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The analysis of coupled models

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The important feature one rapidly realizes when starting the analysis of the behavior of the blood in large or smaller arteries is the fact that two types of phenomenon coexist : the blood flow, whatever model has to be used to represent it and the artery wall displacement, with or without the tissues or muscles that surround the wall. This feature is a new one and it exists independently of how complicated and complete the representations of each independent phenomenon is to be chosen. The model that comes out to represent the fluide-structure interaction enters in the coupled problem category that requires that, first, you know well each model independently and, also, that you know how to treat the way they interact.

Mathematically, the resulting equations are nonlinear, already because any realistic model for the fluid is nonlinear, but the interaction adds a nonlinearity on the top of whatever anterior nonlinearity of the primitive independent model.

This chapter aims at sheding some lights on the mathematical tools that have been adapted or invented to understand if the equations stemming from the model designer provide a mathematically well posed problem. There are many reasons to justify this type of analysis, not including the fact that from the beginning this subject has provided more than 20 research papers, most of them in good journals, some of them being referred to in this chapter. But the main reason is because the various numerical problem that appear at the discretization level for the simulation of such strongly coupled problems are better hinted and understood from the theoretical understanding synthesized in this chapter. Another side result is to guide the intuition on what should be modified if one wants to change the fluid model or the structure model at the level of the interface where the exchange of information take place.

In this contribution we consider mainly the case where whole (resp. part) of the external fluid boundary is composed of an elastic structure (resp. the remaining part being solid or fixed). This means that we shall not integrate in this chapter the many contributions dealing with the interaction of elastic structure floating within a fluid flow itself placed in a rigid container since this is quite far from the scope of this book dealing with flow in elastic vessels.

1.1 A simplified model with half of the interaction

The first model we consider dates already from almost 40 years now and can be found in the book of Jacques-Louis Lions [7]. It takes only part of the interaction into account since both the interface and the fluid domain are assumed to be fixed. This simplifies quite a lot the analysis that is nevertheless not immediate. This model could represent the fluid structure interaction in case one neglects the movement of the interface, the following analysis is directly extracted from [7] and serves as an introduction to the methods that will be used latter in the context of the full interaction problem.

1.1.1 A transmission parabolic-hyperbolic problem

Two domains of $\mathbb{R}^n$ are involved: Ω^f where the fluid flows and Ω^s representing the elastic envelop as described on figure 1.1.

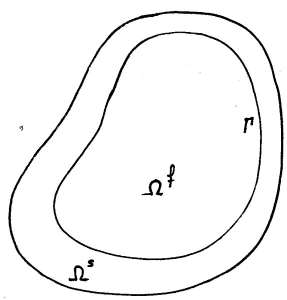

Fig. 1.1. The two domains for fluid-structure transmission.

Here we do not indicate the time dependency nor the "*hat*" of the reference configuration, since, in this model, these domains are fixed. The transmission problem involves a velocity $\boldsymbol{u}_f = (u_1, u_2, u_3)$ and a pressure p defined over $\Omega^f \times (0,T)$ such that

$$\frac{\partial \boldsymbol{u}_f}{\partial t} - 2\,\mathbf{div}(\nu \boldsymbol{D}\boldsymbol{u}_f) + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}_f + \nabla p = \boldsymbol{f}, \quad \text{in } \Omega^f \times (0,T) \tag{1.1}$$

$$\mathbf{div}\,\boldsymbol{u}_f = 0, \quad \text{in } \Omega^f \times (0,T) \tag{1.2}$$

and a displacement $\boldsymbol{\eta}$ over $\Omega^s \times (0,T)$ such that

$$\frac{\partial^2 \boldsymbol{\eta}}{\partial t^2} - \Delta \boldsymbol{\eta} = g, \quad \text{in } \Omega^s \times (0,T). \tag{1.3}$$

These equations are complemented with the transmission conditions over the interface Γ between Ω^f and Ω^s namely

$$\boldsymbol{u}_f = \frac{\partial \boldsymbol{\eta}}{\partial t}, \quad \text{over } \Gamma \times (0,T), \tag{1.4}$$

and

$$2\nu \boldsymbol{D}\boldsymbol{u}_f.\boldsymbol{n} - p\boldsymbol{n} - \frac{1}{2}(\sum_{j=1}^{n} u_j \cos n_j)\boldsymbol{u}_f = \frac{\partial \boldsymbol{\eta}}{\partial \boldsymbol{n}}, \quad \text{over } \Gamma \times (0,T), \tag{1.5}$$

together with the boundary condition over the remaining boundary of Ω^s:

$$\boldsymbol{\eta} = 0, \quad \text{over } [\partial\Omega^s \setminus \Gamma] \times (0,T), \tag{1.6}$$

and the initial conditions

$$\begin{cases} \boldsymbol{u}_f(0) = u_0 & & \text{over } \Omega^f, \\ \boldsymbol{\eta}(0) = 0, & \boldsymbol{\eta}'(0) = \boldsymbol{\eta}_1 & \text{over } \Omega^s. \end{cases} \tag{1.7}$$

It should be noted that the interface between the fluid and the structure does not follow the movement of the structure represented by $\boldsymbol{\eta}$, this represents a major simplification but corresponds fairly well to situations where the displacements we expect are very small and are thus neglected in the modeling of the fluid behavior. Note that this simplification imposes that in (1.5) a nonlinear contribution needs to be added to the normal stress equilibrium (see subsection 1.2.3 for a further comment in the full interaction case). Despite of this simplification, the coupling is still a strong one since neither the fluid flow nor the elastic displacement can be evaluated regardless of the other one. In what follows, we shall present the mathematical analysis of this problem resulting in the

Theorem 1.1. *There exists a unique solution $(\boldsymbol{u}_f, p, \boldsymbol{\eta})$ to the transmission problem*

1.1.2 Frame settings for the analysis of the transmission problem

The analysis starts by introducing the structure velocity $\Phi = \frac{\partial \boldsymbol{\eta}}{\partial t}$ which, from (1.3), satisfies

$$\frac{\partial \Phi}{\partial t} - \Delta\Big(\int_0^t \Phi(s)ds\Big) = g + \Delta\boldsymbol{\eta}_0, \quad \in \Omega^s; \tag{1.8}$$

then, we formulate the problem in a equivalent variational settings : Find $(\boldsymbol{u}_f, \Phi)$ with

$$\begin{aligned} &\boldsymbol{u}_f \in L^2(0,T;V_f) \cap L^\infty(0,T;[L^2(\Omega^f)]^n), \\ &\Phi \in L^\infty(0,T;[L^2(\Omega^s)]^n), \int_0^t \Phi(s)ds \in L^\infty(0,T;V_s), \\ &\boldsymbol{u}_f = \Phi \text{ over } \Gamma \times (0,T), \end{aligned}$$

where the two spaces V_f and V_s are defined as

$$V_f = \{\boldsymbol{v}_f | \boldsymbol{v}_f \in [H^1(\Omega^f)]^n, \operatorname{div} \boldsymbol{v}_f = 0\}, \tag{1.9}$$
$$V_s = \{\varphi | \varphi \in [H^1(\Omega^s)]^n, \varphi = 0 \text{ over } \Gamma\},$$

such that, $\forall \boldsymbol{v}_f \in V_F, \forall \varphi \in V_s$, with $\boldsymbol{v}_f = \varphi$ over Γ

$$\begin{aligned}\int_{\Omega^f} \frac{\partial \boldsymbol{u}_f}{\partial t} \boldsymbol{v}_f d\boldsymbol{x} + \int_{\Omega^s} \frac{\partial \Phi}{\partial t} \varphi d\boldsymbol{x} + \nu a_f(\boldsymbol{u}_f, \boldsymbol{v}_f) + a_s(\int_0^t \Phi(s) ds, \varphi) \\ + b_f(\boldsymbol{u}_f, \boldsymbol{u}_f; \boldsymbol{v}_f) - \frac{1}{2} \sum_{i,j=1}^n \int_\Gamma u_i u_j v_j \cos n_i d\gamma \\ = \int_{\Omega^f} f \boldsymbol{v}_f + \int_{\Omega^s} g\varphi + a_s(\boldsymbol{\eta}_0, \varphi).\end{aligned} \tag{1.10}$$

where

$$a_f(\boldsymbol{u}_f, \boldsymbol{v}_f) = \int_{\Omega^f} \boldsymbol{D}\boldsymbol{u}_f : \boldsymbol{D}\boldsymbol{v}_f, \tag{1.11}$$

$$b_f(\boldsymbol{u}_f, \boldsymbol{v}_f, \boldsymbol{w}_f) = \sum_{i,j=1}^n \int_{\Omega^f} u_i \frac{\partial v_j}{\partial x_i} w_j, \tag{1.12}$$

$$a_s(\boldsymbol{u}_f, \boldsymbol{v}_f) = \sum_{i,j=1}^n \int_{\Omega^s} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}. \tag{1.13}$$

These equations are complemented with the initial conditions (1.7).

In order to check the equivalence of the two formulations (the strong one : (1.1)–(1.7) and the weak one (1.10)), we first notice that the direction strong →weak is obvious by integration by parts, then, assuming that (1.10) holds, we choose in (1.10) $\boldsymbol{v}_f$ with compact support into Ω^f together with $\varphi = 0$ which leads already to equation (1.1) in the sense of distributions, then we choose in (1.10) φ with compact support into Ω^s together with $\boldsymbol{v}_f = 0$ and we get equation (1.8) again in the sense of the distributions. Multiplying now the two equations (1.1) and (1.8) that we have just derived by $\boldsymbol{v}_f$ and φ integrating and taking into account that $\boldsymbol{v}_f = \varphi$ over Γ, we get $\forall \boldsymbol{v}_f \in V_F, \forall \varphi \in V_s$

$$\begin{aligned}\int_{\Omega^f} \frac{\partial \boldsymbol{u}_f}{\partial t} \boldsymbol{v}_f d\boldsymbol{x} + \int_{\Omega^s} \frac{\partial \Phi}{\partial t} \varphi d\boldsymbol{x} + \nu a_f(\boldsymbol{u}_f, \boldsymbol{v}_f) + a_s(\int_0^t \Phi(s) ds, \varphi) \\ + b_f(\boldsymbol{u}_f, \boldsymbol{u}_f; \boldsymbol{v}_f) + \int_\Gamma \Big(-2\nu \boldsymbol{D}\boldsymbol{u}_f.\boldsymbol{n} + p\boldsymbol{n} + \frac{\partial}{\partial \boldsymbol{n}}(\int_0^t \Phi(s) ds)\Big) \boldsymbol{v}_f d\Gamma \\ = \int_{\Omega^f} f \boldsymbol{v}_f + \int_{\Omega^s} g\varphi + a_s(\boldsymbol{\eta}_0, \varphi) - \int_\Gamma \frac{\partial \boldsymbol{\eta}_0}{\partial \boldsymbol{n}} \varphi d\Gamma,\end{aligned}$$

so that recalling (1.10), we derive $\forall \boldsymbol{v}_f \in V_F$:

$$\int_\Gamma \Big(-2\nu \boldsymbol{D}\boldsymbol{u}_f.\boldsymbol{n} + p\boldsymbol{n} + \frac{\partial}{\partial \boldsymbol{n}}\big(\int_0^t \Phi(s)ds + \boldsymbol{\eta}_0\big) + \frac{1}{2}\big(\sum_{j=1}^n u_j.\cos n_j\big)\boldsymbol{u}_f\Big)\boldsymbol{v}_f d\Gamma = 0 \tag{1.14}$$

Since $\operatorname{div} \boldsymbol{v}_f = 0$, we notice $\int_\gamma \boldsymbol{v}_f.\boldsymbol{n}d\gamma = 0$, and reciprocally, to any $\boldsymbol{v}_{f*} \in \big(H^{1/2}(\Gamma)\big)^n$ with $\int_\gamma \boldsymbol{v}_{f*}.\boldsymbol{n}d\gamma = 0$ we can associate a $\boldsymbol{v}_f \in V_f$ with $\boldsymbol{v}_{f|\Gamma} = \boldsymbol{v}_{f*}$, hence (1.14) is equivalent to

$$-2\nu \boldsymbol{D}\boldsymbol{u}_f.\boldsymbol{n} + p\boldsymbol{n} + \frac{\partial}{\partial \boldsymbol{n}}\big(\int_0^t \Phi(s)ds + \boldsymbol{\eta}_0\big) + \frac{1}{2}\big(\sum_{j=1}^n u_j.\cos n_j\big)\boldsymbol{u}_f = \lambda \boldsymbol{n}$$

with $\lambda \in \mathbb{R}$ finally by changing p into $p - \lambda$ we get (1.5).

1.1.3 Analysis of the transmission problem

Approximated solutions

We turn now to the proof of Theorem 1.1. The classical tool that will be introduced here and used over and over in the following sections is known as the Faedo Galerkin method where, by restricting the trial and test functional spaces to a sequence of finite dimensional subspaces, we define a series of systems of differential equations for which the existence of a discrete solution follows from Cauchy Lipschitz's theorem. Different a priori estimates are then derived and provide uniform bounds on the sequences of solutions. This is the corner stone of the proof that allows to state that there exists a subsequence of discrete solutions that converges to a solution of (1.10) when the dimension of the finite dimensional subspaces tends to infinity.

To be more precise, let us introduce a "special basis" of eigenfunctions in W^σ

$$W^\sigma = \{\boldsymbol{v}|\boldsymbol{v} \in (H_0^\sigma(\Omega))^n, \Omega = \overline{\Omega}^f \cup \Omega^s, \operatorname{div} \boldsymbol{v} = 0 \text{ over } \Omega^f\}$$

with $\sigma = n/2$ and scalar product $(.,.)_\sigma$. These are solutions of the following problem

$$(\boldsymbol{w}_j, \boldsymbol{v})_{W^\sigma} = \lambda_j(\boldsymbol{w}_j, \boldsymbol{v}) = \lambda_j \int_\Omega \boldsymbol{w}_j \boldsymbol{v} d\boldsymbol{x} \tag{1.15}$$

(this spectral problem possesses a sequence of eigen-solutions corresponding to an increasing sequence of positive eigenvalues λ_i; they constitute a complete set of orthogonal functions in both $L^2(\Omega)$ and W^σ). With these eigenfunctions, we define the basis $(\boldsymbol{v}_j, \varphi_j)_j$ with $\boldsymbol{v}_j = \boldsymbol{w}_{j|\Omega^f}$ and $\varphi_j = \boldsymbol{w}_{j|\Omega^s}$ and we introduce the discrete subspaces of $V_f \times V_s$

$$W_n = \text{Span}\{(\boldsymbol{v}_j, \varphi_j), j = 1, .., n\}.$$

and, for any m, we then define the discrete solutions $(\boldsymbol{u}_{f_m}, \Phi_m) \in W_m$ of

$$\int_{\Omega^f} \frac{\partial \boldsymbol{u}_{f\,m}}{\partial t} \boldsymbol{v}_f d\boldsymbol{x} + \int_{\Omega^s} \frac{\partial \Phi_m}{\partial t} \varphi d\boldsymbol{x} + \nu a_f(\boldsymbol{u}_{f\,m}, \boldsymbol{v}_f) + a_s(\int_0^t \Phi_m(s)ds, \varphi)$$
$$+ b_f(\boldsymbol{u}_{f\,m}, \boldsymbol{u}_{f\,m}; \boldsymbol{v}_f) - \frac{1}{2} \sum_{i,j=1}^{n} \int_\Gamma u_{m,i} u_{m,j} v_{m,j} \cos n_i d\gamma \tag{1.16}$$
$$= \int_{\Omega^f} f \boldsymbol{v}_f + \int_{\Omega^s} g\varphi + a_s(\boldsymbol{\eta}_0, \varphi), \quad \forall(\boldsymbol{v}_f, \varphi) \in W_m$$

supplemented again with the initial conditions

$$\boldsymbol{u}_{f\,m}(0) = \boldsymbol{u}_{m0} \to \boldsymbol{u}_0 \text{ and } \Phi_m(0) = \boldsymbol{\eta}_{1m} \to \boldsymbol{\eta}_1 \text{ when } m \to \infty$$

these last convergences being in the L^2 sense.

By choosing alternatively $(\boldsymbol{v}_f, \varphi) = (\boldsymbol{v}_j, \varphi_j), j = 1, .., M$, the previous set of equations yields in a system of nonlinear differential equations in the components $g_{jm}(t)$ with $(\boldsymbol{u}_{f\,m}, \Phi_m) = \sum_{j=1}^m g_{jm}(t)(\boldsymbol{v}_j, \varphi_j)$ that possesses a unique solution at least over a time interval $(0, t_m)$.

Derivation of *a priori* estimates

We choose now $(\boldsymbol{v}_f, \varphi) = (\boldsymbol{u}_{f\,m}, \Phi_m)$ in (1.16) which yields

$$\frac{\partial}{\partial t} \int_{\Omega^f} |\boldsymbol{u}_{f\,m}(.,t)|^2 d\boldsymbol{x} + \frac{\partial}{\partial t} \int_{\Omega^s} |\Phi_m(.,t)|^2 d\boldsymbol{x} + 2\nu a_f(\boldsymbol{u}_{f\,m}, \boldsymbol{u}_{f\,m})$$
$$+ \frac{\partial}{\partial t} a_s(\int_0^t \Phi_m(.,s)ds, \int_0^t \Phi_m(.,s)ds) + 2b_f(\boldsymbol{u}_{f\,m}, \boldsymbol{u}_{f\,m}; \boldsymbol{u}_{f\,m}) \tag{1.17}$$
$$- \sum_{i,j=1}^{n} \int_\Gamma u_{m,i} u_{m,j}^2 \cos n_i d\gamma = 2\int_{\Omega^f} f \boldsymbol{u}_{f\,m} + 2\int_{\Omega^s} g\Phi_m + 2a_s(\boldsymbol{\eta}_0, \Phi_m)$$

we notice that, from the divergence free condition over $\boldsymbol{u}_f$ and integration by parts

$$2b_f(\boldsymbol{u}_{f\,m}, \boldsymbol{u}_{f\,m}; \boldsymbol{u}_{f\,m}) = \sum_{i,j=1}^{n} \int_\Gamma u_{m,i} u_{m,j}^2 \cos n_i d\gamma$$

hence from Cauchy Schwarz inequality applied to the right hand side of (1.17) we deduce

$$\frac{\partial}{\partial t} \int_{\Omega^f} |\boldsymbol{u}_{f\,m}(.,t)|^2 d\boldsymbol{x} + \frac{\partial}{\partial t} \int_{\Omega^s} |\Phi_m(.,t)|^2 d\boldsymbol{x} + \nu a_f(\boldsymbol{u}_{f\,m}, \boldsymbol{u}_{f\,m})$$
$$+ \frac{\partial}{\partial t} a_s(\int_0^t \Phi_m(.,s)ds, \int_0^t \Phi_m(.,s)ds) \tag{1.18}$$
$$\leq C[\int_{\Omega^f} f^2 + \int_{\Omega^s} g^2 + a_s(\boldsymbol{\eta}_0, \boldsymbol{\eta}_0)]$$

and integration in times, first tells that $t_m \equiv T$ and

$$\begin{cases} \boldsymbol{u}_{f\,m} \text{ is a bounded sequence in } L^2(0,T;V_f) \cap L^\infty(0,T;(L^2(\Omega^f))^n), \\ \Phi_m \text{ is a bounded sequence in } L^\infty(0,T;L^2(\Omega^s))^n), \\ \int_0^t \Phi_m(.,s)ds \text{ is a bounded sequence in } L^\infty(0,T;V^s). \end{cases} \tag{1.19}$$

In order to be able to pass to the limit in the nonlinear term, we need further bound for derivatives in time. Let P_m denote the projection operator from $L^2(\Omega)$ onto W_n. This operator is self-adjoint and bounded with norm equal to 1 both in $L^2(\Omega)$ and in W^σ due to the use of the special basis, it results, by transposition, that

$$\|P_m\|_{\mathcal{L}(W^{\sigma\prime},W^{\sigma\prime})} \le 1$$

We now propose another way of writing (1.16) through operators

$$\frac{\partial}{\partial t}(\boldsymbol{u}_{f\,m},\Phi_m) = -P_m[A(\boldsymbol{u}_{f\,m},\Phi_m)] - P_m[G(\boldsymbol{u}_{f\,m},\Phi_m)] - P_m[F] \tag{1.20}$$

where $\forall(\boldsymbol{v}_f,\varphi) \in W^\sigma$

$$\begin{cases} < A((\boldsymbol{u}_{f\,m},\Phi_m),(\boldsymbol{v}_f,\varphi) >= \nu a_f(\boldsymbol{u}_{f\,m},\boldsymbol{v}_f) + a_s(\int_0^t \Phi_m(s)ds,\varphi) \\ < G((\boldsymbol{u}_{f\,m},\Phi_m),(\boldsymbol{v}_f,\varphi) >= b_f(\boldsymbol{u}_{f\,m},\boldsymbol{u}_{f\,m};\boldsymbol{v}_f) \\ \qquad -\frac{1}{2}\sum_{i,j=1}^n \int_\Gamma u_{m,i}u_{m,j}v_j \cos n_i d\gamma \\ < F(\boldsymbol{v}_f,\varphi) = \int_{\Omega^f} f\boldsymbol{v}_f + \int_{\Omega^s} g\varphi + a_s(\boldsymbol{\eta}_0,\varphi) \end{cases} \tag{1.21}$$

It is straightforward to note that A is bounded in $\mathcal{L}(W^\sigma, W^{\sigma\prime})$, that F belongs to $W^{\sigma\prime}$ and the following lemma states that G is continuous from W^σ into $W^{\sigma\prime}$:

Lemma 1.1. *There exists a constant c such that for any $\boldsymbol{v}_f \in H^\sigma(\Omega^f)$*

$$|b_f(\boldsymbol{u}_{f\,m},\boldsymbol{u}_{f\,m};\boldsymbol{v}_f)| \le c\|\boldsymbol{u}_{f\,m}\|_{L^2(\Omega^f)}\|\boldsymbol{u}_{f\,m}\|_{H^1(\Omega^f)}\|\boldsymbol{v}_f\|_{H^\sigma(\Omega^f)}$$

$$|\sum_{i,j=1}^n \int_\Gamma u_{m,i}u_{m,j}v_j \cos n_i d\gamma| \le c\|\boldsymbol{u}_{f\,m}\|_{L^2(\Omega^f)}\|\boldsymbol{u}_{f\,m}\|_{H^1(\Omega^f)}\|\boldsymbol{v}_f\|_{H^\sigma(\Omega^f)}$$

that follows from the equality

$$\sum_{i,j=1}^n \int_\Gamma u_{m,i}u_{m,j}v_j \cos n_i d\gamma = b_f(\boldsymbol{u}_{f\,m},\boldsymbol{u}_{f\,m};\boldsymbol{v}_f) + b_f(\boldsymbol{u}_{f\,m},\boldsymbol{v}_f;\boldsymbol{u}_{f\,m})$$

and the bound

$$\|v\|^2_{L^4(\Omega^f)} \le c\|v\|^{1/2}_{L^2(\Omega^f)}\|v\|^{3/2}_{H^1(\Omega^f)}$$

valid in 2D and

$$\|v\|^2_{L^3(\Omega^f)} \le c\|v\|^{1/2}_{L^2(\Omega^f)}\|v\|^{3/2}_{H^1(\Omega^f)}$$

valid in 3D, together with the imbedding of $H^\sigma(\Omega^f)$ into any $L^p(\Omega^f)$ (choose $p = 4$, if $n = 2$ and $p = 6$ if $n = 3$). We deduce from (1.20), (1.21) and Lemma 1.1 that

$$\left(\frac{\partial \boldsymbol{u}_{f\,m}}{\partial t},\frac{\partial \Phi_m}{\partial t}\right) \text{ is a bounded sequence in } L^{4/3}(0,T;W^{\sigma\prime}), \tag{1.22}$$

Passing to the limit

From the boundedness of the sequences $(\boldsymbol{u}_{f\,m}, \Phi_m)$ we know we can extract a subsequence (still indexed by m) that converges weakly

$$\begin{cases} \boldsymbol{u}_{f\,m} \rightharpoonup \boldsymbol{u}_f \text{ in } L^2(0,T;V_f) \cap L^\infty(0,T;(L^2(\Omega^f))^n), \\ \Phi_m \rightharpoonup \Phi \text{ in } L^\infty(0,T;L^2(\Omega^s))^n), \\ \int_0^t \Phi_m(.,s)ds \rightharpoonup \int_0^t \Phi(.,s)ds \text{ in } L^\infty(0,T;V^s) \\ (\frac{\partial \boldsymbol{u}_{f\,m}}{\partial t}, \frac{\partial \Phi_m}{\partial t}) \rightharpoonup (\frac{\partial \boldsymbol{u}_f}{\partial t}, \frac{\partial \Phi}{\partial t}) \text{ in } L^{4/3}(0,T;W^{\sigma\prime}) \end{cases} \tag{1.23}$$

these weak limits allow already to pass to the limit in the linear terms, for any $(\boldsymbol{v}_f, \varphi) \in \cup_{n\in I\!N} W_n$,

$$< A((\boldsymbol{u}_{f\,m}, \Phi_m), (\boldsymbol{v}_f, \varphi) > \to < A((\boldsymbol{u}_f, \Phi), (\boldsymbol{v}_f, \varphi) >$$

and

$$\int_{\Omega^f} \frac{\partial \boldsymbol{u}_{f\,m}}{\partial t} \boldsymbol{v}_f d\boldsymbol{x} + \int_{\Omega^s} \frac{\partial \Phi_m}{\partial t} \varphi d\boldsymbol{x} \to \int_{\Omega^f} \frac{\partial \boldsymbol{u}_f}{\partial t} \boldsymbol{v}_f d\boldsymbol{x} + \int_{\Omega^s} \frac{\partial \Phi}{\partial t} \varphi d\boldsymbol{x}$$

The remaining nonlinear terms require the fundamental lemma: let B_0, B and B_1 be three reflexive Banach spaces with

$$B_0 \hookrightarrow B \hookrightarrow B_1 \text{with continuous imbeddings} \tag{1.24}$$

$$\text{the imbedding } B_0 \hookrightarrow B \text{ is compact} \tag{1.25}$$

let then

$$W = \{v | v \in L^{p_0}(0,T;B_0), \quad \frac{\partial v}{\partial t} \in L^{p_1}(0,T;B_1) \tag{1.26}$$

with $1 < p_i < \infty$, then

Lemma 1.2. *Under the previous hypothesis, the imbedding of W in $L^{p_0}(0,T;B)$ is compact.*

We now use this compactness theorem in the following situation : $p_i = 2$, $B_0 = V^f \times [L^2(\Omega^s)]^n$, $B_1 = W^{\sigma\prime}$ and we choose $B = \{\boldsymbol{v} \in [H^{1-\varepsilon}(\Omega^f)]^n, \operatorname{div} \boldsymbol{v} = 0\} \times [H^{-\varepsilon}(\Omega^s)]^n$ for $0 < \varepsilon < 1/2$. We remind (see e.g. [8]) that the imbedding $H^1(\Omega^f) \to H^{1-\varepsilon}(\Omega^f)$ is compact for any $\varepsilon > 0$ and similarly the imbedding $L^2(\Omega^s) \to H^{-\varepsilon}(\Omega^s)$ is compact, so that there exists a convergent subsequence

$$\boldsymbol{u}_{f\,\mu} \to \boldsymbol{u}_f, \quad \text{in } L^2(0,T;(H^{1-\varepsilon}(\Omega^f))^n) - \text{strong} \tag{1.27}$$

In addition, for any $\varepsilon < \frac{1}{2}$, the imbedding

$$\boldsymbol{v} \mapsto \boldsymbol{v}_{|\Gamma}$$

is continuous from $H^{1-\varepsilon}(\Omega^f) \to L^2(\Gamma)$, hence the convergence

$$\boldsymbol{u}_{f\,m|\Gamma} \to \boldsymbol{u}_{f\,|\Gamma}, \quad \text{is strong in } L^2(0,T;(L^2(\Gamma))^n) \tag{1.28}$$

These two strong convergences allows to prove that

$$< G((\boldsymbol{u}_{f\,m}, \Phi_m), (\boldsymbol{v}_f, \varphi) > \to < G((\boldsymbol{u}_f, \Phi), (\boldsymbol{v}_f, \varphi) > \tag{1.29}$$

which ends the fact that the limit $(\boldsymbol{u}_f, \Phi)$ is a solution to this first transmission problem.

1.2 Some preliminary basic considerations on the full interaction system.

One of the ingredients in the analysis of the interaction problem that has been illustrated in the previous section is the derivation of a priori estimates on the solution of the problem or on the solutions of its approximated versions. These estimates constitute the fundamental argument for the analysis of the couple nonlinear problem since they allow to get bounds on the finite dimensional approximate solutions leading to convergence in weak norms, which is a first step to pass to the limit in most of the linear terms of the problem. Then, combined with some further a priori estimates, compactness results allow to obtain the existence of the solution to the coupled problem. In this section we want to make clear the link between the model of the exchange of informations between the fluid and the structure and the derivation of some a priori estimates. This analysis is actually quite simple to derive in the general settings, unfortunately the following steps are much more difficult to perform and currently the general analysis in this context is not solved yet.

1.2.1 A general settings of the problem

We denote by $\hat{\Omega}_s$ the reference configuration of a structure surrounding a fluid. For each time $t \in (0, T)$, we denote by $\Omega^f(t)$ the fluid domain delimited by the deformed elastic structure. The structure is modeled by the behavior of the displacement $\boldsymbol{\eta}$ such that the structure at time t is the range of $\hat{\Omega}_s$ through deformation $\phi(t,.) = Id + \boldsymbol{\eta}(t,.)$. The outside boundary of $\hat{\Omega}_s$ is assumed to be fixed (just for the sake of simplicity), the inner boundary, denoted as $\hat{\Gamma}$ is assumed to constitute the boundary of the fluid domain. We are thus looking for the displacement $\boldsymbol{\eta}$ over $\hat{\Omega}_s$, a divergence free velocity field $\boldsymbol{u}_f(t,.)$ and a pressure $p(t,.)$ defined over $\Omega^f(t)$ such that, in addition to (1.1), (1.2) valid with $\Omega^f = \Omega^f(t)$, we have the following interface conditions

$$\begin{cases} \boldsymbol{u}_f(t, \phi(t, \hat{\boldsymbol{x}})) = \frac{\partial \boldsymbol{\eta}}{\partial t}(t, \hat{\boldsymbol{x}}), & \forall \hat{\boldsymbol{x}} \in \hat{\Gamma} \\ \left[2\nu \boldsymbol{D}\boldsymbol{u}_f.\boldsymbol{n} - p\boldsymbol{n}\right](t, \phi(t, \hat{\boldsymbol{x}}))\frac{\partial \boldsymbol{\eta}}{\partial \hat{x}} = \left[t_{elastic}(\boldsymbol{\eta})\right](t, \hat{\boldsymbol{x}}), & \forall \hat{\boldsymbol{x}} \in \hat{\Gamma} \end{cases} \tag{1.30}$$

where the traction vector $t_{elastic}(\boldsymbol{\eta})$ is the normal stress to the structure, the expression of which varies according to the definition of the structural energy. For example, going back to a model

$$\rho^s \frac{\partial^2 \boldsymbol{\eta}}{\partial t^2} - \mathbf{div}_{\hat{\boldsymbol{x}}}(\boldsymbol{F}^s \boldsymbol{\Sigma}^s) = 0 \quad \text{in } \hat{\Omega}_s \tag{1.31}$$

the traction is then equal to

$$t_{elastic}(\boldsymbol{\eta})(t, \hat{\boldsymbol{x}}) = [\boldsymbol{F}^s \boldsymbol{\Sigma}^s](t, \hat{\boldsymbol{x}}).\hat{\boldsymbol{n}}(\hat{\boldsymbol{x}}). \tag{1.32}$$

and where $\hat{\boldsymbol{n}}$ denotes the outside normal to $\hat{\Omega}_s$. These equations are then complemented with the necessary initial conditions on the displacement and the fluid velocity.

1.2.2 A basics set of priori estimates

Assuming that this coupled problem possesses a solution, then, by multiplying equation (1.1) by $\boldsymbol{u}_f$, and integrating formally over $\Omega^f(t)$ we get

$$\int_{\Omega^f(t)} \frac{\partial \boldsymbol{u}_f}{\partial t}\boldsymbol{u}_f - \int_{\Omega^f(t)} [2\,\mathbf{div}(\nu \boldsymbol{D}\boldsymbol{u}_f) + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}_f + \nabla p]\boldsymbol{u}_f = \int_{\Omega^f(t)} \boldsymbol{f}\boldsymbol{u}_f \tag{1.33}$$

integrating by parts in the fourth term, with (1.2) yields

$$\int_{\Omega^f(t)} \nabla p \boldsymbol{u}_f = \int_{\Gamma(t)} p\boldsymbol{n}.\boldsymbol{u}_f$$

integrating by parts in the second term yields

$$-\int_{\Omega^f(t)} 2\,\mathbf{div}(\nu \boldsymbol{D}\boldsymbol{u}_f)\boldsymbol{u}_f = 2\nu\int_{\Omega^f(t)} [\boldsymbol{D}\boldsymbol{u}_f]^2 - \int_{\Gamma(t)} 2\nu \boldsymbol{D}\boldsymbol{u}_f.\boldsymbol{n}.\boldsymbol{u}_f$$

Similarly, by multiplying (1.31) by $\frac{\partial \boldsymbol{\eta}}{\partial t}$ integrating over $\hat{\Omega}^s$ and formally integrating by parts, we get

$$\int_{\hat{\Omega}_s} \rho^s \frac{\partial^2 \boldsymbol{\eta}}{\partial t^2}\frac{\partial \boldsymbol{\eta}}{\partial t} + \int_{\hat{\Omega}_s} (\boldsymbol{F}^s\boldsymbol{\Sigma}^s)\nabla[\frac{\partial \boldsymbol{\eta}}{\partial t}] - \int_{\hat{\Gamma}} [\boldsymbol{F}^s\boldsymbol{\Sigma}^s](t,.).\hat{\boldsymbol{n}}\frac{\partial \boldsymbol{\eta}}{\partial t} = 0 \tag{1.34}$$

from (1.30) we now recognize that

$$2\nu\int_{\Omega(t)} [\boldsymbol{D}\boldsymbol{u}_f]^2 - \int_{\Gamma(t)} 2\nu \boldsymbol{D}\boldsymbol{u}_f.\boldsymbol{n}.\boldsymbol{u}_f - \int_{\Gamma(t)} p\boldsymbol{n}.\boldsymbol{u}_f = \int_{\hat{\Gamma}} [\boldsymbol{F}^s\boldsymbol{\Sigma}^s](t,.).\hat{\boldsymbol{n}}\frac{\partial \boldsymbol{\eta}}{\partial t} \tag{1.35}$$

so that by adding (1.33) and (1.34), we get

$$\begin{aligned} \int_{\Omega^f(t)} \tfrac{\partial \boldsymbol{u}_f}{\partial t}\boldsymbol{u}_f &+ \int_{\Omega^f(t)} (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}_f\boldsymbol{u}_f + 2\nu\int_{\Omega(t)}[\boldsymbol{D}\boldsymbol{u}_f]^2 \\ &+ \int_{\hat{\Omega}_s} \rho^s \tfrac{\partial^2 \boldsymbol{\eta}}{\partial t^2}\tfrac{\partial \boldsymbol{\eta}}{\partial t} + \int_{\hat{\Omega}_s} (\boldsymbol{F}^s\boldsymbol{\Sigma}^s)\nabla[\tfrac{\partial \boldsymbol{\eta}}{\partial t}] \\ &= \int_{\Omega^f(t)} \boldsymbol{f}\boldsymbol{u}_f \end{aligned} \tag{1.36}$$

The first remark is now

$$\int_{\hat{\Omega}_s} \rho^s \frac{\partial^2 \boldsymbol{\eta}}{\partial t^2}\frac{\partial \boldsymbol{\eta}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\int_{\hat{\Omega}_s} \rho^s[\frac{\partial \boldsymbol{\eta}}{\partial t}]^2,$$

next, depending on the definition of $\boldsymbol{F}$ and $\boldsymbol{\Sigma}$ we get

$$\int_{\hat{\Omega}_s} (\boldsymbol{F}^s\boldsymbol{\Sigma}^s)\nabla[\frac{\partial \boldsymbol{\eta}}{\partial t}] = \frac{\partial}{\partial t}\mathcal{E}(\boldsymbol{\eta})(t,.),$$

where $\mathcal{E}$ is some mechanical energy related to the displacement and is assumed to be positive. The only terms that do not seem to have an appropriate sign are the two first ones in (1.36). From the Reynolds theorem, nevertheless, we derive

$$\int_{\Omega^f(t)} \frac{\partial \boldsymbol{u}_f}{\partial t} \boldsymbol{u}_f = \frac{1}{2}\frac{\partial}{\partial t}\int_{\Omega^f(t)} |\boldsymbol{u}_f|^2 - \frac{1}{2}\int_{\Gamma(t)} |\boldsymbol{u}_f|^2 \boldsymbol{u}_f.\boldsymbol{n}$$

on the other hand, from the divergence free condition (1.2), we get

$$\int_{\Omega^f(t)} (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}_f\boldsymbol{u}_f = \frac{1}{2}\int_{\Omega^f(t)} (\boldsymbol{u}\cdot\boldsymbol{\nabla})|\boldsymbol{u}_f|^2 = -\frac{1}{2}\int_{\Gamma(t)} |\boldsymbol{u}_f|^2 \boldsymbol{u}_f.\boldsymbol{n}$$

hence, by summarizing up

$$\begin{aligned} \tfrac{1}{2}\tfrac{\partial}{\partial t}\textstyle\int_{\Omega^f(t)} \boldsymbol{u}_f{}^2 + 2\nu \int_{\Omega(t)} [\boldsymbol{D}\boldsymbol{u}_f]^2 + \tfrac{1}{2}\tfrac{\partial}{\partial t}\int_{\hat{\Omega}_s} \rho^s [\tfrac{\partial \boldsymbol{\eta}}{\partial t}]^2 + \tfrac{\partial}{\partial t}\mathcal{E}(\boldsymbol{\eta})(t,.) \\ = \textstyle\int_{\Omega^f(t)} \boldsymbol{f}\boldsymbol{u}_f \end{aligned} \tag{1.37}$$

from which, after integration in time provides the stability, at any time t

$$\begin{aligned} \textstyle\int_{\Omega^f(t)} \boldsymbol{u}_f{}^2 + 4\nu \int_0^t \int_{\Omega(s)} [\boldsymbol{D}\boldsymbol{u}_f]^2 ds + \int_{\hat{\Omega}_s} \rho^s [\tfrac{\partial \boldsymbol{\eta}}{\partial t}(t)]^2 + 2\mathcal{E}(\boldsymbol{\eta})(t,.) \\ = \textstyle\int_0^t \int_{\Omega^f(s)} \boldsymbol{f}\boldsymbol{u}_f ds + \int_{\Omega^f(0)} \boldsymbol{u}_f{}^2 + \int_{\hat{\Omega}_s} \rho^s [\tfrac{\partial \boldsymbol{\eta}}{\partial t}(0)]^2 + 2\mathcal{E}(\boldsymbol{\eta})(0,.) \end{aligned} \tag{1.38}$$

from which we derive, that, if it exists, the solution $(\boldsymbol{u}_f, \boldsymbol{\eta})$ to the interaction problem is stable in the following sense

$$\begin{cases} \boldsymbol{u}_f & \text{is uniformely bounded in } L^\infty(0,T;L^2(\Omega(t)) \\ \boldsymbol{D}\boldsymbol{u}_f & \text{is uniformely bounded in } L^2(0,T;L^2(\Omega(t)) \\ \frac{\partial \boldsymbol{\eta}}{\partial t}(t) & \text{is uniformely bounded in } L^\infty(0,T;L^2(\hat{\Omega}^s) \\ \mathcal{E}(\boldsymbol{\eta}) & \text{is uniformely bounded in } L^\infty(0,T) \end{cases} \tag{1.39}$$

Note that in order to derive some uniform $L^2(0,T;H^1(\Omega(t))$ bound over $\boldsymbol{u}_f$, we need to make use of a Korn-type inequality that may be true only upon some hypothesis on the shape of $\Omega(t)$, in particular the regularity of its boundary, which is one unknown of the problem... this is just a preliminary illustration of the difficulties that have still to be faced.

1.2.3 Some general remarks

One interesting feature of the previous analysis is the different types of balance in the energies. The first one holds between the fluid and the structure, as is illustrated in (1.35) and as it might be expected. It allows to rationally check that the interface conditions that have been proposed are coherent with the model. The second one, less obvious, states that the movement of the interface, i.e. the fluid domain boundary, is balanced by the convection terms. Actually this is not so surprising since, in Lagrangian form, this convection term enter in the total time derivative that commutes with the spacial integration over the original fluid domain.

The corollary is that if, for the sake of simplification, we want to drop some contribution (e.g. the interface movement as we did in the first section, or the convective component in order to arrive to a linear Stokes problem) or if we want to add some effects (e.g. visco-elastic terms in the structure), then a special care should be taken on the way the interface contribution is written up so that the previous balances are maintained. In particular, this is the reason for the incorporation of the term $-\frac{1}{2}\left(\sum_{j=1}^{n} \boldsymbol{u}_{f\,j} \cos n_j\right)$ in (1.5) consequence of the fact that the movement of the interface does not enter the model, yielding in turn the commutation of the time derivative operator with the integration over Ω^f.

The second corollary is that, if the balance is not possible, the consequence of the lack of a priori bound on the solution is not only revealed by the difficulty to get a solution to the coupled problem but it may hurt also at the discretization and simulation level since there may exist no stable scheme for the simulation of this incorrect fluid structure interaction model.

As far as we know, these stability issues where first presented in [6], together with some illustration of the (real versus numerical) instability arriving by getting rid of the nonlinear contribution in the fluid model.

1.3 Weak solution for a full interaction problem with an elastic plate

Let us summarize here the results presented in [3] by Chambolle, Desjardins, Esteban and Grandmont that deals with a more general situation where the domain Ω^f depends on time. The proof of the well-posedness of this problem relies on the a priori estimates and compactness properties that are available thanks to the simple shape of the geometry of the fluid that follows from the plate assumption.

1.3.1 Definition on the interaction problem

The fluid is assumed to fill a three dimensional cavity and to interact with a thin elastic structure sitting on one of its side $\Gamma(t)$. The remaining part of the boundary, denoted as $\gamma_0 = \partial\Omega^f(t) \setminus \Gamma(t)$, is assumed to be rigid. For the sake of simplicity, it is assumed that, in the reference state, the elastic part of the fluid boundary is $\omega \times \{1\}$ where ω denotes a Lipshitz domain in $I\!R^2$; the initial state of the fluid occupies the domain Ω_0^f defined by:

$$\Omega_0^f = \{(x, y, z) \in \omega \times I\!R, 0 < z < 1 + \eta_0(x, y)\},$$

where η_0 is a given initial displacement. The deformation of the elastic part of the boundary is modelled by a classical linear plate theory for transverse motions and the in plane motions are nelected. The transversal displacement of the plate η ($\eta = \eta(t, x, y) \in I\!R$) is supposed to satisfy:

$$\begin{cases} \partial_{tt}\eta + \Delta^2\eta + \mu\Delta^2\partial_t\eta = g + (t_f)_3 & \text{in } \omega, \\ \eta = \frac{\partial\eta}{\partial n} = 0, & \text{over } \partial\omega, \\ \eta(0) = \eta_0, \quad \partial_t\eta(0) = \eta_1 \end{cases} \tag{1.40}$$

Here again g is a given force and t_f is the surface force applied by the fluid onto the structure. The domain occupied by the fluid, at time t is denoted by $\Omega^f(t)$

$$\Omega^f(t) = \{(x,y,z) \in \omega \times I\!R, 0 < z < 1 + \eta(t,x,y)\},$$

the divergence free velocity field $\boldsymbol{u}_f(t,.)$ and a pressure $p(t,.)$ defined over $\Omega^f(t)$ satisfy (1.1), (1.2) over $\Omega^f = \Omega^f(t)$, together with initial and boundary conditions

$$\begin{cases} \boldsymbol{u}_f(t,0) = 0 & \text{over } \gamma_0, \\ \boldsymbol{u}_f(0,.) = \boldsymbol{u}_{f0}, & \text{in } \Omega_0^f, \\ \boldsymbol{u}_f(t,x,y,1+\eta(t,x,y)) = (0,0,\partial_t\eta(t,x,y))^T, & (x,y) \in \omega \end{cases} \tag{1.41}$$

The expression of the surface tension can now be given; it is simpler to express it under a variational form, since it is then obvious that the equilibrium will occur

$$\int_\omega t_f(x,y)\overline{\boldsymbol{v}} = \int_{\Gamma(t)} (2\nu\boldsymbol{D}\boldsymbol{u}_f).\boldsymbol{n} + p\boldsymbol{n}).\boldsymbol{v}, \qquad \forall \boldsymbol{v}$$

where $\overline{\boldsymbol{v}}(t,x,y) = \boldsymbol{v}(t,x,y,\eta(t,x,y))$. Similarly as in the previous section, by multipling equation (1.1) by $\boldsymbol{u}_f$, equation (1.40) by $\partial_t\eta$, integrating by parts over the corresponding domains and taking into account (1.41) we derive stability on the solution $(\boldsymbol{u}_f, \eta)$ (assuming it exists) whenever the datum $\boldsymbol{f} \in L^2(0,T;L^2(I\!R^3))^3$, $g \in L^2(0,T;L^2(\omega))$, $\boldsymbol{u}_{f0} \in L^2(\Omega_0^f)^3$, $\eta_0 \in H_0^2(\omega)$ and $\eta_1 \in L^2(\omega)$. These stability state that

$$\boldsymbol{u}_f \in L^\infty(0,T;L^2(\Omega^f(t))^3), \boldsymbol{D}(\boldsymbol{u}_f) \in L^2(0,T;L^2(\Omega^f(t))^3)$$

and

$$\eta \in W^{1,\infty}(0,T;L^2(\omega)) \cap H^1(0,T;H_0^2(\omega)).$$

It should be noted that the various integrations by parts and trace restriction for the relative boundary terms should be carefully defined since the boundary is an unknown and is evolving in time. This point is checked in [3] and everything is fine over the domain $\Omega^f(t)$ as soon as e.g. the boundary deformation $\eta \in \mathcal{C}^0([0,T];\mathcal{C}^0(\overline{\omega}) \cap H^1(\overline{\omega}))$, we refer to [3], subsection 1.3, for the full details.

In addition to the a priori estimates, the analysis of the problem involves a fixed point procedure together with a regularization process based on two stable regularization operators, $\mathcal{R}_\varepsilon$ from $L^2(0,T;L^2(\omega \times (0,2M))$ onto $\mathcal{C}^\infty([0,T] \times (\overline{\omega} \times [0,2M]))$ and $\mathcal{N}_\varepsilon$ from $\mathcal{C}^0([0,T] \times \overline{\omega})$ into $\mathcal{C}^\infty([0,T] \times \overline{\omega})$, where M is some large enough real number.

1.3.2 A linearized/regularized version of the problem

Let us take $\delta \in H^1(0,T;\mathcal{C}^0(\overline{\omega}) \cap H^1(\overline{\omega}))$; that is a predictor for the plate displacement, and let us regularize it by introducing $\delta_\varepsilon^\star = \mathcal{N}_\varepsilon(\delta)$. This allows the definition of a domain $\Omega_{\delta_\varepsilon^\star}$ as follows

$$\Omega_{\delta_\varepsilon^\star}(t) = \{(x,y,z) \in \omega \times I\!R, 0 < z < 1 + \delta_\varepsilon^\star(t,x,y)\}, \tag{1.42}$$

provided that we assume, e.g. $2M \geq 1 + \delta(t,x,y) \geq \alpha > 0$, $\forall(t,x,y) \in [0,T] \times \overline{\omega}$. Next, similarly as in (1.9), we introduce the space

$$V_{\delta_\varepsilon^\star}(t) = \{\boldsymbol{v}_f | \boldsymbol{v}_f \in [H^1(\Omega_{\delta_\varepsilon^\star}(t))]^n, \operatorname{div} \boldsymbol{v}_f = 0, \boldsymbol{v}_{f\,|\gamma_0} = 0\}.$$

Let now take $\boldsymbol{v} \in L^2(0,T;L^2(\omega \times (0,2M))$, that is a predictor for the fluid convection velocity, and let us regularize it by introducing $\boldsymbol{v}_\varepsilon^\star = \mathcal{R}_\varepsilon(\boldsymbol{v})$.

The intermediate problem we consider reads as follows : Find $(\boldsymbol{u}_{f\,\varepsilon}, \eta_\varepsilon)$ such that

- $\boldsymbol{u}_{f\,\varepsilon} \in L^2(0,T;V_{\delta_\varepsilon^\star}(t)) \cap L^\infty(0,T;L^2(\Omega_{\delta_\varepsilon^\star}(t)))$
- $\eta_\varepsilon \in W^{1,\infty}(0,T;L^2(\omega)) \cap H^1(0,T;H_0^2(\omega))$
- $\boldsymbol{u}_{f\,\varepsilon}(t,x,y,1+\delta_\varepsilon^\star(t,x,y)) = (0,0,\partial_t\eta_\varepsilon(t,x,y))^T$ on ω
- $\frac{\partial \boldsymbol{u}_{f\,\varepsilon}}{\partial t} \in L^2(0,T;L^2(\Omega_{\delta_\varepsilon^\star}(t)))$
- $\frac{\partial^2 \eta_\varepsilon}{\partial t^2} \in L^2(0,T;L^2(\omega))$
- and

$$\begin{aligned}\int_0^t \int_{\Omega_{\delta_\varepsilon^\star}(s)} \frac{\partial \boldsymbol{u}_{f\,\varepsilon}}{\partial t}\phi_\varepsilon + \nu \int_0^t \int_{\Omega_{\delta_\varepsilon^\star}(s)} \nabla \boldsymbol{u}_{f\,\varepsilon} \nabla \phi_\varepsilon + \frac{1}{2}\int_0^t \int_{\Omega_{\delta_\varepsilon^\star}(s)} (\boldsymbol{v}_\varepsilon^\star.\nabla)\boldsymbol{u}_{f\,\varepsilon}.\phi_\varepsilon \\ -\frac{1}{2}\int_0^t \int_{\Omega_{\delta_\varepsilon^\star}(s)} (\boldsymbol{v}_\varepsilon^\star.\nabla)\phi_\varepsilon.\boldsymbol{u}_{f\,\varepsilon} + \frac{1}{2}\int_0^t \int_\omega \frac{\partial \eta_\varepsilon}{\partial t}\frac{\partial \delta_\varepsilon^\star}{\partial t} b + \int_0^t \int_\omega \frac{\partial \eta_\varepsilon}{\partial t^2} b \\ + \int_0^t \int_\omega \frac{\partial \Delta \eta_\varepsilon}{\partial t}\Delta b + \int_0^t \int_\omega \Delta\eta_\varepsilon \Delta b = \int_0^t \int_{\Omega_{\delta_\varepsilon^\star}(s)} \boldsymbol{f}.\phi_\varepsilon \int_0^t \int_\omega gb\end{aligned} \tag{1.43}$$

$$\begin{aligned}&\forall \phi_\varepsilon \in L^2(0,T;V_{\delta_\varepsilon^\star}(t)), \forall b \in L^2(0,T;H_0^2(\omega)) \text{ such that} \\ &\qquad \phi_\varepsilon(t,x,y,1+\delta_\varepsilon^\star(t,x,y)) = (0,0,b(t,x,y))^T, \text{ on } \omega\end{aligned} \tag{1.44}$$

and complemented with initial conditions over $\boldsymbol{u}_{f\,\varepsilon}$, η_ε and $\frac{\eta_\varepsilon}{\partial t}$ that are regularized versions of $\boldsymbol{u}_{f\,0}$, η_0 and η_1.

The regularity of the boundary of the domain $\Omega_{\delta_\varepsilon^\star}(t)$ allows to define a change of variable χ_ε from a reference domain $\mathcal{C} = \omega \times (0,1)$ onto the domains $\Omega_{\delta_\varepsilon^\star}(t)$. Let us denote by $\boldsymbol{w}_\varepsilon$ the time derivative of χ_ε, the problem (1.43) becomes

$$\int_0^t \int_{\mathcal{C}} \frac{\partial \underline{\boldsymbol{u}_f}_\varepsilon}{\partial t} \underline{\phi}_\varepsilon J_\varepsilon + \nu \int_0^t \int_{\mathcal{C}} A_\varepsilon \nabla \underline{\boldsymbol{u}_f}_\varepsilon \nabla \underline{\phi}_\varepsilon + \frac{1}{2} \int_0^t \int_{\mathcal{C}} (\underline{\boldsymbol{v}}_\varepsilon^\star . (B_\varepsilon \nabla)) \underline{\boldsymbol{u}_f}_\varepsilon . \underline{\phi}_\varepsilon$$
$$-\frac{1}{2} \int_0^t \int_{\mathcal{C}} (\underline{\boldsymbol{v}}_\varepsilon^\star . (B_\varepsilon \nabla)) \underline{\phi}_\varepsilon . \underline{\boldsymbol{u}_f}_\varepsilon + \frac{1}{2} \int_0^t \int_\omega \frac{\partial \eta_\varepsilon}{\partial t} \frac{\partial \delta_\varepsilon^\star}{\partial t} b - \int_0^t \int_{\mathcal{C}} (\underline{\boldsymbol{w}}_\varepsilon . (B_\varepsilon \nabla)) \underline{\boldsymbol{u}_f}_\varepsilon . \underline{\phi}_\varepsilon$$
$$+ \int_0^t \int_\omega \frac{\partial^2 \eta_\varepsilon}{\partial t^2} b + \int_0^t \int_\omega \frac{\partial \Delta \eta_\varepsilon}{\partial t} \Delta b + \int_0^t \int_\omega \Delta \eta_\varepsilon \Delta b = \int_0^t \int_{\mathcal{C}} \underline{\boldsymbol{f}} . \underline{\phi}_\varepsilon J_\varepsilon + \int_0^t \int_\omega g b \tag{1.45}$$

$$\forall \underline{\phi}_\varepsilon \in L^2(0,T;V_0), b \in L^2(0,T;H_0^2(\omega)) \text{ such that}$$
$$\underline{\phi}_\varepsilon(t,x,y,1) = (0,0,b(t,x,y))^T, \text{ on } \omega \tag{1.46}$$

where the notation $\underline{\boldsymbol{v}}$ denotes the transported function of $\boldsymbol{v}$ under the flow χ_ε, and J_ε, A_ε, B_ε are the proper geometric factors that result from the change of variables.

In order to prove that there exists a solution to problem (1.45), we introduce, as in equation (1.15) a Galerkin basis of eigenfunctions of the Stokes problem. Restricting the spaces of trial and test functions in (1.45) to the spaces spanned by the first eigenfunctions, allows to get a set of finite dimensional coupled problems that reads as a system of coupled ordinary differential equations that can be shown to possess unique solutions. A similar a priori analysis as the one that was performed in the previous subsection allows to state that the sequence of discrete solutions is uniformly bounded (uniformly with respect to the dimension of the discrete space together with ε). In order to get enough information to pass to the limit in the convection like terms, regularity with respect to time derivatives, similarly as in (1.22), is obtained, and the boundedness of these terms is uniform with respect to the dimension of the discrete space but not with respect to ε. Nevertheless this is enough to pass to the limit in order to get the existence of a solution $(\underline{\boldsymbol{u}_f}_\varepsilon, \eta_\varepsilon)$ to problem (1.45) for every given ε, hence a solution $(\boldsymbol{u}_{f\varepsilon}, \eta_\varepsilon)$ to problem (1.43). This solution satisfy the following energy estimates

$$\|\boldsymbol{u}_{f\varepsilon}\|_{L^\infty(0,T;L^2(\Omega_{\delta_\varepsilon^\star}(t)))} + \|\nabla \boldsymbol{u}_{f\varepsilon}\|_{L^2(0,T;L^2(\Omega_{\delta_\varepsilon^\star}(t)))}$$
$$+ \|\frac{\partial \eta_\varepsilon}{\partial t}\|_{L^\infty(0,T;L^2(\omega))} + \|\Delta \eta_\varepsilon\|_{H^1(0,T;H_0^2(\omega))}$$
$$\leq C(T, \|\boldsymbol{u}_{f0}\|_{L^2(\Omega_{\eta_0})}, \|\boldsymbol{f}\|_{L^2((0,T)\times I\!R^3)}, \|\boldsymbol{g}\|_{L^2((0,T)\times I\!R^3)}, \|\eta_0\|_{H_0^2(\omega)}, \|\eta_1\|_{L_0^2(\omega)}$$

and

$$\|\frac{\partial \underline{\boldsymbol{u}_f}_\varepsilon}{\partial t}\|_{L^2(0,T;L^2(\mathcal{C}))} + \|\frac{\partial \eta_\varepsilon}{\partial t^2}\|_{L^2(0,T;L^2(\omega)} \leq C_{\varepsilon,M,\alpha} \tag{1.47}$$

The fact that this deformation of the boundary is only vertical and regular enough allows to get an easy extension of $\boldsymbol{u}_{f\varepsilon}$ in $L^2(0,T;L^2(\omega \times (0,2M)))$, noted by $\overline{\boldsymbol{u}_f}_\varepsilon$

$$\overline{\boldsymbol{u}_f}_{\varepsilon} = \begin{cases} \boldsymbol{u}_{f_\varepsilon} & \text{in } \Omega_{\delta_\varepsilon^\star}(t) \\ (0,0,\partial_t \eta_\varepsilon)^T & \text{in } (0,2M)\times\omega \setminus \Omega_{\delta_\varepsilon^\star}(t) \end{cases} \tag{1.48}$$

It is important to note that it is divergence free. We can now define the mapping F_ε:

$$F_\varepsilon : (\delta, \boldsymbol{v}) \mapsto (\eta_\varepsilon, \overline{\boldsymbol{u}_f}_\varepsilon) \tag{1.49}$$

The previous estimates allow first to get a set $\mathcal{B}_\varepsilon^M$ such that $F_\varepsilon(\mathcal{B}_\varepsilon^M) \subset \mathcal{B}_\varepsilon^M$ for M large enough. Then the estimates — in particular involving the time derivatives (1.47) — allow to prove that $F_\varepsilon(\mathcal{B}_\varepsilon)$ is relatively compact in $\mathcal{B}_\varepsilon$. The hypothesis of Schauder's fixed point theorem are fulfilled and there exists at least a fixed point to F_ε over a time such that $\min_{[0,T]\times\overline{\omega}}(1+\eta_\varepsilon) > 0$, hence a solution $(\overline{\boldsymbol{u}_f}_\varepsilon, \eta_\varepsilon)$ to the regularized nonlinear problem.

The final step is to get rid about the regularization ingredient. In order to pass to the limit as ε tends to zero, further estimates should be derived since the present available bounds on the time derivatives depends on ε and thus may explode as ε goes to zero. From the estimates already derived, $\overline{\boldsymbol{u}_f}_\varepsilon$ remains uniformely bounded in the $L^2(0,T;H^1)\cap L^\infty(0,T;L^2)$ norm, but this is not enough to pass to the limit in the nonlinear terms. As is often the case for the Navier-Stokes equations uniform estimates do not seem to be available. We replace such a bound by bounds on the "half" derivative of $\overline{\boldsymbol{u}_f}_\varepsilon$, expressed in a weak sense as follows : for any $h > 0$

$$\int_0^T \int_{\omega\times(0,3)} |\overline{\boldsymbol{u}_f}_\varepsilon(t,x) - \overline{\boldsymbol{u}_f}_\varepsilon(t+h,x)|^2 dtdx \\ + \int_0^T \int_\omega (\frac{\partial \eta_\varepsilon}{\partial t}(t,x) - \frac{\partial \eta_\varepsilon}{\partial t}(t+h,x)|^2 dtdx \leq C\sqrt{h}$$

with a constant C uniform in ε.

These estimates allow to prove the relative compactness of the sequences, then allowing to pass to the limit in the equation satisfied by $\boldsymbol{u}_{f_\varepsilon}$ and η_ε written as follows:

$$\int_{\Omega_{\eta_\varepsilon^\star}(t)} \boldsymbol{u}_{f_\varepsilon}(t).\Phi_\varepsilon(t) - \int_0^t \int_{\Omega_{\eta_\varepsilon^\star}(s)} \boldsymbol{u}_{f_\varepsilon}(s).\frac{\partial \Phi_\varepsilon}{\partial t}(s) + \nu \int_0^t \int_{\Omega_{\delta_\varepsilon^\star}(s)} \nabla \boldsymbol{u}_{f_\varepsilon} \nabla \phi_\varepsilon \\ + \frac{1}{2}\int_0^t \int_{\Omega_{\delta_\varepsilon^\star}(s)} (\boldsymbol{u}_{f_\varepsilon}^\star.\nabla)\boldsymbol{u}_{f_\varepsilon}.\phi_\varepsilon - \frac{1}{2}\int_0^t \int_{\Omega_{\delta_\varepsilon^\star}(s)} (\boldsymbol{u}_{f_\varepsilon}^\star.\nabla)\phi_\varepsilon.\boldsymbol{u}_{f_\varepsilon} \\ -\frac{1}{2}\int_0^t \int_\omega \frac{\partial \eta_\varepsilon}{\partial t}\frac{\partial \eta_\varepsilon^\star}{\partial t} b + \int_\omega \frac{\partial \eta_\varepsilon}{\partial t}(t) b(t) - \int_0^t \int_\omega \frac{\partial \eta_\varepsilon}{\partial t}\frac{\partial b}{\partial t} + \int_0^t \int_\omega \frac{\partial \Delta \eta_\varepsilon}{\partial t} \Delta b \\ + \int_0^t \int_\omega \Delta\eta_\varepsilon \Delta b = \int_0^t \int_{\Omega_{\delta_\varepsilon^\star}(s)} \boldsymbol{f}.\phi_\varepsilon + \int_0^t \int_\omega g b + \int_{\Omega_{\eta_\varepsilon^\star}(0)} \boldsymbol{u}_{f_\varepsilon}(0).\Phi_\varepsilon(0) + \int_\omega \eta_1^\varepsilon b(0) \tag{1.50}$$

after verifying that the time for existence of the solutions does not tend to zero as ε converges to zero. Note that this weak treatment of the time derivative, while it require different test functions as they now depend on time, allows to handle the convergence despite the lack of convergence results of $\frac{\partial \boldsymbol{u}_{f_\varepsilon}}{\partial t}$. This allows to state the main result of the paper [3]

Theorem 1.2. *There exists $T^\star \in (0,\infty]$ and a weak solution $(\boldsymbol{u}_f, \eta)$ on $[0,T]$ to the fluid/plate interaction problem in the sense that, for any Φ and b,*

$$\begin{aligned}\int_{\Omega_{\eta^\star}(t)} \boldsymbol{u}_f(t).\Phi(t) - \int_0^t \int_{\Omega_{\eta^\star}(s)} \boldsymbol{u}_f(s).\frac{\partial \Phi}{\partial t}(s) + \nu \int_0^t \int_{\Omega_\delta(s)} \nabla \boldsymbol{u}_f \nabla \phi \\ + \int_0^t \int_{\Omega_\delta(s)} (\boldsymbol{u}_f.\nabla)\boldsymbol{u}_f.\phi - \int_0^t \int_\omega [\frac{\partial \eta}{\partial t}]^2 b + \int_\omega \frac{\partial \eta}{\partial t}(t) b(t) \\ - \int_0^t \int_\omega \frac{\partial \eta}{\partial t}\frac{\partial b}{\partial t} + \int_0^t \int_\omega \frac{\partial \Delta\eta}{\partial t}\Delta b + \int_0^t \int_\omega \Delta\eta \Delta b \\ = \int_0^t \int_{\Omega_\delta(s)} \boldsymbol{f}.\phi + \int_0^t \int_\omega g b + \int_{\Omega_{\eta^\star}(0)} \boldsymbol{u}_f(0).\Phi(0) + \int_\omega \eta_1 b(0)\end{aligned} \tag{1.51}$$

In addition, this solution satisfies the a priori bounds stated at the beginning of this section.

1.4 A strong solution to a two dimensional fluid-vessel interaction

The technique involved in the previous section has used marginally an ALE type change of variable from the actual deformed shape of the fluid domain to a reference domain in one preliminary step of the proof; during all following steps, the fluid solution is considered in Eulerian form and the a priori bounds and the limit process are performed on the Eulerian velocity and pressure. In this section, we report on the paper of Beirao da Veiga [2] that presents a technique that allows to consider both configurations at the same time: the Eulerian form and the Lagrangian one. The change of variable is thus done all through the analysis, this requires more regularity on the solutions, the verification of the equations can be stronger than in the previous analysis.

1.4.1 The fluid vessel coupling

The configuration here is simplified as a two dimensional fluid interacting with a one dimensional membrane. The fluid domain is delimited by a periodic curve, $\Gamma(t)$, $t \in [0,T]$; with equation

$$y = 1 + \eta(t,x), \quad x \in [0,L] \tag{1.52}$$

Without loss of generality, we assume that the initial condition $\eta^0(x) = \eta(0, x)$ satisfies

$$\int_0^L \eta^0(x)dx = 0 \tag{1.53}$$

and $1+\eta_0 \geq 2\delta_0 > 0$. The evolution of these curves is assumed to be governed by the following generalized string model

$$\begin{cases} \frac{\partial^2 \eta}{\partial t^2} - \beta \frac{\partial^2 \eta}{\partial x^2} - \gamma \frac{\partial^3 \eta}{\partial t \partial x^2} + \alpha \frac{\partial^4 \eta}{\partial x^4} + \sigma \eta = \Phi, & \text{over } (0,T) \times (0,L), \\ \eta(0,x) = \eta_0(x), \quad \partial_t \eta(0,x) = \eta_1(x) & \end{cases} \tag{1.54}$$

Here $\gamma > 0$, and α, β, σ are only ≥ 0. The fluid domain

$$\Omega^f(t) = \{(x,y) \in (0,L) \times I\!R, 0 < y < 1 + \eta(t,x)\}. \tag{1.55}$$

The function Φ on the right hand side of (1.54) is defined as follows

$$\Phi[\eta, \boldsymbol{u}_f, p] = \Big(\rho_1 p \boldsymbol{n} - \rho_2 \nu [\boldsymbol{D}\boldsymbol{u}_f].\boldsymbol{n}\Big)_{|\Gamma(t)} \sqrt{1+\eta_x^2}\boldsymbol{e}_y$$

where $\boldsymbol{e}_y$ denotes the unit vector in the y–direction Again the fluid is assumed to be governed by the Navier-Stokes equations (1.1), (1.2) complemented with the initial and boundary conditions

$$\begin{cases} \boldsymbol{u}_f(0,x,y) = \boldsymbol{u}_{f\,0}(x,y), \quad in\ \Omega_0^f \\ \boldsymbol{u}_f(t,x,1+\eta(t,x)) = \frac{\partial \eta}{\partial t}(t,x)\boldsymbol{n}_y, \quad in\ (0,T) \times (0,L) \\ \boldsymbol{u}_f(t,x,0) = 0, \quad in\ (0,T) \times (0,L) \end{cases}$$

and periodicity is assumed in the x direction. The following compatibility condition is required

$$\begin{cases} \nabla.\boldsymbol{u}_{f\,0} = 0, \quad in\ \Omega_0^f \\ \boldsymbol{u}_{f\,0}(x,0) = 0, \quad in\ (0,L) \\ \boldsymbol{u}_{f\,0}(x,1+\eta_0(x)) = \eta_1(x)\boldsymbol{e}_y, \quad in\ (0,L) \\ \int_0^L \eta_1(x)dx = 0 \end{cases}$$

1.4.2 The problem in a ALE form

This geometry allows to use a simple change of variable again

$$x = x, \quad z = \frac{y}{1+\eta(t,x)} \tag{1.56}$$

that transforms the fluid domain $\Omega^f(t)$ into the reference domain $\mathcal{C} = (0,L) \times (0,1)$ and the transformed functions are, as in the previous section, represented as

$$\overline{f}(x,z) = f(x,(1+\eta(t,x))z).$$

The Navier Stokes equation reads now

$$\begin{cases} \frac{\partial \overline{\boldsymbol{u}_f}}{\partial t} - \nu \Delta \overline{\boldsymbol{u}_f} + \nabla \overline{p} = \overline{\boldsymbol{F}}[\eta, \overline{\boldsymbol{u}_f}, \nabla \overline{p}], \\ \mathbf{div}\, \overline{\boldsymbol{u}_f} = \overline{\boldsymbol{g}}[\eta, \overline{\boldsymbol{u}_f}], \\ \overline{\boldsymbol{u}_f}(0,x,z) = \overline{\boldsymbol{u}_f}^0(x,z), \\ \overline{\boldsymbol{u}_f}(0,x,1) = \frac{\partial \eta}{\partial t}(t,x)\boldsymbol{e}_y, \\ \overline{\boldsymbol{u}_f}(0,x,0) = 0 \end{cases} \tag{1.57}$$

where

$$\begin{aligned} \boldsymbol{F}[\eta, \overline{\boldsymbol{u}_f}, \nabla \overline{p}] = & -\eta \frac{\partial \overline{\boldsymbol{u}_f}}{\partial t} + \left[z\frac{\partial \eta}{\partial t} + \nu z \Big(\frac{2\frac{\partial \eta}{\partial x}^2}{1+\eta} - \frac{\partial^2 \eta}{\partial x^2} \Big) \right] \frac{\partial \overline{\boldsymbol{u}_f}}{\partial z} \\ & + \nu \left(-2z \frac{\partial \eta}{\partial x}\frac{\partial^2 \overline{\boldsymbol{u}_f}}{\partial x \partial z} + \eta \frac{\partial^2 \overline{\boldsymbol{u}_f}}{\partial x^2} + \Big[\frac{z^2 \frac{\partial \eta}{\partial x}^2 - \eta}{1+\eta} \Big] \frac{\partial^2 \overline{\boldsymbol{u}_f}}{\partial z^2} \right) \\ & + z \Big(\frac{\partial \eta}{\partial x}\frac{\partial \overline{p}}{\partial z} - \eta \frac{\partial \overline{p}}{\partial x} \Big) \boldsymbol{e}_x - (1+\eta)\overline{u}_{f,1} \frac{\partial \overline{\boldsymbol{u}_f}}{\partial x} \\ & + \big(z\frac{\partial \eta}{\partial x}\overline{u}_{f,1} - \overline{u}_{f,2} \big) \frac{\partial \overline{\boldsymbol{u}_f}}{\partial z} \end{aligned}$$

and

$$\overline{\boldsymbol{g}}[\eta, \overline{\boldsymbol{u}_f}] = -\eta \frac{\partial \overline{u}_{f,1}}{\partial x} + z \frac{\partial \eta}{\partial x}\frac{\partial \overline{u}_{f,1}}{\partial z}. \tag{1.58}$$

The string equation (1.54) has now, as a right hand side, a function in the fluid variables

$$\begin{cases} \frac{\partial^2 \eta}{\partial t^2} - \beta \frac{\partial^2 \eta}{\partial x^2} - \gamma \frac{\partial^3 \eta}{\partial t \partial x^2} + \alpha \frac{\partial^4 \eta}{\partial x^4} + \sigma \eta = \overline{\Phi}[\eta, \overline{\boldsymbol{u}_f}, \overline{p}_0] + \rho_1 \overline{\phi}[\eta, \overline{\boldsymbol{u}_f}], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{over } (0,T)\times(0,L), \\ \eta(0,x) = \eta_0(x), \quad \partial_t \eta(0,x) = \eta_1(x) \end{cases} \tag{1.59}$$

with

$$\overline{\Phi}[\eta, \overline{\boldsymbol{v}}, \overline{p}] = \rho_1 \overline{p} + \nu \rho_2 \left(\frac{1}{1+\eta}\frac{\partial \eta}{\partial x}\frac{\partial \overline{v}_1}{\partial z} + \frac{\partial \eta}{\partial x}\frac{\partial \overline{v}_2}{\partial x} - 2\frac{2+(\partial \eta/\partial x)^2}{1+\eta}\frac{\partial \overline{v}_2}{\partial z} \right) \tag{1.60}$$

$\overline{p}_0$ is such that

$$\int_0^L \overline{p}_0(t,x,1)dx = 0$$

and

$$\overline{\phi}(t) = \overline{\phi}[\eta, \overline{\boldsymbol{u}_f}] = \frac{\nu \rho_2}{L\rho_1} \int_0^L \left(\frac{1}{1+\eta}\frac{\partial \eta}{\partial x}\frac{\partial \overline{v}_1}{\partial z} + \frac{\partial \eta}{\partial x}\frac{\partial \overline{v}_2}{\partial x} - 2\frac{2+(\partial \eta/\partial x)^2}{1+\eta}\frac{\partial \overline{v}_2}{\partial z} \right) dx \tag{1.61}$$

so that the pressure in (1.57) is given by

$$\overline{p}(t,x,z) = \overline{p}_0(t,x,z) + \overline{\phi}(t).$$

1.4.3 The linearized coupled problem

This problem is written in a form well suited to propose a linearized version of it, where the terms in the right hand side are supposed to be given. We are thus faced to a couple of systems :

$$\begin{cases} \frac{\partial^2 \tilde\eta}{\partial t^2} - \beta \frac{\partial^2 \tilde\eta}{\partial x^2} - \gamma \frac{\partial^3 \tilde\eta}{\partial t \partial x^2} + \alpha \frac{\partial^4 \tilde\eta}{\partial x^4} + \sigma\eta = \overline{\Phi}[\hat\eta, \hat{\boldsymbol{u}}_f, \hat p_0] + \rho_1 \overline{\phi}[\hat\eta, \hat{\boldsymbol{u}}_f], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{over } (0,T)\times(0,L), \\ \qquad \tilde\eta(0,x) = \eta_0(x), \quad \partial_t \tilde\eta(0,x) = \eta_1(x) \end{cases} \tag{1.62}$$

corresponding to the string equation, and

$$\begin{cases} \frac{\partial \tilde{\boldsymbol{u}}_f}{\partial t} - \nu \Delta \tilde{\boldsymbol{u}}_f + \nabla \tilde p = \overline{\boldsymbol{F}}[\hat\eta, \hat{\boldsymbol{u}}_f, \nabla \hat p], \\ \qquad\qquad \mathbf{div}\, \tilde{\boldsymbol{u}}_f = \overline{\boldsymbol{g}}[\hat\eta, \hat{\boldsymbol{u}}_f], \\ \qquad \tilde{\boldsymbol{u}}_f(0,x,z) = \hat{\boldsymbol{u}}_f{}^0(x,z), \\ \qquad \tilde{\boldsymbol{u}}_f(0,x,1) = \frac{\partial \hat\eta}{\partial t}(t,x)\boldsymbol{e}_y, \\ \qquad \tilde{\boldsymbol{u}}_f(0,x,0) = 0 \end{cases} \tag{1.63}$$

for the fluid part.

Here the triplet $(\hat\eta, \hat{\boldsymbol{u}}_f, \nabla \hat p)$ is assumed to be given, the solution $(\tilde\eta, \tilde{\boldsymbol{u}}_f, \nabla \tilde p)$ follows and we shall seek a fixed point to the mapping $(\hat\eta, \hat{\boldsymbol{u}}_f, \nabla \hat p) \longrightarrow (\tilde\eta, \tilde{\boldsymbol{u}}_f, \nabla \tilde p)$.

The set $\mathbb{K}$ in which the fixed point procedure will be performed is expressed in terms of the following regularities on

- for the displacement

$$|||\eta|||^2 = \|\eta\|^2_{L^\infty(0,T;H^{7/2})} + \|\frac{\partial \eta}{\partial t}\|^2_{L^\infty(0,T;H^{3/2})} + \|\frac{\partial \eta}{\partial t}\|^2_{L^2(0,T;H^{5/2})} < \infty$$

  and

$$\|\frac{\partial^2 \eta}{\partial t^2}\||^2_{L^2(0,T;H^{-1/2})} < \infty$$

- for the velocity and pressure

$$|||\boldsymbol{u}_f, p|||^2 = \|\boldsymbol{u}_f\|^2_{L^2(0,T;H^2(\Omega))} + \|\frac{\partial \boldsymbol{u}_f}{\partial t}\|^2_{L^2(0,T;L^2(\Omega))} + \|p\|^2_{L^2(0,T;L^2(\Omega))} < \infty$$

which are higher, as announced, than those we have encountered up to now. The generic constants that will appear in the sequel, all denoted by c may depend on $L, \delta_0, \nu, \alpha, \beta, \gamma$ and σ.

We start by analysing the linearized string equation (1.62) for which the existence of a solution is quite simple. By considering now the equation satisfied by $\lambda = \frac{\partial \tilde\eta}{\partial x}$ we deduce by multiplying by $\frac{\partial^2 \tilde\eta}{\partial x \partial t}$ the following stabilities

- $\tilde\eta$ is bounded in $L^\infty(0,T;H^1)$
- $\frac{\partial \tilde\eta}{\partial t}$ is bounded in $L^\infty(0,T;H^1)$
- $\frac{\partial^2 \tilde\eta}{\partial x^2}$ is bounded in $L^\infty(0,T;H^1)$

- $\frac{\partial \tilde{\eta}}{\partial t}$ is bounded in $L^2(0, T; H^2)$

in terms of $\|\eta_0\|_{H^3}$, $\|\eta_1\|_{H^1}$ and the $L^2(0, T; L^2)$-norm of the right hand side in (1.62). By multiplying the equation satisfied by λ by $-\frac{\partial^3 \lambda}{\partial t \partial x^2}$, we get the following stabilities

- $\tilde{\eta}$ is bounded in $L^\infty(0, T; H^2)$
- $\frac{\partial \tilde{\eta}}{\partial t}$ is bounded in $L^\infty(0, T; H^2)$
- $\frac{\partial^2 \tilde{\eta}}{\partial x^2}$ is bounded in $L^\infty(0, T; H^2)$
- $\frac{\partial \tilde{\eta}}{\partial t}$ is bounded in $L^2(0, T; H^3)$

in terms of $\|\eta_0\|_{H^4}$, $\|\eta_1\|_{H^2}$ and the $L^2(0, T; H^1)$-norm of the right hand side in (1.62).

The required stability for $\mathbb{K}$ are derived by interpolation from the two previous stability results that lead to

$$\begin{aligned} &\|\tilde{\eta}\|^2_{L^\infty(0,T;H^{7/2})} + \|\frac{\partial \tilde{\eta}}{\partial t}\|^2_{L^\infty(0,T;H^{3/2})} + \|\frac{\partial \tilde{\eta}}{\partial t}\|^2_{L^2(0,T;H^{5/2})} \\ &\qquad \le c\Big(\|\eta_0\|^2_{H^{7/2}} + \|\eta_1\|^2_{H^{3/2}} + \|\overline{\Phi}[\hat{\eta}, \hat{\boldsymbol{u}}_f, \hat{p}_0] + \rho_1 \overline{\phi}[\hat{\eta}, \hat{\boldsymbol{u}}_f]\|_{L^2(H^{1/2})}\Big). \end{aligned} \tag{1.64}$$

We complement this analysis by noticing that

$$\begin{aligned} &\|\overline{\Phi}[\hat{\eta}, \hat{\boldsymbol{u}}_f, \hat{p}_0] + \rho_1 \overline{\phi}[\hat{\eta}, \hat{\boldsymbol{u}}_f]\|^2_{L^2(H^{1/2})} \\ &\le c\rho_1^2 \|\nabla \hat{p}_0\|^2_{L^2} + c\rho_2^2 \Big(1 + \|\eta_0\|^3_{H^{5/2}} + T^{3/2} \|\frac{\partial \hat{\eta}}{\partial t}\|^3_{L^2(H^{5/2})}\Big)^2 |||\hat{\boldsymbol{u}}_f|||^2 \end{aligned} \tag{1.65}$$

Similarly, we can derive an estimate on $\frac{\partial^2 \tilde{\eta}}{\partial t^2}$:

$$\begin{aligned} \|\frac{\partial^2 \hat{\eta}}{\partial t^2}\|_{L^2(0,T;H^{-1/2})} &\le c\rho_1 \|\nabla \hat{p}_0\|_{L^2} \\ &\quad + c\rho_2 T^{1/8}\Big(1 + \|\eta_0\|^2_{L^\infty} + |||\hat{\eta}|||^2\Big) |||\hat{\boldsymbol{u}}_f||| + cT^{1/2}|||\hat{\eta}||| \end{aligned} \tag{1.66}$$

Let us now consider the linearized fluid problem (1.63) that takes the form of a non homogeneous unsteady Stokes problem. In order to transform the problem under another one that is more classical, Beirao da Veiga proposes to lift the two data $\overline{\boldsymbol{g}}[\hat{\eta}, \hat{\boldsymbol{u}}_f]$ and $\hat{\boldsymbol{u}}_f{}^0(x, z)$. This is done through the definition of a vector field $\boldsymbol{v}$ such that

$$\begin{aligned} \mathbf{div}\, \boldsymbol{v} &= \overline{\boldsymbol{g}}[\hat{\eta}, \hat{\boldsymbol{u}}_f], \\ \boldsymbol{v}(0, x, z) &= \hat{\boldsymbol{u}}_f{}^0(x, z). \end{aligned} \tag{1.67}$$

The construction of such a vector field is rather intricate and involves the resolution of different Poisson problems from which the gradient of the solution is taken. A stability in the $L^2(0,T;H^2)\cap H^1(0,T;L^2)$ norm is natural and is achieved thanks to a careful analysis of the traces. This allows to transform the original problem (1.63) in another one, similar but where the boundary condition is homogeneous and the divergence is free. The regularity and stability of the solution of such a standard Stokes problem is consequence of the regularity of the boundary of the domain $\mathcal{C}$ (remind the periodic condition imposed in the x direction). The stability of the associated solution in the $|||.|||$-norm is governed by the $L^2(0,T;L^2)$ bound on the right hand side that leads to the following statement : the solution $(\tilde{\boldsymbol{u}}_f,\nabla\tilde{p})$ is stable in the following sense

$$\begin{aligned}|||\tilde{\boldsymbol{u}}_f,\nabla\tilde{p}|||^2 \le{}& c\|\hat{\boldsymbol{u}}_f{}^0\|^2_{H^1}\\ &+c\Big(\|\eta_0\|^2_{L^\infty}+\frac{\partial\eta_0}{\partial x}\|^2_{L^\infty}+\frac{\partial\eta_0}{\partial x}\|^4_{L^\infty}+T^{2/3}\|\eta_0\|^2_{H^{5/2}}\\ +T\|\frac{\partial\hat{\eta}}{\partial t}\|^2_{L^\infty(0,T;H^{3/2})}&+T\|\frac{\partial\hat{\eta}}{\partial t}\|^2_{L^2(0,T;H^{5/2})}+T^2\|\frac{\partial\hat{\eta}}{\partial t}\|^4_{L^\infty(0,T;H^{5/2})}\Big)|||\hat{\boldsymbol{u}}_f|||^2\\ &+c\Big(\|\eta_0\|^2_{L^\infty}+\frac{\partial\eta_0}{\partial x}\|^2_{L^\infty}\Big)\|\frac{\partial\hat{\boldsymbol{u}}_f}{\partial t}|||^2_{L^2(0,T;L^2)}\\ &+cT^{1/2}\Big(1+\|\eta_0\|^2_{H^{5/2}}+\|\frac{\partial\hat{\eta}}{\partial t}\|^2_{L^2(0,T;H^{5/2})}\Big)|||\hat{\boldsymbol{u}}_f|||^4\\ &+c\Big(\|\eta_0\|^2_{L^\infty}+\frac{\partial\eta_0}{\partial x}\|^2_{L^\infty}+T\|\frac{\partial\hat{\eta}}{\partial t}\|^2_{L^2(0,T;H^{5/2})}\Big)\|\nabla\hat{p}_0\|^2_{L^2(0,T;L^2)}\\ &+c\Big(\|\frac{\partial\hat{\eta}}{\partial t}\|^2_{L^2(0,T;H^{3/2})}+\|\frac{\partial\hat{\eta}}{\partial t}\|^2_{H^{3/4}(0,T;L^2)}+\|\frac{\partial^2\hat{\eta}}{\partial t^2}\|^2_{L^2(0,T;H^{-1/2})}\Big)\end{aligned}\tag{1.68}$$

1.4.4 The fixed point procedure

From the inequalities

$$\|\frac{\partial\hat{\eta}}{\partial t}\|_{L^2(0,T;H^{3/2})}\le cT^{1/2}\|\frac{\partial\hat{\eta}}{\partial t}\|_{L^\infty(0,T;H^{3/2})}$$

and

$$\|\frac{\partial\hat{\eta}}{\partial t}\|_{H^{3/4}(0,T;L^2)}\le c\|\frac{\partial\hat{\eta}}{\partial t}\|^{1/4}_{H^{3/4}(0,T;L^2)}\|\frac{\partial\hat{\eta}}{\partial t}\|^{3/4}_{H^1(0,T;H^{-1/2})}$$

we first derive that

$$\|\frac{\partial\hat{\eta}}{\partial t}\|_{H^{3/4}(0,T;L^2)}\le cT^{1/8}\Big(\|\hat{\eta}\|_{L^\infty(0,T;H^{3/2})}+\|\frac{\partial\hat{\eta}}{\partial t}\|^{1/4}_{L^\infty(0,T;H^{3/2})}\|\frac{\partial^2\hat{\eta}}{\partial t^2}\|^{3/4}_{L^2(0,T;H^{-1/2})}\Big)$$

We have illustrated here the way to incorporate some dependency in T in the estimates similar as those that have been used to get e.g. (1.68). Summing up, we derive that for T small enough, the mapping $\mathcal{T}:(\hat{\eta},\hat{\boldsymbol{u}}_f,\nabla\hat{p})\longrightarrow(\tilde{\eta},\tilde{\boldsymbol{u}}_f,\nabla\tilde{p})$ maps $I\!\!K$ into itself where

$$I\!\!K=\{(\hat{\eta},\hat{\boldsymbol{u}}_f,\nabla\hat{p})|,|||\hat{\boldsymbol{u}}_f,\nabla\hat{p}|||\le K_0,|||\hat{\eta}|||\le K_1,\|\frac{\partial^2\eta}{\partial t^2}|||^2_{L^2(0,T;H^{-1/2})}\le K_2\}$$

with three constants K_0, K_1 and K_2 appropriately chosen. in addition $I\!\!K$ is a compact subset of $L^2(0,T;L^2) \times L^2(0,T;L^2) \times H^{-1}(0,T;L^2)$.

The Schauder theorem can be applied to get a fixed point of $\mathcal{T}$ once it is checked that $\mathcal{T}$ is continuous with respect to the $L^2(0,T;L^2) \times L^2(0,T;L^2) \times H^{-1}(0,T;L^2)$-topology which is done in details in [2]. This proves the existence of the solution to the problem in the ALE form. The regularities on the solution are sufficient to turn back to the original variables (t,x,y). By using the inverse transform to (1.56), we obtain a solution $(\eta, \boldsymbol{u}_f, p)$ to the fluid vessel coupling problem.

As a final remark, it should be noted that in this section and in the previous one, the regular feature of the structure is a fundamental ingredient for the proof of the existence of a solution. Indeed in both of these analysis, the presence of a visco-elastic contribution has led to increased stability that gave enough compactness to the solutions. Independently of the theoretical question that, at this point, remains unanswered that the coupled problem with standard elasticity (i.e. with no visco-elastic contribution) is well posed or not, this rings a bell at the level of the numerical simulation since most of the times, no such visco-elastic contribution is incorporated in the models. It is well known however that discretization schemes classically add inherent viscosity contributions in the original model and, first this could be the reason why such terms need not be required in the current simulations, second, if the visco-elastic terms would reveal mandatory at the continuous level, this might lead to some problems when the discretization parameters would tend too much to zero. This question, up to date, remains unsolved in this conformation. Nevertheless, we refer to the papers of Coutand and Shkoller [4, 5] where the elastic body, floating *within* a fluid is analyzed and no such viscoelastic term is added. The new ingredient of that paper is the analysis of the fluid part in an hyperbolic-type functional framework that, at the price of increased compatibility assessments between the initial and boundary conditions, allows to get rid about the increased regularization of the elastic behavior. These results are not yet extended to the configuration we are interested in, in this chapter of a fluid inside an elastic envelop.

1.5 A full interaction problem with zero structural mass

The model we consider in this last section corresponds to a more recent analysis, since, at this date, it is still a preprint. This work, due to Cheng, Coutand and Shkoller [1] deals with the full fluid-structure interaction problem where the fluid, modeled by the viscous incompressible equations is enclosed by a moving thin nonlinear elastic shell. The three dimensional fluid interacts here with a structure represented by a two dimensional quasilinear elastic model of Koiter shell type which is directly derived from the asymptotic expansion in the nonlinear three dimensional St Venant-Kirchhoff equations when the thickness of the shell converges to zero. The movement of the structure is assumed to be inertia-free, nevertheless the main difficulty of the coupling is present since the shape of the fluid domain is nonstationary and unknown.

1.5.1 Navier-Stokes/Koiter coupling

We denote again by Ω_0^f an open bounded domain in $\mathbb{R}^3$ with boundary $\Gamma_0 = \partial\Omega_0^f$. For each time $t \in (0,T)$ we look for a volume-preserving transformation $\boldsymbol{\eta}(t,.) : \Omega_0^f \longrightarrow \mathbb{R}^3$, a domain $\Omega^f(t) = \boldsymbol{\eta}(t,\Omega_0^f)$, a divergence free velocity field $\boldsymbol{u}_f(t,.)$ and a pressure $p(t,.)$ defined over $\Omega^f(t,.)$ such that, in addition to (1.1), (1.2) valid with $\Omega^f = \Omega^f(t)$, we have

$$\boldsymbol{\eta}_t(t,x) = \boldsymbol{u}_f(t,\boldsymbol{\eta}(t,x)) \tag{1.69}$$

complemented with the interface conditions

$$2\nu \boldsymbol{D}\boldsymbol{u}_f.\boldsymbol{n} - p\boldsymbol{n} = t_{shell}, \qquad \text{over } \Gamma(t) = \boldsymbol{\eta}(t,\Gamma_0) \tag{1.70}$$

(where the traction vector t_{shell} will be detailed latter) and subject to the initial conditions

$$\begin{cases} \boldsymbol{u}_f(0) = u_0, & \text{over } \Omega_0^f, \\ \boldsymbol{\eta}(0,x) = x, & \forall x \in \Omega_0^f \end{cases} \tag{1.71}$$

In the most generality, the traction vector t_{shell} is derived from the nonlinear Saint Venant-Kirschhoff constitutive law by cancelling out the first variation of the hyperelastic stored energy:

$$E_{shall} = \varepsilon E_{mem} + \varepsilon^3 E_{ben}$$

where ε stands for the thickness of the shell and where the membrane energy satisfies

$$E_{mem} = \int_{\gamma(t)} \Big[\frac{\mu}{4}\sum_{\alpha,\beta=1}^{2}(g_{\alpha\beta} - g_{0\alpha\beta})^2 + \frac{\mu\lambda}{4(2\mu+\lambda)}\Big(\sum_{\alpha=1}^{2}(g_{\alpha\alpha} - g_{0\alpha\alpha})\Big)^2\Big]ds$$

while the bending energy E_{ben} is given by

$$E_{ben} = \int_{\Gamma(t)} \big[(4\mu + 2\lambda)H^2 - 2\mu K\big]ds.$$

In the previous expression, g denotes the induced metric on the surface $\Gamma(t)$, H and K denote the mean and Gauss curvature on $\Gamma(t)$ and $\lambda/2$ and $\mu/2$ are the Lamé constants.

Adopting a local coordinate system in a tubular neighborhood of Γ_0 composed of tangential coordinates y^1 and y^2 and a normal one, the bending traction has the form of a, possibly degenerate, fourth order tangent derivative operator acting on the normal displacement h taking the form

$$\frac{1}{\sqrt{\det(g)}}\frac{\partial^2}{\partial y^\gamma \partial y^\delta}\big[\sqrt{\det(g)}A^{\alpha\beta\gamma\delta}\frac{\partial^2 h}{\partial y^\alpha \partial y^\beta}\big]$$

(where $A^{\alpha\beta\gamma\delta}$ is a fourth-rank tensor) plus some lower order terms, whereas the membrane traction is a second order derivative operator.

1.5.2 Lagrangian formulation of the problem

Following the strategy developped in two former papers of Coutand and Shkoller [4, 5], the analysis is performed on the Lagrangian formulation of the problem. Hence after introducing the Lagrangian velocity $\boldsymbol{v}_f = \boldsymbol{u}_f \circ \boldsymbol{\eta}$ and the Lagrangian pressure $q = p \circ \boldsymbol{\eta}$ and $F = f \circ \boldsymbol{\eta}$ the coupled system can be rewritten as

$$\begin{cases} \boldsymbol{\eta}_t = \boldsymbol{v}_f, & \text{in } (0,T)\times\Omega_0^f, \\ {\boldsymbol{v}_f}_t^i - \nu(a_\ell^j D_{\boldsymbol{\eta}}(\boldsymbol{v}_f)_\ell^i)_{,j} = -(a_i^k q)_{,k} + F^i & \text{in } (0,T)\times\Omega_0^f, \\ a_i^k {\boldsymbol{v}_f}_{,k}^i = 0 & \text{in } (0,T)\times\Omega_0^f, \\ (\nu D_{\boldsymbol{\eta}}(\boldsymbol{v}_f)_\ell^i - q\delta_\ell^i) a_\ell^j \boldsymbol{n}_j = \varepsilon t_{mem} + \varepsilon^3 t_{ben} & \text{on } (0,T)\times\Gamma_0^f, \\ h_t = h_{,\alpha}(\boldsymbol{v}_f \circ \boldsymbol{\eta}^{-\tau})_\alpha + (\boldsymbol{v}_f \circ \boldsymbol{\eta}^{-\tau})_z & \text{on } (0,T)\times\Gamma_0^f, \\ \boldsymbol{v}_f = \boldsymbol{u}_{f0} & \text{on } \Omega_0^f \\ h = 0 & \text{on } \Gamma_0 \\ \boldsymbol{\eta}(0,x) = x, & \forall x \in \Omega_0^f \end{cases} \tag{1.72}$$

As in the previous sections, the interest of this transformation is to work over a fixed domain so that standard imbedding, compactness and Korn or Poincaré type inequalities are available.

The analysis of this nonlinear problem involves as in section 1.3 regularization operators, one for ensuring that the forcing terms and the initial data are smooth enough, another one to regularize the right hand side of the linearized version of the problem. An additional penalization ingredient is incorporated to take care about the transformed incompressibility constraint. The simplest result proved in this rich paper deals with the case where the membrane contribution is neglected and reads

Theorem 1.3. *Assume the data* $\boldsymbol{F}$ *satisfies*

$$\boldsymbol{F} \in L^2(0,T;H^2(\Omega_0^f)) \cap H^1(0,T;L^2(\Omega_0^f)), \quad \boldsymbol{F}(0) \in H^1(\Omega_0^f)$$

and that the initial data $\boldsymbol{u}_{f0} \in H^{5/2}(\Omega_0^f)$ *and its trace* $\boldsymbol{u}_{f0|\gamma_0} \in H^{9/2}$*, and that the tangeantial component of* $\boldsymbol{D}\boldsymbol{u}_{f0}.\boldsymbol{n}$ *vanishes. Suppose in addition that the shell traction is composed only of bending contributions*

$$t_{shell} = t_{ben}.$$

There exists a solution $(\boldsymbol{u}_f, p, h)$ *to the full interaction problem in Lagrangian form, with* $\boldsymbol{u}_f \in L^2(0,T,H^3(\Omega_0^f)) \cap H^1(0,T,H^1(\Omega_0^f))$ *and* $h \in L^2(0,T;H^{11/2}(\Gamma_0)) \cap H^1(0,T;H^{5/2}(\Gamma_0)) \cap H^2(0,T;H^{1/2}(\Gamma_0)) \cap L^\infty(0,T;H^2(\Gamma_0))$*. The solution is unique under appropriate compatibility conditions.*

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