nabla\Phi}(x)^{-1}\boldsymbol{T}(x) = \boldsymbol{\nabla\Phi}(x)^{-1}[\det \boldsymbol{\nabla\Phi}(x)\tilde{\boldsymbol{T}}(\boldsymbol{\Phi}(x))]\boldsymbol{\nabla\Phi}(x)^{-T},$$

which follows from the definition of $\boldsymbol{T}(x)$ and from the expression of the inverse of a matrix in terms of its cofactor matrix, we furthermore deduce from the symmetry of the matrix $\tilde{\boldsymbol{T}}(\tilde{x})$ that the matrix $(\boldsymbol{\nabla\Phi}(x)^{-1}\boldsymbol{T}(x))$ is also symmetric.

It is then clear that the equations of equilibrium in the deformed configuration (see eqns. (1.1)) are equivalent with the following **equations of equilibrium in the reference configuration**:

$$\begin{aligned} -\mathbf{div}\,\boldsymbol{T}(x) &= \boldsymbol{f}(x) && \text{for all } \ x \in \Omega, \\ \boldsymbol{T}(x)\boldsymbol{n}(x) &= \boldsymbol{g}(x) && \text{for all } \ x \in \Gamma_1, \\ \boldsymbol{\nabla\Phi}(x)^{-1}\boldsymbol{T}(x) &\in \mathbb{S}^3 && \text{for all } \ x \in \Omega, \end{aligned} \tag{1.2}$$

where the subset Γ_1 of $\partial\Omega$ is defined by $\tilde{\Gamma}_1 = \boldsymbol{\Phi}(\Gamma_1)$.

Finally, let the **second Piola-Kirchhoff stress tensor field** $\boldsymbol{\Sigma} : \overline{\Omega} \to \mathbb{S}^3$ be defined by

$$\boldsymbol{\Sigma}(x) := \boldsymbol{\nabla\Phi}(x)^{-1}\boldsymbol{T}(x) \ \text{ for all } \ x \in \Omega.$$

Then the *equations of equilibrium defined in the reference configuration* take the equivalent form

$$\begin{aligned} -\mathbf{div}\,(\boldsymbol{\nabla\Phi}(x)\boldsymbol{\Sigma}(x)) &= \boldsymbol{f}(x) \quad \text{for all} \quad x \in \Omega, \\ (\boldsymbol{\nabla\Phi}(x)\boldsymbol{\Sigma}(x))\boldsymbol{n}(x) &= \boldsymbol{g}(x) \quad \text{for all} \quad x \in \Gamma_1, \end{aligned} \tag{1.3}$$

in terms of the symmetric tensor field $\boldsymbol{\Sigma}$.

The unknowns in either system of equations of equilibrium are the deformation of the body defined by the vector field $\boldsymbol{\Phi} : \overline{\Omega} \to \mathbb{R}^3$, and the stress field inside the body defined by the fields $\boldsymbol{T} : \overline{\Omega} \to \mathbb{M}^3$ or $\boldsymbol{\Sigma} : \overline{\Omega} \to \mathbb{S}^3$. In order to determine these unknowns, either system (1.2) or (1.3) has to be supplemented with an equation relating these fields. This is the object of the next section.

1.2. **Constitutive equations of elastic materials.** It is clear that the stress tensor field should depend on the deformation induced by the applied forces. This dependence is reflected by the *constitutive equation* of the material, by means of a *response function*, specific to the material considered. In this article, we will consider one class of such materials, according to the following definition:

Definition 1.3. A material is **elastic** if there exists a function $\boldsymbol{T}^\sharp : \overline{\Omega} \times \mathbb{M}^3_+ \to \mathbb{M}^3$ such that

$$\boldsymbol{T}(x) = \boldsymbol{T}^\sharp(x, \boldsymbol{\nabla\Phi}(x)) \quad \text{for all } x \in \overline{\Omega}.$$

Equivalently, a material is *elastic* if there exists a function $\boldsymbol{\Sigma}^\sharp : \overline{\Omega} \times \mathbb{M}^3_+ \to \mathbb{S}^3$ such that

$$\boldsymbol{\Sigma}(x) = \boldsymbol{\Sigma}^\sharp(x, \boldsymbol{\nabla\Phi}(x)) \quad \text{for all } x \in \overline{\Omega}.$$

Either function $\boldsymbol{T}^\sharp$ or $\boldsymbol{\Sigma}^\sharp$ is called the **response function** of the material.

A response function cannot be arbitrary, because a general axiom in physics asserts that any "observable quantity" must be independent of the particular orthogonal basis in which it is computed. For an elastic material, the "observable quantity" computed through a constitutive equation is the *stress vector field* $\tilde{\boldsymbol{t}}$. Therefore this vector field must be independent of the particular orthogonal basis in which it is computed. This property, which must be satisfied by all elastic materials, is called the **axiom of material frame-indifference**. The following theorem translates this axiom in terms of the response function of the material.

Theorem 1.4. *An elastic material satisfies the axiom of material frame-indifference if and only if*

$$\boldsymbol{T}^\sharp(x, \boldsymbol{QF}) = \boldsymbol{Q}\boldsymbol{T}^\sharp(x, \boldsymbol{F}) \text{ for all } x \in \overline{\Omega} \text{ and } \boldsymbol{Q} \in \mathbb{O}^3_+ \text{ and } \boldsymbol{F} \in \mathbb{M}^3_+,$$

or equivalently, if and only if

$$\boldsymbol{\Sigma}^\sharp(x, \boldsymbol{QF}) = \boldsymbol{\Sigma}^\sharp(x, \boldsymbol{F}) \text{ for all } x \in \overline{\Omega} \text{ and } \boldsymbol{Q} \in \mathbb{O}^3_+ \text{ and } \boldsymbol{F} \in \mathbb{M}^3_+.$$

The second equivalence implies that the response function $\boldsymbol{\Sigma}^\sharp$ depend on $\boldsymbol{F}$ only via the symmetric positive definite matrix $\boldsymbol{U} := (\boldsymbol{F}^T\boldsymbol{F})^{1/2}$, the square root of the symmetric positive definite matrix $(\boldsymbol{F}^T\boldsymbol{F}) \in \mathbb{S}^3_>$. To see this, one uses the polar factorisation $\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}$, where $\boldsymbol{R} := \boldsymbol{F}\boldsymbol{U}^{-1} \in \mathbb{O}^3_+$, in the second equivalence of Theorem 1.4 to deduce that

$$\boldsymbol{\Sigma}^\sharp(x, \boldsymbol{F}) = \boldsymbol{\Sigma}^\sharp(x, \boldsymbol{U}) \text{ for all } x \in \overline{\Omega} \text{ and } \boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} \in \mathbb{M}^3_+.$$

This implies that the second Piola-Kirchhoff stress tensor field $\boldsymbol{\Sigma} : \overline{\Omega} \to \mathbb{S}^3$ depends on the deformation field $\boldsymbol{\Phi} : \overline{\Omega} \to \mathbb{R}^3$ only via the associated metric tensor field $\boldsymbol{C} := \boldsymbol{\nabla}\boldsymbol{\Phi}^T\boldsymbol{\nabla}\boldsymbol{\Phi}$, i.e.,

$$\boldsymbol{\Sigma}(x) = \tilde{\boldsymbol{\Sigma}}(x, \boldsymbol{C}(x)) \quad \text{for all } x \in \overline{\Omega},$$

where the function $\tilde{\boldsymbol{\Sigma}} : \overline{\Omega} \times \mathbb{S}^3_> \to \mathbb{S}^3$ is defined by

$$\tilde{\boldsymbol{\Sigma}}(x, \boldsymbol{C}) := \boldsymbol{\Sigma}^\sharp(x, \boldsymbol{C}^{1/2}) \quad \text{for all } x \in \overline{\Omega} \text{ and } \boldsymbol{C} \in \mathbb{S}^3_>.$$

We just saw how the *axiom* of material frame-indifference restricts the form of the response function. We now examine how its form can be further restricted by other *properties* that a given material *may* possess.

Definition 1.5. An elastic material is **isotropic at a point** x of the reference configuration if the response of the material "is the same in all directions", i.e., if the Cauchy stress tensor $\tilde{\boldsymbol{T}}(\tilde{x})$ is the same if the reference configuration is rotated by an arbitrary matrix of $\mathbb{O}^3_+$ around the point x. An elastic material occupying a reference configuration $\overline{\Omega}$ is **isotropic** if it is isotropic at all points of $\overline{\Omega}$.

The following theorem gives a characterisation of the response function of an isotropic elastic material:

Theorem 1.6. *An elastic material occupying a reference configuration* $\overline{\Omega}$ *is isotropic if and only if*

$$\boldsymbol{T}^\sharp(x, \boldsymbol{F}\boldsymbol{Q}) = \boldsymbol{T}^\sharp(x, \boldsymbol{F})\boldsymbol{Q} \text{ for all } x \in \overline{\Omega} \text{ and } \boldsymbol{Q} \in \mathbb{O}^3_+ \text{ and } \boldsymbol{F} \in \mathbb{M}^3_+,$$

or equivalently, if and only if

$$\boldsymbol{\Sigma}^\sharp(x, \boldsymbol{F}\boldsymbol{Q}) = \boldsymbol{Q}^T\boldsymbol{\Sigma}^\sharp(x, \boldsymbol{F})\boldsymbol{Q} \text{ for all } x \in \overline{\Omega} \text{ and } \boldsymbol{Q} \in \mathbb{O}^3_+ \text{ and } \boldsymbol{F} \in \mathbb{M}^3_+.$$

Another property that an elastic material may satisfy is the property of homogeneity:

Definition 1.7. An elastic material occupying a reference configuration $\overline{\Omega}$ is **homogeneous** if its response function is independent of the particular point $x \in \overline{\Omega}$ considered. This means that the response function $\boldsymbol{T}^\sharp : \overline{\Omega} \times \mathbb{M}^3_+ \to \mathbb{M}^3$, or equivalently the response function $\boldsymbol{\Sigma}^\sharp : \overline{\Omega} \times \mathbb{M}^3_+ \to \mathbb{S}^3$, does not depend on the first variable. In other words, there exist mappings (still denoted) $\boldsymbol{T}^\sharp : \mathbb{M}^3_+ \to \mathbb{M}^3$ and $\boldsymbol{\Sigma}^\sharp : \mathbb{M}^3_+ \to \mathbb{S}^3$ such that

$$\boldsymbol{T}^\sharp(x, \boldsymbol{F}) = \boldsymbol{T}^\sharp(\boldsymbol{F}) \text{ for all } x \in \overline{\Omega} \text{ and } \boldsymbol{F} \in \mathbb{M}^3_+,$$

and

$$\mathbf{\Sigma}^\sharp(x, \boldsymbol{F}) = \mathbf{\Sigma}^\sharp(\boldsymbol{F}) \text{ for all } x \in \overline{\Omega} \text{ and } \boldsymbol{F} \in \mathbb{M}^3_+.$$

The response function of an elastic material can be further restricted if its reference configuration is a *natural state*, according to the following definition:

Definition 1.8. A reference configuration $\overline{\Omega}$ is called a **natural state**, or equivalently is said to be **stress-free**, if

$$\boldsymbol{T}^\sharp(x, \boldsymbol{I}) = \mathbf{0} \text{ for all } x \in \overline{\Omega},$$

or equivalently, if

$$\mathbf{\Sigma}^\sharp(x, \boldsymbol{I}) = \mathbf{0} \text{ for all } x \in \overline{\Omega}.$$

We have seen that the second Piola-Kirchhoff stress tensor field $\mathbf{\Sigma} : \overline{\Omega} \to \mathbb{S}^3$ is expressed in terms of the deformation field $\mathbf{\Phi} : \overline{\Omega} \to \mathbb{R}^3$ as

$$\mathbf{\Sigma}(x) = \tilde{\mathbf{\Sigma}}(x, \boldsymbol{C}(x)), \text{ where } \boldsymbol{C}(x) = \boldsymbol{\nabla}\mathbf{\Phi}^T(x)\boldsymbol{\nabla}\mathbf{\Phi}(x) \text{ for all } x \in \overline{\Omega}.$$

If the elastic material is isotropic, then the dependence of $\mathbf{\Sigma}(x)$ in terms of $\boldsymbol{C}(x)$ can be further reduced in a remarkable way, according to the following **Rivlin-Ericksen theorem**:

Theorem 1.9. *If an elastic material is isotropic and satisfies the principle of material frame-indifference, then there exists functions $\gamma_i^\sharp : \Omega \times \mathbb{R}^3 \to \mathbb{R}$, $i \in \{1, 2, 3\}$, such that*

$$\mathbf{\Sigma}(x) = \gamma_0(x)\boldsymbol{I} + \gamma_1(x)\boldsymbol{C}(x) + \gamma_2(x)\boldsymbol{C}^2(x) \text{ for all } x \in \overline{\Omega},$$

where $\gamma_i(x) = \gamma_i^\sharp(x, \operatorname{tr}\boldsymbol{C}, \operatorname{tr}(\mathbf{Cof}\boldsymbol{C}), \det\boldsymbol{C})$.

Proof. See Rivlin & Ericksen [18] or Ciarlet [6]. □

Note that the numbers $\operatorname{tr}\boldsymbol{C}(x)$, $\operatorname{tr}(\mathbf{Cof}\boldsymbol{C}(x))$, and $\det\boldsymbol{C}(x)$ appearing in the above theorem constitute the three *principal invariants* of the matrix $\boldsymbol{C}(x)$.

Although the Rivlin-Ericksen theorem substantially reduces the range of possible response functions of elastic materials that are isotropic and satisfy the principle of frame-indifference, the expression of $\mathbf{\Sigma}$ is still far too general in view of an effective resolution of the equilibrium equations. To further simplify this expression, we now restrict ourselves to deformations that are "close to the identity".

In terms of the displacement filed $\boldsymbol{u} : \Omega \to \mathbb{R}^3$, which is related to the deformation $\mathbf{\Phi} : \Omega \to \mathbb{R}^3$ by the formula

$$\mathbf{\Phi}(x) = \boldsymbol{x} + \boldsymbol{u}(x) \text{ for all } x \in \overline{\Omega},$$

the metric tensor field $\boldsymbol{C}$ has the expression

$$\boldsymbol{C}(x) = \boldsymbol{I} + 2\boldsymbol{E}(x),$$

where

$$\boldsymbol{E}(x) := \frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{u}^T(x) + \boldsymbol{\nabla}\boldsymbol{u}(x) + \boldsymbol{\nabla}\boldsymbol{u}^T(x)\boldsymbol{\nabla}\boldsymbol{u}(x))$$

denotes the **Green-St Venant strain tensor** at x.

Thanks to the above assumption on the deformation, the matrices $\boldsymbol{E}(x)$ are "small" for all $x \in \overline{\Omega}$, and therefore one can use Taylor expansions to further simplify the expression of the response function given by the Rivlin-Ericksen theorem. Specifically, using the Taylor expansions

$$\begin{aligned}
\operatorname{tr} \boldsymbol{C}(x) &= 3 + 2 \operatorname{tr} \boldsymbol{E}(x), \\
\operatorname{tr}(\mathbf{Cof}\boldsymbol{C}(x)) &= 3 + 4 \operatorname{tr} \boldsymbol{E}(x) + o(|\boldsymbol{E}(x)|), \\
\det \boldsymbol{C}(x) &= 1 + 2 \operatorname{tr} \boldsymbol{E}(x) + o(|\boldsymbol{E}(x)|), \\
\boldsymbol{C}^2(x) &= 1 + 4\boldsymbol{E}(x) + o(|\boldsymbol{E}(x)|),
\end{aligned}$$

and assuming that the functions $\gamma_i^\sharp$ are smooth enough, we deduce from the Rivlin-Ericksen theorem that

$$\boldsymbol{\Sigma}(x) = \boldsymbol{\Sigma}^\sharp(x, \boldsymbol{I}) + \{(\lambda(x) \operatorname{tr} \boldsymbol{E}(x))\boldsymbol{I} + 2\mu(x)\boldsymbol{E}(x)\} + o_x(|\boldsymbol{E}(x)|),$$

where the real-valued functions $\lambda(x)$ and $\mu(x)$ are *independent of* the displacement field $\boldsymbol{u}$. If in addition the material is homogeneous, then λ and μ are *constants*.

To sum up, the constitutive equation of a homogeneous and isotropic elastic material that satisfies the axiom of frame-indifference must be such that

$$\boldsymbol{\Sigma}(x) = \boldsymbol{\Sigma}^\sharp(x, \boldsymbol{I}) + \lambda(\operatorname{tr} \boldsymbol{E}(x))\boldsymbol{I} + 2\mu\boldsymbol{E}(x) + o_x(|\boldsymbol{E}(x)|) \text{ for all } x \in \overline{\Omega}.$$

If in addition Ω is a *natural state*, a natural candidate for a constitutive equation is thus

$$\boldsymbol{\Sigma}(x) = \lambda(\operatorname{tr} \boldsymbol{E}(x))\boldsymbol{I} + 2\mu\boldsymbol{E}(x) \text{ for all } x \in \overline{\Omega},$$

and in this case λ and μ are then called the **Lamé constants** of the material.

A material whose constitutive equation has the above expression is called a **St Venant-Kirchhoff material**. Note that the constitutive equation of a St Venant-Kirchhoff material is invertible, in the sense that the field $\boldsymbol{E}$ can be also expressed as a function of the field $\boldsymbol{\Sigma}$ as

$$\boldsymbol{E}(x) = \frac{1}{2\mu}\boldsymbol{\Sigma}(x) - \frac{\nu}{E}(\operatorname{tr} \boldsymbol{\Sigma}(x))\boldsymbol{I} \text{ for all } x \in \overline{\Omega}.$$

Remark. The Lamé constants are determined experimentally for each elastic material and experimental evidence shows that they are both strictly positive (for instance, $\lambda = 10^6 \text{kg/cm}^2$ and $\mu = 820000 \text{kg/cm}^2$ for steel; $\lambda = 40000 \text{kg/cm}^2$ and $\mu = 1200 \text{kg/cm}^2$ for rubber). Their explicit values do not play any rôle in our subsequent analysis; only their positivity will be used. The Lamé coefficients are sometimes expressed in terms of the *Poisson coefficient* ν and *Young modulus* E through the expressions

$$\nu = \frac{\lambda}{2(\lambda + \mu)} \text{ and } E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}.$$

□

1.3. **The equations of nonlinear elasticity.** It remains to combine the equations of equilibrium (equations (1.3) in Section 1.1) with the constitutive equation of the material considered (Section 1.2) and with boundary conditions on $\Gamma_0 := \partial\Omega \setminus \Gamma_1$. Assuming that the constituting material has a known response function given by $\boldsymbol{T}^\sharp$ or by $\boldsymbol{\Sigma}^\sharp$ and that the body is held fixed on Γ_0, we conclude in this fashion that the deformation arising in the body in response to the applied forces of densities $\boldsymbol{f}$ and $\boldsymbol{g}$ satisfies the nonlinear boundary value problem:

$$\begin{aligned} -\mathbf{div}\, \boldsymbol{T}(x) &= \boldsymbol{f}(x), \quad x \in \Omega, \\ \boldsymbol{\Phi}(x) &= x, \qquad\; x \in \Gamma_0, \\ \boldsymbol{T}(x)\boldsymbol{n}(x) &= \boldsymbol{g}(x), \quad x \in \Gamma_1, \end{aligned} \tag{1.4}$$

where

$$\boldsymbol{T}(x) = \boldsymbol{T}^\sharp(x, \boldsymbol{\nabla\Phi}(x)) = \boldsymbol{\nabla\Phi}(x)\boldsymbol{\Sigma}^\sharp(x, \boldsymbol{\nabla\Phi}(x)) \text{ for all } x \in \overline{\Omega}. \tag{1.5}$$

The equations (1.4) constitute the **equations of nonlinear elasticity**. We will give in Sections 3.1 and 3.2 various sets of assumptions guaranteeing that this problem has solutions.

1.4. **The equations of linear elasticity.** These equations are obtained from the equations of nonlinear elasticity (1.4)-(1.5) under the assumption that the body will undergo a "small" displacement, in the sense that

$$\boldsymbol{\Phi}(x) = x + \boldsymbol{u}(x) \text{ with } |\boldsymbol{\nabla u}(x)| \ll 1 \text{ for all } x \in \overline{\Omega}.$$

It is then reasonable to approach the nonlinear three-dimensional model (1.4)-(1.5) with its linear part, obtained by replacing the response function (1.5) with its affine part in $\boldsymbol{u}$ of the Taylor series expansion

$$\boldsymbol{T}^\sharp(x, \boldsymbol{\nabla\Phi}(x)) = \boldsymbol{T}^\sharp(x, \boldsymbol{I}) + \frac{\partial \boldsymbol{T}^\sharp}{\partial \boldsymbol{F}}(x, \boldsymbol{I})\boldsymbol{\nabla u}(x) + o(|\boldsymbol{\nabla u}(x)|).$$

Therefore, the displacement arising in the body in response to the applied forces of densities $\boldsymbol{f}$ and $\boldsymbol{g}$ satisfies the linear boundary value problem:

$$\begin{aligned} -\mathbf{div}\left(\boldsymbol{T}^\sharp(\cdot, \boldsymbol{I}) + \frac{\partial \boldsymbol{T}^\sharp}{\partial \boldsymbol{F}}(\cdot, \boldsymbol{I})\boldsymbol{\nabla u}\right) &= \boldsymbol{f} \text{ in } \Omega, \\ \boldsymbol{Tn} &= \boldsymbol{g} \text{ on } \Gamma_1, \\ \boldsymbol{u} &= \boldsymbol{0} \text{ on } \Gamma_0. \end{aligned}$$

These equations constitute the **equations of linear elasticity**.

An important particular case of these equations is obtained when the body is made of an *isotropic and homogeneous* elastic material such that its *reference configuration is a natural state*, so that its constitutive equation is (see Section 1.2):

$$\boldsymbol{\Sigma}(x) = \lambda(\mathrm{tr}\, \boldsymbol{E}(x))\boldsymbol{I} + 2\mu\boldsymbol{E}(x) + o(|\boldsymbol{E}(x)|),$$

where $\lambda > 0$ and $\mu > 0$ are the Lamé constants of the material. Then, for all $x \in \overline{\Omega}$,

$$\boldsymbol{E}(x) = \frac{1}{2}(\boldsymbol{\nabla\Phi}^T(x)\boldsymbol{\nabla\Phi}(x) - \boldsymbol{I}) = \frac{1}{2}(\boldsymbol{\nabla u}^T(x) + \boldsymbol{\nabla u}(x)) + o_x(|\boldsymbol{\nabla u}(x)|),$$

and

$$\begin{aligned}\boldsymbol{T}(x) &= \boldsymbol{\nabla\Phi}(x)\boldsymbol{\Sigma}(x) = (\boldsymbol{I} + \boldsymbol{\nabla u}(x))\Big(\lambda(\operatorname{tr}\boldsymbol{E}(x))\boldsymbol{I} + 2\mu\boldsymbol{E}(x)\Big)\\ &= \frac{\lambda}{2}\operatorname{tr}(\boldsymbol{\nabla u}^T(x) + \boldsymbol{\nabla u}(x)) + \mu(\boldsymbol{\nabla u}^T(x) + \boldsymbol{\nabla u}(x)) + o_x(|\boldsymbol{\nabla u}(x)|).\end{aligned}$$

Therefore the **equations of linear elasticity**, which are obtained from (1.4) by replacing $\boldsymbol{T}(x)$ by its linear part with respect to $\boldsymbol{\nabla u}(x)$, are now given by

$$\begin{aligned} -\mathbf{div}\,\boldsymbol{\sigma}(x) &= \boldsymbol{f}(x), \quad && x \in \Omega,\\ \boldsymbol{u}(x) &= \mathbf{0}, && x \in \Gamma_0,\\ \boldsymbol{\sigma}(x)\boldsymbol{n}(x) &= \boldsymbol{g}(x), && x \in \Gamma_1,\end{aligned} \tag{1.6}$$

where

$$\boldsymbol{\sigma}(x) = \lambda(\operatorname{tr}\mathbf{e}(x))\boldsymbol{I} + 2\mu\mathbf{e}(x) \text{ and } \mathbf{e}(x) = \frac{1}{2}(\boldsymbol{\nabla u}^T(x) + \boldsymbol{\nabla u}(x)). \tag{1.7}$$

We show in the next section that this linear system has a unique solution in appropriate function spaces.

2. Linear elasticity

We study here the existence and uniqueness of solutions to the *equations of linear elasticity*. Using a fundamental lemma, due to J.L. Lions, about distributions with derivatives in "negative" Sobolev spaces, we establish the fundamental Korn inequality (Section 2.1), which in turn implies that the *equations of linear elasticity* have a unique, stable, and regular if the data are regular, solution (Section 2.2).

2.1. Korn's inequalities.

We first review some essential definitions and notations, together with a fundamental *lemma of J.L. Lions* (Lemma 2.1). This lemma will play a key rôle in the proofs of Korn's inequality (Theorem 2.2).

Let Ω be a domain in $\mathbb{R}^n$. We recall that, for each integer $m \geq 1$, $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$, and $H^{-m}(\Omega) = W^{-m,2}(\Omega)$ denote the usual *Sobolev spaces*. The norm in $L^2(\Omega)$ is noted $\|\cdot\|_{L^2(\Omega)}$ and the norm in $H^m(\Omega)$, $m \geq 1$, is noted $\|\cdot\|_{H^m(\Omega)}$. In particular,

$$\begin{aligned} H^1(\Omega) &:= \{v \in L^2(\Omega);\, \partial_i v \in L^2(\Omega),\, 1 \leq i \leq n\},\\ H^2(\Omega) &:= \{v \in H^1(\Omega);\, \partial_{ij} v \in L^2(\Omega),\, 1 \leq i,\, j \leq n\},\end{aligned}$$

where $\partial_i v$ and $\partial_{ij} v$ denote partial derivatives in the sense of distributions, and

$$\|v\|_{L^2(\Omega)} := \Big\{ \int_\Omega |v|^2\,dx \Big\}^{1/2} \text{ if } v \in L^2(\Omega),$$

$$\|v\|_{H^1(\Omega)} := \Big\{ \|v\|^2_{L^2(\Omega)} + \sum_{i=1}^n \|\partial_i v\|^2_{L^2(\Omega)} \Big\}^{1/2} \text{ if } v \in H^1(\Omega).$$

Furthermore,

$$H^1_0(\Omega) := \text{closure of } \mathcal{D}(\Omega) \text{ with respect to } \|\cdot\|_{H^1(\Omega)},$$

where $\mathcal{D}(\Omega)$ denotes the space of infinitely differentiable real-valued functions defined over Ω whose support is a compact subset of Ω, and

$$H^{-1}(\Omega) := \text{dual space of } H^1_0(\Omega) \text{ equipped with the norm } \|\cdot\|_{H^1(\Omega)}.$$

Because the boundary of Ω is Lipschitz-continuous,

$$H^1_0(\Omega) = \{v \in H^1(\Omega);\ v = 0 \text{ on } \Gamma\},$$

where the relation $v = 0$ on Γ is to be understood in the sense of trace.

Now it is clear that

$$v \in L^2(\Omega) \implies v \in H^{-1}(\Omega) \text{ and } \partial_i v \in H^{-1}(\Omega),\ 1 \le i \le n,$$

since (the duality between the spaces $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ is denoted by $\langle \cdot, \cdot \rangle$):

$$|\langle v, \varphi\rangle| = \Big| \int_\Omega v\varphi\,dx \Big| \le \|v\|_{L^2(\Omega)} \|\varphi\|_{H^1(\Omega)},$$

$$|\langle \partial_i v, \varphi\rangle| = |-\langle v, \partial_i \varphi\rangle| = \Big| - \int_\Omega v\partial_i\varphi\,dx \Big| \le \|v\|_{L^2(\Omega)} \|\varphi\|_{H^1(\Omega)}$$

for all $\varphi \in \mathcal{D}(\Omega)$. It is remarkable that the converse implication holds:

Lemma 2.1. *Let Ω be a domain in $\mathbb{R}^n$ and let v be a distribution on Ω. Then*

$$\{v \in H^{-1}(\Omega) \text{ and } \partial_i v \in H^{-1}(\Omega),\ 1 \le i \le n\} \implies v \in L^2(\Omega).$$

□

This implication was first proved by J.L. Lions, as stated in Magenes & Stampacchia [15], p. 320, Note ([27]); for this reason, it will be henceforth referred to as the **lemma of J.L. Lions**. Its first published proof for domains with smooth boundaries appeared in Duvaut & Lions [10], p. 111; another proof was also given by Tartar [20]. Various extensions to "genuine" domains, i.e., with Lipschitz-continuous boundaries, are given in Bolley & Camus [4], Geymonat & Suquet [12], and Borchers & Sohr [5]; Amrouche & Girault [2, Proposition 2.10] even proved that the more general implication

$$\{v \in \mathcal{D}'(\Omega) \text{ and } \partial_i v \in H^m(\Omega),\ 1 \le i \le n\} \implies v \in H^{m+1}(\Omega)$$

holds for *arbitrary* integers $m \in \mathbb{Z}$.

We are now in a position to establish the following classical, and fundamental, inequality:

Theorem 2.2 (Korn's inequality). *Let Ω be a domain in $\mathbb{R}^3$ and let $\Gamma_0 \subset \partial\Omega$ be such that* $\operatorname{area}\Gamma_0 > 0$. *Then there exists a constant C such that*

$$\|\mathbf{e}(\boldsymbol{v})\|_{L^2(\Omega;\mathbb{S}^3)} \geq C\|\boldsymbol{v}\|_{H^1(\Omega;\mathbb{R}^3)}$$

for all $\boldsymbol{v} \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^3) := \{\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^3);\ \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_0\}$.

Proof. Several proofs are available in the mathematical literature for this remarkable inequality. We adapt here that given in Duvaut & Lions [10]. We proceed in several steps:

(i) Korn's inequality is a consequence of the identity

$$\partial_{ij}v_k = \partial_i e_{jk}(\boldsymbol{v}) + \partial_j e_{ik}(\boldsymbol{v}) - \partial_k e_{ij}(\boldsymbol{v})$$

relating the matrix fields $\boldsymbol{\nabla v} = (\partial_j v_i)$ and $\mathbf{e}(\boldsymbol{v}) = (e_{ij}(\boldsymbol{v}))$, where

$$e_{ij}(\boldsymbol{v}) := \frac{1}{2}(\partial_i v_j + \partial_j v_i).$$

If $\boldsymbol{v} \in L^2(\Omega;\mathbb{R}^3)$ and $\mathbf{e}(\boldsymbol{v}) \in L^2(\Omega;\mathbb{S}^3)$, the above identity shows that $\partial_{ij}v_k \in H^{-1}(\Omega)$. Since the functions $\partial_j v_k$ also belong to the space $H^{-1}(\Omega)$, the *lemma of J.L. Lions* (Lemma 2.1) shows that $\partial_j v_k \in L^2(\Omega)$. This implies that the space

$$E(\Omega;\mathbb{R}^3) := \{\boldsymbol{v} \in L^2(\Omega;\mathbb{R}^3);\ \mathbf{e}(\boldsymbol{v}) \in L^2(\Omega;\mathbb{S}^3)\}$$

coincides with the Sobolev space $H^1(\Omega;\mathbb{R}^3)$.

(ii) The space $E(\Omega;\mathbb{R}^3)$, equipped with the norm

$$\|\boldsymbol{v}\|_{E(\Omega;\mathbb{R}^3)} := \|\boldsymbol{v}\|_{L^2(\Omega;\mathbb{R}^3)} + \|\mathbf{e}(\boldsymbol{v})\|_{L^2(\Omega;\mathbb{R}^3)},$$

is clearly a Hilbert space, as is the space $H^1(\Omega;\mathbb{R}^3)$ equipped with the norm

$$\|\boldsymbol{v}\|_{H^1(\Omega;\mathbb{R}^3)} := \|\boldsymbol{v}\|_{L^2(\Omega;\mathbb{R}^3)} + \|\boldsymbol{\nabla v}\|_{L^2(\Omega;\mathbb{R}^3)}.$$

Since the identity mapping

$$\mathbf{id} : \boldsymbol{v} \in H^1(\Omega;\mathbb{R}^3) \mapsto \boldsymbol{v} \in E(\Omega;\mathbb{R}^3)$$

is clearly linear, bijective (thanks to the step (i)), and continuous, the open mapping theorem (see, e.g., Yosida [23]) shows that **id** is also an open mapping. Therefore, there exists a constant C such that

$$\|\boldsymbol{v}\|_{H^1(\Omega;\mathbb{R}^3)} \leq C\|\boldsymbol{v}\|_{E(\Omega;\mathbb{R}^3)} \text{ for all } \boldsymbol{v} \in E(\Omega;\mathbb{R}^3),$$

or equivalently, such that

$$\|\boldsymbol{v}\|_{L^2(\Omega;\mathbb{R}^3)} + \|\mathbf{e}(\boldsymbol{v})\|_{L^2(\Omega;\mathbb{S}^3)} \geq C^{-1}\|\boldsymbol{v}\|_{H^1(\Omega;\mathbb{R}^3)}$$

for all $\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^3)$.

(iii) We establish that, if $\boldsymbol{v} \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ satisfies $\mathbf{e}(\boldsymbol{v}) = \mathbf{0}$, then $\boldsymbol{v} = \mathbf{0}$.

This is a consequence of the identity of Part (i), which shows that any field $\boldsymbol{v} \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ that satisfies $\mathbf{e}(\boldsymbol{v}) = \mathbf{0}$ must also satisfy

$$\partial_{ij}v_k = 0 \text{ in } \Omega.$$

Therefore, by a classical result about distributions (see, e.g. Schwartz [19]), the field $\boldsymbol{v}$ must be affine, i.e., of the form $\boldsymbol{v}(x) = \boldsymbol{b} + \boldsymbol{A x}$ for all $x \in \Omega$, where $\boldsymbol{b} \in \mathbb{R}^3$ and $\boldsymbol{A} \in \mathbb{M}^3$. Since the symmetric part of the gradient of $\boldsymbol{v}$, which is precisely $\mathbf{e}(\boldsymbol{v})$, vanishes in Ω, the matrix $\boldsymbol{A}$ must be in addition antisymmetric. Since the rank of a nonzero antisymmetric matrix of order three is necessarily two, the locus of all points x satisfying $\mathbf{a} + \boldsymbol{A}\boldsymbol{x} = \mathbf{0}$ is either a line in $\mathbb{R}^3$ or an empty set, depending on whether the linear system $\mathbf{a} + \boldsymbol{A}\boldsymbol{x} = \mathbf{0}$ has solutions or not. But $\mathbf{a} + \boldsymbol{A}\boldsymbol{x} = \mathbf{0}$ on Γ_0 and $\operatorname{area}\Gamma_0 > 0$. Hence $\boldsymbol{A} = \mathbf{0}$ and $\boldsymbol{b} = \mathbf{0}$, and thus $\boldsymbol{v} = \mathbf{0}$ in Ω.

(iv) The Korn inequality of Theorem 2.2 then follows by contradiction. If the inequality were false, there would exist a sequence $(\boldsymbol{v}_n)_{n\in\mathbb{N}}$ in $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ such that

$$\|\boldsymbol{v}_n\|_{H^1(\Omega;\mathbb{R}^3)} = 1 \text{ for all } n,$$
$$\|\mathbf{e}(\boldsymbol{v}_n)\|_{L^2(\Omega;\mathbb{S}^3)} \to 0 \text{ as } n \to \infty.$$

Because the set Ω is a domain, the inclusion $H^1(\Omega;\mathbb{R}^3) \subset L^2(\Omega;\mathbb{R}^3)$ is compact by the Rellich-Kondrasov theorem. The sequence $(\boldsymbol{v}_n)$ being bounded in $H^1(\Omega;\mathbb{R}^3)$, it contains a subsequence $(\boldsymbol{v}_{\sigma(n)})$, where $\sigma : \mathbb{N} \to \mathbb{N}$ is an increasing function, that converges in $L^2(\Omega;\mathbb{R}^3)$ as $n \to \infty$.

Since the sequences $(\boldsymbol{v}_{\sigma(n)})$ and $(\mathbf{e}(\boldsymbol{v}_{\sigma(n)}))$ converge respectively in the spaces $L^2(\Omega;\mathbb{R}^3)$ and $L^2(\Omega;\mathbb{S}^3)$, they are Cauchy sequences in the same spaces. Therefore the sequence $(\boldsymbol{v}_{\sigma(n)})$ is a Cauchy sequence with respect to the norm $\|\cdot\|_{E(\Omega;\mathbb{R}^3)}$, hence with respect to the norm $\|\cdot\|_{H^1(\Omega;\mathbb{R}^3)}$ by the inequality established in Part (ii).

The space $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ being complete as a closed subspace of $H^1(\Omega;\mathbb{R}^3)$, there exists $\boldsymbol{v} \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ such that

$$\boldsymbol{v}_{\sigma(n)} \to \boldsymbol{v} \text{ in } H^1(\Omega;\mathbb{R}^3).$$

Since its limit satisfies

$$\mathbf{e}(\boldsymbol{v}) = \lim_{n\to\infty} \mathbf{e}(\boldsymbol{v}_{\sigma(n)}) = 0,$$

it follows that $\boldsymbol{v} = 0$ by Part (iii).

But this contradicts the relation $\|\boldsymbol{v}\|_{H^1(\Omega;\mathbb{R}^3)} = \lim_{n\to\infty} \|\boldsymbol{v}_{\sigma(n)}\|_{H^1(\Omega;\mathbb{R}^3)} = 1$, and the proof is complete. $\square$

The inequality established in Part (ii) of the proof is called **Korn's inequality without boundary conditions.**

The uniqueness result established in Part (iii) of the proof is called the **infinitesimal rigid displacement lemma**. It shows that an **infinitesimal rigid displacement field**, i.e., a vector field $\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^3)$ satisfying $\mathbf{e}(\boldsymbol{v}) = \mathbf{0}$, is necessarily of the form

$$\boldsymbol{v}(x) = \mathbf{a} + \boldsymbol{b} \wedge \boldsymbol{x} \text{ for all } x \in \Omega, \text{ where } \mathbf{a}, \boldsymbol{b} \in \mathbb{R}^3.$$

This shows that the space of all infinitesimal rigid displacement fields, denoted

$$\operatorname{Rig}(\Omega;\mathbb{R}^3) := \{\boldsymbol{w} : \Omega \to \mathbb{R}^3;\ \boldsymbol{w}(x) = \mathbf{a} + \boldsymbol{b} \wedge \boldsymbol{x},\ \mathbf{a}, \boldsymbol{b} \in \mathbb{R}^3\},$$

has finite dimension.

Remark. In the special case where $\Gamma_0 = \partial\Omega$, Korn's inequality is a trivial consequence of the identity

$$\int_\Omega |\mathbf{e}(\boldsymbol{v})|^2\,dx = \int_\Omega |\boldsymbol{\nabla}\boldsymbol{v}|^2\,dx \;\text{ for all } \boldsymbol{v} \in H_0^1(\Omega;\mathbb{R}^3),$$

itself obtained by applying twice the formula of integration by parts (see Section "Preliminaries"). □

The next theorem establishes another Korn's inequality:

Theorem 2.3 (Korn's inequality). *Let Ω be a domain in $\mathbb{R}^3$. There exists a constant C such that*

$$\|\mathbf{e}(\boldsymbol{v})\|_{L^2(\Omega;\mathbb{S}^3)} \geq C \inf_{\boldsymbol{w}\in \mathrm{Rig}(\Omega;\mathbb{R}^3)} \|\boldsymbol{v}+\boldsymbol{w}\|_{H^1(\Omega;\mathbb{R}^3)}$$

for all $\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^3)$.

Proof. The proof of this inequality follows that of Theorem 2.2, with the parts (iii) and (iv) adapted as follows. A field $\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^3)$ that satisfies $\mathbf{e}(\boldsymbol{v}) = \mathbf{0}$ necessarily belongs to the space $\mathrm{Rig}(\Omega;\mathbb{R}^3)$. The sequence $(\boldsymbol{v}_n)_{n\in\mathbb{N}}$ is now defined in $H^1(\Omega;\mathbb{R}^3)$ and satisfies

$$\inf_{\boldsymbol{w}\in \mathrm{Rig}(\Omega;\mathbb{R}^3)} \|\boldsymbol{v}_n\|_{H^1(\Omega;\mathbb{R}^3)} = 1 \text{ for all } n,$$
$$\|\mathbf{e}(\boldsymbol{v}_n)\|_{L^2(\Omega;\mathbb{S}^3)} \to 0 \text{ as } n \to \infty.$$

If $\mathrm{Rig}(\Omega;\mathbb{R}^3)^\perp$ denotes the orthogonal complement of $\mathrm{Rig}(\Omega;\mathbb{R}^3)$ in the Hilbert space $H^1(\Omega;\mathbb{R}^3)$, then $\boldsymbol{v}_n = \boldsymbol{u}_n + \boldsymbol{w}_n$ with $\boldsymbol{u}_n \in \mathrm{Rig}(\Omega;\mathbb{R}^3)^\perp$ and $\boldsymbol{w}_n \in \mathrm{Rig}(\Omega;\mathbb{R}^3)$ for all $n \in \mathbb{N}$. Then the above relations imply that

$$\|\boldsymbol{u}_n\|_{H^1(\Omega;\mathbb{R}^3)} = 1 \text{ for all } n,$$
$$\|\mathbf{e}(\boldsymbol{u}_n)\|_{L^2(\Omega;\mathbb{S}^3)} \to 0 \text{ as } n \to \infty.$$

Hence there exists an increasing function $\sigma : \mathbb{N} \to \mathbb{N}$ such that the subsequence $(\boldsymbol{u}_{\sigma(n)})$ is a Cauchy sequence in $H^1(\Omega;\mathbb{R}^3)$. This space being complete, there exists $\boldsymbol{u} \in H^1(\Omega;\mathbb{R}^3)$ such that

$$\boldsymbol{u}_{\sigma(n)} \to \boldsymbol{u} \text{ in } H^1(\Omega;\mathbb{R}^3),$$

and its limit satisfies

$$\mathbf{e}(\boldsymbol{u}) = \lim_{n\to\infty} \mathbf{e}(\boldsymbol{u}_{\sigma(n)}) = \mathbf{0}.$$

Therefore $\boldsymbol{u} \in \mathrm{Rig}(\Omega;\mathbb{R}^3)$. On the other hand, $\boldsymbol{u} \in \mathrm{Rig}(\Omega;\mathbb{R}^3)^\perp$ since $\mathrm{Rig}(\Omega;\mathbb{R}^3)^\perp$ is closed and $\boldsymbol{u}_n \in \mathrm{Rig}(\Omega;\mathbb{R}^3)^\perp$. Thus $\boldsymbol{u} = \mathbf{0}$, which contradicts the relation $\|\boldsymbol{u}_{\sigma(n)}\|_{H^1(\Omega;\mathbb{R}^3)} = 1$ for all n. □

2.2. **Existence, stability and regularity of weak solutions.** We define a *weak* solution to the equations of linear elasticity (Section 1.3) as a solution to the variational equations

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\nabla v}\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx + \int_{\Gamma_1} \boldsymbol{g} \cdot \boldsymbol{v}\, da \tag{2.1}$$

for all smooth vector fields $\boldsymbol{v} : \Omega \to \mathbb{R}^3$ that satisfy $\boldsymbol{v} = \boldsymbol{0}$ on Γ_0, where

$$\boldsymbol{\sigma} = \lambda(\operatorname{tr} \mathbf{e}(\boldsymbol{u}))\boldsymbol{I} + 2\mu \mathbf{e}(\boldsymbol{u}) \text{ and } \mathbf{e}(\boldsymbol{u}) = \frac{1}{2}(\boldsymbol{\nabla u}^T + \boldsymbol{\nabla u}).$$

Note that, because the matrix field $\boldsymbol{\sigma}$ is symmetric, the integrand in the left-hand side can be also written as

$$\boldsymbol{\sigma} : \boldsymbol{\nabla v} = \boldsymbol{\sigma} : \mathbf{e}(\boldsymbol{v}),$$

where

$$\mathbf{e}(\boldsymbol{v}) := \frac{1}{2}(\boldsymbol{\nabla v}^T + \boldsymbol{\nabla v}).$$

The existence of a solution to the above variational problem follows from the Lax-Milgram lemma. We distinguish two cases depending on whether $\operatorname{area} \Gamma_0 > 0$ or not.

Theorem 2.4. *Assume that the Lamé constants satisfy $\lambda \geq 0$ and $\mu > 0$ and that the densities of the applied forces satisfy $\boldsymbol{f} \in L^{6/5}(\Omega; \mathbb{R}^3)$ and $\boldsymbol{g} \in L^{4/3}(\Gamma_1; \mathbb{R}^3)$.*

If $\operatorname{area} \Gamma_0 > 0$*, the variational problem* (2.1) *has a unique solution in the space*

$$H^1_{\Gamma_0}(\Omega; \mathbb{R}^3) := \{\boldsymbol{v} \in H^1(\Omega; \mathbb{R}^3);\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0\}.$$

Proof. It suffices to apply the Lax-Milgram lemma to the variational equation (2.1), since all its assumptions are clearly satisfied. In particular, the coerciveness of the bilinear form appearing in the left-hand side of the equation (2.1) is a consequence of Korn's inequality established in Theorem 2.2 combined with the positiveness of the Lamé constants, which together imply that, for all $\boldsymbol{v} \in H^1_{\Gamma_0}(\Omega; \mathbb{R}^3)$,

$$\begin{aligned}\int_\Omega \boldsymbol{\sigma} : \mathbf{e}(\boldsymbol{v})\, dx &= \int_\Omega (\lambda[\operatorname{tr}(\mathbf{e}(\boldsymbol{v}))]^2 + 2\mu|\mathbf{e}(\boldsymbol{v})|^2)\, dx \\ &\geq 2\mu \int_\Omega |\mathbf{e}(\boldsymbol{v})|^2\, dx \geq C\|\boldsymbol{v}\|^2_{H^1(\Omega;\mathbb{R}^3)}.\end{aligned}$$

□

Theorem 2.5. *Assume that the Lamé constants satisfy $\lambda \geq 0$ and $\mu > 0$ and that the densities of the applied forces satisfy $\boldsymbol{f} \in L^{6/5}(\Omega; \mathbb{R}^3)$ and $\boldsymbol{g} \in L^{4/3}(\partial\Omega; \mathbb{R}^3)$.*

If $\operatorname{area} \Gamma_0 = 0$ *and* $\int_\Omega \boldsymbol{f} \cdot \boldsymbol{w}\, dx + \int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{w}\, da = 0$ *for all* $\boldsymbol{w} \in H^1(\Omega; \mathbb{R}^3)$ *satisfying* $\mathbf{e}(\boldsymbol{w}) = \boldsymbol{0}$*, then the variational problem* (2.1) *has a solution in* $H^1(\Omega; \mathbb{R}^3)$*, unique up to an infinitesimal rigid displacement field.*

Proof. It is again based on the Lax-Milgram lemma applied to the variational equations (2.1), this time defined over the quotient space $H^1(\Omega;\mathbb{R}^3)/\mathrm{Rig}(\Omega;\mathbb{R}^3)$, where $\mathrm{Rig}(\Omega;\mathbb{R}^3)$ is the subspace of $H^1(\Omega;\mathbb{R}^3)$ consisting of all the infinitesimal rigid displacements fields. By the *infinitesimal rigid displacement lemma* (see Part (ii) of the proof of Theorem 2.2), $\mathrm{Rig}(\Omega;\mathbb{R}^3)$ is the finite-dimensional space

$$\{\boldsymbol{w}:\Omega\to\mathbb{R}^3;\ \boldsymbol{w}(x)=\mathbf{a}+\boldsymbol{b}\wedge\boldsymbol{x},\ \mathbf{a},\boldsymbol{b}\in\mathbb{R}^3\}.$$

The compatibility relations satisfied by the applied forces imply that the variational equation (2.1) is well defined over the quotient space

$$H^1(\Omega;\mathbb{R}^3)/\mathrm{Rig}(\Omega;\mathbb{R}^3),$$

which is a Hilbert space with respect to the norm

$$\|\dot{\boldsymbol{v}}\|_{H^1(\Omega;\mathbb{R}^3)/\mathrm{Rig}(\Omega;\mathbb{R}^3)}=\inf_{\boldsymbol{w}\in\mathrm{Rig}(\Omega;\mathbb{R}^3)}\|\boldsymbol{v}+\boldsymbol{w}\|_{H^1(\Omega;\mathbb{R}^3)}.$$

The coerciveness of the bilinear form appearing in the left-hand side of the equation (2.1) is a consequence of Korn's inequality established in Theorem 2.3 combined with the positiveness of the Lamé constants. □

The variational problem (2.1) is called a **pure displacement problem** when $\Gamma_0=\partial\Omega$, a **pure traction problem** when $\Gamma_1=\partial\Omega$, and a **displacement-traction problem** when $\mathrm{area}\,\Gamma_0>0$ and $\mathrm{area}\,\Gamma_1>0$. □

Since the system of partial differential equations associated with the linear variational model is elliptic, we expect the solution of the latter to be regular if the data $\boldsymbol{f}$, $\boldsymbol{g}$, and $\partial\Omega$ are regular and if there is no change of boundary condition along a connected portion of $\partial\Omega$. More specifically, the following regularity results hold (indications about the proof are given in Ciarlet [6, Theorem 6.3-6]).

Theorem 2.6 (pure displacement problem). *Assume that* $\Gamma_0=\partial\Omega$. *If* $\boldsymbol{f}\in W^{m,p}(\Omega;\mathbb{R}^3)$ *and* $\partial\Omega$ *is of class* $\mathcal{C}^{m+2}$ *for some integer* $m\geq 0$ *and real number* $1<p<\infty$ *satisfying* $p\geq\frac{6}{5+2m}$, *then the solution* $\boldsymbol{u}$ *to the variational equation* (2.1) *is in the space* $W^{m+2,p}(\Omega;\mathbb{R}^3)$ *and there exists a constant* C *such that*

$$\|\boldsymbol{u}\|_{W^{m+2,p}(\Omega;\mathbb{R}^3)}\leq C\|\boldsymbol{f}\|_{W^{m+2,p}(\Omega;\mathbb{R}^3)}.$$

Furthermore, $\boldsymbol{u}$ *satisfies the boundary value problem:*

$$\begin{aligned}-\mathbf{div}\,\boldsymbol{\sigma}(x)&=\boldsymbol{f},\quad x\in\Omega,\\ \boldsymbol{u}(x)&=\mathbf{0},\quad x\in\partial\Omega.\end{aligned}$$

Theorem 2.7 (pure traction problem). *Assume that* $\Gamma_1=\partial\Omega$ *and* $\int_\Omega\boldsymbol{f}\cdot\boldsymbol{w}\,dx+\int_{\partial\Omega}\boldsymbol{g}\cdot\boldsymbol{w}\,da=0$ *for all vector fields* $\boldsymbol{v}\in H^1(\Omega;\mathbb{R}^3)$ *satisfying* $\mathbf{e}(\boldsymbol{w})=0$. *If* $\boldsymbol{f}\in W^{m,p}(\Omega;\mathbb{R}^3)$, $\boldsymbol{g}\in W^{m+1-1/p,p}(\Gamma_1;\mathbb{R}^3)$, *and* $\partial\Omega$ *is of class* $\mathcal{C}^{m+2}$ *for some integer* $m\geq 0$ *and real number* $1<p<\infty$ *satisfying*

$p \geq \frac{6}{5+2m}$, *then any solution* $\boldsymbol{u}$ *to the variational equation* (2.1) *is in the space* $W^{m+2,p}(\Omega;\mathbb{R}^3)$ *and there exist a constant* C *such that*

$$\|\dot{\boldsymbol{u}}\|_{W^{m+2,p}(\Omega;\mathbb{R}^3)/\mathrm{Rig}(\Omega;\mathbb{R}^3)} \leq C\Big(\|\boldsymbol{f}\|_{W^{m+2,p}(\Omega;\mathbb{R}^3)} + \|\boldsymbol{g}\|_{W^{m+1-1/p,p}(\partial\Omega;\mathbb{R}^3)}\Big).$$

Furthermore, $\boldsymbol{u}$ *satisfies the boundary value problem:*

$$\begin{aligned} -\mathbf{div}\,\boldsymbol{\sigma}(x) &= \boldsymbol{f}(x), \quad x \in \Omega, \\ \boldsymbol{\sigma}(x)\boldsymbol{n}(x) &= \boldsymbol{g}(x), \quad x \in \partial\Omega. \end{aligned}$$

3. Nonlinear elasticity

We study here the existence of solutions to the *equations of nonlinear elasticity*, which fall into two distinct categories:

If the data are regular, the applied forces are “small”, and the boundary condition does not change its nature along connected portions of the boundary, the *equations of nonlinear elasticity* have a solution by the *implicit function theorem* (Section 3.1).

If the constituting material is hyperelastic and the associated stored energy function satisfies certain conditions of polyconvexity, coerciveness, and growth, the *minimization problem associated with the equations of nonlinear elasticity* has a solution by a fundamental theorem of John Ball (Section 3.2).

3.1. Existence of solutions by the implicit function theorem.

The existence theory based on the implicit function theorem asserts that the equations of nonlinear elasticity have solutions if the solutions to the associated equations of linear elasticity are smooth enough, and the applied forces are small enough. The first requirement essentially means that the bodies are either held fixed along their entire boundary (i.e., $\Gamma_0 = \partial\Omega$), or nowhere along their boundary (i.e., $\Gamma_1 = \partial\Omega$).

We restrict our presentation to the case of elastic bodies made of a St Venant-Kirchhoff material. In other words, we assume throughout this section that

$$\boldsymbol{\Sigma} = \lambda(\mathrm{tr}\,\boldsymbol{E})\boldsymbol{I} + 2\mu\boldsymbol{E} \text{ and } \boldsymbol{E} = \frac{1}{2}\Big(\boldsymbol{\nabla}\boldsymbol{u}^T + \boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}\boldsymbol{u}^T\boldsymbol{\nabla}\boldsymbol{u}\Big), \tag{3.1}$$

where $\lambda > 0$ and $\mu > 0$ are the Lamé constants of the material and $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ is the unknown displacement field. We assume that $\Gamma_0 = \partial\Omega$ (the case where $\Gamma_1 = \partial\Omega$ requires some extra care because the space of infinitesimal rigid displacements fields does not reduce to $\{\mathbf{0}\}$). Hence the equations of nonlinear elasticity assert that the displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ inside the body is the solution to the boundary value problem

$$\begin{aligned} -\mathbf{div}\,((\boldsymbol{I} + \boldsymbol{\nabla}\boldsymbol{u})\boldsymbol{\Sigma}) &= \boldsymbol{f} \text{ in } \Omega, \\ \boldsymbol{u} &= \mathbf{0} \text{ on } \partial\Omega, \end{aligned} \tag{3.2}$$

where the field $\boldsymbol{\Sigma}$ is given in terms of the unknown field $\boldsymbol{u}$ by means of (3.1). The existence result is then the following

Theorem 3.1. *The nonlinear boundary value problem* (3.1)-(3.2) *has a solution* $\boldsymbol{u} \in W^{2,p}(\Omega;\mathbb{R}^3)$ *if* Ω *is a domain with a boundary* $\partial\Omega$ *of class* $\mathcal{C}^2$*, and for some* $p > 3$*,* $\boldsymbol{f} \in L^p(\Omega;\mathbb{R}^3)$ *and* $\|\boldsymbol{f}\|_{L^p(\Omega;\mathbb{R}^3)}$ *is small enough.*

Proof. Define the spaces

$$\begin{aligned}\boldsymbol{X} &:= \{\boldsymbol{v} \in W^{2,p}(\Omega;\mathbb{R}^3); \boldsymbol{v} = \boldsymbol{0} \text{ on } \partial\Omega\},\\ \boldsymbol{Y} &:= L^p(\Omega;\mathbb{R}^3).\end{aligned}$$

Define the nonlinear mapping $\mathcal{F} : \boldsymbol{X} \to \boldsymbol{Y}$ by

$$\mathcal{F}(\boldsymbol{v}) := -\mathbf{div}\,((\boldsymbol{I} + \boldsymbol{\nabla}\boldsymbol{v})\boldsymbol{\Sigma}) \text{ for any } \boldsymbol{v} \in \boldsymbol{X},$$

where

$$\boldsymbol{\Sigma} = \lambda(\operatorname{tr}\boldsymbol{E})\boldsymbol{I} + 2\mu\boldsymbol{E} \text{ and } \boldsymbol{E} = \frac{1}{2}\Big(\boldsymbol{\nabla}\boldsymbol{v}^T + \boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}\boldsymbol{v}^T\boldsymbol{\nabla}\boldsymbol{v}\Big).$$

It suffices to prove that the equation

$$\mathcal{F}(\boldsymbol{u}) = \boldsymbol{f}$$

has solutions in $\boldsymbol{X}$ provided that the norm of $\boldsymbol{f}$ in the space $\boldsymbol{Y}$ is small enough.

The idea for solving the above equation is as follows. If the norm of $\boldsymbol{f}$ is small, we expect the norm of $\boldsymbol{u}$ to be small too, so that the above equation can be written as

$$\mathcal{F}(\boldsymbol{0}) + \mathcal{F}'(\boldsymbol{0})\boldsymbol{u} + o(\|\boldsymbol{u}\|_{\boldsymbol{X}}) = \boldsymbol{f},$$

Since $\mathcal{F}(\boldsymbol{0}) = \boldsymbol{0}$, we expect the above equation to have solution if the linear equation

$$\mathcal{F}'(\boldsymbol{0})\boldsymbol{u} = \boldsymbol{f}$$

has solutions in $\boldsymbol{X}$. But this equation is exactly the *system of equations of linear elasticity*. Hence, as we shall see, this equation has solutions in $\boldsymbol{X}$ thanks to Theorem 2.6.

In order to solve the nonlinear equation $\mathcal{F}(\boldsymbol{u}) = \boldsymbol{f}$, it is thus natural to apply the *inverse function theorem* (see, e.g., Taylor [21]). According to this theorem, if $\mathcal{F} : \boldsymbol{X} \to \boldsymbol{Y}$ is of class $\mathcal{C}^1$ and the Fréchet derivative $\mathcal{F}'(\boldsymbol{0}) : \boldsymbol{X} \to \boldsymbol{Y}$ is an isomorphism (i.e., an operator that is linear, bijective, and continuous with a continuous inverse), then there exist two open sets $U \subset \boldsymbol{X}$ and $V \subset \boldsymbol{Y}$ with $\boldsymbol{0} \in U$ and $\boldsymbol{0} = \mathcal{F}(\boldsymbol{0}) \in V$ such that, for all $\boldsymbol{f} \in V$, there exists a unique element $\boldsymbol{u} \in U$ satisfying the equation

$$\mathcal{F}(\boldsymbol{u}) = \boldsymbol{f}.$$

Furthermore, the mapping

$$\boldsymbol{f} \in V \mapsto \boldsymbol{u} \in U$$

is of class $\mathcal{C}^1$.

It remains to prove that the assumptions of the inverse function theorem are indeed satisfied. First, the function $\mathcal{F}$ is well defined (i.e., $\mathcal{F}(\boldsymbol{u}) \in \boldsymbol{Y}$ for all $\boldsymbol{u} \in \boldsymbol{X}$) since the space $W^{1,p}(\Omega)$ is an algebra (thanks to the assumption $p > 3$). Second, the function $\mathcal{F} : \boldsymbol{X} \to \boldsymbol{Y}$ is of class $\mathcal{C}^1$ since it is multilinear (in fact, $\mathcal{F}$ is even of class $\mathcal{C}^\infty$). Third, the Fréchet derivative of $\mathcal{F}$ is given by

$$\mathcal{F}'(\mathbf{0})\boldsymbol{u} = -\mathbf{div}\, \boldsymbol{\sigma},$$

where

$$\boldsymbol{\sigma} := \lambda(\mathrm{tr}\, \mathbf{e})\boldsymbol{I} + 2\mu\, \mathbf{e} \text{ and } \mathbf{e} := \frac{1}{2}(\boldsymbol{\nabla u}^T + \boldsymbol{\nabla u}),$$

from which we infer that the equation $\mathcal{F}'(\mathbf{0})\boldsymbol{u} = \boldsymbol{f}$ is exactly the *equations of linear elasticity* (see (1.6)-(1.7) with $\Gamma_0 = \partial\Omega$). Therefore, Theorem 2.6 shows that the function $\mathcal{F}'(\mathbf{0}) : \boldsymbol{X} \to \boldsymbol{Y}$ is an isomorphism

Since all the hypotheses of the inverse function theorem are satisfied, the equations of nonlinear elasticity (3.1)-(3.2) have a unique solution in the neighborhood U of the origin in $W^{2,p}(\Omega; \mathbb{R}^3)$ if $\boldsymbol{f}$ belongs to the neighborhood V of the origin in $L^p(\Omega; \mathbb{R}^3)$. In particular, if $\delta > 0$ is the radius of a ball $B(\mathbf{0}, \delta)$ contained in V, then the problem (3.1)-(3.2) has solutions for all $\|\boldsymbol{f}\|_{L^p(\Omega)} < \delta$. □

The unique solution $\boldsymbol{u}$ in the neighborhood U of the origin in $W^{2,p}(\Omega; \mathbb{R}^3)$ of the equations of nonlinear elasticity (3.1)-(3.2) depends continuously on $\boldsymbol{f}$, i.e., with self-explanatory notation

$$\boldsymbol{f}_n \to \boldsymbol{f} \text{ in } L^p(\Omega; \mathbb{R}^3) \Rightarrow \boldsymbol{u}_n \to \boldsymbol{u} \text{ in } W^{2,p}(\Omega; \mathbb{R}^3).$$

This shows that, under the assumptions of Theorem 3.1, the system of equations of nonlinear elasticity is *well-posed*.

Existence results such as Theorem 3.1 can be found in Valent [22], Marsden & Hughes [16], Ciarlet & Destuynder [7], who simultaneously and independently established the existence of solutions to the equations of nonlinear elasticity via the implicit function theorem.

3.2. **Existence of solutions by the minimization of energy.** We begin with the definition of hyperelastic materials. Recall that (see Section 1.2) an elastic material has a constitutive equation of the form

$$\boldsymbol{T}(x) := \boldsymbol{T}^\sharp(x, \boldsymbol{\nabla\Phi}(x)) \text{ for all } x \in \overline{\Omega},$$

where $\boldsymbol{T}^\sharp : \overline{\Omega} \times \mathbb{M}^3_+ \to \mathbb{M}^3$ is the response function of the material and $\boldsymbol{T}(x)$ is the first Piola-Kirchhoff stress tensor at x.

Then an elastic material is **hyperelastic** if there exists a function $W : \overline{\Omega} \times \mathbb{M}^3_+ \to \mathbb{R}$, called the **stored energy function**, such that its response function $\boldsymbol{T}^\sharp$ can be fully reconstructed from W by means of the relation

$$\boldsymbol{T}^\sharp(x, \boldsymbol{F}) = \frac{\partial W}{\partial \boldsymbol{F}}(x, \boldsymbol{F}) \ \text{ for all } (x, \boldsymbol{F}) \in \overline{\Omega} \times \mathbb{M}^3_+,$$

where $\frac{\partial W}{\partial \boldsymbol{F}}$ denotes the Fréchet derivative of W with respect to the variable $\boldsymbol{F}$. In other words, at each $x \in \overline{\Omega}$, $\frac{\partial W}{\partial \boldsymbol{F}}(x, \boldsymbol{F})$ is the unique matrix in $\mathbb{M}^3$ that satisfies

$$W(x, \boldsymbol{F} + \boldsymbol{H}) = W(x, \boldsymbol{F}) + \frac{\partial W}{\partial \boldsymbol{F}}(x, \boldsymbol{F}) : \boldsymbol{H} + o_x(|\boldsymbol{H}|)$$

for all $\boldsymbol{F} \in \mathbb{M}^3_+$ and $\boldsymbol{H} \in \mathbb{M}^3$ (a detailed study of hyperelastic materials can be found in, e.g., Ciarlet [6, Chap. 4]).

John Ball [3] has shown that the *minimization problem* formally associated with the equations of nonlinear elasticity (see (1.4)) when the material constituting the body is *hyperelastic* has solutions if the function W satisfies certain physically realistic conditions of *polyconvexity, coerciveness, and growth.* A typical example of such a function W, which is called the *stored energy function* of the material, is given by

$$W(x, \boldsymbol{F}) = a\|\boldsymbol{F}\|^p + b\|\mathbf{Cof}\,\boldsymbol{F}\|^q + c|\det \boldsymbol{F}|^r - d\log(\det \boldsymbol{F})$$

for all $\boldsymbol{F} \in \mathbb{M}^3_+$, where $p \geq 2$, $q \geq \frac{p}{p-1}$, $r > 1$, $a > 0, , b > 0, c > 0, d > 0$, and $\|\cdot\|$ is the norm defined by $\|\boldsymbol{F}\| := \{\mathrm{tr}(\boldsymbol{F}^T \boldsymbol{F})\}^{1/2}$ for all $\boldsymbol{F} \in \mathbb{M}^3$.

The major interest of hyperelastic materials is that, for such materials, *the equations of nonlinear elasticity are, at least formally, the Euler equation associated with a minimization problem* (this property only holds formally because, in general, the solution to the minimization problem does not have the regularity needed to properly establish the Euler equation associated with the minimization problem). To see this, consider first the equations of nonlinear elasticity (see Section 1.3):

$$\begin{aligned} -\mathbf{div}\, \boldsymbol{T}^\sharp(x, \boldsymbol{\nabla\Phi}(x)) &= \boldsymbol{f}(x), \quad && x \in \Omega, \\ \boldsymbol{\Phi}(x) &= x, && x \in \Gamma_0, \\ \boldsymbol{T}^\sharp(x, \boldsymbol{\nabla\Phi}(x))\boldsymbol{n}(x) &= \boldsymbol{g}(x), && x \in \Gamma_1, \end{aligned} \tag{3.3}$$

where, for simplicity, we have assumed that the applied forces do not depend on the unknown deformation $\boldsymbol{\Phi}$.

A weak solution $\boldsymbol{\Phi}$ to the boundary value problem (3.3) is then the solution to the following variational problem, also known as the **principle of virtual works**:

$$\int_\Omega \boldsymbol{T}^\sharp(\cdot, \boldsymbol{\nabla\Phi}) : \boldsymbol{\nabla v}\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx + \int_{\Gamma_1} \boldsymbol{g} \cdot \boldsymbol{v}\, da \tag{3.4}$$

for all smooth enough vector fields $\boldsymbol{v} : \overline{\Omega} \to \mathbb{R}^3$ such that $\boldsymbol{v} = \mathbf{0}$ on Γ_0.

If the material is hyperelastic, then $\boldsymbol{T}^\sharp(x, \boldsymbol{\nabla\Phi}(x)) = \frac{\partial W}{\partial \boldsymbol{F}}(x, \boldsymbol{\nabla\Phi}(x))$, and the above equation can be written as

$$J'(\boldsymbol{\Phi})\boldsymbol{v} = 0,$$

where J' is the Fréchet derivative of the functional J defined by

$$J(\boldsymbol{\Psi}) := \int_\Omega W(x, \boldsymbol{\nabla\Psi}(x))\, dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\Psi}\, dx - \int_{\Gamma_1} \boldsymbol{g} \cdot \boldsymbol{\Psi}\, da,$$

for all smooth enough vector fields $\boldsymbol{\Psi} : \overline{\Omega} \to \mathbb{R}^3$ such that $\boldsymbol{\Psi} = \mathbf{id}$ on Γ_0. The functional J is called the **total energy**.

Therefore the variational equations associated with the equations of nonlinear three-dimensional elasticity are, at least formally, the Euler equations associated with the minimization problem

$$J(\boldsymbol{\Phi}) = \min_{\boldsymbol{\Psi} \in \mathcal{M}} J(\boldsymbol{\Psi}),$$

where $\mathcal{M}$ is an appropriate set of all *admissible deformations* $\boldsymbol{\Psi} : \Omega \to \mathbb{R}^3$ (an example is given in the next theorem).

John Ball's theory provides an existence theorem for this minimization problem when the function W satisfies the following fundamental definition (see [3]): A stored energy function $W : \overline{\Omega} \times \mathbb{M}^3_+ \to \mathbb{R}$ is said to be **polyconvex** if, for each $x \in \overline{\Omega}$, there exists a *convex* function $\mathcal{W}(x, \cdot) : \mathbb{M}^3 \times \mathbb{M}^3 \times (0, \infty) \to \mathbb{R}$ such that

$$W(x, \boldsymbol{F}) = \mathcal{W}(x, \boldsymbol{F}, \mathbf{Cof}\, \boldsymbol{F}, \det \boldsymbol{F}) \text{ for all } \boldsymbol{F} \in \mathbb{M}^3_+.$$

Theorem 3.2 (John Ball)**.** *Let Ω be a domain in $\mathbb{R}^3$ and let W be a polyconvex function that satisfies the following properties:*

The function $\mathcal{W}(\cdot, \boldsymbol{F}, \boldsymbol{H}, \delta) : \Omega \to \mathbb{R}$ is measurable for all $(\boldsymbol{F}, \boldsymbol{H}, \delta) \in \mathbb{M}^3 \times \mathbb{M}^3 \times (0, \infty)$.

There exist numbers $p \geq 2$, $q \geq \frac{p}{p-1}$, $r > 1$, $\alpha > 0$, and $\beta \in \mathbb{R}$ such that

$$W(x, \boldsymbol{F}) \geq \alpha(\|\boldsymbol{F}\|^p + \|\mathbf{Cof}\, \boldsymbol{F}\|^q + |\det \boldsymbol{F}|^r) - \beta$$

for almost all $x \in \Omega$ and for all $\boldsymbol{F} \in \mathbb{M}^3_+$.

For almost all $x \in \Omega$, $W(x, \boldsymbol{F}) \to +\infty$ if $\boldsymbol{F} \in \mathbb{M}^3_+$ is such that $\det \boldsymbol{F} \to 0$.

Let Γ_1 be a relatively open subset of $\partial\Omega$, let $\Gamma_0 := \partial\Omega \setminus \Gamma_1$, and let there be given fields $\boldsymbol{f} \in L^{6/5}(\Omega; \mathbb{R}^3)$ and $\boldsymbol{g} \in L^{4/3}(\Gamma_1; \mathbb{R}^3)$. Define the functional

$$J(\boldsymbol{\Psi}) := \int_\Omega W(x, \boldsymbol{\nabla}\boldsymbol{\Psi}(x))\, dx - \int_\Omega \boldsymbol{f}(x) \cdot \boldsymbol{\Psi}(x)\, dx - \int_{\Gamma_1} \boldsymbol{g}(x) \cdot \boldsymbol{\Psi}(x)\, da,$$

and the set

$$\mathcal{M} := \{\boldsymbol{\Psi} \in W^{1,p}(\Omega; \mathbb{R}^3);\ \mathbf{Cof}(\boldsymbol{\nabla}\boldsymbol{\Psi}) \in L^q(\Omega; \mathbb{M}^3),\ \det(\boldsymbol{\nabla}\boldsymbol{\Psi}) \in L^r((\Omega),$$
$$\det(\boldsymbol{\nabla}\boldsymbol{\Psi}) > 0 \text{ a.e. in } \Omega,\ \boldsymbol{\Psi} = \mathbf{id} \text{ on } \Gamma_0\}.$$

Finally, assume that $\operatorname{area} \Gamma_0 > 0$ *and that* $\inf_{\boldsymbol{\Psi} \in \mathcal{M}} J(\boldsymbol{\Psi}) < \infty$.

Then there exists $\boldsymbol{\Phi} \in \mathcal{M}$ such that

$$J(\boldsymbol{\Phi}) = \inf_{\boldsymbol{\Psi} \in \mathcal{M}} J(\boldsymbol{\Psi}).$$

Sketch of proof (see Ball [3], or Ciarlet [6], for a detailed proof). Let $\boldsymbol{\Phi}_n$ be a infimizing sequence of the functional J, i.e., a sequence of vector fields $\boldsymbol{\Phi}_n \in \mathcal{M}$ such that

$$J(\boldsymbol{\Phi}_n) \to \inf_{\boldsymbol{\Psi} \in \mathcal{M}} J(\boldsymbol{\Psi}) < \infty.$$

The coerciveness assumption on W implies that the sequences

$$(\boldsymbol{\Phi}_n), (\mathbf{Cof}(\boldsymbol{\nabla}\boldsymbol{\Phi}_n)) \text{ and } (\det(\boldsymbol{\nabla}\boldsymbol{\Phi}_n))$$

are bounded respectively in the spaces $W^{1,p}(\Omega;\mathbb{R}^3)$, $L^q(\Omega;\mathbb{M}^3)$, and $L^r(\Omega)$. Since these spaces are reflexive, there exist subsequences

$$(\boldsymbol{\Phi}_{\sigma(n)}), (\mathbf{Cof}(\boldsymbol{\nabla}\boldsymbol{\Phi}_{\sigma(n)})) \text{ and } (\det(\boldsymbol{\nabla}\boldsymbol{\Phi}_{\sigma(n)}))$$

such that ($\rightharpoonup$ denotes weak convergence)

$$\begin{aligned}
\boldsymbol{\Phi}_{\sigma(n)} &\rightharpoonup \boldsymbol{\Phi} && \text{in } W^{1,p}(\Omega;\mathbb{R}^3),\\
\boldsymbol{H}_{\sigma(n)} := \mathbf{Cof}(\boldsymbol{\nabla}\boldsymbol{\Phi}_{\sigma(n)}) &\rightharpoonup \boldsymbol{H} && \text{in } L^q(\Omega;\mathbb{M}^3),\\
\delta_{\sigma(n)} := \det(\boldsymbol{\nabla}\boldsymbol{\Phi}_{\sigma(n)}) &\rightharpoonup \delta && \text{in } L^r(\Omega).
\end{aligned}$$

For all $\boldsymbol{\Phi} \in W^{1,p}(\Omega;\mathbb{R}^3)$, $\boldsymbol{H} \in L^q(\Omega;\mathbb{M}^3)$, and $\delta \in L^r(\Omega)$ with $\delta > 0$ almost everywhere in Ω, define the functional

$$\begin{aligned}
\mathcal{J}(\boldsymbol{\Phi}, \boldsymbol{H}, \delta) := \int_\Omega \mathcal{W}(x, \boldsymbol{\nabla}\boldsymbol{\Phi}(x), \boldsymbol{H}(x), \delta(x))\,dx \\
- \int_\Omega \boldsymbol{f}(x)\cdot\boldsymbol{\Phi}(x)\,dx - \int_{\Gamma_1} \boldsymbol{g}(x)\cdot\boldsymbol{\Phi}(x)\,da,
\end{aligned}$$

where, for each $x \in \Omega$, $\mathcal{W}(x,\cdot) : \mathbb{M}^3 \times \mathbb{M}^3 \times (0,\infty) \to \mathbb{R}$ is the function given by the polyconvexity assumption on W. Since $\mathcal{W}(x,\cdot)$ is convex, the above weak convergences imply that

$$\mathcal{J}(\boldsymbol{\Phi}, \boldsymbol{H}, \delta) \le \liminf_{n\to\infty} \mathcal{J}(\boldsymbol{\Phi}_{\sigma(n)}, \boldsymbol{H}_{\sigma(n)}, \delta_{\sigma(n)}).$$

But $\mathcal{J}(\boldsymbol{\Phi}_{\sigma(n)}, \boldsymbol{H}_{\sigma(n)}, \delta_{\sigma(n)}) = J(\boldsymbol{\Phi}_{\sigma(n)})$ and $J(\boldsymbol{\Phi}_n) \to \inf_{\boldsymbol{\Psi}\in\mathcal{M}} J(\boldsymbol{\Psi})$. Therefore $\mathcal{J}(\boldsymbol{\Phi}, \boldsymbol{H}, \delta) = \inf_{\boldsymbol{\Psi}\in\mathcal{M}} J(\boldsymbol{\Psi})$.

A compactness by compensation argument applied to the weak convergences above then shows that

$$\boldsymbol{H} = \mathbf{Cof}(\boldsymbol{\nabla}\boldsymbol{\Phi}) \text{ and } \delta = \det(\boldsymbol{\nabla}\boldsymbol{\Phi}).$$

Hence $J(\boldsymbol{\Phi}) = \mathcal{J}(\boldsymbol{\Phi}, \boldsymbol{H}, \delta)$.

It remains to prove that $\boldsymbol{\Phi} \in \mathcal{M}$. The property that $W(\boldsymbol{F}) \to +\infty$ if $\boldsymbol{F} \in \mathbb{M}_+$ is such that $\det \boldsymbol{F} \to 0$, then implies that $\det(\boldsymbol{\nabla}\boldsymbol{\Phi}) > 0$ a.e. in Ω. Finally, since $\boldsymbol{\Phi}_n \rightharpoonup \boldsymbol{\Phi}$ in $W^{1,p}(\Omega;\mathbb{R}^3)$ and since the trace operator is linear, it follows that $\boldsymbol{\Phi} = \mathbf{id}$ on Γ_0. Hence $\boldsymbol{\Phi} \in \mathcal{M}$.

Since $J(\boldsymbol{\Phi}) = \mathcal{J}(\boldsymbol{\Phi}, \boldsymbol{H}, \delta) = \inf_{\boldsymbol{\Psi}\in\mathcal{M}} J(\boldsymbol{\Psi})$, the weak limit $\boldsymbol{\Phi}$ of the sequence $\boldsymbol{\Phi}_{\sigma(n)}$ satisfies the conditions of the theorem.

□

A St Venant-Kirchhoff material with Lamé constants $\lambda > 0$ and $\mu > 0$ is hyperelastic, but not polyconvex. However, Ciarlet & Geymonat [8] have shown that the stored energy function of a St Venant-Kirchhoff material, which is given by

$$W(\boldsymbol{F}) = \frac{\lambda}{8}(\mathrm{tr}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}))^2 + \frac{\mu}{4}\|\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\|^2,$$

can be "approximated" with polyconvex stored energy functions in the following sense: There exists polyconvex stored energy functions of the form

$$W^\flat(\boldsymbol{F}) = a\|\boldsymbol{F}\|^2 + b\|\mathbf{Cof}\,\boldsymbol{F}\|^2 + c|\det \boldsymbol{F}|^2 - d\log(\det \boldsymbol{F}) + e$$

with $a > 0$, $b > 0$, $c > 0$, $d > 0$, $e \in \mathbb{R}$, that satisfy

$$W^\flat(\boldsymbol{F}) = W(\boldsymbol{F}) + \mathcal{O}(\|\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\|^3).$$

A stored energy function of this form possesses all the properties required for applying Theorem 3.2. In particular, it satisfies the coerciveness inequality:

$$W^\flat(\boldsymbol{F}) \geq \alpha(\|\boldsymbol{F}\|^2 + \|\mathbf{Cof}\,\boldsymbol{F}\|^2 + (\det \boldsymbol{F})^2) + \beta, \text{ with } \alpha > 0 \text{ and } \beta \in \mathbb{R}.$$

References

[1] Adams, R.A.: *Sobolev Spaces*, Academic Press, New York, 1975.

[2] Amrouche, C.; Girault, V. : Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension, *Czech. Math. J.* 44 (1994), 109-140.

[3] Ball, J.: Convexity conditions and existence theorems in nonlinear elasticity, *Arch. Rational Mech. Anal.* 63 (1977), 337-403.

[4] Bolley, P.; Camus, J. : Régularité pour une classe de problèmes aux limites elliptiques dégénérés variationnels, *C.R. Acad. Sci. Paris, Sér. A*, 282 (1976), 45-47.

[5] Borchers, W.; Sohr, H. : On the equations $\operatorname{rot} v = g$ and $\operatorname{div} u = f$ with zero boundary conditions, *Hokkaido Math. J.* 19 (1990), 67-87.

[6] Ciarlet, P.G. : *Mathematical Elasticity, Volume I: Three-Dimensional Elasticity*, North-Holland, Amsterdam, 1988.

[7] Ciarlet, P.G.; Destuynder, P.: A justification of the two-dimensional plate model, *J. Mécanique* 18 (1979), 315-344.

[8] Ciarlet, P.G.; Geymonat, G : Sur les lois de comportement en élasticité non-linéaire compressible, *C.R. Acad. Sci. Paris*, Ser. II, 295 (1982), 423-426.

[9] Ciarlet, P.G.; Mardare, C. : An Introduction to Shell Theory, in Differential Geometry: Theory and Applications, Philippe G. Ciarlet & Ta-Tsien Li, Editors, pp. 94-184, Series in Contemporary Applied Mathematics, Vol. 9, Higher Education Press & World Scientific, New Jersey, 2008.

[10] Duvaut, G.; Lions, J.L. : *Les Inéquations en Mécanique et en Physique*, Dunod, Paris, 1972 (English translation: *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin, 1976).

[11] Evans L.C.; Gariepy R.F.: *Measure Theory and Fine Properties of Functions*, CRC Press, 1992.

[12] Geymonat, G.; Suquet, P. : Functional spaces for Norton-Hoff materials, *Math. Methods Appl. Sci.* 8 (1986), 206-222.

[13] Grisvard P.: *Elliptic Problems in Nonsmooth Domains,* Pitman, Boston, 1985.

[14] Gurtin M.E.; Martins L.C.: Cauchy's theorem in classical physics, *Arch. Rational Mech. Anal.* 60 (1976), 305-328.

[15] Magenes, E.; Stampacchia, G. : I problemi al contorno per le equazioni differenziali di tipo ellittico, *Ann. Scuola Norm. Sup. Pisa* 12 (1958), 247-358.

[16] Marsden,J.E.; Hughes, T.J.R.: *Mathematical Foundations of Elasticity*, Prentice-Hall, Englewood Cliffs, 1983.

[17] Nečas, J. : *Les Méthodes Directes en Théorie des Equations Elliptiques*, Masson, Paris, 1967.

[18] Rivlin R.S.; Ericksen J.L.: Stress-deformation relations for isotropic materials, *Arch. rational Mech. Anal.* 4 (1955), 323-425.

[19] Schwartz, L. : *Théorie des Distributions*, Hermann, Paris, 1966.

[20] Tartar, L.: *Topics in Nonlinear Analysis*, Publications Mathématiques d'Orsay No. 78.13, Université de Paris-Sud, Orsay, 1978.

[21] Taylor M.E.: *Partial Differential Equations I: Basic Theory*, Springer, Berlin, 1996.

[22] Valent, T. : *Boundary Value Problems of Finite Elasticity*, Springer Tracts in Natural Philosophy, Vol. 31, Springer-Verlag, Berlin. 1988.

[23] Yosida K.: *Functional Analysis*, Springer, Berlin, 1978.