THÉORIE DES ÉQUATIONS D'ÉVOLUTION

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Chapter 1

A first approach of evolution equations

The purpose of this course is to provide some basic techniques in order to study evolution partial differential equations. In such equations, one variable (namely the time variable) plays a special role. Let us first present three examples of linear partial differential equations which we shall meet later on in the course :

- the heat equation which is a model for so called parabolic equations

$$(HE) \quad \partial_t u - \Delta u = 0$$

- the wave wave equation, which is a model for so called parabolic equations

$$(WE) \quad \partial_t^2 u - \Delta u = 0$$

- Schrödinger equation which is a model for so called dispersive equation

$$(SE) \quad i\partial_t u + \Delta u = 0.$$

To each of this equation, will correspond a non linear model in the following.

The heat equation will appear in the following system which is a model for the description of the evolution of an incompressible viscous fluid. A fluid is described by a time dependant vector fields v which is supposed to describe the speed of a pointwise particle located in x at time t. The system is the following

$$\left\{\begin{array}{rcl} \partial_t v + v\cdot\nabla v - \nu\Delta v & = & -\nabla p \\ \operatorname{div} v & = & 0 \\ v_{|t=0} & = & v_0 \end{array}\right.$$

Here, ν denotes a postive real number which represents the viscosity of the fluid. This system is known as the incompressible Navier-Stokes system.

For the wave equation, the following system is related to gas dynamics. The unknown is the couple (ρ, v) which satisfies

$$\left\{\begin{array}{rcl} \partial_t\rho + v\cdot\nabla\rho + \rho\operatorname{div} v & = & 0 \\ \partial_t v + v\cdot\nabla v + \dfrac{1}{\rho}\nabla p & = & 0 \end{array}\right.$$

with $p = A\rho^\gamma$. Here, ρ is a scalar function with values in $\mathbb{R}_\star^+$ and represents the density of the particules of the gas at time t in the point x and v a time dependant vector field which describes the speed of a particule located in x at time t.

It will be clear later on that we have to change the unknowns defining

$$c \stackrel{\text{def}}{=} \frac{2}{\gamma-1}\left(\frac{\partial p}{\partial \rho}\right)^{\frac{1}{2}} = \frac{(2\gamma A)^{\frac{1}{2}}}{\gamma-1}\rho^{\frac{\gamma-1}{2}}.$$

The first equation becomes

$$\partial_t c + v\cdot\nabla c + \frac{\gamma-1}{2} c \operatorname{div} v = 0.$$

About the second one, let us observe that

$$\frac{\gamma-1}{2} c\nabla c = \frac{1}{\rho}\nabla p.$$

The Euler system related to gas dynamics becomes

$$\left\{\begin{array}{rcl} \partial_t c + v\cdot\nabla c + \dfrac{\gamma-1}{2} c \operatorname{div} v & = & 0 \\ \partial_t v + v\cdot\nabla v + \dfrac{\gamma-1}{2} c\,\nabla c & = & 0. \end{array}\right. \tag{1.1}$$

Let us assume that the solution is "small", i.e. is a perturbation of magnitude ε of a stationnary flat state $v = 0$ and $c = \overline{c}$, by an easy computation of the coefficients of the powers of ε, we infer

$$\left\{\begin{array}{rcl} \partial_t c + \dfrac{\gamma-1}{2} \overline{c} \operatorname{div} v & = & 0 \\ \partial_t v + \dfrac{\gamma-1}{2} \overline{c}\,\nabla c & = & 0. \end{array}\right. \tag{1.2}$$

An obvious computation ensures that

$$\partial_t^2 c - \left(\frac{\gamma-1}{2}\right)^2 \overline{c}^2 \Delta c = 0.$$

This equation is called "acoustic waves equation".

Then, we shall study non linear Schrödinger equations of the type

$$\partial_t u + \frac{i}{2}\Delta u = \pm |u|^{p-1} u$$

for a real number greater or equal to 1.

1.1 A review on ordinary differential equation

Before starting the study of evolution partial differential equation, let us have a look on basic properties of ordinary differential equations.

1.1.1 The linear case

Let E be a Banach space, I an open interval of $\mathbb{R}$ and A a map from I to $\mathcal{L}(E)$, the set of continuous linear maps from E into E. We want to solve the equation

$$(ODE)\begin{cases} \dot{u} \stackrel{\text{def}}{=} \dfrac{du}{dt} &= A(t)u(t) \\ u(0) &= u_0. \end{cases}$$

The proof of the existence and uniqueness of solutions of this equation is very simple. Let λ be a positive real number, let us introduce the space E_λ definied by

$$E_\lambda = \Big\{u \in C(I,E)\,/\,\|u\|_\lambda \stackrel{\text{def}}{=} \sup_{t\in I} \|u(t)\| \exp\Big(-\lambda \int_0^t \|A(t')\|_{\mathcal{L}(E)} dt'\Big) < \infty\Big\}.$$

The solution of (ODE) are the same as the solutions of

$$Lu = u_0 \quad \text{with} \quad Lu(t) \stackrel{\text{def}}{=} u(t) - \int_0^t A(t')u(t')dt'.$$

We have

$$\|(Lu-u)(t)\| \leq \int_0^t \|A(t')\|_{\mathcal{L}(E)} \|u(t')\| dt'.$$

Thus we deduce that

$$\begin{aligned} &\|(Lu-u)(t)\| \exp\Big(-\lambda \int_0^t \|A(t')\|_{\mathcal{L}(E)} dt'\Big) \\ &\leq \int_0^t \exp\Big(-\lambda \int_{t'}^t \|A(t'')\|_{\mathcal{L}(E)} dt''\Big) \|A(t')\|_{\mathcal{L}(E)} \exp\Big(-\lambda \int_0^{t'} \|A(t'')\|_{\mathcal{L}(E)} dt''\Big) \|u(t')\| dt'. \end{aligned}$$

By definition of $\|\cdot\|_\lambda$, we infer that

$$\|(Lu-u)(t)\| \exp\Big(-\lambda \int_0^t \|A(t')\|_{\mathcal{L}(E)} dt'\Big) \leq \frac{1}{\lambda} \|u\|_\lambda$$

and thus that $\|Lu - u\|_\lambda \leq \lambda^{-1}\|u\|_\lambda$. This implies that, for λ greater than 1, L is invertible in $\mathcal{L}(E)$. Then the proof of the existence and uniqueness of solutions is achieved.

1.1.2 The case of non linear equations with almost lipschitz vector field

We still work in a Banach space E and an interval I of $\mathbb{R}$. Let F be a function of $I \times E$ into E.

In the whole of this section, μ will denote a functionfrom $\mathbb{R}^+$ into itsself, vanishing at 0, positive outside 0, continuous and non decreasing.

Definition 1.1.1 *Let (X,d) and (Y,δ) be two metric spaces. We denote by $\mathcal{C}_\mu(X,Y)$ the set of the bounded functions from X into Y such that a constant C exists such that, for any $(x,y) \in X^2$, we have*

$$\delta(u(x), u(y)) \leq C\mu(d(x,y)).$$

Remark If (Y,δ) is a Banach space (which we denote $(E,\|\cdot\|)$ in this case), the space $\mathcal{C}_\mu(X,E)$ is a Banach space equipped with the norm

$$\|u\|_\mu = \|u\|_{L^\infty} + \sup_{(x,y)\in X\times X,\, x\neq y} \frac{\|u(x)-u(y)\|}{\mu(d(x,y))}.$$

The following theorem provides hypotheses for which there is existence and uniqueness of integral curve for an ordinary differential equation.

Theorem 1.1.1 *Let E be a Banach space, Ω an open subset of E , I an open interval of $\mathbb{R}$ and (t_0, x_0) an element of $I \times \Omega$. Let us consider a function F of $L^1_{loc}(I; \mathcal{C}_\mu(\Omega; E))$. Let us assume in addition that*

$$\int_0^1 \frac{dr}{\mu(r)} = +\infty. \tag{1.3}$$

Then an interval J exists such that $t_0 \in J \subset I$ and such that the equation

$$(ODE) \quad x(t) = x_0 + \int_{t_0}^t F(t', x(t'))dt'$$

has a unique continuous solution defined on the interval J.

Remark If $\mu(r) = r$, this theorem is nothing more than the familiar Cauchy-Lipschitz theorem. But let us point out that other functions satisfy the hypotheses of the theorem; for instance the function defined by $\mu(r) = -r \log r$ for $r \leq e^{-1}$ and $\mu(r) = e^{-1}$ if not.

Proof of Theorem 1.1.1 Let us begin by the proof of uniqueness of trajectories. Let $x_1(t)$ and $x_2(t)$ two solutions of (ODE) defined on a neighbourhoood $\widetilde{J}$ of t_0 with this the same initial data x_0. Let us denote

$$\rho(t) \stackrel{\text{def}}{=} ||x_1(t) - x_2(t)||.$$

As F belongs to $L^1_{loc}(I; \mathcal{C}_\mu(\Omega, E))$, we have

$$0 \leq \rho(t) \leq \int_{t_0}^t \gamma(t')\mu(\rho(t'))dt' \quad \text{with} \quad \gamma \in L^1_{loc}(I) \quad \text{and} \quad \gamma \geq 0. \tag{1.4}$$

In the case when $\mu(r) = r$, Gronwall lemma implies that $\rho \equiv 0$. Let us recall a version of Gronwalll lemma which will be useful in the following.

Lemma 1.1.1 *Let f and g be two C^0 (resp. C^1) non negative functions on $[t_0, T]$. Let $\mathcal{A}$ be a continuous functions on $[t_0, T]$. Suppose that, for t in $[t_0, T]$,*

$$\frac{1}{2}\frac{d}{dt} g^2(t) \leq \mathcal{A}(t)\, g^2(t) + f(t)g(t). \tag{1.5}$$

Then for any time t in $[t_0, T]$, we have

$$g(t) \leq g(t_0) \exp \int_{t_0}^t \mathcal{A}(t')\, dt' + \int_{t_0}^t f(t') \exp\Big(\int_{t'}^t \mathcal{A}(t'')\, dt''\Big)\, dt' \,.$$

Proof. Let us define

$$g_\mathcal{A}(t) \stackrel{\text{def}}{=} g(t) \exp\Big(-\int_{t_0}^t \mathcal{A}(t')\, dt'\Big) \quad \text{and} \quad f_\mathcal{A}(t) \stackrel{\text{def}}{=} f(t) \exp\Big(-\int_{t_0}^t \mathcal{A}(t')\, dt'\Big).$$

Obviously, we have $\frac{1}{2}\frac{d}{dt} g^2_\mathcal{A} \leq f_\mathcal{A} g_\mathcal{A}$ so that for any positive ε,

$$\frac{d}{dt}(g^2_\mathcal{A} + \varepsilon^2)^{\frac{1}{2}} \leq \frac{g_\mathcal{A}}{(g^2_\mathcal{A} + \varepsilon^2)^{\frac{1}{2}}} f_\mathcal{A} \leq f_\mathcal{A}.$$

By integration, we get

$$(g^2_\mathcal{A}(t) + \varepsilon^2)^{\frac{1}{2}} \leq (g^2_\mathcal{A}(0) + \varepsilon^2)^{\frac{1}{2}} + \int_0^t f_\mathcal{A}(t')\, dt'.$$

Having ε tend to 0 gives the result. □

The key lemma for the proof of Theorem 1.1.1 is the following.

Lemma 1.1.2 *Let ρ be a measurable non negative function, γ a non negative function locally integrable and μ a continuous non decreasing function. Let us assume that, for a non negative real number a, the function ρ satisfies*

$$\rho(t) \leq a + \int_{t_0}^{t} \gamma(t')\mu(\rho(t'))dt'. \tag{1.6}$$

If a is positive, then we have

$$-\mathcal{M}(\rho(t)) + \mathcal{M}(a) \leq \int_{t_0}^{t} \gamma(t')dt' \quad \textit{with} \quad \mathcal{M}(x) = \int_{x}^{1} \frac{dr}{\mu(r)}\cdot \tag{1.7}$$

If $a = 0$ and if μ satisfies (1.3), then the function ρ is identically 0.

Proof. In order to prove this lemma, let us assume that a is positive and let us define

$$R_a(t) \stackrel{\text{def}}{=} a + \int_{t_0}^{t} \gamma(t')\mu(\rho(t'))dt'.$$

The function R_a is a continuous non decreasing function. Thus, we have,

$$\dot{R}_a(t) = \gamma(t)\mu(\rho(t)).$$

As the function μ is non decreasing, we have

$$\dot{R}_a(t) \leq \gamma(t)\mu(R_a(t)). \tag{1.8}$$

The function R_a is positive. As the function $\mathcal{M}$ is C^1 on $]0,\infty[$, Inequality (1.8) implies that

$$-\frac{d}{dt}\mathcal{M}(R_a(t)) = \frac{\dot{R}_a(t)}{\mu(R_a(t))} \leq \gamma(t).$$

By inegration, we get (1.7) using that the function $-\mathcal{M}$ is increasing and that $\rho \leq R_a$.

Let us assume now that $a = 0$ and let us proceed by contraposition. Let us assume that the function ρ is not identically 0 near t_0. As the function μ is non decreasing, we can substitute $\sup_{t' \in [t_0,t]} \rho(t')$ to the function ρ (we continue to use the same notation ρ). A real number $t_1 > t_0$ exists, such that $\rho(t_1) > 0$. As the function ρ satisfies (1.6) for $a = 0$,it satisfies also this inequality for any positive a'. It comes from (1.7) that

$$\forall a' > 0\,,\ \mathcal{M}(a') \leq \int_{t_0}^{t_1} \gamma(t')dt' + \mathcal{M}(\rho(t_1)).$$

This implies that the integral

$$\int_0^1 \frac{dr}{\mu(r)}$$

is convergent; the proof of the lemma is done. □

Continuation of the proof of Theorem 1.1.1 Thanks to Inequality (1.4), the uniqueness of integral curve issued from a point is an obvious consequence of Lemma 1.1.2.

Let us prove the existence. In order to do so, let us consider the classical Picard scheme

$$x_{k+1}(t) = x_0 + \int_{t_0}^{t} F(t', x_k(t'))dt'.$$

Let us skip the proof of the fact that, for a small enough interval J, the sequence $(x_k)_{k\in\mathbb{N}}$ is bounded in $L^\infty(J)$. Let us prove that this sequence is a Cauchy one in the space of continuous functions from J to E. In order to do so, let us define

$$\rho_{k+1,n}(t) \stackrel{\text{def}}{=} ||x_{k+1+n}(t) - x_{k+1}(t)||.$$

It turns out that

$$0 \leq \rho_{k+1,n}(t) \leq \int_{t_0}^{t} \gamma(t')\mu(\rho_{k,n}(t'))dt'$$

Let us define $\rho_k(t) \stackrel{\text{def}}{=} \sup_n ||x_{k+1+n}(t) - x_{k+1}(t)||$. As the function μ is non decreasing, we have

$$0 \leq \rho_{k+1}(t) \leq \int_{t_0}^{t} \gamma(t')\mu(\rho_k(t'))dt'.$$

Thanks to Fatou lemma, we get, using that the function μ is non decreasing,

$$\widetilde{\rho}(t) \stackrel{\text{def}}{=} \limsup_{k\to+\infty} \rho_k(t) \leq \int_{t_0}^{t} \gamma(t')\mu(\widetilde{\rho}(t'))dt'.$$

Applying again Lemma 1.1.2, we find that $\widetilde{\rho}(t)$ is identically 0 near t_0; this concludes the proof of Theorem 1.1.1. □

Let us point out that the concepts of iterative scheme and of Cauchy sequence plays a key role.

1.1.3 Blow up criteria

The existence and uniqueness theorem for ordinary differential equations is a local theorem. Let us investigate what can be necessary conditions for a blow up phenomena.

Proposition 1.1.1 *Let F be a function of $\mathbb{R}\times E$ in E satisfying the hypothesis of Theorem 1.1.1 in any point x_0 of E. Let us assume in addition that a locally bounded function M from $\mathbb{R}^+$ into $\mathbb{R}^+$ and a locally integrable function β from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that*

$$\|F(t,u)\| \leq \beta(t)M(\|u\|).$$

then, if the maximal interval of definition is $]T_\star, T^\star[$, then, if $T^\star$ is finite,

$$\limsup_{t\to T^\star} \|u(t)\| = \infty.$$

Proof. Let us first prove that, if we consider a time $T > T_0$ such that $\|u(t)\|$ is bounded on the interval $[T_0, T[$, then we can define the solution on a larger interval $[T_0, T_1]$ with $T_1 > T$. As the function u is bounded on the interval $[T_0, T[$, the hypothesis on F that, for any t of the interval $[T_0, T[$, we have

$$\|F(t, u(t))\| \leq C\beta(t).$$

The function β being integrable on the interval $[T_0, T]$, we have deduce que, for any ε strictement positive, it exists a positive real number η such that, pour tout t and t' such that $T-t<\eta$ and $T-t'<\eta$,

$$\|u(t)-u(t')\|<\varepsilon.$$

The space E being complete, an element $u_\star$ of E exists such that

$$\lim_{t\to T^\star} u(t)=u_\star.$$

Applying Theorem 1.1.1, we construct solution of (ODE) on some $[T_+^\star, T_1]$ and the continuous function defined by induction on the interval $[T_0, T_1]$ is a solution of the equation (ODE) on the interval $[T_0, T_1]$. □

Corollary 1.1.1 *Under the hypothesis of Proposition 1.1.1, if we have in addition that*

$$\|F(t,u)\|\leq M\|u\|^2,$$

then, if the interval $]T_\star, T^\star[$ *is the maximal interval of definition of* u *and if* $T^\star$ *is finite, then*

$$\int_{t_0}^{T^\star}\|x(t)\|dt=\infty.$$

Proof. The solution satisfies, for any $t\geq t_0$

$$\|x(t)\|\leq\|x(t_0)\|+M\int_{t_0}^{t}\|x(t')\|^2dt'. \tag{1.9}$$

Gronwall's Lemma implies that

$$\|x(t)\|\leq\|x_0\|\exp\Big(M\int_0^t\|x(t')\|dt'\Big).$$

A more precise way of proving this result is the following.

Let $T\stackrel{\text{def}}{=}\sup\{t\in[t_0,T^\star[\,/\ \|x(t)\|\leq 2\|x(t_0)\|\}$. For any $t\in[t_0,T^\star[$, we have, using (1.9),

$$\|x(t)\|\leq\|x(t_0)\|+4M(t-t_0)\|x(t_0)\|^2.$$

Thus we infer

$$\forall t\in\Big[t_0,\min\Big\{T,t_0+\frac{1}{4M\|x(t_0\|)}\Big\}\Big[\,,\quad \|x(t)\|\leq 2\|x_0\|.$$

Thanks to Proposition 1.1.1, we have

$$T^\star-t_0\geq\frac{c}{\|x_0\|}\cdot$$

Applying again this result at time $t\in[t_0,T^\star[$, we find that

$$\forall t\in[t_0,T^\star[\,,\quad\|x(t)\|\geq\frac{c}{T^\star-t}\cdot$$

□

Exercice 1.1.1 *Let F a function defined on $\mathbb{R}\times E$ such that*

$$\sup_{x\in E}\|F(t,x)\| + \sup_{\substack{(x,y)\in E^2\\ 0<\|x-y\|\leq e^{-1}}} \frac{\|F(t,x)-F(t,y)\|}{-\|x-y\|\log\|x-y\|} \leq \beta(t) \quad \textit{with} \quad \beta\in L^1_{loc}(\mathbb{R}).$$

1) Prove that it existsd a map ψ from $\mathbb{R}\times E$ into E such that

$$\psi(t,x) = x + \int_0^t F(t,\psi(t,x))dt$$

2) Prove that, for any t, $\psi(t,\cdot)$ defines a homeomorphism of E such that

$$|x-y|\leq e^{-\exp\int_0^t\beta(s)ds} \Rightarrow |\psi(t,x)-\psi(t,y)|\leq |x-y|^{\exp-\int_0^t\beta(s)ds} e^{-\exp-\int_0^t\beta(s)ds}.$$

1.1.4 A compactness theorem : Peano's theorem

The theorem is the following.

Theorem 1.1.2 (Peano) *Let I be an open interval of $\mathbb{R}$. Let us consider a function f from $I\times\mathbb{R}^d$ into $\mathbb{R}^d$ such that*

- *For any compact K of $\mathbb{R}^d$, the function $t\mapsto\|f(t)\|_{L^\infty(K)}$ is locally integrable,*
- *For any t of I, the function $x\mapsto f(t,x)$ is continuous on $\mathbb{R}^d$.*

Then, for any point (t_0,x_0) of $I\times\mathbb{R}^d$, an open interval $J\subset I$ containing t_0 and a continuous function x on J exists such that

$$(ODE) \qquad x(t) = x_0 + \int_{t_0}^t f(t',x(t'))dt'.$$

Proof. The structure of the proof is at least as intesting as the result itself. This proof will be a model for the proof of existence of weak solutions for ther incompressible Navier-Stokes equation we shall study in Chater 4.

There are three steps in the proof

- we regularize the function f and we apply Cauchy-Lipschitz's Theorem to the sequence of regularized functions; Proposition 1.1.1 ensures that the solutions of the regularized problem have a commun interval of definition,
- then, we prove that the sequence of those solutions of the regularized problem are relatively compact in the space $C(J,\mathbb{R}^d)$,
- as a conclusion, we pass to the limit.

Let us procede to a classical regularization; let χ a non negative function of $\mathcal{D}(B(0,1))$ the integral of which is 1. Let us define $\chi_n(x)\stackrel{\text{def}}{=} n^d\chi(nx)$ and $f_n(t)=\chi_n\star f(t)$. We have

$$\|f_n(t)\|_{L^\infty(K)}\leq\|f(t)\|_{L^\infty(K+B(0,n^{-1}))}.$$

Morevoer, we have

$$\|\partial_j f_n(t)\|_{L^\infty(K)}\leq C(n+1)\|f(t)\|_{L^\infty(K+B(0,n^{-1}))}.$$

We can apply Cauchy-Lipschitz's Theorem of to the function f_n. Let J_n the maximal interval of definition of x_n. Let J an interval ouvert such that

$$\int_J \|f(t)\|_{L^\infty(B(x_0,2)}dt \leq 1.$$

Let us define $t_n \stackrel{\text{def}}{=} \sup\Big\{t \in [t_0,\infty[\cap J \cap J_n \,/\, \forall t' \leq t\,,\ x(t') \in B(x_0,1)\Big\}$. For any $t \leq t_n$, we have

$$\begin{aligned}
\|x_n(t) - x_0\| &\leq \int_J \|f_n(t)\|_{L^\infty(B(x_0,1))}dt \\
&\leq \int_J \|f(t)\|_{L^\infty(B(x_0,2))}dt \\
&\leq 1.
\end{aligned}$$

Thus $t_n \geq \sup J \cap J_n$. working in the same way for the times less to t_0, we find, using Proposition 1.1.1 that, for any n, $J \subset J_n$. This concludes the first part of the proof.

We have

$$\forall t \in J\,,\ X(t) \stackrel{\text{def}}{=} \{x_n(t),\ n \in \mathbb{N}\} \subset B(x_0,1).$$

As we work on a finite dimensionnal space, $X(t)$ is relatively compact. Moreover, we have

$$\begin{aligned}
\|x_n(t) - x_n(t')\| &\leq \Big|\int_t^{t'} \|f_n(t'')\|_{L^\infty(B(x_0,1))}dt''\Big| \\
&\leq \Big|\int_t^{t'} \|f(t'')\|_{L^\infty(B(x_0,2))}dt''\Big|.
\end{aligned}$$

Thus, for any positive ϵ, it exists a positive real number α such that

$$\forall (t,t') \in J^2\,,\ |t-t'| < \alpha \Longrightarrow \|x_n(t) - x_n(t')\| < \epsilon.$$

In other words, the family $(x_n)_{n\in\mathbb{N}}$ is equicontinuous on J. Ascoli's Theorem ensures that the set of functions x_n is relatively compact in $C(J;\mathbb{R}^d)$. Thus we can extract a subsequence which converge uniformely on J to a function x of $C(J;\mathbb{R}^d)$. Let omit to note the extraction in the following.

Now let us pass to the limit. For any t of J; we have

$$\|f_n(t,x_n(t) - f(t,x(t))\| \leq \|f_n(t) - f(t)\|_{L^\infty(B(x_0,1))} + \|f(t,x_n(t)) - f(t,x(t))\|.$$

Thus for any t of J, we have

$$\lim_{n\to\infty} f_n(t,x_n(t)) = f(t,x(t)).$$

Moreover, $\|f_n(t,x_n(t)\| \leq \|f(t)\|_{L^\infty(B(x_0,2))}$. Lebesgue's Theorem ensures that, for any t, we have

$$\lim_{n\to\infty} \int_{t_0}^t f_n(t',x_n(t'))dt' = \int_{t_0}^t f(t',x(t'))dt'.$$

The theorem is proved. □

1.2 The solution of some classical linear PDE in $\mathbb{R}^d$

In this section, we solve the classical Heat equation and Schrödinger equation in the all space $\mathbb{R}^d$. Let us give a definition of a solution of an evolution PDE for non smooth functions (or distributions).

Definition 1.2.1 *Let u be a continuous function from $\mathbb{R}$ with value in $\mathcal{S}'(\mathbb{R}^d)$, which means exactly that for any ϕ in $\mathcal{S}(\mathbb{R}^d)$, that maps defined by*

$$t \longmapsto \langle u(t), \phi\rangle$$

is continuous. Let f be a locally integrale map from $\mathbb{R}$ into $\mathcal{S}'(\mathbb{R}^d)$, which means exactly that for any ϕ in $\mathcal{S}(\mathbb{R}^d)$, that maps defined by

$$t \longmapsto \langle f(t), \phi\rangle$$

is locally integrable on $\mathbb{R}$. Let u_0 be in $\mathcal{S}'(\mathbb{R}^d)$. We say that u is a solution of (HE) if and only if, for any function ϕ of $\mathcal{S}(\mathbb{R}\times\mathbb{R}^d)$, we have, for any t in $\mathbb{R}$,

$$\langle u(t), \phi(t)\rangle - \langle u_0, \phi(0)\rangle = \int_0^t \big\langle u(t'), \Delta\phi(t') + \partial_t\phi(t')\big\rangle dt' + \int_0^t \langle f(t'), \phi(t')\rangle dt'.$$

Proposition 1.2.1 *If u_0 belongs to $\mathcal{S}'(\mathbb{R}^d)$ and f is locally integrale form $\mathbb{R}$ into $\mathcal{S}'(\mathbb{R}^d)$, there is a unique solution of (LS) with a given by*

$$u(t) = \mathcal{F}^{-1}\bigg(e^{-t|\xi|^2}\widehat{u}_0(\xi) + \int_0^t e^{-(t-t')|\xi|^2}\widehat{f}(t',\xi)dt'\bigg). \tag{1.10}$$

Proof. Let us first prove the result using a duality method. As the equation is linear, uniqueness consists only in proving that if, for all ϕ of $\mathcal{S}(\mathbb{R}\times\mathbb{R}^d)$, if

$$\langle u(T), \phi(T)\rangle = \int_0^T \big\langle u(T), \Delta\phi(t) + \partial_t\phi(t)\big\rangle dt \tag{1.11}$$

then $u(T) = 0$ in $\mathcal{S}'(\mathbb{R}^d)$ for all time T. Let us consider any function ϕ_T in $\mathcal{S}'(\mathbb{R}^d)$. Let us observe that

$$\phi(t) \stackrel{\text{def}}{=} \mathcal{F}^{-1}\big(e^{-(T-t)|\xi|^2}\widehat{\phi}_T\big)$$

satisfies $i\Delta\phi(t) + \partial_t\phi(t) = 0$. Then, Identity (1.13) implies that $\langle u(T), \phi(T)\rangle = 0$ and the uniqueness is proved. This is an excercise on distributions theory to prove that if u is given by the formula (1.10)satisfies the equation. □

The case of Schrödinger equation follows the same lines.

Definition 1.2.2 *Let u be a continuous function from $\mathbb{R}$ with value in $\mathcal{S}'(\mathbb{R}^d)$, which means exactly that for any ϕ in $\mathcal{S}(\mathbb{R}^d)$, that maps defined by*

$$t \longmapsto \langle u(t), \phi\rangle$$

is continuous. Let f be a locally integrale map from $\mathbb{R}$ into $\mathcal{S}'(\mathbb{R}^d)$, which means exactly that for any ϕ in $\mathcal{S}(\mathbb{R}^d)$, that maps defined by

$$t \longmapsto \langle f(t), \phi\rangle$$

is locally integrable on $\mathbb{R}$. Let u_0 be in $\mathcal{S}'(\mathbb{R}^d)$. We say that u is a solution of (LS) if and only if, for any function ϕ of $\mathcal{S}(\mathbb{R}\times\mathbb{R}^d)$, we have, for any t in $\mathbb{R}$,

$$\langle u(t), \phi(t)\rangle - \langle u_0, \phi(0)\rangle = \int_0^t \big\langle u(t'), i\Delta\phi(t') + \partial_t\phi(t')\big\rangle dt' - i\int_0^t \langle f(t'), \phi(t')\rangle dt'.$$

Proposition 1.2.2 *If u_0 belongs to $\mathcal{S}'(\mathbb{R}^d)$ and f is locally integrale form $\mathbb{R}$ into $\mathcal{S}'(\mathbb{R}^d)$, there is a unique solution of (LS) with a given by*

$$u(t) = \mathcal{F}^{-1}\biggl(e^{-it|\xi|^2}\widehat{u}_0(\xi) - i\int_0^t e^{-i(t-t')|\xi|^2}\widehat{f}(t',\xi)dt'\biggr). \tag{1.12}$$

Proof. Let us first prove the result using a duality method. As the equation is linear, uniqueness consists only in proving that if, for all ϕ of $\mathcal{S}(\mathbb{R}\times\mathbb{R}^d)$, if

$$\langle u(T), \phi(T)\rangle = \int_0^T \bigl\langle u(T), i\Delta\phi(t) + \partial_t\phi(t)\bigr\rangle dt \tag{1.13}$$

then $u(T) = 0$ in $\mathcal{S}'(\mathbb{R}^d)$ for all time T. Let us consider any function ϕ_T in $\mathcal{S}'(\mathbb{R}^d)$. Let us observe that

$$\phi(t) \stackrel{\text{def}}{=} \mathcal{F}^{-1}\bigl(e^{-i(T-t)|\xi|^2}\widehat{\phi}_T\bigr)$$

satisfies $i\Delta\phi(t) + \partial_t\phi(t) = 0$. Then, Identity (1.13) implies that $\langle u(T), \phi(T)\rangle = 0$ and the uniqueness is proved. This is an excercise on distributions theory to prove that if u is given by the formula (1.12) satisfies the equation. □

Remarks

- All the theorems and all the proofs of this chapter must be known.
- To know more about ordinary differential equations and their historical aspect of Osgood's theory, see the book by T.M. Fleet, *Differential analysis*, Cambridge University Press, 1980.
- To know more about non lipschiztian vector fields satisfying Osgood condition, see the book by J.-Y. Chemin, *Fluides parfaits incompressibles*, Astérisque, **230**, 1995 or its english version *Incompressible perfect fluids*, Oxford University Press, 1998.

Chapter 2

Sobolev spaces

Introduction

In this course, we shall restrict ourselves to Sobolev spaces modeled on L^2. These spaces definitely play a crucial role in the study of partial differential equations, linear or not. The key tool will be the Fourier transform.

2.1 Definition of Sobolev spaces on $\mathbb{R}^d$

Definition 2.1.1 *Let s be a real number, a tempered distribution u belongs to the Sobolev space of index s, denoted $H^s(\mathbb{R}^d)$, or simply H^s if no confusion is possible, if and only if*

$$\widehat{u} \in L^2_{loc}(\mathbb{R}^d) \quad \text{and} \quad \widehat{u}(\xi) \in L^2(\mathbb{R}^d; (1+|\xi|^2)^s d\xi).$$

and we note

$$\|u\|^2_{H^s} = \int_{\mathbb{R}^d} (1+|\xi|^2)^s |\widehat{u}(\xi)|^2 d\xi.$$

Proposition 2.1.1 *For any s real number, the space H^s, equipped with the norm $\|\cdot\|_{H^s}$, is a Hilbert space.*

Proof. The fact that the norm $\|\cdot\|_{H^s}$ comes from the scalar product

$$(u|v)_{H^s} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} (1+|\xi|^2)^s \widehat{u}(\xi)\overline{\widehat{v}(\xi)} d\xi$$

is obvious. Let us prove that this space is complete. Let $(u_n)_{n\in\mathbb{N}}$ a Cauchy sequence of H^s. By definition of the norm, the sequence $(\widehat{u}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of the space $L^2(\mathbb{R}^d; (1+|\xi|^2)^s d\xi)$. Thus, a function $\widetilde{u}$ exists in the space $L^2(\mathbb{R}^d; (1+|\xi|^2)^s d\xi)$ such that

$$\lim_{n\to\infty} \|\widehat{u}_n - \widetilde{u}\|_{L^2(\mathbb{R}^d;(1+|\xi|^2)^s d\xi)} = 0. \tag{2.1}$$

In particular, the sequence$(\widehat{u}_n)_{n\in\mathbb{N}}$ tends to $\widetilde{u}$ in the space $\mathcal{S}'$ of tempered distribtions. Let $u = \mathcal{F}^{-1}\widetilde{u}$. As the Fourier transform is an isomorphism of $\mathcal{S}'$, the sequence $(u_n)_{n\in\mathbb{N}}$ tends to u in the space $\mathcal{S}'$, and also in H^s thanks to (2.1).

Shortly said, this is nothing more than observing that the Fourier transform is an isometric isomorphism from H^s onto $L^2(\mathbb{R}^d; (1+|\xi|^2)^s d\xi)$. $\square$

Proposition 2.1.2 *Let s be a non negative integer, the space $H^s(\mathbb{R}^d)$ is the space of functions u of L^2 all the derivatives of which of order less or equal to m are distributions which belongs to L^2. Moreover, the space H^m equipped with the norm*

$$\widetilde{\|}u\|_{H^m}^2 \stackrel{\text{def}}{=} \sum_{|\alpha|\leq m} \|\partial^\alpha u\|_{L^2}^2$$

is a Hilbert space and this norm is equivalent to the norme $\|\cdot\|_{H^s}$.

Proof. The fact that

$$\widetilde{\|}u\|_{H^m}^2 = \widetilde{(}u|u)_{H^m} \quad \text{with} \quad \widetilde{(}u|v)_{H^m} \stackrel{\text{déf}}{=} \sum_{|\alpha|\leq m} \int_{\mathbb{R}^d} \partial^\alpha u(x) \overline{\partial^\alpha v(x)} dx.$$

ensures that the norm $\|\cdot\|_{H^m}$ comes from a scalar product. Moreover, a constant C exists such that

$$\forall \xi \in \mathbb{R}^d\,,\ C^{-1}\Big(1 + \sum_{0<|\alpha|\leq m} |\xi|^{2|\alpha|}\Big) \leq (1+|\xi|^2)^s \leq C\Big(1 + \sum_{0<|\alpha|\leq m} |\xi|^{2|\alpha|}\Big). \tag{2.2}$$

As the Fourier transform is, up to a constant, an isometric isomorphism from L^2 onto L^2, we have

$$\partial^\alpha u \in L^2 \Longleftrightarrow \xi^\alpha \widehat{u} \in L^2.$$

Thus, we have deduce that

$$u \in H^m \Longleftrightarrow \forall \alpha \,/\, |\alpha| \leq m\,,\ \partial^\alpha u \in L^2.$$

Inequality (2.2) ensures the equivalence of the two norms using again the fact that the Fourier transform is a isometric isomorphism up to a constant. The proposition is proved. □

Exercice 2.1.1 *Prove that the space $\mathcal{S}$ is continuously included in the space H^s for any real s.*

Exercice 2.1.2 *Prove that the mass of Dirac δ_0 belongs to the space $H^{-\frac{d}{2}-\varepsilon}$ for any positive real number ε. Prove that δ_0 does not belong to the space $H^{-\frac{d}{2}}$.*

Exercice 2.1.3 *Prove that, for any distribution to support compact u, it exists a real number s such that u belongs to the Sobolev space H^s.*

Exercice 2.1.4 *Prove that the constant 1 does not belong to H^s for any real number s.*

Proposition 2.1.3 *Let s a real number of the interval $]0,1[$. Prove that the space H^s is the space des functions u of L^2 such that*

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(x+y)-u(x)|^2}{|y|^{d+2s}} dxdy.$$

Moreover, a constant C exists such that, for any function u of H^s, we have

$$C^{-1}\|u\|_{H^s}^2 \leq \|u\|_{L^2}^2 + \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(x+y)-u(x)|^2}{|y|^{d+2s}} dxdy \leq C\|u\|_{H^s}^2.$$

Proof. Thanks to Fourier-Plancherel identity,we can write that

$$\int_{\mathbb{R}^d} \frac{|u(x+y)-u(x)|^2}{|y|^{d+2s}} dx = \int_{\mathbb{R}^d} \frac{|e^{i(y|\xi)}-1|^2}{|y|^{d+2s}} |\widehat{u}(\xi)|^2 d\xi < \infty.$$

It turns out that

$$\begin{aligned} \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(x+y)-u(x)|^2}{|y|^{d+2s}} dxdy &= \int_{\mathbb{R}^d} F(\xi)|\widehat{u}(\xi)|^2 d\xi \quad \text{with} \\ F(\xi) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \frac{|e^{i(y|\xi)}-1|^2}{|y|^{2s}} \frac{dy}{|y|^d} \cdot \end{aligned}$$

By an obvious change of variable, we see that the function F is radial and homogeneous of degree $2s$. Thus

$$F(\xi) = |\xi|^{2s} \int \frac{|e^{iy_1}-1|^2}{|y|^{2s}} \frac{dy}{|y|^d} \cdot$$

This conclude the proof of the proposition. □

Let us prove now an interpolation inequality which will be very useful.

Proposition 2.1.4 *If $s = \theta s_1 + (1-\theta)s_2$ with θ in $[0,1]$, then, we have*

$$\|u\|_{H^s} \leq \|u\|_{H^{s_1}}^\theta \|u\|_{H^{s_2}}^{1-\theta}.$$

The proof consits in applying Hölder inequality with the measure $|\widehat{u}(\xi)|^2 d\xi$ and the two functions $(1+|\xi|^2)^{\theta s_1}$ and $(1+|\xi|^2)^{(1-\theta)s_2}$.

Theorem 2.1.1 *Let s a real quelconque;*

- *the space $\mathcal{D}(\mathbb{R}^d)$ is dense in $H^s(\mathbb{R}^d)$,*
- *the multiplication by a function of $\mathcal{S}$ is a continuous function of H^s into itself.*

Proof. In order to prove the first point of this theorem, let us consider a distribution u of H^s such that, for any test function φ, we have $(\varphi|u)_{H^s} = 0$. This means that, for any test function φ, we have

$$\int_{\mathbb{R}^d} \widehat{\varphi}(\xi)(1+|\xi|^2)^s \overline{\widehat{u}}(\xi) d\xi = 0.$$

which means that $(1+|\cdot|^2)^s \overline{\widehat{u}} = 0$ as a tempered distribution. As the multiplication by the function $(1+|\cdot|^2)^{-s}$ is continuous in $\mathcal{S}'$, we have that $\overline{\widehat{u}} = 0$ as a tempered distribution. Thus $u \equiv 0$.

Let us prove now the second second point of the theorem. This proof is presented here just for culture. We know that

$$\widehat{\varphi u} = (2\pi)^{-d} \widehat{\varphi} \star \widehat{u}.$$

The point is to estimate the L^2 norm of the function defined by

$$U(\xi) = (1+|\xi^2|)^{\frac{s}{2}} \int_{\mathbb{R}^d} |\widehat{\varphi}(\xi-\eta)| \times |\widehat{u}(\eta)| d\eta.$$

We shall use the following lemma.

Lemma 2.1.1 *For any (a,b) in $\mathbb{R}^d$, for any $s\in\mathbb{R}$, we have*

$$(1+|a+b|^2)^{\frac{s}{2}} \leq 2^{\frac{|s|}{2}}(1+|a|^2)^{\frac{|s|}{2}}(1+|b|^2)^{\frac{s}{2}}.$$

Proof. Let us first observe that

$$1+|a+b|^2 \leq 1+2(|a|^2+|b|^2) \leq 2(1+|a|^2)(1+|b|^2).$$

Taking the power $s/2$ of this inequality, we find the result for non negative $s\geq 0$. In the case when s is negative, we have

$$(1+|b|^2)^{-\frac{s}{2}} \leq 2^{-\frac{s}{2}}(1+|a+b|^2)^{-\frac{s}{2}}(1+|a|^2)^{-\frac{s}{2}}.$$

Thus the result is proved. □

Continuation of the proof of Theorem 2.1.1 Lemma 2.1.1 implies that

$$U(\xi) \leq \int_{\mathbb{R}^d} (1+|\xi-\eta|^2|^{\frac{|s|}{2}} |\widehat{\varphi}(\xi-\eta)| \times (1+|\eta|^2|^{\frac{s}{2}} |\widehat{u}(\eta)| d\eta.$$

Young's law implies that $\|U_2\|_{L^2} \leq C\|u\|_{H^s}$; this concludes the proof of the theorem. □

Exercice 2.1.5 *Let $\mathcal{F}L^1 = \{u \in \mathcal{S}' \,/\, \widehat{u} \in L^1\}$. Prove that, for any non negative real number s, the product is a bilinear continuous map from $\mathcal{F}L^1\cap H^s \times \mathcal{F}L^1\cap H^s$ into $\mathcal{F}L^1\cap H^s$. What happens when s is greater than $d/2$?*

Exercice 2.1.6 *Let s a real number greater than $1/2$. Prove that the map γ defined by*

$$\gamma \left\{ \begin{array}{ccl} \mathcal{D}(\mathbb{R}^d) & \longrightarrow & \mathcal{D}(\mathbb{R}^{d-1}) \\ \varphi & \longmapsto & \gamma(\varphi) : (x_2,\cdots,x_d) \mapsto \varphi(0,x_2,\cdots,x_d) \end{array} \right.$$

can be extended in a continuous onto map from $H^s(\mathbb{R}^d)$ onto $H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})$.
Hint : *Write*

$$\mathcal{F}_{\mathbb{R}^{d-1}}\varphi(0,\xi_2,\cdots,\xi_d) = (2\pi)^{-1}\int_{\mathbb{R}} \widehat{\varphi}(\xi_1,\xi_2,\cdots,\xi_d)d\xi_1.$$

and for the fact that the map in onto, observe that, if

$$u = (2\pi)^{-(n-1)} C_s \mathcal{F}^{-1}\left(\frac{(1+|\xi'|^2)^{s-\frac{1}{2}}}{(1+|\xi|^2)^s}\widehat{v}(\xi')\right),$$

then $u\in H^s$ and $\gamma(u)=v$.

Let us prove a theorem which describes the dual of the space H^s.

Theorem 2.1.2 *The bilinear form B defined by*

$$B \left\{ \begin{array}{ccl} \mathcal{S}\times\mathcal{S} & \to & \mathbb{C} \\ (u,\varphi) & \mapsto & \displaystyle\int_{\mathbb{R}^d} u(x)\varphi(x)dx \end{array} \right.$$

can b extended as a bilinear form continuous from $H^{-s}\times H^s$ to $\mathbb{C}$. Moreover, the map δ_B defined by

$$\delta_B \left\{ \begin{array}{ccl} H^{-s} & \longrightarrow & (H^s)' \\ u & \longmapsto & \delta_B(u) : (\varphi) \mapsto B(u,\varphi) \end{array} \right.$$

is a linear and isometric isomorphism (up to a constant), which means that the bilinear form B identifies the space H^{-s} to the dual space of H^s.

Proof. The important point of the proof of this theorem is inverse Fourier formula which ensures that, for any couple (u,φ) of functions of $\mathcal{S}$, we have

$$\begin{aligned} B(u,\varphi) &= \int_{\mathbb{R}^d} u(x)\varphi(x)dx \\ &= \int_{\mathbb{R}^d} u(x)\mathcal{F}(\mathcal{F}^{-1}\varphi)(x)dx \\ &= \int_{\mathbb{R}^d} \widehat{u}(\xi)(\mathcal{F}^{-1}\varphi)(\xi)d\xi \\ &= (2\pi)^{-d}\int_{\mathbb{R}^d} \widehat{u}(\xi)\widehat{\varphi}(-\xi)d\xi. \qquad (2.3) \end{aligned}$$

Multiplying and dividing by $(1+|\xi|^2)^{\frac{s}{2}}$, we immediately get thanks to Cauchy-Schwarz inequality,

$$|B(u,\varphi)| \leq (2\pi)^{-d}\|u\|_{H^s}\|\varphi\|_{H^{-s}}.$$

Thus the first point of the theorem. The fact that the map δ_B is one to one comes from the fact that if, for any function $\varphi \in \mathcal{S}$, we have $B(u,\varphi)=0$, then u is 0. We shall prove that δ_B is one to one and onto.

Let ϕ and φ be in $\mathcal{S}_0$. One can write that

$$\begin{aligned} \left|\int_{\mathbb{R}^d} \phi(x)\varphi(x)\,dx\right| &= \left|\int_{\mathbb{R}^d} (\mathcal{F}^{-1}\phi)(\xi)(\mathcal{F}\varphi)(\xi)\,d\xi\right| \\ &= (2\pi)^{-d}\left|\int_{\mathbb{R}^d} |\xi|^{-s}\widehat{\phi}(-\xi)|\xi|^{s}\widehat{\varphi}(\xi)\,d\xi\right| \\ &\leq (2\pi)^{-d}\|\phi\|_{\dot{H}^{-s}}\|\varphi\|_{\dot{H}^{s}}. \end{aligned}$$

As $\mathcal{S}_0$ is dense in $\dot{H}^\sigma$ when $|\sigma|<d/2$, then one can extend B to $\dot{H}^{-s}\times\dot{H}^{s}$. Of course, if (u,ϕ) is in $\dot{H}^{-s}\times\mathcal{S}$ then $\mathcal{B}(u,\phi)=\langle u,\phi\rangle$.

Let L be a linear functional on $\dot{H}^s$. Consider the linear functional L_s defined by

$$L_s : \left\{ \begin{array}{ccc} L^2(\mathbb{R}^d) & \longrightarrow & \mathbb{C} \\ f & \longmapsto & \langle L, \mathcal{F}^{-1}\big((1+|\cdot|^2)^{-\frac{s}{2}}f\big)\rangle. \end{array} \right.$$

It is obvious that

$$\begin{aligned} \sup_{\|f\|_{L^2}=1} |\langle L_s, f\rangle| &= \sup_{\|f\|_{L^2}=1} |\langle L, \mathcal{F}^{-1}\big((1+|\cdot|^2)^{-\frac{s}{2}}f\big)\rangle| \\ &= \sup_{\|\phi\|_{\dot{H}^s}=1} |\langle L,\phi\rangle| \\ &= \|L\|_{(\dot{H}^s)'}. \end{aligned}$$

Riesz representation theorem implies that a function g exists in L^2 such that

$$\forall h\in L^2\,,\ \langle L_s,h\rangle = \int_{\mathbb{R}^d} g(\xi)h(\xi)\,d\xi.$$

We obviously have $|\cdot|^s g \in L^2(\mathbb{R}^d;|\xi|^{-2s}\,d\xi)$. Now, as $|s|<d/2$, this implies that $|\cdot|^s g$ is in $\mathcal{S}_h'(\mathbb{R}^d)$; thus we can define $u \stackrel{\text{def}}{=} \mathcal{F}(|\cdot|^s g)$ which belongs to $\mathcal{S}_h'(\mathbb{R}^d)$ Then, for any ϕ in $\mathcal{S}(\mathbb{R}^d)$, we have

$$\langle u,\phi\rangle = \int_{\mathbb{R}^d} g(\xi)|\xi|^s\widehat{\phi}(\xi)\,d\xi = \langle L_s, |\cdot|^s\widehat{\phi}\rangle.$$

By definition of L_s, we have $\langle u,\phi\rangle = \langle L,\phi\rangle$ and the theorem is proved.

2.2 Sobolev embeddings

The purpose of this section is the study of embedding properties of Sobolev spaces $H^s(\mathbb{R}^d)$ into L^p spaces . Let us prove the following theorem.

Theorem 2.2.1 *If s is greater than $d/2$, then the space H^s is continuously included in the space of continuous functions which tend to 0 at infinity. If s is a positive real number less than $d/2$, then the space H^s is continuously included in $L^{\frac{2d}{d-2s}}$ and we have*

$$\|f\|_{L^p} \leq C\|f\|_{\dot{H}^s} \quad \textit{with} \quad \|f\|_{\dot{H}^s} \stackrel{\text{def}}{=} \Big(\int_{\mathbb{R}^d} |\xi|^{2s}|\widehat{f}(\xi)|^2 d\xi\Big)^{\frac{1}{2}}.$$

Proof. The first point of this theorem is very easy to prove. Let us use the fact that

$$\|u\|_{L^\infty} \leq (2\pi)^{-d}\|\widehat{u}\|_{L^1} \tag{2.4}$$

Indeed, if s is greater than $d/2$, we have,

$$|\widehat{u}(\xi)| \leq (1+|\xi|^2)^{-s/2}(1+|\xi|^2)^{s/2}|\widehat{u}(\xi)|. \tag{2.5}$$

The fact that s is greater than $d/2$ implies that the function

$$\xi \mapsto (1+|\xi|^2)^{-s/2}$$

belongs to L^2. Thus, we have

$$\|\widehat{u}\|_{L^1} \leq \left(\int (1+|\xi|^2)^{-s}d\xi\right)^{\frac{1}{2}} \|u\|_{H^s}.$$

The first point of the theorem is proved.

The proof of the second point is more delicate. A way to unterstand the index $p = 2d/(d-2s)$ is the use of a scaling argument. Let us consider a function v on $\mathbb{R}^d$ and let us denote by v_λ the function $v_\lambda(x) = v(\lambda x)$. We have

$$\|v_\lambda\|_{L^p} = \lambda^{-\frac{d}{p}}\|v\|_{L^p}$$

and also

$$\begin{aligned} \int |\xi|^{2s}|\widehat{v_\lambda}(\xi)|^2 d\xi &= \lambda^{-2d}\int |\xi|^{2s}|\widehat{v}(\lambda^{-1}\xi)|^2 d\xi \\ &= \lambda^{-d+2s}\|v\|^2_{\dot{H}^s}, \end{aligned}$$

with

$$\|v\|^2_{\dot{H}^s} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} |\xi|^{2s}|\widehat{v}(\xi)|^2 d\xi.$$

The two quantities $\|\cdot\|_{L^p}$ and $\|\cdot\|_{\dot{H}^s}$ have the same scaling, which means that they have the same behaviour with respect to changes of unit. Thus, it make sense to compare them.

Multiplying f by a positive real number, it is enough to prove the inequality in the case when $\|f\|_{\dot{H}^s} = 1$. On utilise then the fact that for any p de the interval $]1, +\infty[$, we have, for any function measurable f,

$$\|f\|^p_{L^p} = p\int_0^\infty \lambda^{p-1} m(|f| > \lambda)d\lambda.$$

Let us decompose f in a low and in a high frequencies by writing

$$f = f_{1,A} + f_{2,A} \quad \text{with} \quad f_{1,A} = \mathcal{F}^{-1}(\mathbf{1}_{B(0,A)}\widehat{f}) \quad \text{and} \quad f_{2,A} = \mathcal{F}^{-1}(\mathbf{1}_{B^c(0,A)}\widehat{f}). \tag{2.6}$$

As the support of the Fourier transform of $f_{1,A}$ is compact, the function $f_{1,A}$ is bounded and more precisely,

$$\begin{aligned}
\|f_{1,A}\|_{L^\infty} &\leq (2\pi)^{-d}\|\widehat{f_{1,A}}\|_{L^1} \\
&\leq (2\pi)^{-d}\int_{B(0,A)} |\xi|^{-s}|\xi|^{s}|\widehat{f}(\xi)|d\xi \\
&\leq (2\pi)^{-d}\left(\int_{B(0,A)} |\xi|^{-2s}d\xi\right)^{\frac{1}{2}} \\
&\leq \frac{C}{(d-2s)^{\frac{1}{2}}}A^{\frac{d}{2}-s}.
\end{aligned} \tag{2.7}$$

The triangle inequality implies that, for any positive real number A,

$$(|f| > \lambda) \subset (2|f_{1,A}| > \lambda) \cup (2|f_{2,A}| > \lambda).$$

Using Inequality (2.7), we have

$$A = A_\lambda \stackrel{\text{def}}{=} \left(\frac{\lambda(d-2s)^{\frac{1}{2}}}{4C}\right)^{\frac{p}{d}} \Longrightarrow m\left(|f_{1,A}| > \frac{\lambda}{2}\right) = 0.$$

Thus we deduce that

$$\|f\|_{L^p}^p = p\int_0^\infty \lambda^{p-1} m(2|f_{2,A_\lambda}| > \lambda)d\lambda.$$

it is well known (this is Bienaimé-Tchebychev inequality) that

$$\begin{aligned}
m\left(|f_{2,A_\lambda}| > \frac{\lambda}{2}\right) &= \int_{(|f_{2,A_\lambda}|>\frac{\lambda}{2})} dx \\
&\leq \int_{(|f_{2,A_\lambda}|>\frac{\lambda}{2})} \frac{4|f_{2,A_\lambda}(x)|^2}{\lambda^2}dx \\
&\leq 4\frac{\|f_{2,A_\lambda}\|_{L^2}^2}{\lambda^2}.
\end{aligned}$$

For such a choice of A, we have

$$\|f\|_{L^p}^p \leq 4p\int_0^\infty \lambda^{p-3}\|f_{2,A_\lambda}\|_{L^2}^2 d\lambda. \tag{2.8}$$

As the Fourier transform is (up to a constant) an isometric isomorphism of L^2, we have

$$\|f_{2,A_\lambda}\|_{L^2}^2 = (2\pi)^{-d}\int_{(|\xi|\geq A_\lambda)} |\widehat{f}(\xi)|^2 d\xi.$$

Thanks to Inequality (2.8), we get

$$\|f\|_{L^p}^p \leq 4p(2\pi)^{-d}\int_{\mathbb{R}_+\times\mathbb{R}^d} \lambda^{p-3}\mathbf{1}_{\{(\lambda,\xi)\,/\,|\xi|\geq A_\lambda\}}(\lambda,\xi)|\widehat{f}(\xi)|^2 d\xi d\lambda.$$

By definition of A_λ, we have

$$|\xi| \geq A_\lambda \Longleftrightarrow \lambda \leq C_\xi \stackrel{\text{déf}}{=} \frac{4C}{(d-2s)^{\frac{1}{2}}} |\xi|^{\frac{d}{p}}.$$

Fubini's theorem implies that

$$\begin{aligned} \|f\|_{L^p}^p &\leq 4p(2\pi)^{-d} \int_{\mathbb{R}^d} \left(\int_0^{C_\xi} \lambda^{p-3} d\lambda \right) |\widehat{f}(\xi)|^2 d\xi \\ &\leq 4 \frac{p(2\pi)^d}{p-2} \left(\frac{4C}{(d-2s)^{\frac{1}{2}}} \right)^{p-2} \int_{\mathbb{R}^d} |\xi|^{\frac{d(p-2)}{p}} |\widehat{f}(\xi)|^2 d\xi. \end{aligned}$$

As $2s = \dfrac{d(p-2)}{p}$, the theorem is proved. □

Corollary 2.2.1 *Let p be in $]2,\infty[$, and s gretar than $s_p \stackrel{\text{def}}{=} d\Big(\dfrac{1}{2} - \dfrac{1}{p}\Big)$. We have*

$$\|u\|_{L^p} \leq C \|u\|_{L^2}^{1-\theta} \|u\|_{\dot{H}^s}^{\theta} \quad \textit{with} \quad \theta = \frac{s_p}{s} \cdot$$

Proof. It is an application of the above theorem together with the fact that

$$\|u\|_{\dot{H}^{\theta s_1 + (1-\theta)s_2}} \leq \|u\|_{\dot{H}_1^s}^{\theta} \|u\|_{\dot{H}^{s_2}}^{1-\theta}.$$

2.3 Homogeneous Sobolev spaces

Definition 2.3.1 *Let s be a real number, the homogeneous Sobolev space $\dot{H}^s$ is the space of tempered distribtions such that $\widehat{u}$ belongs to L^1_{loc} and satisfies*

$$\|u\|_{\dot{H}^s}^2 \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} |\xi|^{2s} |\widehat{u}(\xi)|^2 d\xi < \infty.$$

These spaces (or at least theirs norms) naturally appeared in the proof of Theorem 2.2.1. The $\|\cdot\|_{\dot{H}^s}$ norm has the following scaling property

$$\|f(\lambda\cdot)\|_{\dot{H}^s} = \lambda^{-\frac{d}{2}+s} \|f\|_{\dot{H}^s}.$$

These spaces are different from the inhomogeneous H^s spaces. Let us notice that if s is positive, then H^s is included in $\dot{H}^s$ but that if s is negative, then $\dot{H}^s$ is included in H^s. The inhomongeneous spaces is a decreasing family of spaces (with respect to the index s). The homogeneous ones are not comparable together.

We shall only consider theses homogeneous spaces in the case when s is less than the half dimension.

Proposition 2.3.1 *If $s < d/2$, then the space $\dot{H}^s$ is a Banach space.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ a Cauchy sequence of $\dot{H}^s$. The sequence $(\widehat{u}_n)_{n\in\mathbb{N}}$ is a Cauchy one in the Banach space $L^2(\mathbb{R}^d \setminus \{0\}; |\xi|^{2s} d\xi)$. Let f be its limit. It is clear that f belongs to $L^1_{loc}(\mathbb{R}^d \setminus \{0\})$. Moreover,

$$\int_{B(0,1)} |f(\xi)| d\xi \leq \Big(\int_{\mathbb{R}^d} |\xi|^{2s} f(\xi)|^2 d\xi \Big)^{\frac{1}{2}} \Big(\int_{B(0,1)} |\xi|^{-2s} d\xi \Big)^{\frac{1}{2}} < \infty$$

because s is less than the half-dimension. Thus $\widehat{f}$ belongs to $\mathcal{S}'$ and to L^1_{loc}. Thus $u \stackrel{\text{déf}}{=} \mathcal{F}^{-1} f$ is well defined, belongs to $\dot{H}^s$, and is the limit of the sequence $(u_n)_{n\in\mathbb{N}}$ in the sense of the norm $\dot{H}^s$. □

Exercice 2.3.1 *1) Prove that the space*

$$\mathcal{B} \stackrel{\text{def}}{=} \{u \in \mathcal{S}'(\mathbb{R}^d)\,,\ \widehat{u} \in L^1(B(0,1); d\xi) \cap L^2(\mathbb{R}^d; |\xi|^{2s} d\xi)\}$$

equipped with the norme $N(u) \stackrel{\text{def}}{=} \|\widehat{u}\|_{L^1(B(0,1))} + \|u\|_{\dot{H}^s}$ *is a Banach space.*

2) Let $s \geq d/2$. *Give an example of a sequence* $(f_n)_{n\in\mathbb{N}}$ *of* $\mathcal{B}$, *bounded in* $\dot{H}^s(\mathbb{R}^d)$, *such that*

$$\lim_{n\to\infty} N(f_n) = +\infty.$$

3) Then deduce that $(\dot{H}^s, \|\cdot\|_{\dot{H}^s})$ *is not a Banach space.*

Exercice 2.3.2 *Prove that, if* $k \in \mathbb{N}$, *then we have*

$$\dot{H}^{-k}(\mathbb{R}^d) = \Big\{u \in \mathcal{S}'(\mathbb{R}^d)\,,\ u = \sum_{|\alpha|=k} \partial^\alpha f_\alpha \quad \text{with} \quad f_\alpha \in L^2\Big\}.$$

Prove that a constant C *exists such that*

$$C^{-1}\|u\|_{\dot{H}^{-k}} \leq \inf\bigg\{\Big(\sum_{|\alpha|=k} \|f_\alpha\|_{L^2}^2\Big)^{\frac{1}{2}} /\ u = \sum_{|\alpha|=k} \partial^\alpha f_\alpha\bigg\} \leq C\|u\|_{\dot{H}^{-k}}.$$

2.4 The spaces $H_0^1(\Omega)$ and $H^{-1}(\Omega)$

Definition 2.4.1 *Let* Ω *a domain of* $\mathbb{R}^d$, *the space* $H_0^1(\Omega)$ *is defined as the closure of* $\mathcal{D}(\Omega)$ *in the sense of the norm* $H^1(\mathbb{R}^d)$.

The space $H^{-1}(\Omega)$ *is the set of distributions* u *on* Ω *such that*

$$\|u\|_{H^{-1}(\Omega)} \stackrel{\text{def}}{=} \sup_{\substack{f\in\mathcal{D}(\Omega)\\ \|f\|_{H_0^1(\Omega)}\leq 1}} |\langle u, f\rangle| < \infty.$$

Proposition 2.4.1 *The space* $H_0^1(\Omega)$ *is a Hilbert space equipped with the norm*

$$\Big(\|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2\Big)^{\frac{1}{2}}.$$

The proof is an easy exercice left to the reader. The space $H^{-1}(\Omega)$ can be indentified to the dual space of $H_0^1(\Omega)$ thanks to the following theorem.

Theorem 2.4.1 *The bilinear map defined by*

$$B\left\{\begin{array}{ccc} H^{-1}(\Omega)\times\mathcal{D}(\Omega) & \longrightarrow & \mathbb{C}\\ (u,\varphi) & \longmapsto & \langle u,\varphi\rangle\end{array}\right.$$

can be extended to a bilinear continuous map from $H^{-1}(\Omega) \times H_0^1(\Omega)$ *into* $\mathbb{C}$, *still denoted by* B. *Moreover, the map* δ_B *defined by*

$$\delta_B\left\{\begin{array}{ccc} H^{-1}(\Omega) & \longrightarrow & (H_0^1(\Omega))'\\ u & \longmapsto & \delta_B(u)(\varphi) \stackrel{\text{def}}{=} B(u,\varphi)\end{array}\right.$$

is a linear isometric isomorphism between the space $H^{-1}(\Omega)$ *and the dual space of* $H_0^1(\Omega)$.

Proof. The fact that the bilinear map B can be extended because B is uniformely continuous. Let ℓ a linear form continuous on $H^1_0(\Omega)$. Its restriction on $\mathcal{D}(\Omega)$ is a distribution u on Ω such that

$$\forall \varphi \in \mathcal{D}(\Omega)\,,\ \langle u, \varphi \rangle = \langle \ell, \varphi \rangle.$$

By definition of the norm on $(H^1_0(\Omega))'$, the theorem is proved. □

Theorem 2.4.2 (Poincaré Inequality) *Let* Ω *be* **bounded** *open subset of* $\mathbb{R}^d$. *A constant* C *exists such that*

$$\forall \varphi \in H^1_0(\Omega)\,,\ \|\varphi\|_{L^2} \leq C \left(\sum_{j=1}^{d} \|\partial_j \varphi\|^2_{L^2} \right)^{\frac{1}{2}}.$$

Proof. Let R a positive real number such that Ω is included in $]-R, R[\times \mathbb{R}^{d-1}$.Then, for any test function φ, we have

$$\varphi(x_1, \cdots, x_d) = \int_{-R}^{x_1} \frac{\partial \varphi}{\partial y_1}(y_1, x_2, \cdots, x_d) dy_1.$$

Cauchy-Schwarz Inequality implies that

$$|\varphi(x_1, \cdots, x_d)|^2 \leq 2R \int_{-R}^{x_1} \left| \frac{\partial \varphi}{\partial y_1}(y_1, x_2, \cdots, x_d) \right|^2 dy_1.$$

By integration in x_1, we get

$$\int_{\Omega} |\varphi(x_1, \cdots, x_d)|^2 dx_1 \leq 2R \int_{\Omega \times]-R,R[} \left| \frac{\partial \varphi}{\partial y_1}(y_1, x_2, \cdots, x_d) \right|^2 dy_1.$$

Then, integrating with respect to the other $d-1$ variables, we find

$$\begin{aligned} \int_{\Omega} |\varphi(x_1, \cdots, x_d)|^2 dx &\leq 2R \int_{\Omega \times]-R,R[} \left| \frac{\partial \varphi}{\partial y_1}(y_1, x_2, \cdots, x_d) \right|^2 dy_1 dx_2 \cdots dx_d \\ &\leq 4R^2 \sum_{j=1}^{d} \|\partial_j \varphi\|^2_{L^2}. \end{aligned}$$

As $\mathcal{D}(\Omega)$ is dense in $H^1_0(\Omega)$, the theorem is proved. □

It obviously implies the following corollary.

Corollary 2.4.1 *The space* $H^1_0(\Omega)$ *equipped with the norm*

$$u \longmapsto \left(\sum_{j=1}^{d} \|\partial_j u\|^2_{L^2} \right)^{\frac{1}{2}} \stackrel{\text{def}}{=} \|\nabla u\|_{L^2}$$

is a Hilbert space and the this norm is equivalent to the previous one.

In order to conclude this chapter, let us prove the following very important compactness theorem.

Theorem 2.4.3 *Let K be a compact of $\mathbb{R}^d$ and (s, s') a couple of real numbers such that $s' < s$. Let us denote by H^s_K the space of distributions of $H^s(\mathbb{R}^d)$ the support of which is included in K. The embedding of H^s_K in $H^{s'}_K$ is compact.*

Before proving this theorem, let us give some immediate corollaries.

Theorem 2.4.4 *Let Ω be a bounded open subset of $\mathbb{R}^d$ with $d \geq 2$. If p is a real number less than*

$$p_c \stackrel{\text{def}}{=} \frac{2d}{d-2},$$

then the embedding of $H^1_0(\Omega)$ in $L^p(\Omega)$ is compact.

Proof of Theorem 2.4.3. Without any loss on generalitry, we can assume that the compact K is included in the interior of the cube $[0, 2\pi]^d$. Now let us introduce the linear map defined by

$$P \left\{ \begin{array}{ccc} \mathcal{D}_K & \longrightarrow & C^\infty(\mathbb{T}^d) \\ u & \longmapsto & \displaystyle\sum_{k \in 2\pi\,\mathbb{Z}^d} u(x-k) \end{array} \right.$$

where $\mathcal{D}_K$ denotes the set of smooth compactly supported functions the support of which is included in K. This map can be extended to a linear continuous map of H^s_K in $H^s(\mathbb{T}^d)$ defined by

$$H^s(\mathbb{T}^d) = \{u \in \mathcal{D}(\mathbb{T}^d) \;/\; \|u\|^2_{H^s(\mathbb{T}^d)} \stackrel{\text{def}}{=} \sum_{n \in \mathbb{Z}} (1 + |n|^2)^s |\widehat{u}(n)|^2 < +\infty\}.$$

This comes from the fact that a constant C exists such that

$$\forall \varphi \in \mathcal{D}_K\,,\; \|P\varphi\|_{H^s(\mathbb{T}^d)} \leq C\|\varphi\|_{H^s}. \tag{2.9}$$

In order to prove this, let us consider a function χ of $\mathcal{D}]0, 2\pi[^d)$ with value 1 near K. Then, let us write

$$\begin{aligned} \widehat{\varphi}(n) &= \int \varphi(x) e^{-inx} dx \\ &= \int \varphi(x)\chi(x) e^{-inx} dx \\ &= (2\pi)^{-d} \int \widehat{\varphi}(\xi) \mathcal{F}(\chi e^{-inx})(\xi) d\xi \\ &= (2\pi)^{-d} \int \widehat{\varphi}(\xi) \widehat{\chi}(n - \xi) d\xi. \end{aligned}$$

As χ is a smooth compactly supported function, for any integer N, a constant C_N exists such that

$$|\widehat{\varphi}(n)| \leq C_N \int_{\mathbb{R}^d} \frac{|\widehat{\varphi}(\xi)|}{(1 + |n - \xi|)^{-N}} d\xi.$$

The result is an obvious consequence of Lemma 2.1.1 and of the following lemma.

Lemma 2.4.1 *Let (X_{j,μ_j}) two measured spaces and k a function measurable from $X_1 \times X_2$ into $\mathbb{R}$ such that*

$$M \stackrel{\text{déf}}{=} \max\Big\{ \sup_{x_2 \in \mathcal{X}_2} \int_{X_1} |k(x_1, x_2)| d\mu_1(x_1),\; \sup_{x_1 \in \mathcal{X}_1} \int_{X_2} |k(x_1, x_2)| d\mu_2(x_2) \Big\} < \infty.$$

Then the map defined by

$$(Kf)(x_2) = \int_{X_1} k(x_1, x_2) f(x_1) d\mu_1(x_1),$$

maps $L^p(X_1, d\mu_1)$ *into* $L^p(X_2, d\mu_2)$ *for any* $p \in [1, \infty]$. *More precisely, we have*

$$\|Kf\|_{L^p(X_2, d\mu_2)} \leq M \|f\|_{L^p(X_1, d\mu_1)}.$$

Proof.

Let g be an element of norm 1 of $L^{p'}(X_2, d\mu_2)$. We have

$$\int_{X_2} |Kf(x_2)|\, |g(x_2)| d\mu_2(x_2) \leq \int_{X_1 \times X_2} |k(x_1, x_2)|\, |f(x_1)|\, |g(x_2)| d\mu_1(x_1) d\mu_2(x_2).$$

Hölder Inequality for the measure $|k(x_1, x_2)| d\mu_1(x_1) d\mu_2(x_2)$ implies that

$$\begin{aligned}\int_{X_2} |Kf(x_2)|\, |g(x_2)| d\mu_2(x_2) \leq & \left(\int_{X_1 \times X_2} |f(x_1)|^p |k(x_1, x_2)| d\mu_1(x_1) d\mu_2(x_2) \right)^{\frac{1}{p}} \\ & \times \left(\int_{X_1 \times X_2} |g(x_2)|^{p'} |k(x_1, x_2)| d\mu_1(x_1) d\mu_2(x_2) \right)^{\frac{1}{p'}}.\end{aligned}$$

Then Fubini's Theorem ensures the result. □

Continuation of the proof of Theorem 2.4.3. Let us observe that, if we define

$$i_N(\varphi) \stackrel{\text{def}}{=} (2\pi)^{-d} \sum_{n \leq |N|} \widehat{\varphi}(n) e^{i(n|x)},$$

where N denotes any integer, then we have

$$\begin{aligned}|\varphi - i_N(\varphi)|^2_{H^{s'}(\mathbb{T}^d)} &= \sum_{n > |N|} (1 + |n|^2)^{s'} |\widehat{\varphi}(n)|^2 \\ &= (1 + |N|^2)^{s'-s} \sum_{n > |N|} (1 + |n|^2)^s |\widehat{\varphi}(n)|^2 \\ &= (1 + |N|^2)^{s'-s} |\varphi|^2_{H^{s'}(\mathbb{T}^d)}.\end{aligned}$$

Thus, the embedding i of $H^s(\mathbb{T}^d)$ in $H^{s'}(\mathbb{T}^d)$ is compact as a limit of finite rank operators.

Let us consider the map defined by

$$M_\chi \left\{ \begin{array}{ccc} H^{s'}(\mathbb{T}^d) & \longrightarrow & H^{s'}(\mathbb{R}^d) \\ u & \longmapsto & \chi u \end{array} \right.$$

Using the Fourier transform of u, we get

$$\begin{aligned}\mathcal{F}(\chi u)(\xi) &= \int_{\mathbb{R}^d} e^{-i(x|\xi)} \chi(x) u(x) dx \\ &= \sum_{n \in \mathbb{Z}^d} \widehat{u}(n) \int_{\mathbb{R}^d} e^{-i(x|\xi - n)} \chi(x) dx \\ &= \sum_{n \in \mathbb{Z}^d} \widehat{u}(n) \widehat{\chi}(n - \xi).\end{aligned}$$

The proof of the continuity of M_χ is strictly analogous to the proof of the continuity of P,. Moreover, it is clear that the embedding of H^s_K in $H^{s'}_K$ is equal to $P \circ i \circ M_\chi$; the theorem is proved. □

Remarks

The proofs of this chapter must be known except the proof of the second point of Theorem 2.1.1.

To know more about Sobolev spaces, the reader can conslut the classical book

R. A. Adams, *Sobolev spaces*, Pure and Applied Mathematics, Vol. 65, Academic Press, 1975.

Chapter 3

Extrema problem and the least action principle

3.1 The problem of Dirichlet seen as an extremum problem

In this section and also in the following one, Ω denotes a bounded domain of $\mathbb{R}^d$. Let f be an element of $H^{-1}(\Omega)$, let us consider the functionnal F defined par

$$F \left\{ \begin{array}{ccc} H_0^1(\Omega) & \longrightarrow & \mathbb{R} \\ u & \longmapsto & \frac{1}{2}\|\nabla u\|_{L^2}^2 - \langle f, u\rangle. \end{array} \right.$$

Dirichlet Theorem is the following:

Theorem 3.1.1 *The functionnal F has a unique minimum which is the unique solution in $H_0^1(\Omega)$ of $-\Delta u = f$ in the distribution sense in Ω.*

Proof. In order to prove this theorem, let us observe that the functionnal F bounded from below because

$$\begin{aligned} F(u) & \geq \frac{1}{2}\|\nabla u\|_{L^2}^2 - \|\nabla u\|_{L^2}\|f\|_{H^{-1}(\Omega)} \\ & \geq \frac{1}{2}\big(\|\nabla u\|_{L^2} - \|f\|_{H^{-1}(\Omega)}\big)^2 - \frac{1}{2}\|f\|_{H^{-1}(\Omega)}^2. \end{aligned} \tag{3.1}$$

The functionnal F has a lower bound m. Let us consider a minimizing sequence $(u_n)_{n\in\mathbb{N}}$ i.e. a sequence $(u_n)_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to\infty} F(u_n) = m$. Using lnequality (3.1), we have

$$\|\nabla u_n\|_{L^2} \leq \big(2F(u_n) + \|f\|_{H^{-1}(\Omega)}\big)^{\frac{1}{2}} + \|f\|_{H^{-1}(\Omega)}.$$

The sequence $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence of the space $H_0^1(\Omega)$ which is complete. Thus it exists a function u in $H_0^1(\Omega)$ and a subsequence of $(u_n)_{n\in\mathbb{N}}$ (which we still denote by $(u_n)_{n\in\mathbb{N}}$) such that $(u_n)_{n\in\mathbb{N}}$ tends weakly to u. Moreover, we know that the sequence $(\|\nabla u_n\|_{L^2})_{n\in\mathbb{N}}$ converges to $m + \langle f, u\rangle$. Thanks to the properties of the weak limit we have

$$\lim_{n\to\infty} \|\nabla u_n\|_{L^2} \geq \|\nabla u\|_{L^2}.$$

Let us assume that $\nabla u\|_{L^2} < \lim\limits_{n\to\infty} \|\nabla u_n\|_{L^2}$. Then, we have $F(u) < m$ which is in contradiction with the fact that m is the infimum of F. Thus

$$\lim_{n\to\infty} \|\nabla u_n\|_{L^2} = \|\nabla u\|_{L^2}$$

and then the lower bound is a minimum and the sequence $(u_n)_{n\in\mathbb{N}}$ converges strongly to u in $H^1_0(\Omega)$).

Now let us prove that u is a solution of Laplace Equation. The functionnal F is the sum of the quadratic functionnal (the norm to the square) and of of a linear functionnal (both continuous). We have, for any function h of $H^1_0(\Omega)$,

$$F(u+h) = F(u) + 2(\nabla u|\nabla h)_{L^2} - \langle f, h\rangle + \|\nabla h\|^2_{L^2}. \tag{3.2}$$

If u is a minumum, then the differential vanishes at u and thus u is a solution of Laplace Equation. Moreover, Relation (3.2) implies that the minimum is unique and it is characterised by the fact that, for any h in $H^1_0(\Omega)$, we have $(\nabla u|\nabla h)_{L^2} - \langle f, h\rangle = 0$. Thus the theorem is proved. □

Exercice 3.1.1 *Let Ω a bounded domain of $\mathbb{R}^d$ and f a distribution of $H^{-1}(\Omega)$. Prove that a vector field v exists in $L^2(\Omega)$ such that $\operatorname{div} v = f$.*

Let us prove now a result about the spectral strucutre of the Laplacian in a bounded domain.

Theorem 3.1.2 *It exists a non decreasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ of positive real numbers which thends to infinity and a hilbertian basis of $L^2(\Omega)$ denoted by $(e_k)_{k\in\mathbb{N}}$ such that the sequence $(\lambda_k^{-1}e_k)_{k\in\mathbb{N}}$ is an orthonormal basis of $H^1_0(\Omega)$ such that*

$$-\Delta e_k = \lambda_k e_k.$$

Moreover, if f belongs to $H^{-1}(\Omega)$, then

$$\|f\|^2_{H^{-1}(\Omega)} = \sum_k \lambda_k^{-2}(\langle f, e_k\rangle)^2.$$

Remark Thus, the space $H^{-1}(\Omega)$ is a Hilbert space and $(\lambda_k e_k)_{k\in\mathbb{N}}$ is a hilbertian basis of $H^{-1}(\Omega)$.

Proof of Theorem 3.1.2. As the space L^2 is continuously included in $H^{-1}(\Omega)$, we can define an operator B as follows:

$$B \left\{ \begin{array}{lcl} L^2 & \longrightarrow & H^1_0(\Omega) \subset L^2(\Omega) \\ f & \longmapsto & u \end{array} \right.$$

such that u is the solution in $H^1_0(\Omega)$ of $-\Delta u = f$. The operator B is of course continuous from $L^2(\Omega)$ into $H^1_0(\Omega)$. Thanks to Theorem 2.4.3, the operator B is compact from $L^2(\Omega)$ into $L^2(\Omega)$. Let us prove it is self adjoint and positive. Let us write that for any couple of functions (f, g) in $L^2(\Omega)$, we have

$$(Bf|g)_{L^2} = \langle g, Bf\rangle_{H^{-1}\times H^1_0}.$$

By definition of B, we have for any g in $L^2(\Omega)$, $g = \Delta Bg$. Thus we infer that

$$(Bf|g)_{L^2} = \langle \Delta Bg, Bf\rangle_{H^{-1}\times H^1_0}.$$

As for any u and v in $H^1_0(\Omega)$, we have

$$\langle \Delta u, v\rangle_{H^{-1}\times H^1_0} = (u|v)_{H^1_0}\,,$$

we deduce that

$$(Bf|g)_{L^2} = (Bf|Bg)_{H_0^1}.$$

Thus the operator B is compact, selfadjoint and positive. The spectral theorem applied to B implies the existence of a non increasing sequence $(\mu_k)_{k\in\mathbb{N}}$ of positive real numbers which tends to 0 and a hilbertian basis of $L^2(\Omega)$ denoted $(e_k)_{k\in\mathbb{N}}$ such that , for any k, the function e_k belongs to $L^2(\Omega)$ and such that $Be_k = \mu_k e_k$. This implies that $-\Delta e_k = \mu_k^{-1} e_k$. We have,

$$\|f\|_{H^{-1}(\Omega)} = \sup_{(c_k)\in B_f} \langle f, \sum_k \lambda_k^{-1} c_k e_k\rangle$$

where B_f denotes the set of sequences having only a finite number of non zero terms and of ℓ^2 norm less or equal to to 1. Thus

$$\|f\|_{H^{-1}(\Omega)} = \sup_{(c_k)\in B_f} \sum_k \lambda_k^{-1}\langle f, e_k\rangle c_k = \|(\lambda_k^{-1}\langle f, e_k\rangle)_{k\in\mathbb{N}}\|_{\ell^2(\mathbb{N})}.$$

Theorem 3.1.2 is proved. □

Let us apply this result to solve the heat equation with Dirichlet boudary condition.

3.2 The stationnary Stokes's problem

This problem is analogous to the Dirichlet problem, but we work on the set of divergence free vector field. Nevertheless, the fact that we impose a constrain (even a linear one) of the space on which we search the minimum will introduce an important change. The Laplace eqaution will become the Stokes equation. Let us first define of the space we are going to work with.

Definition 3.2.1 *Let us denote by $\mathcal{V}_\sigma(\Omega)$ the set of divergence free vector fields whose componants are in $H_0^1(\Omega)$ and by $\mathcal{H}(\Omega)$ the closure in $(L^2(\Omega))^d$ de $\mathcal{V}_\sigma(\Omega)$.*

Let us state the analogous of Dirichlet theorem in this framework. As in the preceeding section, let us consider a vector feld f whose componants are in $H^{-1}(\Omega)$; then we define the functionnal F

$$F \left\{ \begin{array}{ccc} \mathcal{V}_\sigma(\Omega) & \longrightarrow & \mathbb{R} \\ u & \longmapsto & \dfrac{1}{2}\|\nabla u\|_{L^2}^2 - \langle f, u\rangle. \end{array} \right.$$

Theorem 3.2.1 *Let $f \in \mathcal{V}'(\Omega)$. It exists a unique minimum of the functionnal F which is also the unique solution of following equation*

$$-\Delta u - f \in (\mathcal{V}_\sigma(\Omega))^\circ$$

which means that, for any vector field v of $\mathcal{V}_\sigma(\Omega)$, we have

$$-\langle \Delta u, v\rangle = \langle f, v\rangle. \tag{3.3}$$

The existence and the uniqueness of a minimum u for the functionnal F can be proved following exactly the same lines as in the case of Dirichlet problem. The fact that the differential of F vanishes at point u implies the relation (3.3).

Remarks

- The fact that a vector field g of $H^{-1}(\Omega)$ belongs the polar set (in the sense of the duality) of $H^1_0(\Omega)$ implies in particular that, for any function φ of $\mathcal{D}(\Omega)$, we have

$$\langle g^i, -\partial_j\varphi\rangle + \langle g^j, \partial_i\varphi\rangle = 0$$

  which implies that $\partial_j g^i - \partial_i g^j = 0$, i.e. the curl of g is identically 0.

- Very simple domains exist sucht that it exists a vector field of $H^{-1}(\Omega)$ which are of divergence and of curl identically 0 and which are not gradients of functions.

Let us consider the domain of the plan $\Omega \stackrel{\text{def}}{=} \{x \in \mathbb{R}^2 \ / \ 0 < R_1 < |x| < R_2\}$ and the vector field f defined by $(-\partial_2 \log|x|, \partial_1 \log|x|)$. We have the following lemma.

Proposition 3.2.1 *The vector field f is curl free, as but it is not the gradient of a function.*

Proof. The fact that f is of divergence free is obvious. The fact that its curl is 0 follows from the fact that that the function $x \mapsto \log|x|$ is harmonic on Ω. Let us assume that f is a gradient of some distribution $-p$. As f is smooth, p is also smooth. Let us consider the flow of $f = -\nabla p$. By definition of f, its trajectories are periodic. Let us consider a trajectory γ from of a point of Ω such that $f \neq 0$ (here all points are like this). We have

$$\frac{d}{dt}(p\circ\gamma)(t) = \Big(\frac{d\gamma}{dt}\Big|\nabla p(\gamma(t))\Big)_{L^2} = -|\nabla p(\gamma(t))|^2 \leq 0.$$

The fact that the derivative en $t = 0$ is negative is in contradiction with the periodicity of the trajectory γ. Proposition 3.2.1 is proved. □

As shown by the following proposition, belonging to $(\mathcal{V}_\sigma(\Omega))^\circ$ the polar space (in the sens of the duality $H^{-1}, H^1_0(\Omega)$) of $\mathcal{V}_\sigma(\Omega)$ is stronger than being curl free. Let us admit the following proposition.

Proposition 3.2.2 *Let f in $\mathcal{V}'(\Omega)$. If f belongs à $(\mathcal{V}_\sigma(\Omega))^\circ$ i.e. if*

$$\forall v \in \mathcal{V}_\sigma(\Omega)\,,\ \sum_{j=1}^d \langle f^j, v^j\rangle = 0,$$

then it exists p in $\mathcal{D}'(\Omega)$ such that $f = -\nabla p$. If the boundary of Ω is a C^1 hypersurface, then $p \in L^2(\Omega)$.

As in the case of Dirichlet problem, the spectral structure of selfadjoint compact operators will give the follwing result.

Theorem 3.2.2 *A non decreasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ of positive numbers which tends to infinity and an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of $\mathcal{H}(\Omega)$ exist such that the sequence $(\lambda_k^{-1}e_k)_{k\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{V}_\sigma(\Omega)$ and such that*

$$-\Delta e_k - \lambda_k^2 e_k \in (\mathcal{V}_\sigma(\Omega))^\circ.$$

Moreover, if $f \in \mathcal{V}'(\Omega)$, alors

$$\|f\|^2_{\mathcal{V}'_\sigma(\Omega)} = \sum_{k\in N} \lambda_k^{-2}(\langle f, e_k\rangle)^2 \quad \text{and} \quad \lim_{n\to\infty}\Big\|f - \sum_{k=0}^n \langle f, e_k\rangle e_k\Big\|_{(\mathcal{V}_\sigma(\Omega))'} = 0.$$

Proof. It is very close to the proof of Theorem 3.1.2. As the space $\mathcal{H}(\Omega)$ is continuously included in $H^{-1}(\Omega)$, we can define the operator B

$$B \left\{ \begin{array}{ccc} \mathcal{H}(\Omega) & \longrightarrow & \mathcal{V}_\sigma(\Omega) \subset \mathcal{H}(\Omega) \\ f & \longmapsto & u \end{array} \right.$$

such that u is the solution in $H^1_0(\Omega)$ of $-\Delta u - f \in (\mathcal{V}_\sigma(\Omega))^\circ$. The following of the proof is strictement analogous to the one of Dirichlet problem. □

The orthogonal projection of $L^2(\Omega)$ on $\mathcal{H}(\Omega)$, denoted by $\mathbb{P}$, is the Leray projection.

3.3 The time dependent Stokes problem

Given a positive viscosity ν, the evolution Stokes problem reads as follows:

$$(ES_\nu) \left\{ \begin{array}{rcl} \partial_t u - \nu\Delta u & = & f - \nabla p \\ \operatorname{div} u & = & 0 \\ u_{|\partial\Omega} & = & 0 \\ u_{|t=0} & = & u_0 \in \mathcal{H}. \end{array} \right.$$

Let us define what a solution of this problem is.

Definition 3.3.1 *Let u_0 be in $\mathcal{H}$ and f in $L^2_{loc}(\mathbb{R}^+;\mathcal{V}')$. We shall say that u is a solution of (ES_ν) with initial data u_0 and external force f if and only if u belongs to the space*

$$C(\mathbb{R}^+;\mathcal{V}'_\sigma) \cap L^\infty_{loc}(\mathbb{R}^+;\mathcal{H}) \cap L^2_{loc}(\mathbb{R}^+;\mathcal{V}_\sigma)$$

and satisfies, for any Ψ in $C^1(\mathbb{R}^+;\mathcal{V}_\sigma)$,

$$\langle u(t),\Psi(t)\rangle + \int_{[0,t]\times\Omega} \Big(\nu\nabla u : \nabla\Psi - u\cdot\partial_t\Psi\Big)(t',x)\,dxdt'$$
$$= \int_\Omega u_0(x)\cdot\Psi(0,x)\,dx + \int_0^t \langle f(t'),\Psi(t')\rangle\,dt'.$$

The following theorem holds.

Theorem 3.3.1 *The problem (ES_ν) has a unique solution in the sense of the above definition. Moreover this solution belongs to $C(\mathbb{R}^+;\mathcal{H})$ and satisfies*

$$\frac{1}{2}\|u(t)\|^2_{L^2} + \nu\int_0^t \|\nabla u(t')\|^2_{L^2}dt' = \frac{1}{2}\|u_0\|^2_{L^2} + \int_0^t \langle f(t'),u(t')\rangle dt'.$$

Proof. In order to prove uniqueness, let us consider some function u in $C(\mathbb{R}^+;\mathcal{V}'_\sigma)\cap L^2_{loc}(\mathbb{R}^+;\mathcal{V}_\sigma)$ such that, for all Ψ in $C^1(\mathbb{R}^+;\mathcal{V}_\sigma)$,

$$\langle u(t),\Psi(t)\rangle + \int_0^t\int_\Omega \Big(\nu\nabla u : \nabla\Psi - u\cdot\partial_t\Psi\Big)(t',x)\,dxdt' = 0.$$

This is valid in particular for the time independent function $\Psi(t) \equiv e_k$ where the family vector fields $(e_k)_{k\in\mathbb{N}}$ is given by Theorem 3.2.2. This gives

$$\begin{array}{rcl} \langle u(t),e_k\rangle & = & -\nu\displaystyle\int_0^t\int_\Omega \nabla u(t',x) : \nabla e_k(x)dxdt' \\ & = & \nu\displaystyle\int_0^t \langle u(t'),\Delta\,\mathbb{P}_k\,\Psi\rangle. \end{array}$$

Thanks to the spectral Theorem 3.2.2 together with the fact that, for almost every t', $u(t)$ belongs to $\mathcal{V}_\sigma$, we have

$$-\int_\Omega \nabla u(t',x) : \nabla e_k(x)dx = \langle \Delta e_k, u(t')\rangle = \lambda_k^2 \langle e_k, u(t')\rangle.$$

This gives

$$\langle u(t), e_k\rangle = \int_0^t \lambda_k^2 \langle e_k, u(t')\rangle dt'.$$

The fact that $\langle u(0), e_k\rangle = 0$ implies that, for any k, $\langle u(t), e_k\rangle = (u|e_k)_{\mathcal{H}} = 0$. Thus $u \equiv 0$.

In order to prove existence, let us consider a sequence $(f_k)_{k\in\mathbb{N}}$ associated with f by Lemma 4.2.2 page 46 and then the approximated problem

$$(ES_{\nu,k}) \left\{ \begin{array}{rcl} \partial_t u_k - \nu\, \mathbb{P}_k\, \Delta u_k & = & f_k \\ u_{k|t=0} & = & \mathbb{P}_k\, u_0 \end{array} \right. \quad \text{with} \quad \mathbb{P}_k\, f \stackrel{\text{def}}{=} \sum_{j\leq k} \langle f, e_j\rangle e_j. \tag{3.1}$$

Again thanks to Theorem 3.2.2 page 34, it is a linear ordinary differential equation on $\mathcal{H}_k$ which has a global solution u_k which is $C^1(\mathbb{R}^+; \mathcal{H}_k)$. By an energy estimate in $(ES_{\nu,k})$ we get that

$$\frac{1}{2}\frac{d}{dt}\|u_k(t)\|_{L^2}^2 + \nu \|\nabla u_k(t)\|_{L^2}^2 = \langle f_k(t), u_k(t)\rangle.$$

A time integration gives

$$\frac{1}{2}\|u_k(t)\|_{L^2}^2 + \nu \int_0^t \|\nabla u_k(t')\|_{L^2}^2 dt' = \frac{1}{2}\|\, \mathbb{P}_k\, u(0)\|_{L^2}^2 + \int_0^t \langle f_k(t'), u_k(t')\rangle dt'. \tag{3.2}$$

In order to pass to the limit, we write an energy estimate for $u_k - u_{k+\ell}$, which gives

$$\begin{aligned} \delta_{k,\ell}(t) &\stackrel{\text{def}}{=} \frac{1}{2}\|(u_k - u_{k+\ell})(t)\|_{L^2}^2 + \nu \int_0^t \|\nabla(u_k - u_{k+\ell})(t')\|_{L^2}^2 dt' \\ &= \frac{1}{2}\|(\mathbb{P}_k - \mathbb{P}_{k+\ell})u(0)\|_{L^2}^2 + \int_0^t \langle (f_k - f_{k+\ell})(t'), u_k(t')\rangle dt' \\ &\leq \frac{1}{2}\|(\mathbb{P}_k - \mathbb{P}_{k+\ell})u(0)\|_{L^2}^2 + \frac{C}{\nu}\int_0^t \|(f_k - f_{k+\ell})(t')\|_{\mathcal{V}'_\sigma}^2 dt' \\ &\quad + \frac{\nu}{2}\int_0^t \|\nabla(u_k - u_{k+\ell})(t')\|_{L^2}^2 dt' + \frac{\nu}{2}\int_0^t \|(u_k - u_{k+\ell})(t')\|_{L^2}^2 dt'. \end{aligned}$$

Using Poincaré's inequality, this implies that

$$\delta_{k,\ell}(t) \leq \|(\mathbb{P}_k - \mathbb{P}_{k+\ell})u(0)\|_{L^2}^2 + \frac{C}{\nu}\int_0^t \|(f_k - f_{k+\ell})(t')\|_{\mathcal{V}'_\sigma}^2 dt'.$$

This implies that the sequence $(u_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $C(\mathbb{R}^+; \mathcal{H}) \cap L^2_{loc}(\mathbb{R}^+; \mathcal{V}_\sigma)$. Let us denote by u the limit and prove that u is a solution in the sense of Definition 3.3.1. As u_k is a C^1 solution of the ordinary differential equation $(ES_{\nu,k})$, we have, for a Ψ in $C^1(\mathbb{R}^+; \mathcal{V}_\sigma)$,

$$\frac{d}{dt}\langle u_k(t), \Psi(t)\rangle = \nu \langle \Delta u_k(t), \Psi(t)\rangle + \langle f_k(t), \Psi(t)\rangle + \langle u_k(t), \partial_t \Psi(t)\rangle.$$

By time integration, we get

$$\begin{aligned} \langle u_k(t), \Psi(t)\rangle = -\nu \int_0^t \int_\Omega \nabla u_k(t',x) : \nabla \Psi(t',x)\, dxdt' \\ + \int_0^t \langle f_k(t'), \Psi(t')\rangle dt' + \langle \mathbb{P}_k\, u(0), \Psi(0)\rangle + \int_0^t \langle u_k(t'), \partial_t \Psi(t')\rangle dt'. \end{aligned}$$

Passing to the limit in the above equality and in (3.2) gives the theorem. □

Remark. The solution is given by the explicit formula

$$\begin{aligned} u(t) &= \sum_{j\in\mathbb{N}} U_j(t) e_j \quad \text{with} \\ U_j(t) &\stackrel{\text{def}}{=} e^{-\nu\mu_j^2 t}(u_0|e_j)_{L^2} + \int_0^t e^{-\nu\mu_j^2(t-t')}\langle f(t'), e_j\rangle\, dt'. \end{aligned} \tag{3.3}$$

In the case of the whole space $\mathbb{R}^d$, we have the following analogous formula

$$\begin{aligned} u(t,x) &= (2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\widehat{u}(t,\xi)d\xi \quad \text{with} \\ \widehat{u}(t,\xi) &\stackrel{\text{def}}{=} e^{-\nu|\xi|^2 t}\widehat{u}_0(\xi) + \int_0^t e^{-\nu|\xi|^2(t-t')}\widehat{f}(t',\xi)\, dt'. \end{aligned} \tag{3.4}$$

3.4 How to modelize fluids using the least action principle

In this section, we shall always assume that the fluid extends to the whole space $\mathbb{R}^d$, which means that we neglect boundary effects. We want to describe the evolution of a perfect fluid between two times t_0 and t_1. A particule of fluid located at point x to l'time t_0 sera located at point $\psi_1(x)$ at the time t_1. The possible incompressibility of the fluid will be described by the fact that the map ψ_1, assumed to be a diffeomorphism, will preserved the measure, i.e. its jacobian is of determinant 1.

Let us precise the functionnal spaces we are going to work with. In all this section, we consider a diffeomorphism ψ_1 which preserves the volume if the fluid is assumed incompressible.

Definition 3.4.1 *Let us denote by $\mathcal{L}$ the space of C^1 functions from $[t_0,t_1]\times\mathbb{R}^d$ in $\mathbb{R}^d$ such that $\psi(t_0) = \mathrm{Id}$ and $\psi(t_1) = \psi_1$, such that, each time t, the function $\psi(t)$ is a diffeomorphism of $\mathbb{R}^d$ and then such that $\partial_t\psi(t)$ is continuous from $[t_0,t_1]$ into L^2.*

Let us denote by $\mathcal{L}_0$ the space des functions of $\mathcal{L}$ such that, for any time t, the diffeomorphism $\psi(t)$ preserves the measure.

A possible evolution of a compressible fluid between the situation at time t_0 and the one at time t_1 by the diffeomorphism ψ_1, is modelized by a function ψ of the space $\mathcal{L}$. A possible evolution of an incompressible fluid between the situation at time t_0 and the situation at time t_1 described by the diffeomorphism ψ_1, is modelized by a function ψ of the space $\mathcal{L}_0$. This is the point of view is called the lagragian one.

Let us defined a functionnal the extremal points of which will decribes the evolutions of the fluid.

Definition 3.4.2 *Let F a C^∞ function and ρ_0 a C^∞ positive function; Let us define the action by the map $\mathcal{A}$ defined from $\mathcal{L}$ into $\mathbb{R}$*

$$\begin{aligned} \mathcal{A}(\psi) &\stackrel{\text{def}}{=} \mathcal{A}_1(\psi) + \mathcal{A}_2(\psi) \quad \textit{with} \\ \mathcal{A}_1(\psi) &\stackrel{\text{def}}{=} \frac{1}{2}\int_{t_0}^{t_1}\int_{\mathbb{R}^d} |\partial_t\psi(t,x)|^2 dxdt \quad \textit{and} \\ \mathcal{A}_2(\psi) &\stackrel{\text{def}}{=} = \frac{1}{2}\int_{t_0}^{t_1}\int_{\mathbb{R}^d} F((J\psi(t,x))^{-1}\rho_0(x))J\psi(t,x)dxdt. \end{aligned}$$

where $J\psi$ denotes the jacobian determinant of ψ, i.e. $J\psi \stackrel{\text{def}}{=} \det D\psi$.

Remark The term $\mathcal{A}_1$ modelize cinetic energy of the system, the term $\mathcal{A}_2$ of internal energy.

Proposition 3.4.1 *The functionnals $\mathcal{A}_1$ and $\mathcal{A}_2$ are differentiables of $\mathcal{L}$ in $\mathbb{R}$ and we have*

$$\begin{aligned} D\mathcal{A}_1(\psi)h &= \int_{t_0}^{t_1}\int_{\mathbb{R}^d} \rho_0(x)\partial_t\psi(t,x)\cdot\partial_t h(t,x)dxdt \quad \text{and} \\ D\mathcal{A}_2(\psi)h &= \int_{t_0}^{t_1} G((J\psi(t,x))^{-1}\rho_0(x))(\operatorname{div}\tau)(t,\psi(t,x))J\psi(t,x)dtdx \quad \text{with} \\ G(y) &\stackrel{\text{def}}{=} F(y) - yF'(y) \quad \text{and} \quad \tau(t,x) \stackrel{\text{def}}{=} h(t,\psi^{-1}(t,x)) \quad \text{and} \\ \operatorname{div} h(t,x) &= \sum_{j=1}^{d} \frac{\partial}{\partial x_j} h^j(t,x)\,; \end{aligned}$$

la differential being defined sur

$$\overrightarrow{\mathcal{L}} \stackrel{\text{def}}{=} \{h \in C^\infty([0,1]\times\mathbb{R}^d;\mathbb{R}^d) \;/\; \forall x \in \mathbb{R}^d\,,\quad h(0,x) = h(1,x) = 0\}.$$

Proof. As the functionnal $\mathcal{A}_1$ is quadratic, the computation of $D\mathcal{A}_1$ is trivial. The computation of $D\mathcal{A}_2$ comes from the chain rule. The proof of the following lemma is left to the reader as an exercise.

Lemma 3.4.1 *We have the following formula*

$$DJ(\psi)\cdot h = \sum_{j=1}^{d} \det\left(D_x\psi^1, \cdots, D_x\psi^{j-1}, D_x h^j, D_x\psi^{j+1}\cdots, D_x\psi^d\right).$$

If we assume that $\tau(t,x) \stackrel{\text{def}}{=} h(t,\psi^{-1}(t,x))$, then we have

$$(DJ(\psi)\cdot h)(t,x) = (\operatorname{div}\tau)(t,\psi(t,x))J(\psi)(t,x).$$

Continuation of the proof of Proposition 3.4.1. The chain rule implies that

$$D(J\psi)^{-1}h(t,x) = -(J\psi(t,x))^{-1}(\operatorname{div}\tau)(t,\psi(t,x))\,;$$

the proposition is proved. □

Now we can define perfect compressible fluids .

Definition 3.4.3 *A perfect compressible fluid is a a fluid whose evolution follows extremals of the functional $\mathcal{A}$, i.e. following an element ψ of $\mathcal{L}$ such that*

$$\forall h \in \overrightarrow{\mathcal{L}}\,,\; D\mathcal{A}(\psi)h = 0.$$

3.5 The eulerian point of view in the compressible case

The above definition seems rather implicit. The purpose of this section is to from a lagrangian description , i.e. a description which is based on the evolution of each pointwise particule x to a eulerian description, i.e. to a description of the fluid based on the knowledge of the speed of the fluid fluid in whole of its points.

Mathematically, the link between these two points of view is the theory of ordinary differential equations. Indeed, let us consider an element ψ of $\mathcal{L}$. Then the associated vector field is defined by

$$v(t,x) = \partial_t\psi(t,\psi^{-1}(t,x)). \tag{3.5}$$

It is clear that v belongs to $\overrightarrow{\mathcal{L}}$ and that ψ is solution of

$$\begin{cases} \partial_t\psi(t,x) &= v(t,\psi(t,x)) \\ \psi(0,x) &= x. \end{cases} \tag{3.6}$$

Conversely, if v belongs to $\overrightarrow{\mathcal{L}}$, Cauchy-Lipschitz Theorem allows to define a flow ψ belonging to $\mathcal{L}$ by the above system (3.6).

Now let us state the theorem which justifies this approach and derives the Euler system of an incompressible fluid.

Theorem 3.5.1 *Let ψ be an extremal of the functionnal $\mathcal{A}$, i.e. an element of $\mathcal{L}$ such that*

$$\forall h \in \overrightarrow{\mathcal{L}}\,,\ D\mathcal{A}(\psi)h = 0.$$

Let us consider time dependant vector field vdefined by the above relation (3.5); let us define

$$\rho(t,x) \stackrel{\text{def}}{=} \rho_0(\psi^{-1}(t,x))J\psi(t,\psi^{-1}(t,x))^{-1},$$

then the couple (ρ, v) satisfies the following system, called compressible Euler system

$$(E_{comp})\left\{\begin{array}{rcl} \partial_t\rho + v\cdot\nabla\rho + \rho\,\mathrm{div}\,v & = & 0 \\ \rho(\partial_t v + v\cdot\nabla v) + \nabla p & = & 0 \end{array}\right. \quad \textit{with} \quad v\cdot\nabla a \stackrel{\text{def}}{=} \sum_{j=1}^{d} v^j \frac{\partial a}{\partial x_j} \quad \textit{and} \quad p = G(\rho).$$

Proof. First, the equation on ρ, which comes form the mass conservation, is nothing more than the translation in terms of partial differential equations of the fact that, by definition of ρ, we have

$$\rho(t,\psi(t,x)) = \rho_0(x)(J\psi(t,x))^{-1}.$$

By time derivation of this formula, it comes frome the chain rule and from Lemma 3.4.1

$$\begin{array}{rcl} (\partial_t\rho + v\cdot\nabla\rho)(t,\psi(t,x)) & = & -\rho_0(x)(J\psi(t,x))^{-2}DJ(\psi)(t,\psi(t,x))\cdot\partial_t\psi(t,x) \\ & = & -\rho_0(x)(J\psi(t,x))^{-2}DJ(\psi)\cdot v(t,\psi(t,x)) \\ & = & -\rho_0(x)(J\psi(t,x))^{-1}(\mathrm{div}\,v)(t,\psi(t,x)) \\ & = & -(\rho\,\mathrm{div}\,v)(t,\psi(t,x)). \end{array}$$

For the second equation, we use the hypothesis $D\mathcal{A}(\psi)h = 0$. Using an integration by parts and the definition of v and ρ, we find that

$$\begin{array}{rcl} D\mathcal{A}_1(\psi)h & = & -\displaystyle\int_{t_0}^{t_1}\int_{\mathbb{R}^d}\rho_0(x)\partial_t^2\psi(t,x)h(t,x)dxdt \\ & = & -\displaystyle\int_{t_0}^{t_1}\int_{\mathbb{R}^d}\rho_0(x)\partial_t v(t,\psi(t,x))h(t,x)dxdt \\ & = & -\displaystyle\int_{t_0}^{t_1}\int_{\mathbb{R}^d}\rho_0(x)(\partial_t v + v\cdot\nabla v)(t,\psi(t,x))\tau(t,\psi(t,x))dxdt \end{array}$$

Performing the change of variable $y = \psi(t,x)$, we find, by definition of ρ,

$$D\mathcal{A}_1(\psi)h = \int_{t_0}^{t_1}\int_{\mathbb{R}^d}\rho(x)(\partial_t v + v\cdot\nabla v)(t,x)\tau(t,x)dxdt. \tag{3.7}$$

Performing the change of variable $y = \psi(t,x)$ into the formula of $D\mathcal{A}_2$, it comes

$$D\mathcal{A}_2(\psi)h = -\int_{t_0}^{t_1}\int_{\mathbb{R}^d}\nabla p(t,x)\tau(t,x)dtdx. \tag{3.8}$$

Applying these two formulas (3.7) and (3.8), we find that

$$D\mathcal{A}(\psi)h = -\int_{t_0}^{t_1}\int_{\mathbb{R}^d}\Big(\rho(x)(\partial_t v + v\cdot\nabla v)(t,x) + \nabla p(t,x)\Big)\tau(t,x)dtdx$$

Thus the theorem is proved. □

3.6 The incompressible case

As above, the idea is to define a perfect incompressible fluid as an incompressible fluid whose evolution follows extremals of the functionnal action $\mathcal{A}_1$ restricted to the space $\mathcal{L}_0$. In order to define the notion of infinitesimal variation on the space $\mathcal{L}_0$ which is included in an affine space. Following the classical definition of the tangent space of a submanifold of $\mathbb{R}^d$, we give the following definition.

Definition 3.6.1 *An infinitesimal variation at a point ψ of the space $\mathcal{L}_0$ is the derivative at 0 of any function Θ continuously differentiable from $[0,1]$ into $\mathcal{L}_0$ such that $\Theta(0) = \psi$.*

Because of some regularity problem, it is not very easy to describe exactly the set of infinitesimal variations. Moreover, the intuition of finite dimensionnal spaces must be used with a lot of care always because of regularity problems. The following proposition will be enough for our purpose here.

Proposition 3.6.1 *Let us denote by $\overrightarrow{\mathcal{L}}_0$ the set of vector fields τ whose coefficients are continuously differentiables on $[t_0, t_1] \times \mathbb{R}^d$ and such that*

$$\tau(t_0) = \tau(t_1) = 0 \quad \text{and} \quad \forall\, t \in [t_0, t_1]\,,\ \operatorname{div} \tau(t) = 0.$$

Let θ be an infinitesimal variation at a point ψ of $\mathcal{L}_0$; it exists a vector field τ of the space $\overrightarrow{\mathcal{L}}_0$ such that $\theta(t,x) = \tau(t, \psi(t,x))$.

Conversely, let α a smooth compactly supported function on $]t_0, t_1[$ and τ a divergence free vector field whose componants belong to the space $\mathcal{S}$; if $\theta(t,x) = \alpha(t)\tau(\psi(t,x))$ then θ is an infinitesimal variation at point ψ.

Proof. Let θ an infinitesimal variation at point ψ. By definition, it exists a function Θ continuously differentiable of $[0,1]$ in $\mathcal{L}_0$ such that

$$\partial_s \Theta(s,t,x)_{|s=0} = \theta(t,x) \quad \text{et} \quad \Theta(0,t,x) = \psi(t,x). \tag{3.9}$$

As for any s and any t, $\Theta(s,t)$ is a diffeomorphism, we can define a vector field $\widetilde{\tau}(s,t,x)$ by

$$\widetilde{\tau}(s,t,x) = \partial_s \Theta(s,t,\Theta^{-1}(s,t,x)).$$

This means that

$$\partial_s \Theta(s,t,x) = \widetilde{\tau}(s,t,\Theta(s,t,x))$$

Thanks to Lemma 3.4.1, it is enough to apply the chain rule, which gives

$$\partial_s \det D_x \Theta(s,t,x) = \operatorname{div} \widetilde{\tau}(s,t,\Theta(s,t,x)) \times \det(D_x \theta(s,x)). \tag{3.10}$$

As $\det D_x \Theta(s,t,x) = 1$, we have

$$\forall (s,t) \in [0,1] \times [t_0, t_1]\ ,\ \operatorname{div} \widetilde{\tau}(s,t) = 0.$$

As $\partial_s \theta(0,t,x) = \widetilde{\tau}(0,t,\psi(t,x))$, we have the first point of the proposition defining

$$\tau(t,x) \stackrel{\text{def}}{=} \widetilde{\tau}(0,t,x).$$

The second point is very easy. It is enough to solve the following differential equation

$$\begin{cases} \partial_s \Theta(s,t,x) &= \alpha(t)\tau(t,\Theta(s,t,x)), \\ \Theta(0,t,x) &= \psi(t,x). \end{cases}$$

This concludes the proof □

Now let us define mathematically what a perfect incompressible fluid is.

Definition 3.6.2 *A fluid is perfect and incompressible if its evolution between the time t_0 and situation ψ_1 at time t_1 if its follows an element ψ of $\mathcal{L}_0$ such that, for any infinitesimal variation θ at point ψ, we have $D\mathcal{A}_1(\psi)\cdot\theta = 0$.*

The above definition can be formulated as the fact perfect incompressible fluids follows extremales of the action functionnal action, which is defined on the space of curves of measure preserving diffeomorphisms.

The above description of the evolution of an incompressible fluid by a curve on the spaceof measure preserving diffeomorphisms; it is the lagrangien point of view. Let us take now the Eulerian point of view.

Theorem 3.6.1 *Let ψ an evolution of a pecfect imcompressible fluid and v the divergence free vector field associated with ψ by (3.5). It exists then a tempered distribution p such that, if we definee*

$$v\cdot\nabla \stackrel{\mathrm{def}}{=} \sum_{j=1}^{d} v^j\partial_j,$$

we have

$$\partial_t v + v\cdot\nabla v \in V_\sigma(R^d)^\circ$$

where $V_\sigma(R^d)^\circ$ is the set of all (vector valued distributions) tempered distributions w such that, for all divergence free vector field ϕ with componants in $\mathcal{S}(\mathbb{R}^d)$, we have

$$\langle w,\phi\rangle \stackrel{\mathrm{def}}{=} \sum_{j=1}^{d}\langle w^j,\phi^j\rangle = 0.$$

Proof. Let us suppose that ψ is an evolution of a perfect incompressible fluid. Definition 3.6.1 and Proposition 3.6.1 imply that, for any $\alpha\in\mathcal{D}(]t_0,t_1[)$ and any divergence free vector field, we have $\tau\in C^\infty([t_0,t_1];\mathcal{S}(\mathbb{R}^d;\mathbb{R}^d))$,

$$\int_{t_0}^{t_1}\int_{\mathbb{R}^d}\partial_t\psi(t,x)\partial_t(\alpha(t)\tau(t,\psi(t,x)))dtdx = 0.$$

Form (3.6), it comes

$$\int_{t_0}^{t_1}\int_{\mathbb{R}^d} v(t,\psi(t,x))\partial_t(\alpha(t)\tau(t,\psi(t,x)))dtdx = 0.$$

An integration by parts with respect the time variable ensures that

$$\int_{t_0}^{t_1}\int_{\mathbb{R}^d}\partial_t(v(t,\psi(t,x)))\alpha(t)\tau(t,\psi(t,x))dtdx = 0.$$

As $\partial_t(v(t,\psi(t,x)) = (\partial_t v + v\cdot\nabla v)(t,\psi(t,x))$, and as $\psi(t)$ is a measure preserving diffeomorphism, we have, for any $\alpha\in\mathcal{D}(]t_0,t_1[)$ and any divergence free vector field τ,

$$\int_{t_0}^{t_1}\int_{\mathbb{R}^d}(\partial_t v + v\cdot\nabla v)(t,x)\alpha(t)\tau(t,x)dtdx = 0. \tag{3.11}$$

As $\partial_t v + v\cdot\nabla v$ is a continuous function of the time, we have, for any divergence free vector field τ and for any time t,

$$\int_{\mathbb{R}^d}(\partial_t v + v\cdot\nabla v)(t,x)\tau(t,x)dx = 0. \tag{3.12}$$

Conversely, if v is a vector field satisfying (3.12). Let us consider the flow ψ of v defined by (3.6) et θ an infinitesimal variation in ψ. Then, we have

$$\int_{t_0}^{t_1}\int_{\mathbb{R}^d}\partial_t\psi(t,x)\partial_t\theta(t,x)dtdx = -\int_{t_0}^{t_1}\int_{\mathbb{R}^d}\partial_t^2\psi(t,x)\theta(t,x)dtdx.$$

Proposition 3.6.1 ensures the existence of a divergence free vector field τ such that $\theta = \tau(t, \psi(t,x))$. It turns out that

$$\int_{t_0}^{t_1} \int_{\mathbb{R}^d} \partial_t \psi(t,x) \partial_t \theta(t,x) dt dx = \int_{t_0}^{t_1} \int_{\mathbb{R}^d} (\nabla p)(\psi(t,x)) \tau(t, \psi(t,x)) dt dx.$$

For any time t, the diffeomorphism $\psi(t)$ preserves the measure; this concludes the proof the theorem. □

The proof of the following proposition is an exercise of distribution theory of distributions which is left to the reader.

Proposition 3.6.2 *Let w a vector field the coefficients of which are tempered distribtions. The existence of a tempered distribution p such that $w = \nabla p$ is equivalent to the fact that the curl of w is 0, i.e. $\partial_j w^i = \partial_i w^j$.*

Now let us give a weak formulation to Equation (3.12).This formulation is equivalent to the one of Relation (3.12) in the case when the vector field solution is smooth enough. Nevertheless, it can be important to have a weak formulation. If v is a divergence free vector field continuously differentiable, we have $v \cdot \nabla a = \operatorname{div}(av)$ for any function a continuously differentiable. This gives

$$\partial_t v + v \cdot \nabla v = \partial_t v + \operatorname{div} v \otimes v,$$

where $\operatorname{div} v \otimes v$ denotes the vector field whose coordiante of order j is $\displaystyle\sum_{j=1}^{d} \partial_j (v^i v^j)$. Thus we get the following formulation of incompressible Euler equations.

$$(E) \begin{cases} \partial_t v + \operatorname{div} v \otimes v &= -\nabla p \\ \operatorname{div} v &= 0 \\ v_{|t=0} &= v_0. \end{cases}$$

Remarks

- The proofs of sections 3.1 3.2 and 3.3 must be known. The sections 3.4, 3.5 and 3.6 are important in the point of view of modelization. It is not necessary to know the proofs of these sections.

Chapter 4

Leray's Theorem on Navier-Stokes equations

In this chapitre, we shall prove the existence of global solutions for the incompressible Navier-Stokes system in a bounded domain with Dirichlet boundary conditions which means

$$\begin{cases} \partial_t v + v\cdot\nabla v - \nu\Delta v &= -\nabla p \\ \operatorname{div} v &= 0 \\ v_{|t=0} &= v_0 \\ v_{|\partial\Omega} &= 0. \end{cases}$$

In all this chapter, Ω will denoted a bounded domain of $\mathbb{R}^d$ with d in $\{2,3\}$. We want to solve the above system for initial data of finite kinetic energy (i.e. for u_0 in $\mathcal{H}$). The purpose is the proof existence (but not uniqueness) of global solution in the energy space which is the space

$$\mathcal{E} \stackrel{\text{def}}{=} L^\infty_{loc}(\mathbb{R}^+;\mathcal{H}) \cap L^2_{loc}(\mathbb{R}^+;\mathcal{V}_\sigma).$$

4.1 The concept of turbulent solution

Let us state now the weak formulation of the incompressible Navier–Stokes system (NS_ν).

Definition 4.1.1 *Given a domain Ω in $\mathbb{R}^d$, we shall say that u is a weak solution of the Navier–Stokes equations on $\mathbb{R}^+\times\Omega$ with an initial data u_0 in $\mathcal{H}$ and an external force f in $L^2_{loc}(\mathbb{R}^+;\mathcal{V}')$ if and only if u belongs to the space*

$$C(\mathbb{R}^+;\mathcal{V}'_\sigma) \cap L^\infty_{loc}(\mathbb{R}^+;\mathcal{H}) \cap L^2_{loc}(\mathbb{R}^+;\mathcal{V}_\sigma)$$

and for any function Ψ in $C^1(\mathbb{R}^+;\mathcal{V}_\sigma)$, the vector field u satisfies the following condition:

$$\begin{aligned} \int_\Omega (u\cdot\Psi)(t,x)\,dx &= \int_\Omega u_0(x)\cdot\Psi(0,x)\,dx - \int_0^t\int_\Omega \Big(\nu\nabla u : \nabla\Psi - u\cdot\partial_t\Psi\Big)(t',x)\,dxdt' \\ &+ \int_0^t\int_\Omega u\otimes u : \nabla\Psi(t',x)dt'dx + \int_0^t \langle f(t'),\Psi(t')\rangle\,dt' \quad \textit{with} \\ \nabla u : \nabla\Psi &= \sum_{j,k=1}^d \partial_j u^k \partial_j \Psi^k \quad \textit{and} \quad u\otimes u : \nabla\Psi = \sum_{j,k=1}^d u^\ell u^k \partial_j\Psi^k. \end{aligned}$$

Let us point tout that this definition make sense because, using Gagliardo–Nirenberg's inequality stated in Corollary 2.2.1 we get

$$\forall u \in \mathcal{V}\,,\ \|u\|_{L^4} \leq \|\nabla u\|_{L^2}^{\frac{d}{4}} \|u\|_{L^2}^{1-\frac{d}{4}}.$$

Thus the energy space is included in the space $L^{\frac{8}{d}}_{loc}(\mathbb{R}^+; L^4)$ and as d is less than 4, product of coordinates of u belongs to $L^1_{loc}(\mathbb{R}^+; L^2)$ The non linear term

$$-\int_0^t \int_\Omega u \otimes u : \nabla \Psi(t', x) dt' dx$$

is well defined for u in the energy space $\mathcal{E}$.

Let us remark that the above relation means that the equality in (NS_ν) must be understood as an equality in the sense of $\mathcal{V}'_\sigma$. Now let us state the Leray theorem.

Theorem 4.1.1 *Let Ω be a domain of $\mathbb{R}^d$ and u_0 a vector field in $\mathcal{H}$. Then, there exists a global weak solution u to (NS_ν) in the sense of Definition 4.1.1. Moreover, this solution satisfies the energy inequality for all $t \geq 0$,*

$$\begin{aligned} \frac{1}{2}\int_\Omega |u(t,x)|^2 dx \quad &+ \quad \nu \int_0^t \int_\Omega |\nabla u(t',x)|^2 dx dt' \\ &\leq \quad \frac{1}{2}\int_\Omega |u_0(x)|^2 dx + \int_0^t \langle f(t',\cdot), u(t',\cdot)\rangle dt'. \end{aligned} \tag{4.1}$$

It is convenient to state the following definition.

Definition 4.1.2 *A solution of (NS_ν) in the sense of the above Definition 4.1.1 which moreover satisfies the energy inequality (4.1) is called a Leray solution (or a turbulent solution of (NS_ν).*

Let us remark that the energy inequality implies a control on the energy.

Proposition 4.1.1 *Any Leray solution u of (NS_ν) satisfies*

$$\|u(t)\|_{L^2}^2 + \nu \int_0^t \|\nabla u(t')\|_{L^2}^2 dt' \leq \|u_0\|_{L^2}^2 + \frac{C}{\nu}\int_0^t \|f(t')\|_{\mathcal{V}'_\sigma}^2 dt'.$$

Proof. By definition of the norm $\|\cdot\|_{\mathcal{V}'_\sigma}$, we have

$$\langle f(t,\cdot), u(t,\cdot)\rangle \leq \|f(t,\cdot)\|_{\mathcal{V}'_\sigma} \|u(t,\cdot)\|_{\mathcal{V}_\sigma}.$$

Inequality (4.1) becomes

$$\|u(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u(t')\|_{L^2}^2 dt' \leq \|u_0\|_{L^2}^2 + \int_0^t \|f(t')\|_{\mathcal{V}'_\sigma} \|u(t',\cdot)\|_{\mathcal{V}}^2 dt'.$$

As $\|u(t',\cdot)\|_{\mathcal{V}}^2 = \|\nabla u(t',\cdot)\|_{L^2}^2$, we get, using the fact that $2ab \leq a^2 + b^2$,

$$\|u(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u(t')\|_{L^2}^2 dt' \leq \|u_0\|_{L^2}^2 + \frac{C}{\nu}\int_0^t \|f(t')\|_{\mathcal{V}'_\sigma}^2 dt'.$$

Thus the proposition is proved.

The outline of this section is now the following:

- first approximate solutions are built in spaces with finite frequencies by using simple ordinary differential equations results in L^2–type spaces.
- Next, a compactness result is derived.
- Finally the conclusion is obtained by passing to the limit in the weak formulation, taking especially care of the nonlinear terms.

4.2 Construction of approximate solutions

In this section, we intend to build approximate solutions of the Navier–Stokes equations. We use the projections $\mathbb{P}_k$ defined in (3.1) and denote by $\mathcal{H}_k$ the space $\mathbb{P}_k \mathcal{H} = \mathbb{P}_k \mathcal{V}'$ equipped with the norm L^2. Let us introduce the function

$$F_k \quad \left\{ \begin{array}{ccc} \mathcal{H}_k & \longrightarrow & \mathcal{H}_k \\ v & \longmapsto & \mathbb{P}_k(\operatorname{div}(v \otimes v)). \end{array} \right.$$

The properties of F_k will be the consequence of the following lemma which will be very useful.

Lemma 4.2.1 *Let us define the bilinear map*

$$Q \left\{ \begin{array}{ccc} \mathcal{V} \times \mathcal{V} & \longrightarrow & \mathcal{V}' \\ (u, v) & \longmapsto & -\operatorname{div}(u \otimes v). \end{array} \right. \tag{4.2}$$

For any u and v in $\mathcal{V}$, the following estimates hold. For d in $\{2, 3\}$, a constant C exists such that, for any $\varphi \in \mathcal{V}$,

$$\langle Q(u, v), \varphi \rangle \leq C \|\nabla u\|_{L^2}^{\frac{d}{4}} \|\nabla v\|_{L^2}^{\frac{d}{4}} \|u\|_{L^2}^{1-\frac{d}{4}} \|v\|_{L^2}^{1-\frac{d}{4}} \|\nabla \varphi\|_{L^2}.$$

Moreover for any u in $\mathcal{V}_\sigma$ and any v in $\mathcal{V}$, $\langle Q(u, v), v \rangle = 0$.

Proof. The first two inequalities follow directly from Gagliardo–Nirenberg's inequality stated in Corollary 2.2.1 page 24, once noticed that

$$\begin{aligned} \langle Q(u, v), \varphi \rangle &\leq \|u \otimes v\|_{L^2} \|\nabla \varphi\|_{L^2} \\ &\leq \|u\|_{L^4} \|v\|_{L^4} \|\nabla \varphi\|_{L^2}. \end{aligned}$$

In order to prove the third assertion, let us assume that u and v are two vector fields the components of which belong to $\mathcal{D}(\Omega)$. Then we deduce from integrations by parts that

$$\begin{aligned} \langle Q(u, v), v \rangle &= -\int_\Omega (\operatorname{div}(u \otimes v) \cdot v)(x)\, dx \\ &= -\sum_{\ell, m=1}^{d} \int_\Omega \partial_m (u^m(x) v^\ell(x)) v^\ell(x)\, dx \\ &= \sum_{\ell, m=1}^{d} \int_\Omega u^m(x) v^\ell(x) \partial_m v^\ell(x)\, dx \\ &= -\int_\Omega |v(x)|^2 \operatorname{div} u(x)\, dx - \langle Q(u, v), v \rangle. \end{aligned}$$

Thus, we have

$$\langle Q(u,v), v\rangle = -\frac{1}{2}\int_\Omega |v(x)|^2 \operatorname{div} u(x)\, dx.$$

The two expressions are continuous on $\mathcal{V}$ and by definition, $\mathcal{D}$ is dense in $\mathcal{V}$. Thus the formula is true for any (u,v) in $\mathcal{V}_\sigma \times \mathcal{V}$, which completes the proof. □

As a corolllary , we get that F_k is locally Lipschitz in $\mathcal{H}_k$ and that F_k satisfies

$$\|F_k(v)\|_{\mathcal{H}_k} \lesssim \lambda_k^{\frac{d}{4}} \|v\|^2_{\mathcal{H}_k}. \tag{4.3}$$

Lemma 4.2.2 *For any force f in $L^2_{loc}(\mathbb{R}^+;\mathcal{V}')$, a sequence $(f_k)_{k\in\mathbb{N}}$ exists in $C^1(\mathbb{R}^+;\mathcal{V}_\sigma)$ such that for any $k\in\mathbb{N}$ and for any $t>0$, the vector field $f_k(t)$ belongs to $\mathcal{H}_k$, and*

$$\lim_{k\to\infty} \|f_k - f\|_{L^2([0,T];\mathcal{V}'_\sigma)} = 0.$$

Proof. Thanks to Theorem 3.2.2 and to the Lebesgue Theorem, a sequence $(\widetilde{f}_k)_{k\in\mathbb{N}}$ exists in $L^2_{loc}(\mathbb{R}^+;\mathcal{V}_\sigma)$ such that for any positive integer k and for almost all positive t, the vector field $\widetilde{f}_k(t)$ belongs to $\mathcal{H}_k$ and

$$\forall T>0\,,\ \lim_{k\to\infty} \|\widetilde{f}_k - f\|_{L^2([0,T];\mathcal{V}'_\sigma)} = 0.$$

A standard (and omitted) time regularization procedure concludes the proof of the lemma.

Thanks to Theorem 3.2.2 and to the above lemma, we can solve the following ordinary differential equation

$$(NS_{\nu,k}) \quad \left\{ \begin{array}{rcl} \dot{u}_k(t) & = & \nu\,\mathbb{P}_k\,\Delta u_k(t) + F_k(u_k(t)) + f_k(t) \\ u_k(0) & = & \mathbb{P}_k\, u_0. \end{array} \right.$$

Theorem 3.2.2 implies that $\mathbb{P}_k\,\Delta$ is a linear map from $\mathcal{H}_k$ into itself. Thus the continuity properties on Q and $\mathbb{P}_k$ allow to apply the Cauchy–Lipschitz theorem. This gives the existence of $T_k \in]0,+\infty]$ and a unique maximal solution u_k of $(NS_{\nu,k})$ in $C^\infty([0,T_k[;\mathcal{H}_k)$. In order to prove that $T_k=+\infty$, let us use the energy estimate

$$\frac{1}{2}\frac{d}{dt}\|u_k(t)\|^2_{L^2} = -\nu\|\nabla u_k(t)\|^2_{L^2} + (F_k(u_k)|u_k)_{L^2} + (f_k(t')|u_k(t))_{L^2}.$$

Because of Lema 4.2.1, we get

$$\frac{1}{2}\frac{d}{dt}\|u_k(t)\|^2_{L^2} = -\nu\|\nabla u_k(t)\|^2_{L^2} + \langle f_k(t'), u_k(t)\rangle.$$

By integration in time, we infer that

$$\frac{1}{2}\|u_k(t)\|^2_{L^2} + \nu\int_0^t \|\nabla u_k(t')\|^2_{L^2} dt' = \frac{1}{2}\|u_k(0)\|^2_{L^2} + \int_0^t (f_k(t')|u_k(t'))_{L^2} dt'. \tag{4.4}$$

Using the (well known) fact that $2ab \leq a^2+b^2$, we get

$$\|u_k(t)\|^2_{L^2} + \nu\int_0^t \|\nabla u_k(t')\|^2_{L^2}\, dt' \leq \|u_k(0)\|^2_{L^2} + \frac{C}{\nu}\int_0^t \|f_k(t')\|^2_{\mathcal{V}'_\sigma} dt' \tag{4.5}$$

Using corollary 1.1.1 page 11, we get that, for all k, T_k is infinite and the above estimate is valid for any time t.

Now let us write the ODE on $\mathcal{H}_k$ in terms of Definition 4.1.1. We have, for any function Ψ in $C^1(\mathbb{R}^+;\mathcal{V}_\sigma)$, the vector field u satisfies the following condition:

$$\begin{aligned}\int_\Omega (u_k\cdot\Psi)(t,x)\,dx = \int_\Omega (\mathbb{P}_k\,u_0)(x)\cdot\Psi(0,x)\,dx - \int_0^t\!\!\int_\Omega \Big(\nu\nabla u_k:\nabla\Psi - u_k\cdot\partial_t\Psi\Big)(t',x)\,dxdt' \\ = -\int_0^t\int_\Omega u_k\otimes u_k:\nabla\,\mathbb{P}_k\,\Psi(t',x)dt'dx + \int_0^t \langle f_k(t'),\Psi(t')\rangle\,dt'\,.\end{aligned}\tag{4.6}$$

The problem consists now to pass to the limit in the above formula. The first step is the definiton of some weak limit u. Letus first work on a fixed time interval $[0,T]$. As $(u_k)_{k\in\mathbb{N}}$ is bounded in the Hilbert space $L^2([0,T];\mathcal{V}_\sigma)$, up to an extraction we can assume that a function$u$ exists in $L^2([0,T];\mathcal{V}_\sigma)$ such that the sequence $(u_k)_{k\in\mathbb{N}}$ convergence to u in the sense of the weak convergence in Hilbert space, which means, fora any θ in $L^2([0,T];\mathcal{V}_\sigma)$,

$$\lim_{k\to\infty}\int_0^T\int_\Omega \nabla u_k(t,x)\cdot\nabla\theta(t,x)dtdx = \int_0^T\int_\Omega \nabla u(t,x)\cdot\nabla\theta(t,x)dtdx. \tag{4.7}$$

Moreover, as $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $L^\infty([0,T];\mathcal{H})$, it is a bouded sequence of the Hilbert space $L^2([0,T]\times\Omega)$. Thus again up to an extraction, we can assume that $(u_k)_{k\in\mathbb{N}}$ is wealky convergent to u in the sense that for any φ in $L^2([0,T]\times\Omega)$, we have

$$\lim_{k\to\infty}\int_0^T\int_\Omega u_k(t,x)\cdot\theta(t,x)dtdx = \int_0^T\int_\Omega u(t,x)\cdot\theta(t,x)dtdx. \tag{4.8}$$

This impiles in particular that $(u_k)_{k\in\mathbb{N}}$ tends to u in the sense of distribution. Now let us justify the limit in the linear term. Indeed, Assertions (4.7) and (4.8) imply that

$$\lim_{n\to\infty}\int_0^t\!\!\int_\Omega\Big(\nu\nabla u_k:\nabla\Psi - u_k\cdot\partial_t\Psi\Big)(t',x)\,dxdt' = \int_0^t\!\!\int_\Omega\Big(\nu\nabla u:\nabla\Psi - u\cdot\partial_t\Psi\Big)(t',x)\,dxdt'. \tag{4.9}$$

Now let us get rid of the term $\mathbb{P}_k$ in the non linear term. It relies on the following lemma.

Lemma 4.2.3 *Let* H *be a Hilbert space, and let* $(A_n)_{n\in\mathbb{N}}$ *be a bounded sequence of linear operators on* H *such that*

$$\forall h\in \mathrm{H},\quad \lim_{n\to\infty}\|A_nh-h\|_{\mathrm{H}}=0.$$

Then if $\psi\in C([0,T];\mathrm{H})$ *we have* $\lim_{n\to\infty}\sup_{t\in[0,T]}\|A_n\psi(t)-\psi(t)\|_{\mathrm{H}}=0.$

Proof. The function ψ is continuous in time with values in H, so for all positive ε, the compact $\psi([0,T])$, can be covered by a finite number of balls of radius

$$\frac{\varepsilon}{2(\mathcal{A}+1)}\quad\text{with}\quad \mathcal{A}\stackrel{\text{def}}{=}\sup_n\|A_n\|_{\mathcal{L}(\mathrm{H})}.$$

and center $(\psi(t_\ell))_{0\leq\ell\leq N}$. Then we have, for all t in $[0,T]$ and ℓ in $\{0,\cdots,N\}$,

$$\|A_n\psi(t)-\psi(t)\|_{\mathrm{H}}\leq\|A_n\psi(t)-A_n\psi(t_\ell)\|_{\mathrm{H}}+\|A_n\psi(t_\ell)-\psi(t_\ell)\|_{\mathrm{H}}+\|\psi(t_\ell)-\psi(t)\|_{\mathrm{H}}.$$

The assumption on A_n implies that for any ℓ, the sequence $(A_n\psi(t_\ell))_{n\in\mathbb{N}}$ tends to $\psi(t_\ell)$. Thus, an integer n_N exists such that, if $n\geq n_N$,

$$\forall\ell\in\{0,\cdots N\}\,,\ \|A_n\psi(t_\ell)-\psi(t_\ell)\|_{\mathrm{H}}<\frac{\varepsilon}{2}.$$

We infer that, if $n \geq n_N$, for all t in $[0,T]$ and all ℓ in $\{0,\cdots,N\}$,

$$\|A_n\psi(t)-\psi(t)\|_{\mathrm{H}} \leq \|A_n\psi(t)-A_n\psi(t_\ell)\|_{\mathrm{H}} + \|\psi(t_\ell)-\psi(t)\|_{\mathrm{H}} + \frac{\varepsilon}{2}\cdot$$

For any t, let us choose ℓ such that

$$\|\psi(t)-\psi(t_\ell)\|_{\mathrm{H}} \leq \frac{\varepsilon}{2(\mathcal{A}+1)}\,\cdot$$

The lemma is proved. □

Continuation of the proof of Theorem Using (4.7), (4.7) and Lemma 4.2.3, Identity (4.6) becomes

$$\begin{aligned}\int_\Omega (u_k\cdot\Psi)(t,x)\,dx &= \int_\Omega u_0(x)\cdot\Psi(0,x)\,dx - \int_0^t\int_\Omega \Big(\nu\nabla u:\nabla\Psi - u\cdot\partial_t\Psi\Big)(t',x)\,dxdt' \\ &= -\int_0^t\int_\Omega u_k\otimes u_k:\nabla\Psi(t',x)dt'dx + \int_0^t \langle f(t'),\Psi(t')\rangle\,dt' + o_T(1)\,.\end{aligned} \tag{4.10}$$

Now the main step in the proof of the following proposotion.

Proposition 4.2.1

$$\lim_{k\to\infty}\|u_k-u\|_{L^\infty([0,T];\mathcal{V}'_\sigma)} = 0.$$

Proof. Let us establish that

$$\forall\varepsilon\,,\ \exists k_0\,/\,\forall k\,,\ \|u_k-\mathbb{P}_{k_0}\,u_k\|_{L^\infty([0,T];\mathcal{V}'_\sigma)} < \frac{\varepsilon}{2}\,\cdot \tag{4.11}$$

The proof of the claim is based on Theorem 3.2.2. Using this result, we can write that for any t in $[0,T]$, we get

$$\|u_k(t)-\mathbb{P}_{k_0}\,u_k(t)\|_{\mathcal{V}'_\sigma}^2 \quad = \quad \sum_{j\geq k_0+1}\lambda^{-2j}\langle u_k(t),e_j\rangle^2.$$

Using that the sequence $(\lambda_j)_j$ is increasing, we get, by Theorem 3.2.2,

$$\begin{aligned}\|u_k(t)-\mathbb{P}_{k_0}\,u_k(t)\|_{\mathcal{V}'_\sigma}^2 \quad &\leq \quad \lambda_{k_0+1}^{-2}\sum_j \lambda_j^2\langle u_k(t),e_j\rangle)^2 \\ &\leq \quad \lambda_{k_0+1}^{-2}\|u_k\|_{L^\infty([0,T];\mathcal{H})}^2.\end{aligned}$$

The fact that $\lim\limits_{j\to\infty}\lambda_j = +\infty$ ensures (4.11).

Now, let us prove that the sequence $(\mathbb{P}_{k_0}\,u_k)_k$ is relativement compact in $\mathcal{H}_{k_0}$. As all the norm are equivalent on $\mathcal{H}_{k_0}$, let us consider $\mathcal{H}_{k_0}$equipped with the norm $\mathcal{V}'_\sigma$. It turns out that

$$\|\,\mathbb{P}_{k_0}\,\dot u_k(t)\|_{\mathcal{V}'_\sigma} \leq \|u_k(t)\|_{\mathcal{V}_\sigma} + \|u_k(t)\|_{L^2}^{2-\frac{d}{2}}\|\nabla u_k(t)\|_{L^2}^{\frac{d}{2}}.$$

Using energy estimate (4.5), we infer that $(\mathbb{P}_{k_0}\,\dot u_k)_k$ is a bounded sequence of $L^{\frac{4}{d}}([0,T];L^2)$ which means that

$$\forall k\,,\ \|\,\mathbb{P}_{k_0}\,\dot u_k\|_{L^{\frac{4}{d}}([0,T];\mathcal{V}'_\sigma)} \leq T^{1-\frac{2}{d}}\|u_0\|_{\mathcal{H}} + C\|u_0\|_{\mathcal{H}}^2.$$

Thus, by integration and Hölder estimate, we get, for any (t,t') in $[0,T]^2$,

$$\| \mathbb{P}_{k_0} u_k(t) - \mathbb{P}_{k_0} u_k(t)\|_{\mathcal{V}'_\sigma} \leq |t-t'|^{1\frac{4}{d}} \big(T^{1-\frac{2}{d}}\|u_0\|_{\mathcal{H}} + C\|u_0\|^2_{\mathcal{H}}\big). \tag{4.12}$$

Moreover, for any t in $[0,T]$, the set $\{\mathbb{P}_{k_0} u_k(t),\ k\in\mathbb{N}\}$ in bounded in the finite dimensionnal space $\mathcal{H}_{k_0}$. Thus it is relatively compact. Together with (4.12), this allows to apply Ascoli's compactness theorem. Thus, in particular, the set $\{\mathbb{P}_{k_0} u_k,\ k\in\mathbb{N}\}$ can be recovered by a finite number of balls of radius $\varepsilon/2$ in the norm $L^\infty([0,T];\mathcal{V}'_\sigma)$. Together with (4.11), this proves the sequence $(u_k)_{k\in\mathbb{N}}$ is relatively compact in $C([0,T];\mathcal{V}'_\sigma)$. As the sequence $(u_k)_{k\in\mathbb{N}}$ is bounded implies the convergence in the sense of distributions, we get the result. □

Conclusion of the proof the proof Theorem 4.1.1 Because of Relation (4.10), it is enough to prove that

$$\lim_{n\to\infty} \|u_k - u\|_{L^2([0,T];L^4)} = 0. \tag{4.13}$$

It will be a consequence of Proposition 4.2.1 and the energy bound (4.5). We have

$$\|u_k(t)-u(t)\|^2_{L^4} \leq C\|u_k(t)-u(t)\|^{2-\frac{d}{2}}_{L^2}\|\nabla u_k(t)-\nabla u(t)\|^{\frac{d}{2}}_{L^2} \tag{4.14}$$

Let us observe that for any v in $\mathcal{H}$, we have, using Cauchy Schwarz inequality,

$$\begin{aligned}\|v\|^2_{\mathcal{H}} &= \sum_{j=0}^{\infty} \lambda_j^{-1}\langle v,e_j\rangle \lambda_j \langle v,e_j\rangle \\ &\leq \Big(\sum_{j=0}^{\infty}\lambda_j^{-2}\langle v,e_j\rangle^2\Big)^{\frac12}\Big(\sum_{j=0}^{\infty}\lambda_j^{2}\langle v,e_j\rangle^2\Big)^{\frac12} = \|v\|_{\mathcal{V}'_\sigma}\|v\|_{\mathcal{V}_\sigma}.\end{aligned}$$

Plugging this in (4.14) gives

$$\|u_k(t)-u(t)\|^2_{L^4} \leq C\|u_k(t)-u(t)\|^{1-\frac{d}{4}}_{\mathcal{V}'_\sigma}\|\nabla u_k(t)-\nabla u(t)\|^{1+\frac{d}{4}}_{L^2}$$

By times integration this gives

$$\|u_k(t)-u(t)\|^2_{L^2([0,T];L^4)} \leq C\|u_k-u\|^{1-\frac{d}{4}}_{L^\infty([0,T];\mathcal{V}'_\sigma)}\|\nabla u_k-\nabla u\|^{1+\frac{d}{4}}_{L^{1+\frac{d}{4}}([0,T];L^2)}.$$

As d is less than 4, Hölder estimate implies that

$$\|u_k(t)-u(t)\|^2_{L^2([0,T];L^4)} \leq CT^{\frac12-\frac{d}{8}}\|u_k-u\|^{1-\frac{d}{4}}_{L^\infty([0,T];\mathcal{V}'_\sigma)}\|\nabla u_k-\nabla u\|^{1+\frac{d}{4}}_{L^2([0,T];L^2)}.$$

As $(u_k)_{k\in\mathbb{N}}$ is bounded in $L^2([0,T];\mathcal{V}_\sigma)$, this proves (4.13).

It remains to prove the energy inequality (4.1). Proposition 4.2.1 implies in particular that for any time $t\geq 0$ and any $v\in\mathcal{V}_\sigma$,

$$\lim_{k\to\infty}(u_k(t)|v)_{\mathcal{H}} = \lim_{k\to\infty}\langle u_k(t),v)\rangle = \langle u(t),v)\rangle = (u(t)|v)_{\mathcal{H}}.$$

As $\mathcal{V}_\sigma$ is dense in $\mathcal{H}$, we get that for any $t\geq 0$, the sequence $(u_k(t))_{k\in\mathbb{N}}$ converges weakly towards $u(t)$ in the Hilbert space $\mathcal{H}$. Hence

$$\|u(t)\|^2_{L^2} \leq \liminf_{k\to\infty}\|u_k(t)\|^2_{L^2} \quad \text{for all} \quad t\geq 0.$$

On the other hand, $(u_k)_{k\in\mathbb{N}}$ converges weakly to u in $L^2_{loc}([0,T];\mathcal{V})$, so that for all non negative t, we have

$$\int_0^t \|\nabla u(t')\|^2_{L^2}\,dt' \leq \liminf_{k\to\infty}\int_0^t \|\nabla u_k(t')\|^2_{L^2}\,dt'.$$

Taking the $\liminf_{k\to\infty}$ in the energy equality for approximate solutions (4.4) yields the energy inequality (4.1). In order to get a global solution, we consider an increasing sequence of positive time $(T_k)_{k\in\mathbb{N}}$ which tends to infinity. Using the diagonal process gives the result on the whiole interval $[0,\infty[$.

4.3 Stability of Leray solutions in dimension two

In a two dimensional domain, the Leray weak solutions are unique and even stable. More precisely, we have the following theorem.

Theorem 4.3.1 *For any data u_0 in $\mathcal{H}$ and f in $L^2_{loc}(\mathbb{R}^+;\mathcal{V}')$, the Leray weak solution is unique. Moreover, it belongs to $C(\mathbb{R}^+;\mathcal{H})$ and satisfies, for any (s,t) such that $0\leq s\leq t$,*

$$\frac{1}{2}\|u(t)\|^2_{L^2}+\nu\int_s^t \|\nabla u(t')\|^2_{L^2}\,dt' = \frac{1}{2}\|u(s)\|^2_{L^2}+\int_s^t \langle f(t'),u(t')\rangle\,dt'. \tag{4.1}$$

Furthermore, the Leray solutions are stable in the following sense. Let u (resp. v) be the Leray solution associated with u_0 (resp. v_0) in $\mathcal{H}$ and f (resp. g) in the space $L^2_{loc}(\mathbb{R}^+;\mathcal{V}')$ then,

$$\begin{aligned}\|(u-v)(t)\|^2_{L^2}&+\nu\int_0^t\|\nabla(u-v)(t')\|^2_{L^2}\,dt'\\ &\leq\Big(\|u_0-v_0\|^2_{L^2}+\frac{1}{\nu}\int_0^t\|(f-g)(t')\|^2_{\mathcal{V}'_\sigma}\,dt'\Big)\exp\Big(\frac{CE^2(t)}{\nu^4}\Big)\quad \textit{with}\end{aligned}$$

$$E(t)\stackrel{\text{def}}{=}\min\Big\{\|u_0\|^2_{L^2}+\frac{1}{\nu}\int_0^t\|f(t')\|^2_{\mathcal{V}'_\sigma}\,dt'\,,\|v_0\|^2_{L^2}+\frac{1}{\nu}\int_0^t\|g(t')\|^2_{\mathcal{V}'_\sigma}\,dt'\Big\}.$$

Proof. As u belongs to $L^\infty_{loc}(\mathbb{R}^+;\mathcal{H})\cap L^2_{loc}(\mathbb{R}^+;\mathcal{V}_\sigma)$, thanks to Lemma 4.2.1 page 45, the non linear term $Q(u,u)$ belongs to $L^2_{loc}(\mathbb{R}^+;\mathcal{V}')$. Thus u is the solution of (ES_ν) with initial data u_0 and external force $f+Q(u,u)$. Theorem 3.3.1 immediately implies that u belongs to $C(\mathbb{R}^+;\mathcal{H})$ and satisfies, for any (s,t) such that $0\leq s\leq t$,

$$\begin{aligned}\frac{1}{2}\|u(t)\|^2_{L^2}+\nu\int_s^t\|\nabla u(t')\|^2_{L^2}\,dt' &= \frac{1}{2}\|u(s)\|^2_{L^2}\\ &+\int_s^t\langle f(t'),u(t')\rangle\,dt'+\int_s^t\langle Q(u(t'),u(t')),u(t')\rangle\,dt'.\end{aligned}$$

Using Lemma 4.2.1, we get the energy equality (4.1).

To prove the stability, let us observe that, by difference $w\stackrel{\text{def}}{=}u-v$ is the solution of (ES_ν) with data u_0-v_0 and external force $f-g+Q(u,u)-Q(v,v)$, Theorem 3.3.1 implies that

$$\begin{aligned}\|w(t)\|^2_{L^2}+2\nu\int_0^t\|\nabla w(t')\|^2_{L^2}\,dt' &= \|w(0)\|^2_{L^2}\\ &+2\int_0^t\langle(f-g)(t'),w(t')\rangle\,dt'+2\int_0^t\langle(Q(u,u)-Q(v,v))(t'),w(t')\rangle\,dt'.\end{aligned}$$

The non linear term is estimated thanks to the following lemma.

Lemma 4.3.1 *In two dimensional domains, if a and b belong to $\mathcal{V}_\sigma$, we have*

$$|\langle Q(a,a)-Q(b,b),a-b\rangle| \leq C\|\nabla(a-b)\|_{L^2}^{\frac{3}{2}}\|a-b\|_{L^2}^{\frac{1}{2}}\|\nabla a\|_{L^2}^{\frac{1}{2}}\|a\|_{L^2}^{\frac{1}{2}}.$$

Proof. Lemma 4.2.1 implies that the quantity $\langle Q(a,b),a\rangle$ is well defined. As

$$\langle Q(b,b-a),b-a\rangle = 0,$$

we have

$$\begin{aligned}\langle Q(a,a)-Q(b,b),a-b\rangle &= \langle Q(a,a)-Q(b,a),a-b\rangle\\ &= \langle Q(a-b,a),a-b\rangle. \end{aligned}\tag{4.2}$$

Using again Lemma 4.2.1, we get the result. □

Continuation of the proof of Theorem 4.3.1. Using that $2ab \leq a^2+b^2$, we get

$$\begin{aligned}\|w(t)\|_{L^2}^2 + \frac{3}{2}\nu\int_0^t \|\nabla w(t')\|_{L^2}^2\,dt' &\leq \|w(0)\|_{L^2}^2 + \frac{2}{\nu}\int_0^t \|(f-g)(t')\|_{\mathcal{V}_\sigma'}^2\,dt'\\ &\quad + C\int_0^t \|\nabla w(t')\|_{L^2}^{\frac{3}{2}}\|w(t')\|_{L^2}^{\frac{1}{2}}\|\nabla u(t')\|_{L^2}^{\frac{1}{2}}\|u(t')\|_{L^2}^{\frac{1}{2}}\,dt'.\end{aligned}$$

Using (with $\theta = 1/4$) the convexity inequality

$$ab \leq \theta a^{\frac{1}{\theta}} + (1-\theta)b^{1-\frac{1}{\theta}} \tag{4.3}$$

we infer that

$$\begin{aligned}\|w(t)\|_{L^2}^2 + \nu\int_0^t \|\nabla w(t')\|_{L^2}^2\,dt' &\leq \|w(0)\|_{L^2}^2 + \frac{2}{\nu}\int_0^t \|(f-g)(t')\|_{\mathcal{V}_\sigma'}^2\,dt'\\ &\quad + \frac{C}{\nu^3}\int_0^t \|w(t')\|_{L^2}^2\left(\|\nabla u(t')\|_{L^2}^2\|u(t')\|_{L^2}^2\right)\,dt'.\end{aligned}$$

Gronwall's lemma implies that

$$\begin{aligned}\|w(t)\|_{L^2}^2 + \nu\int_0^t \|\nabla w(t')\|_{L^2}^2\,dt' \quad &\leq \quad \Big(\|w(0)\|_{L^2}^2 + \frac{2}{\nu}\int_0^t \|(f-g)(t')\|_{\mathcal{V}_\sigma'}^2\,dt'\Big)\\ &\quad \times \exp\bigg(\frac{C}{\nu^3}\sup_{t'\in[0,t]}\|u(t')\|_{L^2}^2\int_0^t \|\nabla u(t')\|_{L^2}^2\,dt'\bigg).\end{aligned}$$

The energy estimate tells us that

$$\sup_{t'\in[0,t]}\|u(t')\|_{L^2}^2\int_0^t \|\nabla u(t')\|_{L^2}^2\,dt' \leq \frac{1}{\nu}\bigg(\|u_0\|_{L^2}^2 + \frac{2}{\nu}\int_0^t \|f(t')\|_{\mathcal{V}_\sigma'}^2\,dt'\bigg)^2.$$

As u and v play the same role, the theorem is proved. □

Remarks

- This chapter must be known.
- If you want to know more about the basis of the subject, you can read the seminal paper of J. Leray "Essai on the mouvement of a liquide visqueux emplissant the space, *Acta Matematica*, **63**, 1933, pages 193–248.

- To have a more recent review of results on incompressible Navier-Stokes, we can see te books of P. Constantin and C. Foias *Navier-Stokes equations*, Chigago University Press, 1988 and of P.-G. Lemarié-Rieusset, *Recent developments in the Navier-Stokes problem.* Chapman & Hall/CRC, Research Notes in Mathematics, **431**, 2002.

- If you are interested in developments related to geophysical fluids, you can see the book of J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, *Mathematical Geophysics; an introduction to rotating fluids and Navier-Stokes equations*, Oxford Lecture series in Mathematics and its maps, **32**, Oxford University Press, 2006.

Chapter 5

Stability of Navier-Stokes equations in dimension 3

5.1 A sufficient condition of 3D stability

The stability result is the following.

Theorem 5.1.1 *Let u be a Leray solution associated with initial velocity u_0 in $\mathcal{H}$ and external force f in $L^2([0,T];\mathcal{V}')$. We assume that u belongs to the space $L^4([0,T];\mathcal{V}_\sigma)$ for some positive T. Then u is unique, belongs to $C([0,T];\mathcal{H})$ and satisfies, for any (s,t) such that $0 \leq s \leq t \leq T$,*

$$\frac{1}{2}\|u(t)\|_{L^2}^2 + \nu \int_s^t \|\nabla u(t')\|_{L^2}^2 \, dt' = \frac{1}{2}\|u(s)\|_{L^2}^2 + \int_s^t \langle f(t'), u(t') \rangle \, dt'. \tag{5.1}$$

Let v be any solution associated with v_0 in $\mathcal{H}$ and g in $L^2_{loc}([0,T];\mathcal{V}')$. Then, for all t in $[0,T]$,

$$\begin{aligned}
&\|(u-v)(t)\|_{L^2}^2 + \nu \int_0^t \|\nabla(u-v)(t')\|_{L^2}^2 \, dt' \\
&\qquad \leq \left(\|u_0 - v_0\|_{L^2}^2 + \frac{2}{\nu} \int_0^t \|(f-g)(t')\|_{\mathcal{V}'_\sigma}^2 \, dt' \right) \exp\left(\frac{C}{\nu^3} \int_0^t \|\nabla u(t')\|_{L^2}^4 \, dt' \right).
\end{aligned}$$

Proof. Thanks to Lemma 4.2.1, the fact that u belongs to $L^4([0,T];\mathcal{V}_\sigma)$ implies that

$$\begin{aligned}
\|Q(u,u)\|_{L^2([0,T];\mathcal{V}')} &\leq C\|u\|_{L^\infty([0,T];L^2)}^{\frac{1}{2}} \|u\|_{L^3([0,T];H_0^1)}^{\frac{3}{2}} \\
&\leq CT^{\frac{1}{8}} \|u\|_{L^\infty([0,T];L^2)}^{\frac{1}{2}} \|u\|_{L^4([0,T];H_0^1)}^{\frac{3}{2}}.
\end{aligned} \tag{5.2}$$

Hence the non linear term $Q(u,u)$ belongs to $L^2([0,T];\mathcal{V}')$. Thus, exactly as in the two dimensional case, u is the solution of (ES_ν) with initial data u_0 and external force $f + Q(u,u)$. Theorem 3.3.1 immediately implies that u belongs to $C([0,T];\mathcal{H})$ and satisfies, for any (s,t) such that $0 \leq s \leq t$,

$$\begin{aligned}
\frac{1}{2}\|u(t)\|_{L^2}^2 + \nu \int_s^t \|\nabla u(t')\|_{L^2}^2 \, dt' &= \frac{1}{2}\|u(s)\|_{L^2}^2 \\
&\quad + \int_s^t \langle f(t'), u(t') \rangle \, dt' + \int_s^t \langle Q(u(t'), u(t')), u(t') \rangle \, dt'.
\end{aligned}$$

Using Lemma 4.2.1, we get the energy equality (5.1).

As u and v are two Leray solutions, we can write that

$$\begin{aligned}\delta_\nu(t) &\stackrel{\text{def}}{=} \|(u-v)(t)\|_{L^2}^2 + 2\nu\int_0^t \|\nabla(u-v)(t')\|_{L^2}^2\,dt'\\ &= \|u(t)\|_{L^2}^2 + 2\nu\int_0^t \|\nabla u(t')\|_{L^2}^2 dt' + \|v(t)\|_{L^2}^2 + 2\nu\int_0^t \|\nabla v(t')\|_{L^2}^2\,dt'.\end{aligned}$$

This can be written

$$\begin{aligned}\delta_\nu(t) &\leq \|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 - 2P(t) \quad \text{with}\\ P(t) &\stackrel{\text{def}}{=} (v(t)|u(t))_{L^2} + 2\nu\int_0^t (\nabla v(t')|\nabla u(t'))_{L^2}\,dt'\\ &\qquad + 2\int_0^t \langle g(t'), v(t')\rangle\,dt' + 2\int_0^t \langle f(t'), u(t')\rangle\,dt'.\end{aligned} \tag{5.3}$$

The quantity P can be evaluated thanks to the following lemma.

Lemma 5.1.1 *Let v be a weak solution of (NS) in the sense of Definition 4.1.1 page 43. Then for any function ψ such that*

$$\partial_t\psi \in L^2_{loc}(\mathbb{R}^+;\mathcal{V}_\sigma')\,,\ \psi\in C(\mathbb{R}^+;\mathcal{H}) \quad \text{and} \quad \psi\in L^4_{loc}(\mathbb{R}^+;\mathcal{V}_\sigma) \tag{5.4}$$

we have

$$\begin{aligned}\int_\Omega (v\cdot\Psi)(t,x)\,dx = \int_\Omega v_0(x)\cdot\Psi(0,x)\,dx - \int_0^t\int_\Omega \Big(\nu\nabla v:\nabla\Psi - v\cdot\partial_t\Psi\Big)(t',x)\,dxdt'\\ + \int_0^t\int_\Omega v\otimes v:\nabla\Psi(t',x)dt'dx + \int_0^t \langle g(t'),\Psi(t')\rangle\,dt'\,.\end{aligned}$$

Proof. It relies on density argument. If a function ψ satisfies (5.4), then, by Lebesgue convergence theorem, we get, for any time T,

$$\lim_{k\to\infty} \|\partial_t\,\mathbb{P}_k\,\psi - \partial_t\psi\|_{L^2([0,T];\mathcal{V}_\sigma')} = \lim_{k\to\infty} \|\,\mathbb{P}_k\,\psi - \psi\|_{L^\infty([0,T];\mathcal{H})} = \lim_{k\to\infty} \|\,\mathbb{P}_k\,\psi - \psi\|_\psi\|_{L^4([0,T];\mathcal{V}_\sigma)} = 0.$$

Then the following time regularization

$$\psi_{\varepsilon,k} \stackrel{\text{def}}{=} \frac{1}{\varepsilon}\chi\Big(\frac{\cdot}{\varepsilon}\Big) \star \mathbf{1}_{[0,T]}\,\mathbb{P}_k\,\psi$$

approximate ψ by C^1 functions with value in $\mathcal{V}_\sigma$. Then, we have to ckeck the wecan pass to the limit all terms of the definition of weal solution. All the terms are obvious except the term

$$\int_0^t\int_\Omega v\otimes v:\nabla\Psi(t',x)dt'dx.$$

From Gagliardo–Nirenberg's inequality stated in Corollary 2.2.1 page 24, we get that v belongs $L^{\frac{8}{3}}_{loc}(\mathbb{R}^+,L^4)$ and thus $v\otimes v$ belongs to $L^{\frac{4}{3}}_{loc}(\mathbb{R}^+;L^2)$. As we have convergence in $L^4(\mathcal{V}_\sigma)$, we can pass to the limit. □

Continuation of the proof of Theorem 5.1.1 The idea is to use u as a test function for the (weak) solution v. This gives

$$(v(t)|u(t))_{L^2} = (v_0|u_0)_{L^2} + \int_0^t \Big(-\nu(v(t')|u(t'))_{\mathcal{V}_\sigma} + \langle \partial_t u(t'), v(t')\rangle\Big)dt'$$
$$- \int_0^t \langle \mathrm{div}\, v\otimes v, u(t')\rangle dt' + \int_0^t \langle g(t'), u(t')\rangle\, dt'\,.$$

The fact that u is a solution of Navier-Stokes with initial data u_0 and external force f means exactly that, for any t',

$$\partial_t u(t') = \nu\Delta u(t') - \mathrm{div}\, u\otimes u(t') + f(t')$$

in the space $\mathcal{V}'_\sigma$. As for almost every t', $v(t')$ belongs to $\mathcal{V}_\sigma$, we infer that

$$\int_0^t \langle \partial_t u(t'), v(t')\rangle dt' = \int_0^t \big\langle \nu\Delta u(t') - \mathrm{div}\, u\otimes u(t') + f(t'), v(t')\big\rangle dt'.$$

Moreover, for almost every t', we have $(v(t')|u(t'))_{\mathcal{V}_\sigma} = -\langle \Delta u(t'), v(t')\rangle$. This gives

$$(v(t)|u(t))_{L^2} = (v_0|u_0)_{L^2} - 2\int_0^t (v(t')|u(t'))_{\mathcal{V}_\sigma} dt'$$
$$- \int_0^t \big(\nu\langle \mathrm{div}\, v\otimes v, u\rangle + \langle \mathrm{div}\, u\otimes u, v\rangle\big)dt' + \int_0^t \big(\langle f(t'), v(t')\rangle + \langle g(t'), u(t')\rangle\big)dt'\,.$$

Using (5.3), this gives

$$\begin{aligned}\delta_\nu(t) \leq \|u_0 - v_0\|^2_{L^2} - &\int_0^t \big(\langle \mathrm{div}\, v\otimes v, u\rangle + \langle \mathrm{div}\, u\otimes u, v\rangle\big)dt' \\ &\qquad + 2\int_0^t \langle (f-g)(t'), (u-v)(t')\rangle\, dt'\,.\end{aligned} \tag{5.5}$$

As $\langle \mathrm{div}\, v\otimes v, v\rangle = \langle \mathrm{div}\, u\otimes u, u\rangle = 0$, we have

$$\begin{aligned}\langle \mathrm{div}\, v\otimes v, u\rangle + \langle \mathrm{div}\, u\otimes u, v\rangle &= \langle \mathrm{div}\, v\otimes v - u\otimes u, u-v\rangle \\ &= \langle \mathrm{div}\, v\otimes (v+u-v) - u\otimes u, u-v\rangle \\ &= \langle \mathrm{div}(v-u\otimes u, u-v\rangle.\end{aligned}$$

Using the divergence free condition, we

$$\big|\langle \mathrm{div}\, v\otimes v, u\rangle + \langle \mathrm{div}\, u\otimes u, v\rangle\big| \leq \|u-v\|^2_{L^4}\|\nabla u\|_{L^2}.$$

Using Gagliardo–Nirenberg's inequality stated in Corollary 2.2.1 page 24 and the convexity inequality 4.3 page 51, we infer

$$\begin{aligned}\langle \mathrm{div}\, v\otimes v, u\rangle + \langle \mathrm{div}\, u\otimes u, v\rangle &\leq \nu^{\frac34}\|\nabla(u-v)\|^{\frac32}_{L^2} C\nu^{-\frac34}\|\nabla u\|_{L^2}\|u-v\|^{\frac14}_{L^2} \\ &\leq \frac12\nu\|\nabla u\|^2_{L^2} + \frac{C}{\nu^3}\|\nabla u(t)\|^4_{L^2}\|u-v\|^2_{L^2}.\end{aligned}$$

Moreover, we have

$$\langle (f-g)(t'), (u-v)(t')\rangle \leq \frac12\nu\|\nabla(u-v)(t')\|^2_{L^2} + \frac{1}{2\nu}\|f(t')-g(t')\|^2_{\mathcal{V}'}.$$

Plugging the above two inequalities in (5.5) gives

$$\|(u-v)(t)\|_{L^2}^2 + \nu\int_0^t \|\nabla(u-v)(t')\|_{L^2}^2\,dt' \leq \|u_0-v_0\|_{L^2}^2$$
$$+\frac{C}{\nu^3}\int_0^t \|\nabla u(t')\|_{L^2}^4\|(u-v)(t')\|_{L^2}^2 + \frac{1}{2\nu}\int_0^t \|f(t')-g(t')\|_{\mathcal{V}'}^2\,dt'\,.$$

Gronwall lemma gives the result. □

5.2 Stable solutions in a bounded domain

The purpose of this section is the proof of the existence of solutions of the system (NS_ν) which are L^4 in time with values in $\mathcal{V}_\sigma$. In order to state (and prove) a sharp theorem, we shall introduce intermediate spaces between the spaces $\mathcal{V}'_\sigma$ and $\mathcal{V}_\sigma$. Then, we shall prove a global existence theorem for small data and then a local in time theorem for large data.

5.2.1 Intermediate spaces

We shall define a family of intermediate spaces between the spaces $\mathcal{V}'_\sigma$ and $\mathcal{V}_\sigma$. This can be done by abstract interpolation theory but we prefer to do it here in an explicit way.

Definition 5.2.1 *Let s be in $[-1,1]$. We shall denote by $\mathcal{V}^s_\sigma$ the space of vector fields u in $\mathcal{V}'$ such that*

$$\|u\|_{\mathcal{V}^s_\sigma}^2 \stackrel{\text{def}}{=} \sum_{j\in\mathbb{N}} \mu_j^{2s}\langle u,e_j\rangle^2 < +\infty.$$

Theorem 3.2.2 implies that $\mathcal{V}^0_\sigma = \mathcal{H}$ and $\mathcal{V}^1_\sigma = \mathcal{V}_\sigma$. Moreover, it is obvious that, when s is non negative, $\mathcal{V}^s_\sigma$ endowed with the norm $\|\cdot\|_{\mathcal{V}^s_\sigma}$ is a Hilbert space.

The following proposition will be important in the following two paragraphs.

Proposition 5.2.1 *The space $\mathcal{V}^{\frac12}_\sigma$ is embedded in L^3 and the space $\mathbb{P}\,L^{\frac32}$ is embedded in $\mathcal{V}^{-\frac12}_\sigma$.*

Proof. This proposition can be proved using abstract interpolation theory. We prefer to present here a self contained proof in the spirit of the proof of Theorem 2.2.1. Let us consider a in $\mathcal{V}^{\frac12}_\sigma$. Without any loss of generality, we can assume that $\|a\|_{\mathcal{V}^{\frac12}_\sigma} \leq 1$. Let us define, for a positive real number Λ,

$$a_\Lambda \stackrel{\text{def}}{=} \sum_{j\,/\,\mu_j<\Lambda} \langle a,e_j\rangle e_j \quad\text{and}\quad b_\Lambda \stackrel{\text{def}}{=} a-a_\Lambda.$$

Using the fact that $\big\{x\in\Omega\,/\,|a(x)|>\Lambda\big\} \subset \big\{x\in\Omega\,/\,|a_\Lambda(x)|>\Lambda/2\big\}\cup\big\{x\in\Omega\,/\,|b_\Lambda(x)|>\Lambda/2\big\}$, we can write

$$\begin{aligned}\|a\|_{L^3}^3 &\leq 3\int_0^{+\infty}\Lambda^2\text{meas}\big(\big\{x\in\Omega\,/\,|a_\Lambda(x)|>\Lambda/2\big\}\big)\,d\Lambda\\ &\qquad\qquad + 3\int_0^{+\infty}\Lambda^2\text{meas}\big(\big\{x\in\Omega\,/\,|b_\Lambda(x)|>\Lambda/2\big\}\big)\,d\Lambda\\ &\leq 3\times 2^6\int_0^{+\infty}\Lambda^{-4}\|a_\Lambda\|_{L^6}^6\,d\Lambda + 3\times 2^2\int_0^{+\infty}\|b_\Lambda\|_{L^2}^2\,d\Lambda.\end{aligned}$$

Thanks to Theorem 2.2.1, we have, by definition of the $\|\cdot\|_{\mathcal{V}_\sigma^s}$ norm,

$$\begin{aligned}
\|a_\Lambda\|_{L^6}^2 &\leq C\|a_\Lambda\|_{\mathcal{V}_\sigma}^2 \\
&\leq C \sum_{j\,/\,\mu_j<\Lambda} \mu_j^2 \langle a, e_j\rangle^2 \\
&\leq C\Lambda \sum_{j\,/\,\mu_j<\Lambda} \mu_j \langle a, e_j\rangle^2 \leq C\Lambda.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\|a\|_{L^3}^3 &\leq C\int_0^{+\infty} \Lambda^{-2}\|a_\Lambda\|_{\mathcal{V}_\sigma}^2 \, d\Lambda + C\int_0^{+\infty} \|b_\Lambda\|_{L^2}^2 \, d\Lambda \\
&\leq C\sum_{j\in\mathbb{N}} \int_{\mu_j}^{+\infty} \Lambda^{-2}\mu_j^2 \langle a, e_j\rangle^2 \, d\Lambda + C\sum_{j\in\mathbb{N}} \int_0^{\mu_j} \langle a, e_j\rangle^2 \, d\Lambda \\
&\leq C\sum_{j\in\mathbb{N}} \mu_j \langle a, e_j\rangle^2 \\
&\leq C.
\end{aligned}$$

This proves the first part of the proposition.

The second part is obtained by a duality argument. By definition, we have, for any a in $\mathcal{V}'$,

$$\begin{aligned}
\|\mathbb{P}a\|_{\mathcal{V}_\sigma^{-\frac{1}{2}}} &= \|(\mu_j^{-\frac{1}{2}}\langle a, e_j\rangle)_{j\in\mathbb{N}}\|_{\ell^2} \\
&= \sup_{\substack{(\alpha_j)_{j\in\mathbb{N}} \\ \|(\alpha_j)_{j\in\mathbb{N}}\|_{\ell^2}\leq 1}} \sum_{j\in\mathbb{N}} \alpha_j \mu_j^{-\frac{1}{2}} \langle a, e_j\rangle. \qquad (5.1)
\end{aligned}$$

The map L defined by

$$L \left\{ \begin{array}{ccc} \ell^2 & \longrightarrow & \mathcal{V}_\sigma^{\frac{1}{2}} \\ (\alpha_j)_{j\in\mathbb{N}} & \longmapsto & \displaystyle\sum_{j\in\mathbb{N}} \alpha_j \mu_j^{-\frac{1}{2}} e_j \end{array} \right.$$

is an onto isometry. Thus, thanks to (5.1), we have

$$\|\mathbb{P}a\|_{\mathcal{V}_\sigma^{-\frac{1}{2}}} = \sup_{\|\varphi\|_{\mathcal{V}_\sigma^{\frac{1}{2}}}\leq 1} \sum_{j\in\mathbb{N}} (L^{-1}\varphi)_j \mu_j^{-\frac{1}{2}} \langle a, e_j\rangle.$$

For any φ in $\mathcal{V}_\sigma$, we have

$$\sum_{j\in\mathbb{N}} (L^{-1}\varphi)_j \mu_j^{-\frac{1}{2}} \langle a, e_j\rangle = \langle a, \varphi\rangle.$$

If we assume that a is in $L^{\frac{3}{2}}$, we have, because φ is in L^3,

$$\langle a, \varphi\rangle = \int_\Omega a(x)\cdot\varphi(x)\, dx.$$

Hölder's inequality and the first part of Proposition 5.2.1 imply that

$$|\langle a, \varphi\rangle| \leq \|a\|_{L^{\frac{3}{2}}} \|\varphi\|_{L^3} \leq C\|a\|_{L^{\frac{3}{2}}} \|\varphi\|_{\mathcal{V}_\sigma^{\frac{1}{2}}}.$$

Thus we have

$$\| \mathbb{P}\, a\|_{\mathcal{V}_\sigma^{-\frac{1}{2}}} \leq \sup_{\|\varphi\|_{\mathcal{V}_\sigma^{\frac{1}{2}}}\leq 1} \langle a, \varphi\rangle \leq C\|a\|_{L^{\frac{3}{2}}}.$$

This completes the proof of Proposition 5.2.1. □

5.2.2 The wellposedness result in $\mathcal{V}_\sigma^{\frac{1}{2}}$

The aim of this paragraph is the proof of the following existence theorem with data in $\mathcal{V}_\sigma^{\frac{1}{2}}$.

Theorem 5.2.1 *If the initial data u_0 belongs to $\mathcal{V}_\sigma^{\frac{1}{2}}$ and the external force f belongs to the space $L^2_{loc}(\mathbb{R}_+; \mathcal{V}_\sigma^{-\frac{1}{2}})$, then a positive time T exists such that a solution u of (NS_ν) exists in $L^4([0,T]; \mathcal{V}_\sigma)$. This solution is unique and belongs to $C([0,T]; \mathcal{V}_\sigma^{\frac{1}{2}})$.*

Moreover, a constant c exists (which can be chosen independent of the domain Ω) such that, if

$$\|u_0\|_{\mathcal{V}_\sigma^{\frac{1}{2}}} + \frac{1}{\nu}\|f\|_{L^2(\mathbb{R}^+; \mathcal{V}_\sigma^{-\frac{1}{2}})} \leq c\nu,$$

then the above solution is global.

Proof. For the sake of simplicity, we shall ignore the external force in the proof. Let us observe that the map

$$\mathcal{Q} \quad \begin{cases} \mathcal{V}_\sigma \times \mathcal{V}_\sigma & \longrightarrow \quad \mathcal{V}_\sigma^{-\frac{1}{2}} \\ (u,v) & \longmapsto \quad \mathbb{P}\operatorname{div}(u\otimes v) \end{cases}$$

is a bilinear continuous map. Indeed, as u is divergence free, we can write

$$\operatorname{div}(u\otimes v)^\ell = \sum_{k=1}^{3} u^k \partial_k v^\ell.$$

Then Hölder inequality gives

$$\| \operatorname{div}(u\otimes v)^\ell\|_{L^{\frac{3}{2}}} \lesssim \|u\|_{L^6}\|\nabla v\|_{L^2}.$$

Sobolev embedding $\mathcal{V}_\sigma \hookrightarrow L^6$ and dual Sobolev embedding of Proposition 5.2.1 implies that

$$\| \mathbb{P}\operatorname{div}(u\otimes v)\|_{\mathcal{V}_\sigma^{-\frac{1}{2}}} \lesssim \|u\|_{\mathcal{V}_\sigma}\|v\|_{\mathcal{V}_\sigma}. \tag{5.2}$$

Now let us define the followong bilienar operator

$$B \quad \begin{cases} L^4([0,T];\mathcal{V}_\sigma)\times L^4([0,T];\mathcal{V}_\sigma) & \longrightarrow \quad L^4([0,T];\mathcal{V}_\sigma) \\ (u,v) & \longmapsto \quad B(u,v) \end{cases}$$

where $B(u,v)$ is the solution of the linear Stokes problem with initial data 0 and external force $\frac{1}{2}\big(\mathcal{Q}(u,v)+\mathcal{Q}(v,u)\big)$. Using (5.2), it turns out that

$$\big\|\mathbb{P}\big(\mathcal{Q}(u,v)+\mathcal{Q}(v,u)\big)\big\|_{L^2([0,T];\mathcal{V}_\sigma^{-\frac{1}{2}})} \lesssim \|u\|_{L^4([0,T];\mathcal{V}_\sigma)}\|v\|_{L^4([0,T];\mathcal{V}_\sigma)}.$$

As $\mathcal{V}_\sigma^{-\frac{1}{2}}$ is by contruction a supspace of $\mathcal{V}'_\sigma$, the term $\mathcal{Q}(u,v)+\mathcal{Q}(v,u)$ can be considered as an external force for the Stokes problem of evolution. Thus the blinear operator B is well defined. The fact that it maps continuously $L^4([0,T];\mathcal{V}_\sigma)\times L^4([0,T];\mathcal{V}_\sigma)$ into $L^4([0,T];\mathcal{V}_\sigma)$ will be a consequence of the following lemma.

Lemma 5.2.1 *Let us consider f in $L^2([0,T];\mathcal{V}_\sigma^{-\frac{1}{2}})$. A constant C exists, such that, for any p in $[4,\infty]$, the solution $\mathcal{L}f$ of the Stokes problem with external force f and initial data 0 satisfies*

$$\sum_j \mu_j^{1+\frac{4}{p}} \|\|\langle \mathcal{L}f(t), e_k\rangle\|^2_{L^p([0,T])} \lesssim \|f\|^2_{L^2([0,T];\mathcal{V}_\sigma^{-\frac{1}{2}})}.$$

Proof. Using Formula (3.3) page 37, we get

$$\langle \mathcal{L}f(t), e_j\rangle = \int_0^t e^{-\nu\mu_j^2(t-t')}\langle f(t'), e_j\rangle\, dt'.$$

Using the Young's inequality, we get

$$\|\langle \mathcal{L}f(t), e_k\rangle\|_{L^p([0,T]} \leq \frac{1}{\nu^{\frac{1}{2}+\frac{1}{p}}}\mu_j^{-\frac{1}{2}-\frac{2}{p}}\|\langle f(t), e_j\rangle\|_{L^2([0,T])}.$$

By definition of the $\mathcal{V}_\sigma^{-\frac{1}{2}}$ norm, we get the result. □

Continuation of the proof of Theorem 5.2.1 Let us denote by $S(t)u_0$ the solution of the linear Stokes problem with intial data u_0 and external force 0. Using again Formula (3.3) page 37, we get

$$\langle S(t)u_0, e_j\rangle = \langle u_0, e_j\rangle e^{-\nu\mu_j^2 t}.$$

Thus, wfor any p in $[4,\infty]$, we get

$$\sum_j \mu_j^{1+\frac{4}{p}} \|\langle S(t)u_0(t), e_j\rangle\|^2_{L^p([0,T])} \lesssim \frac{1}{\nu^{\frac{2}{p}}}\sum_j \mu_j(\langle u_0, e_j\rangle^2 \tag{5.3}$$

The idea of the Kato theory is the following: u is a solution of incompressible Navier-Stokes equation in the space $L^4([0,T];\mathcal{V}_\sigma)$ if and only if u satisfies

$$u = S(t)u_0 + B(u,u).$$

In other terms, u is a fixed point of the map

$$u \longmapsto S(t)u_0 + B(u,u).$$

Now let us observe that, thanks to the Cauchy–Schwarz inequality, for any a in $\ell^2(L^4[0,T])$,

$$\begin{aligned}
\int_0^T \|a_j(t)\|^4_{\ell^2(\mathbb{N})}dt &= \int_0^T \Big(\sum_{j\in\mathbb{N}} a_j^2(t)\Big)^2 dt \\
&= \sum_{j\in\mathbb{N},k\in\mathbb{N}} \int_0^T a_j^2(t)a_k^2(t)dt \\
&\leq \sum_{j\in\mathbb{N},k\in\mathbb{N}} \|a_j\|^2_{L^4([0,T])}\|a_k\|^2_{L^4([0,T])} \\
&\leq \Big\|(\|a_j\|_{L^4([0,T])})_{j\in\mathbb{N}}\Big\|^4_{\ell^2}
\end{aligned}$$

Let us notice that this is a particular case of the Minkowski inequality. Then we deduce from Lemma 5.2.1 that

$$\|B(u,u)\|_{L^4([0,T];\mathcal{V}_\sigma)} \leq \frac{C}{\nu^{\frac{3}{4}}}\|u\|^2_{L^4([0,T];\mathcal{V}_\sigma)}.$$

Using (5.3) with $p=4$ and Minkowski inequality gives

$$\|S(t)u_0\|_{L^4([0,T];\mathcal{V}_\sigma)} \leq \frac{1}{\nu^{\frac14}}\|u_0\|_{\mathcal{V}_\sigma^{\frac12}}\cdot \tag{5.4}$$

Using Picard fixed point theorem, if we prove that $\lim_{T\to 0}\|u_L\|_{L^4([0,T];\mathcal{V}_\sigma)}=0$ we conclude the proof of the theorem up to the continuity of u. Let us observe that, for any positive ε, an integer j_ε exsits such that

$$\|(\mathrm{Id}-\mathbb{P}_{j_\varepsilon})u_0\|_{\mathcal{V}_\sigma} \leq \frac{\nu^{\frac14}\varepsilon}{2} \quad \text{with} \quad \mathbb{P}_k\, a \stackrel{\text{déf}}{=} \sum_{k'\leq k}\langle a, e_{k'}\rangle e_{k'}.$$

Then Inequality (5.4) implies that

$$\|S(t)\,\mathbb{P}_{j_\varepsilon}\,u_0\|_{L^4([0,T];\mathcal{V}_\sigma)} \leq \frac{\varepsilon}{2}\cdot \tag{5.5}$$

Then we have

$$\begin{aligned}\|S(t)\,\mathbb{P}_{j_\varepsilon})u_0\|_{L^4([0,T];\mathcal{V}_\sigma)} &\leq T^{\frac14}\|S(t)\,\mathbb{P}_{j_\varepsilon})u_0\|_{L^\infty([0,T];\mathcal{V}_\sigma)}\\ &\leq T^{\frac14}\mu_{j_\varepsilon}^{\frac12}\|u_0\|_{\mathcal{V}_\sigma^{\frac12}}\cdot\end{aligned}$$

Together with (5.5), this implies that

$$\lim_{T\to 0}\|u_L\|_{L^4([0,T];\mathcal{V}_\sigma)}=0.$$

In order to prove the continuity in time with value in $\mathcal{V}_\sigma^{\frac12}$ of u, let us observe that, as

$$\partial_t u = \Delta u + \mathbb{P}\,\mathrm{div}(u\otimes u)$$

we have that $\partial_t u$ belongs to

$$L^4([0,T];\mathcal{V}'_\sigma) + L^2([0,T];\mathcal{V}_\sigma^{-\frac12}) \hookrightarrow L^2([0,T];\mathcal{V}').$$

Thus u is continuous with value in $\mathcal{V}'_\sigma$. Using Lemma 5.2.1 and Inequality (5.2.1) we infer that

$$\sum_j \mu_j\|\langle u(t),e_j\rangle\|^2_{L^\infty([0,T])}<\infty$$

Thus for any positive real number ε, an integer j_ε exists such that

$$\sum_{j>j_\varepsilon}\mu_j\|\langle u(t),e_j\rangle\|^2_{L^\infty([0,T])}<\frac{\varepsilon^2}{4}\cdot$$

Now, it turns out that for all (t_1,t_2) in $[0,T]^2$, one has

$$\begin{aligned}\|u(t_1)-u(t_2)\|_{\mathcal{V}_\sigma^{\frac12}} &\leq \Big(\sum_{j>j_\varepsilon}\mu_j\|\langle u(t),e_j\rangle\|^2_{L^\infty([0,T])}\Big)^{\frac12}+\Big(\sum_{j\leq j_\varepsilon}\mu_j\langle u(t_1)-u(t_2),e_j\rangle^2\Big)^{\frac12}\\ &\leq \frac{\varepsilon}{2}+\mu_{j_\varepsilon}\|u(t_1)-u(t_2)\|_{\mathcal{V}'_\sigma}.\end{aligned}$$

As u is continuous in time with value in$\mathcal{V}'_\sigma$, the whole Theorem 5.2.1 is proved. □

5.2.3 Some remarks about stable solutions

In this paragraph, we shall assume that the external force f is identically 0. We shall establish some results about the maximal existence time of the solution constructed in the preceding paragraph.

Proposition 5.2.2 *Let us assume that the initial data u_0 belongs to $\mathcal{V}_\sigma$. Then the maximal time of existence $T^\star$ of the solution u in the space $C([0,T^\star[;\mathcal{V}_\sigma^{\frac{1}{2}})\cap L^4_{loc}([0,T^\star[;\mathcal{V}_\sigma)$ satisfies*

$$T^\star \geq \frac{c\nu^3}{\|\nabla u_0\|_{L^2}^4}\cdot$$

Proof. Let us observe that , if u_0 belongs to $\mathcal{V}_\sigma$, we have

$$\|S(t)u_0\|_{L^4([0,T];\mathcal{V}_\sigma)} \leq T^{\frac{1}{4}}\|u_0\|_{\mathcal{V}_\sigma}$$

This ensures the proposition. □

From this proposition, we infer the following corollary.

Corollary 5.2.1 *Let $T^\star$ be the maximal time of existence for a solution u of the system (NS_ν) in the space $C([0,T^\star);\mathcal{V}_\sigma^{\frac{1}{2}})\cap L^4_{loc}([0,T^\star);\mathcal{V}_\sigma)$. If $T^\star$ is finite, then*

$$\int_0^{T^\star}\|\nabla u(t)\|_{L^2}^4 dt = +\infty \quad \text{and} \quad T^\star \leq \frac{c}{\nu^5}\|u_0\|_{L^2}^4.$$

Proof. For almost every t, $u(t)$ belongs to $\mathcal{V}_\sigma$. Then, thanks to the above proposition, the maximal time of existence of the solution starting at time t, which is of course $T^\star - t$, satisfies

$$T^\star - t \geq \frac{c\nu^3}{\|\nabla u(t)\|_{L^2}^4}\cdot$$

This can be written as

$$\|\nabla u(t)\|_{L^2}^4 \geq \frac{c\nu^3}{T^\star - t}\cdot$$

This gives the first part of the corollary. Taking the square root of the above inequality gives, thanks to the energy estimate,

$$c\nu^{\frac{5}{2}}\int_0^{T^\star}\frac{dt}{(T^\star-t)^{\frac{1}{2}}} \leq \frac{1}{2}\|u_0\|_{L^2}^2.$$

The corollary is proved.

Remarks

- Sections 4.3 and 5.2 must be known.
- Again books of P. Constantin et C. Foias *Navier-Stokes equations*, Chigago University Press, 1988, de P.-G. Lemarié-Rieusset, *Recent developments in the Navier-Stokes problem*. Chapman & Hall/CRC, Research Notes in Mathematics, **431**, 2002 and of J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, *Mathematical Geophysics; an introduction to rotating fluids and Navier-Stokes equations*, Oxford Lecture series in Mathematics and its maps, **32**, Oxford University Press, 2006 give more details.

Chapter 6

Linear symmetric systems

6.1 Definition and examples

Let us first define the concept in the framework of linear system with variable coefficients. Let I be a closed interval of $\mathbb{R}$ which has 0 as an interior point. Let us consider a family $\mathcal{A}$ of smooth bounded functions $(\mathcal{A}_k)_{0\leq k\leq d}$ from $I\times\mathbb{R}^d$ into the space of $N\times N$ matrices with real coefficients. Let us assume that all their derivatives in the x variable are bounded. We consider the system

$$(LS)\qquad \begin{cases} \partial_t U + \displaystyle\sum_{k=1}^d \mathcal{A}_k\partial_k U + \mathcal{A}_0 U = F \\ U_{|t=0} = U_0. \end{cases}$$

Let us introduce the following notations. If U in $\mathcal{D}'(\mathbb{R}^d)$ and φ in $\mathcal{D}(\mathbb{R}^d)$,

$$\langle U,\varphi\rangle = \sum_{i=1}^N \langle U^i,\varphi^i\rangle$$

and if U and V belongs to $(L^2(\mathbb{R}^d))^N$,

$$(U|V)_{L^2} = \sum_{i=1}^N \int_{\mathbb{R}^d} U^i(x)V^i(x)dx.$$

Definition 6.1.1 *A function U in $(C(I;L^2))^N$ is a solution of (LS) if and only if for any φ in $\mathcal{D}(\mathbb{R}\times\mathbb{R}^d)$, for any t in I,*

$$\begin{aligned}\langle U(t),\varphi(t)\rangle &= \int_0^t \langle U(t'),\partial_t\varphi(t')\rangle dt' + \langle U_0,\varphi(0)\rangle \\ &+ \int_0^t \sum_{k=1}\langle(\mathcal{A}_k U)(t'),\partial_k\varphi(t')\rangle dt' + \int_0^t \langle(\operatorname{div}\mathcal{A}U)(t'),\varphi(t')\rangle dt' + \int_0^t \langle F(t'),\varphi(t')\rangle dt'.\end{aligned}$$

Let us define the concept of symmetric system.

Definition 6.1.2 *The above system (LS) is symmetric if and only if for any $k\in\{1,\dots,d\}$ and any (t,x) in $I\times\mathbb{R}^d$ the matrices $\mathcal{A}_k(t,x)$ are symmetric, which means that for any k, we have $\mathcal{A}_{k,i,j}(t,x) = \mathcal{A}_{k,j,i}(t,x)$.*

An example of such a system is given by (1.2) page 6. The reason why this definition is fundamental is the following. Let us consider a solution of (LS) and let us look to the evolution of its energy. This question leads to the following formal computation which will be made rigourous in the following section.

$$\frac{1}{2}\frac{d}{dt}\|U(t)\|_{L^2}^2 = -\sum_{k=1}^{d}\Big(\mathcal{A}_k\partial_k U\big|U\Big)_{L^2} - (\mathcal{A}_0 U|U)_{L^2} + (F|U)_{L^2}$$

By integration by part, we get that

$$\begin{aligned} -\Big(\mathcal{A}_k\partial_k U\big|U\Big)_{L^2} &= -\sum_{i,j}\int_{\mathbb{R}^d}\mathcal{A}_{k,i,j}\partial_k U^i U^\ell dx\\ &= \sum_{i,j}\int_{\mathbb{R}^d}\mathcal{A}_{k,i,j}U^i\partial_k U^\ell dx + \sum_{i,j}\int_{\mathbb{R}^d}\partial_k\mathcal{A}_{k,i,j}U^i U^\ell dx. \end{aligned}$$

If the system (LS) is symmetric, then we have

$$-\sum_{k=1}^{d}\Big(\mathcal{A}_k\partial_k U\big|U\Big)_{L^2} = \frac{1}{2}((\operatorname{div}\mathcal{A})U|U)_{L^2} \quad\text{with}\quad (\operatorname{div}\mathcal{A})_{i,j} \stackrel{\text{def}}{=} \sum_{k=1}^{d}\partial_k\mathcal{A}_{k,i,j}$$

This implies that

$$\left|\sum_{k=1}^{d}\Big(\mathcal{A}_k(t)\partial_k U(t)\big|U(t)\Big)_{L^2}\right| \leq \frac{1}{2}\|\operatorname{div}\mathcal{A}(t)\|_{L^\infty}\|U(t)\|_{L^2}^2.$$

Thus we get that

$$\frac{d}{dt}\|U(t)\|_{L^2}^2 \leq a_0(t)\|U\|_{L^2}^2 + (F|U)_{L^2} \quad\text{with}\quad a_0(t) \stackrel{\text{def}}{=} \|\operatorname{div}\mathcal{A}(t,\cdot)\|_{L^\infty} + 2\|\mathcal{A}_0(t,\cdot)\|_{L^\infty}. \tag{6.1}$$

The purpose of this section is to study linear symmetric systems. First, we want to solve them and then to study basic properties of their solutions. In this section, for $s\in\mathbb{N}$ we shall state

$$|U(t)|_s^2 \stackrel{\text{def}}{=} \sum_{\substack{1\leq j\leq N\\ 1\leq|\alpha|\leq d}}\|\partial_x^\alpha U^\ell(t)\|_{L^2}^2.$$

6.2 The wellposedness of linear symmetric systems

The goal of this paragraph is the proof of the following wellposedness theorem.

Theorem 6.2.1 *Let (LS) be a linear symmetric system. Then, if U_0 belongs to H^s and if F is a continuous function with value in H^s, then a unique solution of (S) exists in the space $C^0(I,H^s)\cap C^1(I;H^{s-1})$.*

Proof. It requires four steps:

- We first prove a-priori estimates for smooth enough solutions of the system (S).
- Then we apply Friedrichs method.

- Then we pass to the limit in the case of smooth enough initial data and we get existence in any case by smoothing of the initial data.

- Finally, we get uniqueness using existence for the adjoint system.

A priori estimates use in a crucial way the symmetry hypothesis and are true only for smooth enough solutions.

Lemma 6.2.1 *For any non negative integer s, a locally bounded function a_s exists such that for any function U in $C^0(I, H^{s+1}) \cap C^1(I, H^s)$, we have for any t in I,*

$$|U(t)|_s \leq |U(0)|_s \exp \int_0^t a_s(t')dt' + \int_0^t |F(t')|_s \exp \left(\int_{t'}^t a_s(t'')dt'' \right) dt',$$

with

$$F = \partial_t U + \sum_{k=1}^{d} \mathcal{A}_k \partial_k U + \mathcal{A}_0 u\,.$$

Proof. To start with, let us prove this lemma for $s = 0$. Let us consider a function U in the space $C^0(I; H^1) \cap C^1(I; L^2)$. By definition of F, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |U(t)|_0^2 &= (\partial_t U | U)_{L^2} \\ &= (F|U)_{L^2} - (\mathcal{A}_0 U|U)_{L^2} - \sum_{k=1}^{d} (\mathcal{A}_k \partial_k U|U)_{L^2}\,. \end{aligned}$$

As the system (LS) is symmetric and U belongs to $C^0(I; H^1) \cap C^1(I; L^2)$, computations done page 64 which lead to (6.1) are rigourous. Thus we have

$$\frac{d}{dt} |U(t)|_0^2 \leq a_0(t)|U(t)|_0^2 + 2|F(t)|_0|U(t)|_0 \tag{6.2}$$

with $a_0(t) \stackrel{\text{def}}{=} \| \operatorname{div} \mathcal{A}(t, \cdot)\|_{L^\infty} + 2\|\mathcal{A}_0(t, \cdot)\|_{L^\infty}$. By Gronwall lemma, we get

$$|U(t)|_0 \leq |U(0)|_0 \exp \int_0^t a_0(t')dt' + \int_0^t |F(t')|_0 \exp \left(\int_{t'}^t a_0(t'')dt'' \right) dt' \tag{6.3}$$

and the lemma is proved is the case when $s = 0$.

Remark Let us point out that the above computations are also valid when the matrices $(\mathcal{A}_k)$ have C^1 coefficients and $\mathcal{A}_0$ has C^0 coefficients.

Let us study the case when s is any non negative integer. To do so, we shall proceed by induction on the integer s. Let us assume that Lemma 6.2.1 is proved for some s. Let U be a function in $C^0(I, H^{s+2}) \cap C^1(I; H^{s+1})$. Let us introduce the function (with $N(d+1)$ components) $\widetilde{U}$ defined by

$$\widetilde{U} = (U, \partial_1 U, \cdots, \partial_d U)\,.$$

As, for any j in $\{1, \ldots, d\}$,

$$F = \partial_t U + \sum_{k=1}^{d} \mathcal{A}_k \partial_k U + \mathcal{A}_0 U,$$

we obtain by differentation of the equation,

$$\partial_t\left(\partial_j U\right) = -\sum_{k=1}^{d} \mathcal{A}_k \partial_k \partial_j U - \sum_{k=1}^{d} (\partial_j \mathcal{A}_k) \cdot \partial_k U - \partial_j(\mathcal{A}_0 U) - \partial_j F.$$

We may write

$$\partial_t \widetilde{U} \; = \; -\sum_{k=1}^{d} \mathcal{B}_k \,\partial_k \widetilde{U} \; - \; \mathcal{B}_0 \widetilde{U} + \widetilde{F} \quad \text{with} \tag{6.4}$$

$$\widetilde{F} \; = \; (F, \partial_1 F, \cdots, \partial_d F) \quad \text{and} \tag{6.5}$$

$$\mathcal{B}_k \; = \; \begin{pmatrix} \mathcal{A}_k & & 0 \\ & \mathcal{A}_k & \\ 0 & & \mathcal{A}_k \end{pmatrix}. \tag{6.6}$$

The coefficients of $\mathcal{B}_0$ are computed from those of $\mathcal{A}_k$ $(k = 0, \ldots, d)$ and their first order derivatives. The induction hypothesis allows to conclude the proof of Lemma 6.2.1. □

Remark Let us point out that the proof of the inequalities of Lemma 6.2.1 done above demands exactly one more derivative than in the statement of Theorem 6.2.1.

Second step of the proof of Thoerem 6.2.1 . This leads us to use a smoothing method, the Friedrich method. It consists in smoothing both the initial data and the system itself. More precisely, let us consider the system (LS_n) defined by

$$(LS_n) \quad \begin{cases} \partial_t U_n + \displaystyle\sum_{k=1}^{d} \mathbb{E}_n\left(\mathcal{A}_k \partial_k U_n\right) + \mathbb{E}_n(\mathcal{A}_0 U_n) = \mathbb{E}_n\, F \\ \mathbb{E}_n\, U_{|t=0} = \mathbb{E}_n\, U_0 \end{cases}$$

where $\mathbb{E}_n$ is the cutoff operator defined on L^2 by

$$\mathbb{E}_n\, u \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\mathbf{1}_{B(0,n)}\widehat{u}). \tag{6.7}$$

This is nothing else that the orthogonal projection of L^2 on the closed space L^2_n of the L^2 functions the Fourier transform of which are supported in the ball of center 0 and radius n. Lemma 7.1.1 tells us in particular that the operator ∂_k is a continuous on L^2_n. As the functions $\mathcal{A}_k$ are bounded it turns out that the linear operator

$$V \longmapsto \sum_{k=1}^{d} \mathbb{E}_n\left(\mathcal{A}_k \partial_k V\right) + \mathbb{E}_n(\mathcal{A}_0 V)$$

is continuous on L^2_n. Thus the system (LS_n) is a linear sytstem of ordinary differential equations on L^2_n. This implies the existence of a unique function U_n continuous on I with value in L^2_n which is a solution of (LS_n). Moreover as the functions $\mathcal{A}_k$ are smooth functions in (t, x), using the equation (LS_n), we get that U_n is a smooth function on I with value in H^s for any integer s.

Let us prove that the functions U_n satisfy the energy estimates of Lemma 6.2.1. More precisely, we have the following lemma.

Lemma 6.2.2 *For any non negative integer s, a locally bounded function a_s exists such that for any $n \in N$ and any t in I we have,*

$$|U_n(t)|_s \leq |\,\mathbb{E}_n\, U(0)|_s \exp \int_0^t a_s(t')dt' + \int_0^t |\,\mathbb{E}_n\, F(t')|_s \exp\left(\int_{t'}^t a_s(t'')dt''\right) dt',$$

Proof. Taking the scalar product of (LS_n) with U_n in L^2, we get using the fact that the operator $\mathbb{E}_n$ is selfadjoint on L^2 and that $\mathbb{E}_n\, U_n = U_n$,

$$\frac{d}{dt}|U_n(t)|_0^2 = -2\sum_{k=1}^d (\mathcal{A}_k \partial_k U_n | U_n)_{L^2} - 2(\mathcal{A}_0 U_n)|U_n)_{L^2} - 2(\mathbb{E}_n\, F|U_n)_{L^2}.$$

We proceed exactly as in the proof of Lemma 6.2.1. As the system (LS) is symmetric and U_n belongs to $C^0(I;H^1) \cap C^1(I;L^2)$, computations done page 64 which lead to (6.1) are rigourous. Thus we have

$$\frac{d}{dt}\,|U_n(t)|_0^2 \leq a_0(t)|U_n(t)|_0^2 + 2|\,\mathbb{E}_n\, F(t)|_0 |U_n(t)|_0 \tag{6.8}$$

with $a_0(t) \stackrel{\text{def}}{=} \|\operatorname{div} \mathcal{A}(t,\cdot)\|_{L^\infty} + 2\|\mathcal{A}_0(t,\cdot)\|_{L^\infty}$. Gronwall Lemma implies that

$$|U_n(t)|_0 \leq |\,\mathbb{E}_n\, U_0|_0 \exp \int_0^t a_0(t')dt' + \int_0^t |\,\mathbb{E}_n\, F(t')|_0 \exp\left(\int_{t'}^t a_0(t'')dt''\right) dt'.$$

The proof of the lemma for any integer s works exactly as the one of Lemma 6.2.1 and is omitted. □

Third step of the proof of Theorem 6.2.1 . The third step consists in the proof of the following wellposedness result.

Proposition 6.2.1 *Let $s \geq 3$. We consider the linear symmetric system*

$$(LS) \begin{cases} \partial_t U + \displaystyle\sum_{k=1}^d \mathcal{A}_k\, \partial_k U \; + \mathcal{A}_0 U = F \\ U(0) = U_0 \end{cases}$$

with F in $C(I;H^s)$ and U_0 in H^s. A unique solution U exists in $C(I, H^{s-2}) \cap C^1(I;H^{s-3})$ which moreover satisfies the energy estimate

$$\forall \sigma \leq s\,,\ \forall\, t \in I\,,\ |U(t)|_\sigma \leq |U_0|_\sigma \exp \int_0^t a_s(t')dt' + \int_0^t |F(t')|_\sigma \exp\left(\int_{t'}^t a_s(t'')dt''\right) dt'.$$

Proof. Let us consider the sequence $(U_n)_{n\in\mathbb{N}}$ of solution of (LS_n). We shall prove that $(U_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty(I;H^{s-2})$. In order to do so, let us state $V_{n,p} \stackrel{\text{def}}{=} U_{n+p} - U_n$. We have

$$\begin{cases} \partial_t V_{n,p} + \displaystyle\sum_{k=1}^d \mathbb{E}_{n+p}\,(\mathcal{A}_k\, \partial_k V_{n,p}) \; + \mathbb{E}_{n+p}(\mathcal{A}_0 V_{n,p}) = F_{n,p} \\ V_{n,p}(0) = (\mathbb{E}_{n+p} - \mathbb{E}_n) U_0 \end{cases} \tag{6.9}$$

with

$$F_{n,p} \stackrel{\text{def}}{=} \sum_{k=1}^d (\mathbb{E}_{n+p} - \mathbb{E}_n)\,(\mathcal{A}_k\, \partial_k U_n) \; - (\mathbb{E}_{n+p} - \mathbb{E}_n)(\mathcal{A}_0 U_n) + (\mathbb{E}_{n+p} - \mathbb{E}_n) F.$$

Lemma 6.2.2 tells us that the sequence $(U_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(I;H^s)$. Moreover we have

$$|(\mathbb{E}_{n+p}-\mathbb{E}_n)a|_{\sigma-1}\leq \frac{C}{n}|a|_\sigma.$$

Thus we have

$$\begin{aligned}|(\mathbb{E}_{n+p}-\mathbb{E}_n)\,(\mathcal{A}_k\,\partial_k U_n(t))|_{s-2} &\leq \frac{C}{n}\sup_k|(\mathbb{E}_{n+p}-\mathbb{E}_n)\,(\mathcal{A}_k\,\partial_k U_n(t))|_{s-1}\\ &\leq \frac{C}{n}\,|U_n(t)|_s.\end{aligned}$$

The sames arguments give

$$\Big|(\mathbb{E}_{n+p}-\mathbb{E}_n)(\mathcal{A}_0U_n(t))+(\mathbb{E}_{n+p}-\mathbb{E}_n)F(t)\Big|_{s-2}\leq\frac{C}{n^2}\,(|U_n(t)|_s+|F(t)|_s). \tag{6.10}$$

Energy estimate implies that

$$|V_{n,p}(t)|_{s-2}\leq\frac{C}{n}\,\exp\Big(t\int_0^t a_s(t')dt'\Big).$$

Thus the sequence $(U_n)_{n\in\mathbb{N}}$ is a Cauchy one in $L^\infty(I,H^{s-2})$. Moreover, using (6.9) and (6.10), we infer that $(\partial_t U_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty(I;H^{s-3})$. Let us denote by U the limit of $(U_n)_{n\in\mathbb{N}}$. Of course, U belongs to $C(I;H^{s-2})\cap C^1(I;H^{s-3})$. Let us check that this function U is solution of (LS). As F belongs to $C(I;H^s)$, we have that

$$\lim_{n\to\infty}\mathbb{E}_n\,F=F\quad\text{in}\quad L^\infty(I;H^s). \tag{6.11}$$

As the sequence $(U_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(I;H^s)$, we have that

$$\|(\mathbb{E}_n-\mathrm{Id})\mathcal{A}_k(U)\partial_k U_n\|_{L^\infty(I;H^{s-2})}\leq\frac{C}{n}\cdot$$

Thus U is a solution of (LS). To conclude, let us point out that the sequence $(U_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(I;H^s)$. Using interpolation inequality, we get that for any $s'<s$, the sequence $(U_n)_{n\in\mathbb{N}}$ is a Cauchy one in $C(I,H^{s'})$. Thus U belongs to $C(I,H^{s'})$. Using the fact that U is a solution of (LS), we get that U belongs to $C(I,H^{s'})\cap C^1(I;H^{s'-1})$. But as $(U_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(I;H^s)$, it weakly converges to U in $L^\infty(I;H^s)$. Using (6.11), the fact that $(\mathbb{E}_n\,U_0)_{n\in\mathbb{N}}$ converges to U_0 in H^s and that

$$\|U\|_{L^\infty([0,t];H^s)}\leq\limsup_{n\to\infty}\|U_n\|_{L^\infty([0,t];H^s)}$$

we get, passing to the limit in Lemma 6.2.2 that

$$|U_n(t)|_s\leq|\,\mathbb{E}_n\,U(0)|_s\exp\int_0^t a_s(t')dt'+\int_0^t|\,\mathbb{E}_n\,F(t')|_s\exp\left(\int_{t'}^t a_s(t'')dt''\right)dt'.$$

Proposition 6.2.1 is proved. □

Last step of the proof of Theorem 6.2.1 . Let us consider the sequence $(\widetilde{U}_n)_{n\in\mathbb{N}}$ of solutions of

$$\begin{cases}\dfrac{\partial\widetilde{U}_n}{\partial t}+\displaystyle\sum_{k=1}^d\mathcal{A}_k\partial_k\widetilde{U}_n+\mathcal{A}_0\widetilde{U}_n=\mathbb{E}_n\,F\\ \widetilde{U}_{n|t=0}=\mathbb{E}_n\,U_0.\end{cases}$$

Thanks to Proposition 6.2.1 this solution does exist in $C^1(I, H^s)$ for any positive real number s. Let us state $V_{n,p} \stackrel{\text{def}}{=} U_{n+p} - U_n$. It satisfies

$$\begin{cases} \partial_t \widetilde{V}_{n,p} + \displaystyle\sum_{k=1}^{d} \mathcal{A}_k \partial_k \widetilde{V}_{n,p} + \mathcal{A}_0 \widetilde{V}_{n,p} = (\mathbb{E}_{n+p} - \mathbb{E}_n) F \\ \widetilde{V}_{n,p|t=0} = (\mathbb{E}_{n+p} - \mathbb{E}_n) U_0. \end{cases}$$

Lemma 6.2.1 implies that

$$\begin{aligned} |\widetilde{V}_{n,p}(t)|_s \leq |(\mathbb{E}_{n+p} - \mathbb{E}_n) U(0)|_s \exp \int_0^t a_s(t')dt' \\ + \int_0^t |(\mathbb{E}_{n+p} - \mathbb{E}_n) F(t')|_s \exp\left(\int_{t'}^t a_s(t')dt'\right) dt. \end{aligned}$$

As the function F is continuous from I into H^s, the sequence $(\mathbb{E}_n\, F)_{n\in\mathbb{N}}$ converges to F in the space $L^\infty([0,T]; H^s)$. As U_0 belongs to H^s, the sequence $(\mathbb{E}_n\, U_0)_{n\in\mathbb{N}}$ converges to U_0 in H^s. Thus the sequence $(\widetilde{U}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty(I; H^s)$. It converges to a function U of $C(I; H^s)$ which is of course solution of the system (LS). The fact that $\partial_t U$ belongs to $C(I; H^{s-1})$ comes immediately from the fact that U is solution of (S).

The existence part of Theorem 6.2.1 and also the uniqueness when $s \geq 1$ is now proved. The following proposition will conclude the proof of Theorem 6.2.1.

Proposition 6.2.2 *Let U be a solution $C^0(I; L^2)$ of the symmetric system (LS).*

$$(LS) \quad \begin{cases} \partial_t U + \displaystyle\sum_{k=1}^{d} \mathcal{A}_k\, \partial_k U\, + \mathcal{A}_0 U = 0 \\ U_{|t=0} = 0. \end{cases}$$

Then $U \equiv 0$.

Proof. In order to prove this proposition, we shall use a duality method. Let ψ be a function of $\mathcal{D}(]0,T[\times \mathbb{R}^d)$. Let us consider the solution of

$$({}^tLS) \quad \begin{cases} -\partial_t \varphi - \displaystyle\sum_{k=1}^{d} \partial_k(\mathcal{A}_k \varphi) + {}^t\mathcal{A}_0 \varphi = \psi \\ \varphi_{|t=T} = 0. \end{cases}$$

The system $({}^tLS)$ can be understood as the adjoint system of the system (LS). As we have

$$\partial_k(\mathcal{A}_k \varphi) = \mathcal{A}_k \partial_k \varphi + \partial_k \mathcal{A}_k \varphi,$$

the system $({}^tLS)$ becomes

$$({}^tS) \quad \begin{cases} -\partial_t \varphi - \displaystyle\sum_{k=1}^{d} \mathcal{A}_k \partial_k \varphi + \widetilde{\mathcal{A}}_0 \varphi = \psi \\ \varphi_{|t=T} = 0 \end{cases} \qquad \text{with} \quad \widetilde{\mathcal{A}}_0 \stackrel{\text{def}}{=} {}^t\mathcal{A}_0 - \sum_{k=1}^{d} \partial_k \mathcal{A}_k.$$

This is obviously a linear symmetric system. The existence part of Theorem 6.2.1 tells us that a solution φ of $({}^tLS)$ exists in $C^1(I,H^s)$ for any s in $\mathbb{N}$. Thus we have

$$\begin{aligned}\langle U,\psi\rangle &= \Big\langle U, -\partial_t\varphi - \sum_{k=1}^d \mathcal{A}_k\partial_k\varphi + \widetilde{\mathcal{A}}_0\varphi\Big\rangle\\ &= -\int_I \langle U(t,\cdot),\partial_t\varphi(t,\cdot\rangle dt - \sum_{k=1}^d\int_I\langle U(t),\partial_k(\mathcal{A}_k\varphi)(t)\rangle dt\\ &\qquad\qquad + \int_{I\times\mathbb{R}^d} U(t,x)^t\mathcal{A}_0\varphi(t,x)dtdx.\end{aligned}$$

Considering the weak regularity of U, each integration by part must be justified. Using Theorem 6.3.2 page 72 below (the proof of which is totally independant of Proposition 6.2.2), we have that for any t in I, the function $\varphi(t,\cdot)$ belongs to $\mathcal{D}(\mathbb{R}^d)$. By definition of the derivative of distributions, we have

$$\begin{aligned}\langle U(t),\partial_k(\mathcal{A}_k\varphi)(t)\rangle &= \sum_{i,j}\langle U^i(t),\partial_k(\mathcal{A}_{k,i,j}\varphi^j)(t)\rangle\\ &= \sum_{i,j}\langle\partial_k U^i(t),\mathcal{A}_{k,i,j}\varphi^j(t)\rangle.\end{aligned}$$

Because the matrices $\mathcal{A}_k$ are symmetric, we have for any t in I,

$$\langle U(t),\partial_k(\mathcal{A}_k\varphi)(t)\rangle = \langle\mathcal{A}_k\frac{\partial U(t)}{\partial x_k},\varphi(t)\rangle.$$

It turns ot that

$$\langle U,\psi\rangle = -\int_I\langle U(t,\cdot),\partial_t\varphi(t,\cdot\rangle dt - \Big\langle\sum_{k=1}^d\mathcal{A}_k\partial_k U - \widetilde{\mathcal{A}}_0 U,\varphi\Big\rangle.$$

In order to justify the time integration by part, let us observe that U belongs to $C^1(I,H^{-1})$. Indeed, as for smooth function we have

$$\begin{aligned}\langle\mathcal{A}_k\partial_k V,\varphi\rangle &= -\langle V,\partial_k{}^t\mathcal{A}_k\varphi\rangle - \langle V,{}^t\mathcal{A}_k\partial_k\varphi\rangle\\ &\leq (\|\mathcal{A}_k(t,\cdot)\|_{L^\infty}+a_0(t))\|V\|_{L^2}\|\varphi\|_{H^1}.\end{aligned}$$

This implies that $\partial_t U$ belongs to $C^0(I;H^{-1})$. Now, let us use the smoothing operator $\mathbb{E}_n$ defined by (6.7). The function $\mathbb{E}_n U$ belongs to $C^1(I;H^s)$ for any s. Using this with s greater than $d/2$ implies that for any x, the function

$$t\mapsto\mathbb{E}_n U(t,x)$$

exists and is a C^1 function on I. This impies that

$$\begin{aligned}-\int_I\mathbb{E}_n U(t,x)\frac{\partial\varphi}{\partial t}(t,x)dt &= -\mathbb{E}_n U(T,x)\varphi(T,x)+\mathbb{E}_n U(0,x)\varphi(0,x)\\ &\qquad + \int_I\frac{\partial\,\mathbb{E}_n U}{\partial t}(t,x)\varphi(t,x)dt.\end{aligned}$$

Using the fact that $U_0=0$ and that $\varphi(T,\cdot)=0$, we get that

$$-\int_I\mathbb{E}_n U(t,x)\partial_t\varphi(t,x)dt = \int_I\partial_t(\mathbb{E}_n U)(t,x)\varphi(t,x)dt.$$

By integration in the variable x and interchanging time and space integration, we get that

$$-\int_I \langle \mathbb{E}_n U(t,\cdot), \partial_t\varphi(t,\cdot)\rangle dt = \int_I \langle \partial_t(\mathbb{E}_n U)(t,\cdot), \varphi(t,\cdot)\rangle dt.$$

As U is a function of $C(I;L^2)\cap C^1(I;H^{-1})$, we have

$$\lim_{n\to\infty} \mathbb{E}_n U = U \quad \text{in} \quad L^\infty(I,L^2) \quad \text{and} \quad \lim_{n\to\infty} \mathbb{E}_n \partial_t U = \partial_t U \quad \text{in} \quad L^\infty(I,H^{-1}).$$

Passing to the limit in the above equality gives

$$-\langle U, \partial_t\varphi\rangle = \int_I \langle \partial_t U(t,\cdot), \varphi(t,\cdot)\rangle dt$$

and thus

$$\langle U, \psi\rangle = \int_I \Big\langle \partial_t U(t,\cdot) + \sum_{k=1}^d \mathcal{A}_k\partial_k U(t,\cdot) + \mathcal{A}_0 U(t,\cdot), \varphi(t,\cdot)\Big\rangle.$$

Ass U is solution of (LS) with $F=0$, then $U\equiv 0$ which ends the proof of the proposition □

The whole Theorem 6.2.1 is nowproved. □

6.3 Finite propagation speed

The phenomena of finite propagation speed is describe by the following theorem.

Theorem 6.3.1 *Let (LS) be a symmetric system. A constant C_0 exists such that, for any positive real number R and any data $F\in C^0(I,L^2)$ and $U_0\in L^2$ such that*

$$F(t,x)\equiv 0 \quad \textit{when} \quad |x|<R-C_0t \quad \textit{and} \quad U_0(x)\equiv 0 \quad \textit{when} \quad |x|<R. \tag{6.12}$$

then the unique solution U of the system (LS) in $C^0(I,L^2)$ with data F and U_0 satisfies

$$U(t,x)\equiv 0 \quad \textit{when} \quad |x|<R-C_0t.$$

An other form of this statement is given by the following corollary.

Corollary 6.3.1 *If the data F and U_0 satisfy*

$$F(t,x)\equiv 0 \quad \textit{for} \quad |x|>R+C_0t \quad \textit{and} \quad U_0(x)\equiv 0 \quad \textit{for} \quad |x|>R,$$

then the solution U satisfies

$$U(t,x)\equiv 0 \quad \textit{when} \quad |x|>R+C_0t.$$

Proof of Theorem 6.3.1. To start with, let us regularize the data U_0 and F perturbing their support as less as possible. Let χ be a function of $\mathcal{D}(B(0,1))$ the integral of which is 1. For any positive ϵ, we state

$$\chi_\varepsilon(x) \stackrel{\text{def}}{=} \frac{1}{\varepsilon^d}\chi\left(\frac{x}{\varepsilon}\right)\cdot$$

Now let us consider the data

$$U_{0,\varepsilon} \stackrel{\text{def}}{=} \chi_\varepsilon \star U_0 \quad \text{and} \quad F_\varepsilon(t,\cdot) \stackrel{\text{def}}{=} \chi_\varepsilon \star F(t,\cdot).$$

Of course, we have

$$\text{Supp } U_{0,\varepsilon} \subset \text{Supp } U_0 + B(0,\varepsilon) \quad \text{and} \quad F_\varepsilon(t,\cdot) \subset \text{Supp } F(t,\cdot) + B(0,\varepsilon).$$

The support hypothesis are satisfies for $U_{0,\varepsilon}$ and F_ε with $R+\varepsilon$ instead of R and the associated solution U_ε is $C^1(I;H^s)$ for any $s\in\mathbb{N}$. Thus it is enough to prove Theorem 6.3.1 with those regular solutions, namely the following statement.

Theorem 6.3.2 *Let (LS) be a symmetric system. A constant C_0 exists such that, for any positive real number R and any data $F \in C^0(I, H^1) \cap C^1(I, L^2)$ and $U_0 \in H^1$ such that*

$$F(t,x) \equiv 0 \quad \text{when} \quad |x| < R - C_0 t \quad \text{and} \quad U_0(x) \equiv 0 \quad \text{when} \quad |x| < R. \tag{6.13}$$

then the unique solution U of the system (LS) in $C^0(I, H^1) \cap C^1(I, L^2)$ with data F and U_0 satisfies

$$U(t,x) \equiv 0 \quad \text{when} \quad |x| < R - C_0 t.$$

Proof. The method used is weighted energy estimates. More precisely, for τ greater than 1, let us introduce

$$U_\tau(t,x) \stackrel{\text{def}}{=} e^{\tau\phi(t,x)} U(t,x).$$

with $\phi(t,x) = -t + \psi(x)$. The function ψ is a smooth real valued function on $\mathbb{R}^d$ which will be choosen later on.

$$\partial_t U_\tau + \sum_{k=1}^{d} \mathcal{A}_k \, \partial_k U_\tau + \mathcal{B}_\tau U_\tau = F_\tau$$

with

$$\begin{aligned} F_\tau(t,x) &\stackrel{\text{def}}{=} e^{\tau\phi(t,x)} F(t,x) \quad \text{and} \\ \mathcal{B}_\tau &\stackrel{\text{def}}{=} \mathcal{A}_0 + \tau \left(\partial_t \phi \mathrm{Id} + \sum_{k=1}^{d} \partial_k \phi \mathcal{A}_k \right) \end{aligned}$$

Considering the form of the function ϕ, we have

$$\mathcal{B}_\tau = \mathcal{A}_0 - \tau \left(\mathrm{Id} - \sum_{k=1}^{d} \partial_k \psi \mathcal{A}_k \right)$$

Thus a constant $K > 0$ exists such that for any $(t,x) \in I \times \mathbb{R}^d$, any vector $W \in \mathbb{R}^N$ and any positive real number τ, we have

$$\|\nabla\psi\|_{L^\infty} \leq K \Rightarrow (\mathcal{B}_\tau(t,x) W | \bar{W}) \leq (\mathcal{A}_0(t,x) W | W).$$

Then let us write the energy estimate and use the above inequality and relation (6.1); we get

$$\begin{aligned} \frac{d}{dt} |U_\tau(t)|_0^2 &= -2 \sum_{k=1}^{d} (\mathcal{A}_k \partial_k U | U_\tau)_{L^2} - 2(\mathcal{B}_\tau U_\tau | U_\tau)_{L^2} + 2(F_\tau | U_\tau)_{L^2} \\ &\leq a_0(t) |U_\tau|_0^2 + (F_\tau | U_\tau)_{L^2} \end{aligned}$$

Using Gronwall Lemma, we get

$$|U_\tau(t)|_0 \leq |U_\tau(0)|_0 \exp \int_0^t a_0(t') dt' + \int_0^t |F_\tau(t')|_0 \exp\Big(\int_{t'}^t a_0(t'') dt'' \Big) dt'. \tag{6.14}$$

Let us point out that the above inequality is independant of τ. Now let us state $C_0 = 1/K$ and let us pick up a smooth function $\psi = \psi(|x|)$ such that

$$-2\varepsilon + K(R - |x|) \leq \psi(x) \leq -\varepsilon + K(R - |x|) \quad \text{and} \quad \|\nabla\psi\|_{L^\infty} \leq K. \tag{6.15}$$

Then we have

$$\forall (t,x)\in I\times\mathbb{R}^d\,,\ |x|\geq R-C_0t\Longrightarrow -t+\psi(x)\leq -\varepsilon.$$

When τ tends to $+\infty$ in the inequality (6.14), we get that

$$\forall t\in I\,,\ \lim_{\tau\to\infty}\int_{\mathbb{R}^d} e^{2\tau\phi(t,x)}|u(t,x)|^2dx=0.$$

Thus $U(t,x)\equiv 0$ on the open set $t<\psi(x)$. But, if (t_0,x_0) satisfies $|x_0|<R-C_0t_0$, it is possible to pick up a function ψ satisfying (6.15) and such that $t_0<\psi(x_0)$. This proves the theorem. □

6.4 A final remark about Gronwall's lemma

In this chapter, we often use Gronwall lemma. Let us state and prove a genreral version of it.

Lemma 6.4.1 *Let us consider f a C^1 function and a and b two functions from an open interval I of $\mathbb{R}$ (which contains 0) into $\mathbb{R}^+$. Let us assume that*

$$(f^2)'\leq af^2+bf.$$

Then, we have, for any positive t,

$$f(t)\leq f(0)\exp\Bigl(\frac12\int_0^t a(t')dt'\Bigr)+\int_0^t b(t')\exp\Bigl(\frac12\int_{t'}^t a(t'')'dt'\Bigr)dt'.$$

Proof. Let us define

$$f_a(t)\stackrel{\rm def}{=}f(t)\exp\Bigl(\frac12\int_0^t a(t')dt'\Bigr)\quad\text{and}\quad b_a(t)\stackrel{\rm def}{=}b(t)\exp\Bigl(\frac12\int_0^t a(t')dt'\Bigr).$$

We have

$$(f_a^2(t))'=\bigl((f^2(t))'-a(t)f^2(t)\bigr)\exp\Bigl(\frac12\int_0^t a(t')dt'\Bigr)\leq b_a(t)f_a(t).$$

By integration, we get

$$f_a^2(t)\leq f^2(0)+\int_0^t b_a(t')f_a(t')dt'.$$

Defining $M_a(t)\stackrel{\rm def}{=}\sup\limits_{0\leq t'\leq t} f_a(t')$, we infer that

$$M_a^2(t)\leq f^2(0)+M_a(t)\int_0^t b_a(t')dt'.$$

Thus we have

$$\Bigl(M_a(t)-\frac12\int_0^t b_a(t')dt'\Bigr)^2\leq f^2(0)+\frac14\Bigl(\int_0^t b_a(t')dt'\Bigr)^2\leq\Bigl(f(0)+\frac12\int_0^t b_a(t')dt'\Bigr)^2.$$

We deduce that

$$M_a(t)\leq f(0)+\int_0^t b_a(t')dt',$$

which concludes the proof of the lemma. □

6.5 Réferences and remarques

This chapter must be known except Section 6.3.

Chapter 7

Littlewood-Paley theory

7.1 Localization in frequency space

The very basic idea of this theory consists in a localization procedure in the frequency space. The interest of this method is that the derivatives (or more generally the Fourier multipliers) act in a very special way on distributions the Fourier transform of which is supported in a ball or a ring. More precisely, we have the following lemma.

7.1.1 Bernstein inequalities

Lemma 7.1.1 (of localization) *Let $\mathcal{C}$ be a ring, B a ball. A constant C exists so that, for any non negative integer k, any smooth homogeneous function σ of degree m, any couple of real (a,b) so that $b \geq a \geq 1$ and any function u of L^a, we have*

$$\text{Supp } \widehat{u} \subset \lambda B \Rightarrow \sup_{\alpha = k} \|\partial^\alpha u\|_{L^b} \leq C^{k+1} \lambda^{k+d(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a};$$
$$\text{Supp } \widehat{u} \subset \lambda \mathcal{C} \Rightarrow C^{-k-1} \lambda^k \|u\|_{L^a} \leq \sup_{\alpha = k} \|\partial^\alpha u\|_{L^a} \leq C^{k+1} \lambda^k \|u\|_{L^a};$$

Proof. Using a dilation of ifze λ, we can assume all along the proof that $\lambda = 1$. Let ϕ be a function of $\mathcal{D}(\mathbb{R}^d)$ the value of which is 1 near B. As $\widehat{u}(\xi) = \phi(\xi)\widehat{u}(\xi)$, we can write, if g denotes the inverse fourier transform of ϕ,

$$\partial^\alpha u = \partial^\alpha g \star u.$$

Applying Young inequalities the result follows through

$$\begin{aligned} \|\partial^\alpha g\|_{L^c} &\leq \|\partial^\alpha g\|_{L^\infty} + \|\partial^\alpha g\|_{L^1} \\ &\leq 2\|(1+|\cdot|^2)^d \partial^\alpha g\|_{L^\infty} \\ &\leq 2\|(\mathrm{Id} - \Delta)^d \big((\cdot)^\alpha \phi\big)\|_{L^1} \\ &\leq C^{k+1}. \end{aligned}$$

To prove the second assertion, let us consider a function $\widetilde{\phi}$ which belongs to $\mathcal{D}(\mathbb{R}^d \setminus \{0\})$ the value of which is identically 1 near the ring $\mathcal{C}$. Using the algebraic Using the following algebraic identity

$$\begin{aligned} |\xi|^{2k} &= \sum_{1 \leq j_1, \cdots, j_k \leq d} \xi_{j_1}^2 \cdots \xi_{j_k}^2 \\ &= \sum_{|\alpha| = k} (i\xi)^\alpha (-i\xi)^\alpha, \end{aligned} \tag{7.1}$$

and stating $g_\alpha \stackrel{\text{def}}{=} \mathcal{F}^{-1}(i\xi_j)^\alpha|\xi|^{-2k}\widetilde{\phi}(\xi)$, we can write, as $\widehat{u} = \widetilde{\phi}\widehat{u}$ that

$$\widehat{u} = \sum_{|\alpha|=k} (-i\xi)^\alpha \widehat{g}_\alpha \widehat{u},$$

which implies that

$$u = \sum_{|\alpha|=k} g_\alpha \star \partial^\alpha u \tag{7.2}$$

and then the result. This proves the whole lemma. □

7.1.2 Dyadic partition of unity

Now, let us define a dyadic partition of unity. We shall use it all along this text.

Proposition 7.1.1 *Let us define by $\mathcal{C}$ the ring of center 0, of small radius $3/4$ and great radius $8/3$. It exists two radial functions χ and φ the values of which are in the interval $[0,1]$, belonging respectively to $\mathcal{D}(B(0,4/3))$ and to $\mathcal{D}(\mathcal{C})$ such that*

$$\forall \xi \in \mathbb{R}^d\,,\ \chi(\xi) + \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1, \tag{7.3}$$

$$\forall \xi \in \mathbb{R}^d \backslash\{0\}\,,\ \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \tag{7.4}$$

$$|j-j'| \geq 2 \Rightarrow \text{Supp }\ \varphi(2^{-j}\cdot) \cap \text{Supp }\ \varphi(2^{-j'}\cdot) = \emptyset, \tag{7.5}$$

$$q \geq 1 \Rightarrow \text{Supp }\ \chi \cap \text{Supp }\ \varphi(2^{-j}\cdot) = \emptyset, \tag{7.6}$$

If $\widetilde{\mathcal{C}} = B(0,2/3) + \mathcal{C}$, then $\widetilde{\mathcal{C}}$ is a ring and we have

$$|j-j'| \geq 5 \Rightarrow 2^{j'}\widetilde{\mathcal{C}} \cap 2^j\mathcal{C} = \emptyset, \tag{7.7}$$

$$\forall \xi \in \mathbb{R}^d\,,\ \frac{1}{3} \leq \chi^2(\xi) + \sum_{j\geq 0} \varphi^2(2^{-j}\xi) \leq 1, \tag{7.8}$$

$$\forall \xi \in \mathbb{R}^d \backslash\{0\}\,,\ \frac{1}{2} \leq \sum_{j\in\mathbb{Z}} \varphi^2(2^{-j}\xi) \leq 1. \tag{7.9}$$

Proof. Let us choose α in the interval $]1,4/3[$ let us denote by $\mathcal{C}'$ the ring of small radius α^{-1} and big radius 2α. Let us choose a smooth function θ, radial with value in $[0,1]$, supported in $\mathcal{C}$ with value 1 in the neighbourhood of $\mathcal{C}'$. The important point is the following. For any couple of integers (p,q) we have

$$|j-j'| \geq 2 \Rightarrow 2^j\mathcal{C} \cap 2^{j'}\mathcal{C} = \emptyset. \tag{7.10}$$

Let us suppose that $2^{j'}\mathcal{C} \cap 2^j\mathcal{C} \neq \emptyset$ and that $p \geq q$. It turns out that $2^{j'} \times 3/4 \leq 4 \times 2^{j+1}/3$, which implies that $j'-j \leq 1$. Now let us state

$$S(\xi) = \sum_{j\in\mathbb{Z}} \theta(2^{-j}\xi).$$

Thanks to (7.10), this sum is locally finite on the space $\mathbb{R}^d \backslash \{0\}$. Thus the function S is smooth on this space. As α is greater than 1,

$$\bigcup_{j \in \mathbf{Z}} 2^j \mathcal{C}' = \mathbb{R}^d \backslash \{0\}.$$

As the function θ is non negative and has value 1 near $\mathcal{C}'$, it comes from the above covering property that the above function is positive. Then let us state

$$\varphi = \frac{\theta}{S}. \tag{7.11}$$

Let us check that φ fits. It is obvious that $\varphi \in \mathcal{D}(\mathcal{C})$. The function $1 - \sum_{j \geq 0} \varphi(2^{-j}\xi)$ is smooth thanks to (7.10). As the support of θ is included in $\mathcal{C}$, we have

$$|\xi| \geq \frac{4}{3} \Rightarrow \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1. \tag{7.12}$$

thus stating

$$\chi(\xi) = 1 - \sum_{j \geq 0} \varphi(2^{-j}\xi), \tag{7.13}$$

we get Identites (7.3)and (7.5). Identity (7.6) is a obvious consequence of (7.10) and of (7.12). Now let us prove (7.7) which will be useful in Section 7.4. It is clear that the ring $\widetilde{\mathcal{C}}$ is the ring of center 0, of small radius 1/12 and of big radius 10/3. Then it turns out that

$$2^{j'}\widetilde{\mathcal{C}} \cap 2^j \mathcal{C} \neq \emptyset \Rightarrow \Big(\frac{3}{4} \times 2^j \leq 2^{j'} \times \frac{10}{3} \quad \text{ou} \quad \frac{1}{12} \times 2^{j'} \leq 2^j \frac{8}{3} \Big),$$

and (7.7) is proved. Now let us prove (7.8). As χ and φ have their values in $[0,1]$, it is clear that

$$\chi^2(\xi) + \sum_{j \geq 0} \varphi^2(2^{-j}\xi) \leq 1. \tag{7.14}$$

Let us bound from below the sum of squares. The notation $a \equiv b(2)$ means that $a - b$ is even. So we have

$$1 = (\chi(\xi) + \Sigma_0(\xi) + \Sigma_1(\xi))^2 \quad \text{with}$$
$$\Sigma_0(\xi) = \sum_{j \equiv 0(2), q \geq 0} \varphi(2^{-j}\xi) \quad \text{and} \quad \Sigma_1(\xi) = \sum_{j \equiv 1(2), q \geq 0} \varphi(2^{-j}\xi).$$

From this it comes that $1 \leq 3(\chi^2(\xi) + \Sigma_0^2(\xi) + \Sigma_1^2(\xi))$. But thanks to (7.5), we get

$$\Sigma_i^2(\xi) = \sum_{j \geq 0, q \equiv i(2)} \varphi^2(2^{-j}\xi)$$

and the proposition is proved. □

We shall consider all along this book two fixed functions χ and φ satisfying the assertions (7.3)–(7.8). Now let us to fix the notations that will be used in all the following of this text.

Notations

$$h = \mathcal{F}^{-1}\varphi \quad \text{and} \quad \widetilde{h} = \mathcal{F}^{-1}\chi,$$
$$\Delta_{-1}u = \chi(D)u = \mathcal{F}^{-1}(\chi(\xi)\widehat{u}(\xi)),$$
$$\text{if} \quad j \geq 0\,,\ \Delta_j u = \varphi(2^{-j}D)u = 2^{jd}\int_{\mathbb{R}^d} h(2^j y)u(x-y)dy,$$
$$\text{if} \quad j \leq -2\,,\ \Delta_j u = 0,$$
$$S_j u = \sum_{j'\leq j-1} \Delta_{j'}u = \chi(2^{-j}D)u = 2^{jd}\int_{\mathbb{R}^d} \widetilde{h}(2^j y)u(x-y)dy,$$
$$\text{if} \quad j \in \mathbb{Z}\,,\ \dot{\Delta}_j u = \varphi(2^{-j}D)u = 2^{jd}\int_{\mathbb{R}^d} h(2^j y)u(x-y)dy,$$
$$\text{if} \quad j \in \mathbb{Z}\,,\ \dot{S}_j u = \sum_{j'\leq j-1} \dot{\Delta}_{j'}u.$$

Remark Let us point that all the above operators Δ_j and S_j maps L^p into L^p with norms which do not depend on j. This fact will be used all along this book.

Now let us have a look of the case when we may write

$$\mathrm{Id} = \sum_j \Delta_j.$$

This is described by the following proposition.

Proposition 7.1.2 *Let u be in $\mathcal{S}'(\mathbb{R}^d)$. Then, we have, in the sense of the convergence in the space $\mathcal{S}'(\mathbb{R}^d)$,*

$$u = \lim_{j\to\infty} S_j u.$$

Proof. Let $f \in \mathcal{S}(\mathbb{R}^d)$. We have $< u - S_j u, f >=< u, f - S_j f >$. Thus it is enough to prove that in the space $\mathcal{S}(\mathbb{R}^d)$, we have

$$f = \lim_{j\to\infty} S_j f.$$

We shall use the family of semi norms $\|\cdot\widehat{\|}_{k,\mathcal{S}}$ of $\mathcal{S}$ defined by

$$\|f\widehat{\|}_{k,\mathcal{S}} \stackrel{\text{def}}{=} \sup_{\substack{|\alpha|\leq k\\ \xi\in\mathbb{R}^d}} (1+|\xi|)^k |\partial^\alpha \widehat{f}(\xi)|.$$

Thanks to Leibnitz formula, we have

$$\|f - S_j f\widehat{\|}_{k,\mathcal{S}} \leq \sup_{\substack{|\alpha|\leq k\\ \xi\in\mathbb{R}^d}} \biggl\{(1+|\xi|)^k \Bigl(|1-\chi(2^{-j}\xi)| \times |\partial^\alpha \widehat{f}(\xi)|)$$
$$+ \sum_{0<\beta\leq\alpha} C_\alpha^\beta 2^{-|\beta|} |(\partial^\beta\chi)(2^{-j}\xi)| \times |\partial^{\alpha-\beta}\widehat{f}(\xi)|\Bigr)\biggr\}.$$

As χ equals to 1 near the origin it turns out that

$$\|f - S_j f\widehat{\|}_{k,\mathcal{S}} \leq C_\alpha 2^{-j} \|f\widehat{\|}_{k+1,\mathcal{S}}.$$

The proposition is proved. □

The following proposition tells us that the condition of convergence in $\mathcal{S}'$ is somehow weak for series, the Fourier transform of which is supported in dyadic rings.

Proposition 7.1.3 *Let $(u_j)_{j\in\mathbb{N}}$ be a sequence of bounded functions such that the Fourier transform of u_j is supported in $2^j\widetilde{\mathcal{C}}$ where $\widetilde{\mathcal{C}}$ is a given ring. Let us assume that*

$$\|u_j\|_{L^\infty} \leq C2^{jN}.$$

Then the series $(u_j)_{j\in\mathbb{N}}$ is convergent in $\mathcal{S}'$.

Proof. Let us use the relation (7.2). After rescaling it can be written as

$$u_j = 2^{-jk} \sum_{|\alpha|=k} 2^{jd} g_\alpha(2^j \cdot) \star \partial^\alpha u_j.$$

Then for any test function ϕ in $\mathcal{S}$, let us write that

$$\begin{aligned} \langle u_j, \phi\rangle &= -2^{-jk} \sum_{|\alpha|=k} \langle u_j, 2^{jd} g_\alpha(2^j \cdot) \star \partial^\alpha \phi\rangle \qquad (7.15)\\ &\leq C2^{-jk} \sum_{|\alpha|=k} 2^{jN} \|\partial^\alpha \phi\|_{L^1}. \end{aligned}$$

Let us choose $k > N$. Then $(\langle u_j, \phi\rangle)_{j\in\mathbb{N}}$ is a convergent series, the sum of which is less than $C\|\phi\|_{M,\mathcal{S}}$ for some integer M. Thus the formula

$$\langle u, \phi\rangle \stackrel{\text{def}}{=} \lim_{j\to\infty} \sum_{j'\leq j} \langle \Delta_{j'} u, \phi\rangle$$

defines a tempered distribution. □

7.2 Inhomogeneous Besov spaces

7.2.1 Definition and examples

Definition 7.2.1 *Let s be a real number, and p and r two reals numbers greater than 1. The Besov spaces $B^s_{p,r}$ is the space of all tempered distributions so that*

$$\|u\|_{B^s_{p,r}} \stackrel{\text{def}}{=} \left\|(2^{js}\|\Delta_j u\|_{L^p})_{j\in\mathbb{Z}}\right\|_{\ell^r(\mathbb{Z})} < +\infty.$$

Lemma 7.2.1 *If r is finite, then for any u in $B^s_{p,r}$, we have*

$$\lim_{j\to\infty} \|S_j u - u\|_{B^s_{p,r}} = 0$$

The proof of this proposition is an easy exercise left to the reader. Let us give the first example for Besov space, the Sobolev spaces H^s. We have the following result.

Theorem 7.2.1 *The two spaces H^s and $B^s_{2,2}$ are equal and the two norms satisfies*

$$\frac{1}{C^{|s|+1}}\|u\|_{B^s_{2,2}} \leq \|u\|_{H^s} \leq C^{|s|+1}\|u\|_{B^s_{2,2}}.$$

As the support of the Fourier transform of $\Delta_j u$ is included in the ring $2^j\mathcal{C}$, it is clear, as $j \geq 0$, that a constant C exists such that, for any real s and any u such that $\widehat{u}$ belongs to L^2_{loc},

$$\frac{1}{C^{|s|+1}} 2^{js}\|\Delta_j u\|_{L^2} \leq \|\Delta_j u\|_{H^s} \leq C^{|s|+1} 2^{js}\|\Delta_j u\|_{L^2}. \tag{7.16}$$

Using Identity (7.8), we get

$$\frac{1}{3}\|u\|^2_{H^s} \leq \int \chi^2(\xi)(1+|\xi|^2)^s|\widehat{u}(\xi)|^2 d\xi + \sum_{j\geq 0}\int \varphi^2(2^{-j}\xi)(1+|\xi|^2)^s|\widehat{u}(\xi)|^2 d\xi \leq \|u\|^2_{H^s}$$

which proves the theorem. □

Proposition 7.2.1 *The space $B^0_{p,1}$ is continuously embedded in L^p and the space L^p is continuously embedded in $B^0_{p,\infty}$.*

The proof is trivial. The first inclusion comes from the fact that the series $(\Delta_j u)_{j\in\mathbb{Z}}$ is convergent in L^p. The second one comes from the fact that for any p, we have $\|\Delta_j u\|_{L^p} \leq C\|u\|_{L^p}$.

7.2.2 Basic properties

The first point to look at is the invariance with respect to the choice of the dyadic partition of unity choosen to define the space. Most of the properties of the Besov spaces are based on the following lemma.

Lemma 7.2.2 *Let $\mathcal{C}'$ be a ring in $\mathbb{R}^d$; let s be a real number and p and r two real numbers greater than 1. Let $(u_j)_{j\in\mathbb{N}}$ be a sequence of smooth functions such that*

$$\mathrm{Supp}\ \ \widehat{u}_j \subset 2^j\mathcal{C}' \quad \textit{and} \quad \left\|(2^{js}\|u_j\|_{L^p})_{j\in\mathbb{N}}\right\|_{\ell^r} < +\infty.$$

Then we have

$$u = \sum_{j\in\mathbb{N}} u_j \in B^s_{p,r} \quad \textit{and} \quad \|u\|_{B^s_{p,r}} \leq C_s\left\|(2^{js}\|u_j\|_{L^p})_{j\in\mathbb{N}}\right\|_{\ell^r}.$$

This immediately implies the following corollary.

Corollary 7.2.1 *The space $B^s_{p,r}$ does not depend on the choice of the functions χ and φ used in the Definition 7.2.1.*

Proof of Lemma 7.2.2 Let us first observe that $(u_j)_{j\in\mathbb{N}}$ is a convergente series in $\mathcal{S}'$. Indeed using Lemma 7.1.1, we get that $\|u_j\|_{L^\infty} \leq C2^{j\left(\frac{d}{p}-s\right)}$. Proposition 7.1.3 implies that $(u_j)_{j\in\mathbb{N}}$ is a convergente series in $\mathcal{S}'$. Then, let us study $\Delta_{j'}u$. As $\mathcal{C}$ and $\mathcal{C}'$ are two rings, an integer N_0 exists so that

$$|j'-j| \geq N_0 \Longrightarrow 2^j\mathcal{C} \cap 2^{j'}\mathcal{C}' = \phi.$$

Here $\mathcal{C}$ is the ring defined in the Proposition 7.1.1. Now, it is clear that

$$\begin{aligned} |j'-j| \geq N_0 \quad &\Longrightarrow \quad \mathcal{F}(\Delta_{j'}u_j) = 0 \\ &\Longrightarrow \quad \Delta_{j'}u_j = 0. \end{aligned}$$

Now, we can write that

$$
\begin{aligned}
\|\Delta_{j'} u\|_{L^p} &= \|\sum_{|j-j'|<N_0} \Delta_{j'} u_j\|_{L^p} \\
&\leq C \sum_{|j-j'|<N_0} \|u_j\|_{L^p}.
\end{aligned}
$$

So, we obtain that

$$
\begin{aligned}
2^{j's}\|\Delta_{j'} u\|_{L^p} &\leq C \sum_{\substack{j'\geq -1 \\ |j-j'|\leq N_0}} 2^{j's}\|u_j\|_{L^p} \\
&\leq C \sum_{\substack{j'\geq -1 \\ |j-j'|\leq N_0}} 2^{js}\|u_j\|_{L^p}.
\end{aligned}
$$

We deduce from this that

$$
2^{j's}\|\Delta_{j'} u\|_{L^p} \leq (c_k)_{k\in\mathbb{Z}} \star (d_\ell)_{\ell\in\mathbb{Z}} \quad \text{with} \quad c_k = \mathbf{1}_{[-N_0,N_0]}(k) \quad \text{and} \quad d_\ell = \mathbf{1}_{\mathbb{N}}(\ell) 2^{\ell s}\|u_\ell\|_{L^p}.
$$

The classical property of convolution between $\ell^1(\mathbb{Z})$ and $\ell^r(\mathbb{Z})$ gives that

$$
\|u\|_{B^s_{p,r}} \leq C\Big(\sum_{j\in\mathbb{N}} 2^{rqs}\|u_j\|^r_{L^p}\Big)^{\frac{1}{r}},
$$

which proves the lemma. □

The following theorem is the equivalent of Sobolev embedding (see Theorem 2.2.1 page 22).

Theorem 7.2.2 *Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$. Then for any real number s the space $B^s_{p_1,r_1}$ is continuously embedded in $B^{s-d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}_{p_2,r_2}$.*

Proof. In order to prove this result, we again apply Lemma 7.1.1 which tells us that

$$
\begin{aligned}
\|S_0 u\|_{L^{p_2}} &\leq C\|u\|_{L^{p_1}} \quad \text{and} \\
\|\Delta_j u\|_{L^{p_2}} &\leq C 2^{jd\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|\Delta_j u\|_{L^{p_1}}.
\end{aligned}
$$

Considering that $\ell^{r_1}(\mathbb{Z}) \subset \ell^{r_2}(\mathbb{Z})$, the theorem is proved.

Proposition 7.2.2 *The space $B^s_{p,r}$ is continuously embedded in $\mathcal{S}'$.*

By definition $B^s_{p,r}$ is a subspace of $\mathcal{S}'$. Thus we have only to proof of a constant C and an integer M exists such that for any test function ϕ in $\mathcal{S}$ we have

$$
\langle u, \phi\rangle \leq C\|u\|_{B^s_{p,r}}\|\phi\|_{M,\mathcal{S}}. \tag{7.17}
$$

Using the above Theorem 7.2.2 and the relation (7.15), we can write, if N is a large enough integer,

$$
\begin{aligned}
\langle \Delta_j u, \phi\rangle &= -2^{-j(N+1)} \sum_{|\alpha|=N+1} \langle \Delta_j u, 2^{jd} g_\alpha(2^j\cdot) \star \partial^\alpha \phi\rangle \\
&\leq 2^{-j}\|u\|_{B^{-N}_{\infty,\infty}} \sup_{|\alpha|=N+1} \|\partial^\alpha\phi\|_{L^1} \qquad (7.18) \\
&\leq C2^{-j}\|u\|_{B^s_{p,r}}\|\phi\|_{M,\mathcal{S}}.
\end{aligned}
$$

Now Proposition 7.1.2 implies the Inequality (7.17). □

Theorem 7.2.3 *The space $B^s_{p,r}$ equipped with the norm $\|\cdot\|_{B^s_{p,r}}$ is a Banach space and satisfies the Fatou properties, i.e. if $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence of $B^s_{p,r}$, then an element u of $B^s_{p,r}$ and a subsequence $u_{\psi(n)}$ exist such that*

$$\lim_{n\to\infty} u_n = u \text{ in } \mathcal{S}' \quad \text{and} \quad \|u\|_{B^s_{p,r}} \leq \liminf_{n\to\infty} \|u_n\|_{B^s_{p,r}}.$$

Proof. Let us first prove the Fatou property. Using Lemma 7.1.1, we claim that, for any j, the sequence $(\Delta_j u_n)_{n\in\mathbb{N}}$ is bounded in $L^p \cap L^\infty$. Then, using Cantor's diagonal process, we infer the existence of a subsequence $(u_{\psi(n)})_{n\in\mathbb{N}}$ and a sequence $(\widetilde{u}_j)_{j\in\mathbb{Z}}$ such that, for any $j\in\mathbb{Z}$ and any $\phi\in\mathcal{S}$,

$$\lim_{n\to\infty}\int \Delta_j u_{\psi(n)}(x)\phi(x)dx = \int \widetilde{u}_j(x)\phi(x)dx \quad \text{and} \quad \|\widetilde{u}_j\|_{L^p} \leq \lim_{n\to\infty}\|\Delta_j u_n\|_{L^p}.$$

As the Fourier transform of $(\Delta_j u_n)_{n\in\mathbb{N}}$ is supported in $2^j\widetilde{\mathcal{C}}$, the same holds for $\widetilde{u}_j$. Then, let us observe that the sequence $\big((2^{js}\|\Delta_j u_n\|_{L^p})_j\big)_{n\in\mathbb{N}}$ is a bounded sequence of ℓ^r, an element $(\widetilde{c}_j)_j$ of ℓ^r such that (up to an omitted extraction), we have, for any sequence $(d_j)_j$ of non negative real numbers different from 0 only for a finite number of indices j,

$$\lim_{n\to\infty}\sum_j 2^{js}\|\Delta_j u_{\psi(n)}\|_{L^p} d_j = \sum_j \widetilde{c}_j d_j \quad \text{and} \quad \|(\widetilde{c}_j)_j\|_{\ell^r} \leq \lim_{n\to\infty}\|u_{\psi(n)}\|_{\dot{B}^s_{p,r}}.$$

Passing to the limit in the sum gives that $(2^{js}\|\widetilde{u}_j\|_{L^p})_j$ belongs to $\ell^r(\mathbb{Z})$. Using Lemma 7.1.1 and Proposition 7.2.2 implies that the series $(\widetilde{u}_j)_{j\in\mathbb{Z}}$ converges to some u in $B^s_{p,r}$ such that

$$\|u\|_{B^s_{p,r}} \leq C_s \Big\|(2^{js}\|\widetilde{u}_j\|_{L^p})_j\Big\|_{\ell^r}.$$

This is proves the first part of the theorem.

Now, let us check that $\dot{B}^s_{p,r}$ is complete. Let us consider a Cauchy sequence $(u_n)_{n\in\mathbb{N}}$. This sequence is of course bounded. Thus u exists in $\dot{B}^s_{p,r}$ such that a subsequence $(u_{\psi(n)})_{n\in\mathbb{N}}$ converges to u in $\mathcal{S}'$. Using that, for any positive ε, an integer n_ϵ exists such that

$$\forall (n,m)\in [n_\varepsilon,\infty[\,,\ \|u_m - u_{\psi(n)}\|_{B^s_{p,}} < \varepsilon.$$

Applying the above method to the sequence $(u_m - u_{\psi(n)})_{n\in\mathbb{N}}$, we infer that

$$\forall m \geq n_\varepsilon\,,\ \|u_m - u\|_{\dot{B}^s_{p,r}} \leq \varepsilon.$$

The theorem is proved. □

7.3 The case of Hölder type spaces

Another relevant example of Besov spaces are Hölder spaces.

Definition 7.3.1 *Let (k,ρ) be in $\mathbb{N}\times]0,1]$. The* Hölder space $C^{k,\rho}(\mathbb{R}^d)$, *(or $C^{k,\rho}$ if no confusion is possible) is the space of C^k functions u on $\mathbb{R}^d$ such that*

$$\|u\|_{C^{k,\rho}} = \sup_{|\alpha|\leq k}\Big(\|\partial^\alpha u\|_{L^\infty} + \sup_{x\neq y}\frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^\rho}\Big) < \infty.$$

We have the following result.

Proposition 7.3.1 *For any k in $\mathbb{N}$, a constant C_k exists such that for any $\rho \in]0,1]$ and any function u belonging to $C^{k,\rho}$, we have*

$$\sup_j 2^{j(k+\rho)} \|\Delta_j u\|_{L^\infty} \leq C_k \|u\|_{C^{k,\rho}}.$$

Proof. To prove this, let us first observe that, when $j=-1$, we have that $\|S_0 u\|_{L^\infty} \leq C\|u\|_{L^\infty}$. When j is non negative, let us write the operator Δ_j of the convolution form

$$\Delta_j u(x) = 2^{jd} \int h(2^j(x-y))u(y)dy.$$

The fact that the function φ is identically 0 near the origin implies that for any $\alpha \in \mathbb{N}^d$,

$$\int x^\alpha h(x)dx = 0.$$

Thus we can write

$$\Delta_j u(x) = 2^{jd} \int h(2^j(x-y))\Big(u(y) - u(x) - \sum_{0<|\alpha|\leq k} \frac{1}{\alpha!}\partial^\alpha u(x)(y-x)^\alpha\Big)dy. \tag{7.19}$$

Taylor formula at order $k-1$ implies that

$$\begin{aligned} u(y) - u(x) - &\sum_{0<|\alpha|\leq k} \frac{1}{\alpha!}\partial^\alpha u(x)(y-x)^\alpha \\ &= \int_0^1 k(1-t)^{k-1} \sum_{|\alpha|=k} \frac{(y-x)^\alpha}{\alpha!}\Big(\partial^\alpha u(x+t(y-x)) - \partial^\alpha u(x)\Big)dt. \end{aligned}$$

The fact that the functions $\partial^\alpha u$ belong to the space C^ρ implies that

$$\Big|u(y) - u(x) - \sum_{0<|\alpha|\leq k} \frac{1}{\alpha!}\partial^\alpha u(x)(y-x)^\alpha\Big| \leq C_k |y-x|^{k+\rho}\|u\|_{C^{k,\rho}}.$$

Then it comes from (7.19) that

$$|\Delta_j u(x)| \leq C_k \|u\|_{C^{k,\rho}} 2^{jd} \int |x-y|^{k+\rho} |h(2^j(x-y))|dy$$

and the proposition is proved. □

Let us study the reciproque of Proposition 7.3.1. We have the following proposition.

Proposition 7.3.2 *Let r be in $\mathbb{R}^+ \setminus \mathbb{N}$ and let u be in $B^r_{\infty,\infty}$. Then u is in $C^{k,\rho}$ with $k=[r]$ and $\rho = r-[r]$. Moreover we have*

$$\|u\|_{C^{k,\rho}} \leq C_r \|u\|_{B^r_{\infty,\infty}} \quad \textit{with} \quad C_r = C_k\Big(\frac{1}{\rho} + \frac{1}{1-\rho}\Big).$$

Proof. To star with, let us observe that, thanks to Lemma 7.1.1, we have, for any α the length of which is less than r,

$$\|\Delta_j \partial^\alpha u\|_{L^\infty} \leq C^{k+1} 2^{-q(r-|\alpha|)} \|u\|_{B^r_{\infty,\infty}}.$$

Thus the series $(\Delta_j \partial^\alpha u)_{j\in\mathbf{N}}$ is convergent in the space L^∞ and we have

$$\|\partial^\alpha u\|_{L^\infty} \le C^{k+1}\frac{1}{\rho}\|u\|_{B^r_{\infty,\infty}}. \tag{7.20}$$

Now let us study the derivative of order k. We can write, for a positive integer N which will be chosen later on, that

$$|\partial^\alpha u(x) - \partial^\alpha u(y)| \le \sum_{j=0}^{N-1} |\partial^\alpha \Delta_j u(x) - \partial^\alpha \Delta_j u(y)| + \sum_{j\ge N} |\partial^\alpha \Delta_j u(x) - \partial^\alpha \Delta_j u(y)|.$$

By Taylor inequality, we have that

$$|\partial^\alpha \Delta_j u(x) - \partial^\alpha \Delta_j u(y)| \le C|x-y| \sup_{|\beta|=k+1} \|\partial^\beta \Delta_j u\|_{L^\infty}.$$

Using Lemma 7.1.1, we get

$$|\partial^\alpha \Delta_j u(x) - \partial^\alpha \Delta_j u(y)| \le C_k \|u\|_{B^r_{\infty,\infty}} |x-y| 2^{-q(\rho-1)}. \tag{7.21}$$

The high frequency terms are estimated very roughly writing

$$|\partial^\alpha \Delta_j u(x) - \partial^\alpha \Delta_j u(y)| \le C_k 2^{-q\rho} \|u\|_{B^r_{\infty,\infty}}.$$

Then it comes from (7.21) that

$$|\partial^\alpha u(x) - \partial^\alpha u(y)| \le C_k \|u\|_{B^r_{\infty,\infty}} \Big(\sum_{j=0}^{N} 2^{-q(\rho-1)}|x-y| + \sum_{j\ge N+1} 2^{-q\rho} \Big).$$

Thanks to (7.20), we may assume that $|x-y| \le 1$. Choosing

$$N = [-\log_2 |x-y|] + 1,$$

in the above inequality, we conclude the proof of the proposition. □

Propositions 7.3.1 and 7.3.2 together imply the following theorem.

Theorem 7.3.1 *Let r be in $\mathbb{R}^+ \setminus \mathbb{N}$. Then the spaces $B^r_{\infty,\infty}$ and $C^{[r],r-[r]}$ are equal and we have*

$$C_{[r]}^{-1}\|u\|_{B^r_{\infty,\infty}} \le \|u\|_{C^{[r],r-[r]}} \le C_{[r]}\Big(\frac{1}{r-[r]} + \frac{1}{1-(r-[r])}\Big)\|u\|_{B^r_{\infty,\infty}}.$$

7.4 Paradifferential calculus

In this section, we are going to study the way how the product acts on Besov spaces. Of course, we shall use the dyadic decomposition constructed in the Section 7.1.2.

7.4.1 Bony's decomposition

Let us consider two tempered distributions u and v, we write

$$u = \sum_{j'} \Delta_{j'} u \quad \text{and} \quad v = \sum_{j} \Delta_j v.$$

Formally, the product can be written as

$$uv = \sum_{j,j'} \Delta_{j'} u \Delta_j v.$$

Now, let us introduce Bony's decomposition.

Definition 7.4.1 *We shall designate paraproduct by u and shall denote by $T_u v$ the following bilinear operator:*

$$T_u v \stackrel{\text{def}}{=} \sum_j S_{j-1} u \Delta_j v.$$

We shall designate remainder of u and v and shall denote by $R(u,v)$ the following bilinear operator:

$$R(u,v) = \sum_{|j-j'|\leq 1} \Delta_{j'} u \Delta_j v.$$

Just by looking at the definition, it is clear that

$$uv = T_u v + T_v u + R(u,v). \tag{7.22}$$

The way how paraproduct and remainder act on Besov spaces is described by the following theorem.

Lemma 7.4.1 *For any s, a constant C exists such that, for any (p,r) in $[1,+\infty]^2$, we have*

$$\forall (u,v) \in L^\infty \times B^s_{p,r}\,,\ \|T_u v\|_{B^s_{p,r}} \leq C \|u\|_{L^\infty} \|v\|_{B^s_{p,r}}.$$

Proof. From the assertion (7.7), the Fourier transform of $S_{j-1} u \Delta_j v$ is supported in $2^j \widetilde{\mathcal{C}}$. Then, let us write that

$$\|S_{j-1} u \Delta_j v\|_{L^p} \leq C \|u\|_{L^\infty} \|\Delta_j v\|_{L^p}.$$

Lemma 7.2.2 implies the result. □

Now we shall study the behaviour of operators R. Here we have to consider terms of the type $\Delta_j u \Delta_j v$. The Fourier transform of such terms is not supported in ring but in balls of the type $2^j B$. Thus to prove that remainder terms belong to some Besov spaces, we need the following lemma.

Lemma 7.4.2 *Let B be a ball of $\mathbb{R}^d$, s a positive real number and (p,r) in $[1,\infty]^2$. Let $(u_j)_{j\in\mathbb{N}}$ be a sequence of smooth functions such that*

$$\text{Supp}\ \ \widehat{u}_j \subset 2^j B \quad \text{and} \quad \Big\| (2^{js} \|u_j\|_{L^p})_{j\in\mathbb{N}} \Big\|_{\ell^r} < +\infty.$$

Then we have

$$u = \sum_{j\in\mathbb{N}} u_j \in B^s_{p,r} \quad \text{and} \quad \|u\|_{B^s_{p,r}} \leq C_s \Big\| (2^{js} \|u_j\|_{L^p})_{j\in\mathbb{N}} \Big\|_{\ell^r}.$$

Proof. We have for any j

$$\|u_j\|_{L^p} \leq C 2^{-js}.$$

As s is positive, $(u_j)_{j\in\mathbb{N}}$ is a convergent series in L^p. We then study $\Delta_{j'} u_j$. As $\mathcal{C}$ is a ring (defined in the proposition 7.1.1) and B is a ball, an integer N_1 exists so that

$$j' \geq j + N_1 \Longrightarrow 2^{j'}\mathcal{C} \cap 2^j B = \phi.$$

So it is clear that

$$\begin{aligned} j' \geq j + N_1 \quad &\Longrightarrow \quad \mathcal{F}(\Delta_{j'} u_j) = 0 \\ &\Longrightarrow \quad \Delta_{j'} u_j = 0. \end{aligned}$$

Now, we write that

$$\begin{aligned} \|\Delta_{j'} u\|_{L^p} &= \Big\| \sum_{j \geq j' - N_1} \Delta_{j'} u_j \Big\|_{L^p} \\ &\leq \sum_{j \geq j' - N_1} \|\Delta_{j'} u_j\|_{L^p} \\ &\leq \sum_{j \geq j' - N_1} \|u_j\|_{L^p}. \end{aligned}$$

So, we get that

$$\begin{aligned} 2^{j's}\|\Delta_{j'} u\|_{L^p} &\leq \sum_{j \geq j' - N_1} 2^{j's}\|u_j\|_{L^p} \\ &\leq \sum_{j \geq j' - N_1} 2^{(j'-j)s} 2^{js}\|u_j\|_{L^p}. \end{aligned}$$

So, we deduce from this that

$$2^{j's}\|\Delta_{j'} u\|_{L^p} \leq (c_k) \star (d_\ell) \quad \text{with} \quad c_k = \mathbf{1}_{[-N_1, +\infty[}(k) 2^{-ks} \quad \text{and} \quad d_\ell = 2^{\ell s}\|u_\ell\|_{L^p}.$$

So, the lemma is proved. □

Lemma 7.4.3 *For any (s_1, s_2) such that $s_1 + s_2 > 0$ a constant C exists such that , any (p_1, p_2, r_1, r_2) in $[1, \infty]^4$ such that*

$$, \ \frac{1}{p} \stackrel{\text{def}}{=} \frac{1}{p_1} + \frac{1}{p_2} \leq 1 \quad \text{and} \quad \frac{1}{r_1} + \frac{1}{r_2} \stackrel{\text{def}}{=} \frac{1}{r} \leq 1,$$

we have that

$$\forall (u, v) \in B^{s_1}_{p_1, r_1} \times B^{s_2}_{p_2, r_2}\,,\ \|R(u, v)\|_{B^{s_1+s_2}_{p,r}} \leq C\|u\|_{B^{s_1}_{p_1, r_1}} \|v\|_{B^{s_2}_{p_2, r_2}}.$$

Proof. By definition of the remainder operator, we have

$$R(u, v) = \sum_j R_j \quad \text{with} \quad R_j = \sum_{\ell=-1}^{1} \Delta_{j-\ell} u \Delta_j v.$$

By definition of Δ_j, the support of the Fourier transform of R_j is included in $2^j B(0, 24)$. Moreover, Hölder inequalities implies that

$$2^{j(s_1+s_2)}\|R_j\|_{L^p} \leq \sum_{\ell=-1}^{1} \|\Delta_{j-\ell} u\|_{L^{p_1}} \|\Delta_j v\|_{L^{p_2}}.$$

Thus $2^{j(s_1+s_2)}\|R_j\|_{L^p}$ appears to be a sum of three series which are the product of a ℓ^{r_1} series by a ℓ^{r_2} series. Thus the lemma is proved. □

Now, we are going to infer the following corollary.

Corollary 7.4.1 *For any positive s, the space $L^\infty \cap B^s_{p,r}$ is an algebra. More precilesely, a constant C exists such that*

$$\|uv\|_{B^s_{p,r}} \leq C(\|u\|_{L^\infty}\|v\|_{B^s_{p,r}} + \|u\|_{B^s_{p,r}}\|v\|_{L^\infty}).$$

The proof is nothing but the use of Bony's decomposition and the map of Theorems 7.4.1 and 7.4.3.

7.4.2 Action of smooth functions

In this paragraph we shall study the action of smooth functions on the space $B^s_{p,r}$. More precisely, if f is a smooth function vanishing at 0, and u a function of $B^s_{p,r}$, does $f \circ u$ belongs to $B^s_{p,r}$? The answer is given by the following theorem.

Theorem 7.4.1 *Let f be a smooth function and s a positive real number and (p,r) in $[1,\infty]^2$. If u belongs to $B^s_{p,r} \cap L^\infty$, then $f \circ u$ belongs to $B^s_{p,r}$ and we have*

$$\|f \circ u\|_{B^s_{p,r}} \leq C(s, f, \|u\|_{L^\infty})\|u\|_{B^s_{p,r}}.$$

Before proving this theorem, let us notice that if $s > d/p$ or if $s = d/p$ and $r = 1$, then the space $B^s_{p,r}$ is included into L^∞. Thus in those cases, the space $B^s_{p,r}$ is stable under the action of f by composition. This is in particular the case for the Sobolev space H^s with $s > d/2$.

Proof of Theorem 7.4.1 We shall use the argument of the so called "telescopic series". As the sequence $(S_j u)_{j\in\mathbb{N}}$ converges to u in L^p and $f(0) = 0$, then we have

$$f(u) = \sum_j f_j \quad \text{with} \quad f_j \stackrel{\text{def}}{=} f(S_{j+1}u) - f(S_j u). \tag{7.23}$$

Using Taylor formula at order 1, we get

$$f_j = m_j \Delta_j u \quad \text{with} \quad m_j \stackrel{\text{def}}{=} \int_0^1 f'(S_j u + t\Delta_j u)dt. \tag{7.24}$$

At this point of the proof, let us point out that there is no hope for the Fourier transform of f_j to be compactly supported. Thus Lemma 7.4.2 is not efficient in this case. We shall prove the following improvement of this lemma. □

Lemma 7.4.4 *Let s be a positive real number and (p,r) in $[1,\infty]^2$. A constant C_s exists such that if $(u_j)_{j\in\mathbb{N}}$ is a sequence of smooth functions which satisfies*

$$\Big(\sup_{|\alpha|\leq[s]+1} 2^{j(s-|\alpha|)}\|\partial^\alpha u_j\|_{L^p}\Big)_j \in \ell^r,$$

then we have

$$u = \sum_{j\in\mathbb{N}} u_j \in B^s_{p,r} \quad \text{and} \quad \|u\|_{B^s_{p,r}} \leq C_s \Big\|\Big(\sup_{|\alpha|\leq[s]+1} 2^{j(s-|\alpha|)}\|\partial^\alpha u_j\|_{L^p}\Big)_j\Big\|_{\ell^r}.$$

Proof. As s is positive, the series $(u_q)_{q\in\mathbb{N}}$ is convergent in L^p. Let us denote its sum by u and let us write that

$$\Delta_j u = \sum_{j'\leq j} \Delta_j u_{j'} + \sum_{j'>j} \Delta_j u_{j'}.$$

Using that $\|\Delta_j u_{j'}\|_{L^p} \leq \|u_{j'}\|_{L^p}$, we get that

$$\begin{aligned} 2^{js}\Big\|\sum_{j'>j} \Delta_j u_{j'}\Big\|_{L^p} &\leq 2^{js}\sum_{j'>j}\|u_{j'}\|_{L^p} \\ &\leq \sum_{j'>j} 2^{-(q'-q)s} 2^{j's}\|u_{j'}\|_{L^p}. \end{aligned} \tag{7.25}$$

Then using Lemma 7.1.1, we write that

$$\|\Delta_j u_{j'}\|_{L^p} \leq C 2^{-j([s]+1)} \sup_{|\alpha|=[s]+1} \|\partial^\alpha u_{j'}\|_{L^p}.$$

Then we write

$$2^{js}\Big\|\sum_{j'\leq j} \Delta_j u_{j'}\Big\|_{L^p} \leq \sum_{j'\leq j} 2^{(j'-j)([s]+1-s)} \sup_{|\alpha|=[s]+1} 2^{j'(s-|\alpha|)}\|\partial^\alpha u_{j'}\|_{L^p}.$$

This inequality together with (7.25) implies that

$$\begin{aligned} 2^{js}\|\Delta_j u\|_{L^p} &\leq (a\star b)_j \quad \text{with} \\ a_j &\stackrel{\text{def}}{=} \mathbf{1}_{\mathbb{N}}(j)2^{-js} + \mathbf{1}_{\mathbb{N}}(j)2^{-j([s]+1-s)} \quad \text{and} \\ b_j &\stackrel{\text{def}}{=} 2^{js}\|u_j\|_{L^p} + \sup_{|\alpha|=[s]+1} 2^{j(s-|\alpha|)}\|\partial^\alpha u_j\|_{L^p}. \end{aligned}$$

This proves the lemma. □

Continuation of the proof of Theorem 7.4.1 Let us admit for a while that

$$\forall \alpha\in\mathbb{N}^d,\ \|\partial^\alpha m_j\|_{L^\infty} \leq C_\alpha(f,\|u\|_{L^\infty})2^{j|\alpha|}. \tag{7.26}$$

Thus using Leibnitz formula and Lemma 7.1.1, we get that

$$\|\partial^\alpha f_j\|_{L^p} \leq \sum_{\beta\leq\alpha} C_\beta^\alpha 2^{j|\beta|} C_\beta(f,\|u\|_{L^\infty}) 2^{j(|\alpha|-|\beta|)}\|\Delta_j u\|_{L^p}$$

Thus we get that

$$\begin{aligned} \|\partial^\alpha f_j\|_{L^p} &\leq C_\alpha(f,\|u\|_{L^\infty})2^{j|\alpha|}\|\Delta_j u\|_{L^p} \\ &\leq c_j C_\alpha(f,\|u\|_{L^\infty}) 2^{-j(s-|\alpha|)}\|u\|_{B^s_{p,r}} \end{aligned} \tag{7.27}$$

with $\|(c_j)\|_{\ell^r} = 1$. We apply Lemma 7.4.4 and the theorem is proved provided we check (7.26). In order to do it, let us recall Faa-di-Bruno's formula.

$$\partial^\alpha g(a) = \sum_{\substack{\alpha_1+\cdots+\alpha_p=\alpha \\ |\alpha_j|\geq 1}} \Big(\prod_{k=1}^p \partial^{\alpha_k} a\Big) g^{(p)}(a).$$

From this formula, we infer that

$$\partial^\alpha m_j = \sum_{\substack{\alpha_1+\cdots+\alpha_p=\alpha \\ |\alpha_j|\geq 1}} \int_0^1 \Big(\prod_{k=1}^p \partial^{\alpha_k}(S_j u + t\Delta_j u)\Big) f^{(p+1)}(S_j u + t\Delta_j u)dt.$$

Using Lemma 7.1.1, we get that

$$\begin{aligned} \|\partial^\alpha m_j\|_{L^\infty} &\leq C_\alpha(f) \sum_{\substack{\alpha_1+\cdots+\alpha_p=\alpha \\ |\alpha_j|\geq 1}} \int_0^1 \Big(\prod_{k=1}^p 2^{j|\alpha_k|}\|u\|_{L^\infty}\Big) \\ &\leq C_\alpha(f, \|u\|_{L^\infty})2^{j|\alpha|}. \end{aligned}$$

This proves (7.26) and thus the theorem.

Chapter 8

Quasilinear symmetric systems

8.1 Definition and examples

Let us now define what a quasilinear symmetric system is. First a quasilinear system is a system of the form

$$(S)\begin{cases} \partial_t U + \displaystyle\sum_{k=1}^{d} A_k(U)\partial_k U &= 0 \\ U_{|t=0} &= U_0 \end{cases}$$

where $A = (A_k)_{1\leq k\leq d}$ are smooth functions from $\mathbb{R}^N$ into the space of $N \times N$ matrices with real coefficients.

Definition 8.1.1 *The above system (S) is symmetric if and only if for any k in $\{1,\ldots,d\}$ the function A_k takes its value in the space of symmetric $N \times N$ matrices.*

An example of such a system is given by (1.1) page 6.

8.2 The resolution of quasilinear symmetric systems

The purpose of this section is to prove a local wellposedness for quasilinear symmetric systems

$$(S)\begin{cases} \partial_t U + \displaystyle\sum_{k=1}^{d} A_k(U)\partial_k U &= 0 \\ U_{|t=0} &= U_0 \end{cases}$$

The basic theorem is the following

Theorem 8.2.1 *If U_0 belongs to H^s with $s > d/2+1$, a positive time T exists such that a unique solution U of (S) exists in*

$$C([0,T];H^s) \cap C^1([0,T];H^{s-1})$$

Moreover T can be bounded from below by $c\|\nabla U_0\|_{H^{s-1}}$, where c depends only on the family $A = (A_k)_{1\leq k\leq d}$.

Finally, if $T^\star$ denotes the maximal time of existence of such a solution, then $T^\star$ does not depend on s and

$$T^\star < \infty \Longrightarrow \int_0^{T^\star} \|\nabla U(t,\cdot)\|_{L^\infty} dt.$$

For sake of technical simplicity, we shall assume that the functions are of the type

$$A_k(U) = A_k^{(0)} + \sum_{j=1}^{N} A_k^{\ell} U_j$$

8.2.1 Basic shemes of resolution for quasilinear symmetric systems

In order to prove this theorem, we shall use an iterative scheme. Let us consider the sequence $(U^{(n)})_{n\in\mathbb{N}}$ defined by $U^{(0)} = 0$ and

$$\begin{cases} \partial_t U^{(n+1)} + \displaystyle\sum_{k=1}^{d} A_k(U^{(n)})\partial_k U^{(n+1)} & = & 0 \\ U^{(n+1)}_{|t=0} & = & S_{n+1}U_0. \end{cases}$$

Theorem 6.2.1 ensures that this sequence is well defined and that $U^{(n)}$ belongs to $C^1(\mathbb{R}; H^s)$ for any s. The proof proceeds in three steps:

- first, we shall prove that for T small enough, the sequence $(U^{(n)})_{n\in\mathbb{N}}$ is bounded in the space $L^\infty([0,T]; H^s)$,
- then we shall prove that for T small enough, the sequence $(U^{(n)})_{n\in\mathbb{N}}$ is a Cauchy one in $L^\infty([0,T]; H^{s'})$ for any $s' < s$,
- finally we shall prove that the limit of this sequence is a solution of (S) and that it belongs to $C([0,T]; H^s)$.

The two main tools of the proof will be Littlewood-Paley theory and paradifferential computationus.

8.2.2 Paralinearization and energy estimates

The main lemma is the following

Lemma 8.2.1 *Let s be a positive real number. Let us consider U and V two functions of $L^\infty([0,T]; H^s)$ the gradient of which is in $L^\infty([0,T]\times\mathbb{R}^d)$. Let us assume that*

$$\partial_t V + \sum_{k=1}^{d} A_k(U)\partial_k V = F.$$

Stating $V_j \stackrel{\text{def}}{=} \Delta_j V$, we have

$$\partial_t V_j + \sum_{k=1}^{d} (S_{j-1}A_k(U))\partial_k V_j = \Delta_j F + R_j(U,V)$$

where $R_j(U,V)$ satisfies

$$\|R_j(U,V)(t)\|_{L^2} \leq C c_j(t) 2^{-js}\Big(\|\nabla U(t)\|_{L^\infty}\|V(t)\|_{H^s} + \|\nabla V(t)\|_{L^\infty}\|U(t)\|_{H^s}\Big) \qquad (8.1)$$

with, as along this section $\sum_j c_j^2(t) = 1$.

Proof. First of all, let us write that

$$\partial_t V_j + \Delta_j \sum_{k=1}^{d} A_k(U)\partial_k V = \Delta_j F.$$

The main point of the proof of the lemma is the commutation between a multiplication and the operator Δ_j. Let us point out that the constant part of $A_k(U)$ does not play any role here because it obviously commutes with Δ_j. Thus we have

$$\Delta_j \sum_{k=1}^{d} A_k(U)\partial_k V = A_k^{(0)}\Delta_j V + \sum_{\ell,k} A_k^\ell \Delta_j \left(U^\ell \partial_k V\right).$$

Let us use a simplified version of Bony's decomposition used in Section 7.4. Let us write

$$\begin{aligned} U^\ell \partial_k V &= T_{U^\ell}\partial_k V + T'_{\partial_k V} U^\ell \quad \text{with} \\ T_{U^\ell}\partial_k V &= \sum_{j'} S_{j'-1}U^\ell \Delta_{j'}\partial_k V \quad \text{and} \\ T'_{\partial_k V} U^\ell &= \sum_{j'} S_{j'+2}\partial_k V \Delta_{j'} U^\ell. \end{aligned}$$

As the support of the Fourier transform of $S_{j'-1}U^\ell \Delta_{j'}\partial_k V$ is included in a ring of the type $2^{q'}\widetilde{\mathcal{C}}$, we have

$$\begin{aligned} \Delta_j \sum_{j'} S_{j'-1}U^\ell \Delta_{j'}\partial_k V &= \Delta_j \sum_{|j'-j|\leq N_1} S_{j'-1}U^\ell \Delta_{j'}\partial_k V \\ &= R_j^1(U,V) + \sum_{|j'-j|\leq N_1} S_{j'-1}U^\ell \Delta_j \Delta_{j'}\partial_k V \\ &= R_j^1(U,V) + R_j^2(U,V) + S_{j-1}U^\ell \partial_k V_j \quad \text{with} \\ R_j^1(U,V) &\stackrel{\text{def}}{=} \sum_{|j'-j|\leq N_1} \left[\Delta_j, S_{j'-1}U^\ell\right]\Delta_{j'}\partial_k V \quad \text{and} \\ R_j^2(U,V) &\stackrel{\text{def}}{=} \sum_{|j'-j|\leq N_1} (S_{j'-1}U^\ell - S_{j-1}U^\ell)\Delta_j\Delta_{j'}\partial_k V \cdot \end{aligned}$$

The commutation between the operator Δ_j and the equation can be described by the following formula:

$$\begin{aligned} \partial_t V_j + \sum_{k=1}^{d} S_{j-1}A_k(U)\frac{\partial V_j}{\partial x_k} &= \Delta_j F + \sum_{m=1}^{3} R_q^m(U,V) \quad \text{with} \qquad (8.2) \\ R_j^1(U,V) &\stackrel{\text{def}}{=} \sum_{\substack{|j'-j|\leq N_1 \\ j,k}} A_k^\ell \left[\Delta_j, S_{j'-1}U^\ell\right]\Delta_{j'}\partial_k V, \\ R_j^2(U,V) &\stackrel{\text{def}}{=} \sum_{\substack{|j'-j|\leq N_1 \\ j,k}} A_k^\ell (S_{j'-1}U^\ell - S_{j-1}U^\ell)\Delta_j\Delta_{j'}\partial_k V \quad \text{and} \\ R_j^3(U,V) &\stackrel{\text{def}}{=} \Delta_j \sum_{\ell,k} A_k^\ell T'_{\partial_k V} U^\ell. \end{aligned}$$

The first term is estimated thanks to the following lemma.

Lemma 8.2.2 *A constant C exists such that, for any $p \in [1, +\infty]$, any lipschitz function a and any function b in L^p,*

$$\|[\Delta_j, a]b\|_{L^p} \leq C 2^{-j} \|\nabla a\|_{L^\infty} \|b\|_{L^p}.$$

Proof. Let us think Δ_j as a convolution and let us write

$$\begin{aligned}\Big([\Delta_j, a]b\Big)(x) &= \Delta_j(ab)(x) - a(x)\Delta_j b(x) \\ &= 2^{jd} \int_{\mathbb{R}^d} h(2^j(x-y))(a(y) - a(x))b(y)dy.\end{aligned}$$

As the function a is supposed to be lipschitz, we have

$$|a(y) - a(x)| \leq \|\nabla a\|_{L^\infty} |y - x|.$$

It turns out that

$$|\Big([\Delta_j, a]b\Big)(x)| \leq 2^{jd} \|\nabla a\|_{L^\infty} \int_{\mathbb{R}^d} |h(2^j(x-y))|\,|y-x|\,|b(y)|dy.$$

Then Young inequality implies that

$$\|[\Delta_j, a]b\|_{L^p} \leq 2^{-j} \|h(\cdot)|\cdot|\,\|_{L^1} \|\nabla a\|_{L^\infty} \|b\|_{L^p}.$$

This concludes the proof of the lemma. □

The lemma implies that

$$\|R^1_j(U,V)\|_{L^2} \leq C 2^{-j} \sum_{\substack{|j'-j|\leq N_1 \\ 1\leq k\leq d}} \|\nabla S_{j'-1} U\|_{L^\infty} \left\|\Delta_{j'} \partial_k V\right\|_{L^2}.$$

Lemma 7.1.1 page 75 and the fact that $\|S_j a\|_{L^\infty} \leq C\|a\|_{L^\infty}$ imply that

$$\|R^1_j(U,V)\|_{L^2} \leq C \sum_{\substack{|j'-j|\leq N_1 \\ 1\leq k\leq d}} 2^{-(j-j')'(s-1)} \|\nabla U\|_{L^\infty} 2^{j'(s-1)} \left\|\Delta_{j'} \partial_k V\right\|_{L^2}.$$

But as we have $|j' - j| \leq N_1$, we get by definition of the H^s norm

$$\|R^1_j(U,V)\|_{L^2} \leq C c_j 2^{-js} \|\nabla U\|_{L^\infty} \|\nabla V\|_{H^{s-1}}. \tag{8.3}$$

In order to estimate $R^2_j(U,V)$, let us use again Lemma 7.1.1 page 75 which tells us that

$$\|\Delta_p U\|_{L^\infty} \leq C 2^{-p} \|\nabla U\|_{L^\infty}.$$

This implies that

$$\|R^2_j(U,V)\|_{L^2} \leq C \sum_{\substack{|j'-j|\leq N_1 \\ p\in[j-1,j'-1]}} 2^{-p} c_{j'} 2^{-j'(s-1)} \|\nabla U\|_{L^\infty} \|\nabla V\|_{H^{s-1}}.$$

It turns out that

$$\|R^2_j(U,V)\|_{L^2} \leq C c_j 2^{-js} \|\nabla U\|_{L^\infty} \|\nabla V\|_{H^{s-1}}. \tag{8.4}$$

Theorems 7.4.1 page 85 and 7.4.3 page 86 implies that

$$\|T'_{\partial_k V} U\|_{H^s} \leq C \|\nabla V\|_{L^\infty} \|U\|_{H^s}.$$

By definition of H^s norm, we get that

$$\|R^3_j(U,V)\|_{L^2} \leq C c_j 2^{-js} \|\nabla V\|_{L^\infty} \|U\|_{H^s}. \tag{8.5}$$

Putting the three estimates (8.3)–(8.5) together implies the lemma. □

Let us apply this lemma to achieve the first step of the proof of Theorem 8.2.1. In order to do so, let us prove by induction that for a suitable constant C_0, we have

$$4C_0T\|U_0\|_{H^s} < 1 \Longrightarrow \forall n \in \mathbb{N}\,,\ \|U^{(n)}\|_{L^\infty([0,T];H^s)} \leq 2\|U_0\|_{H^s}. \tag{8.6}$$

The above assertion is of course true for $n = 0$. Let us assume it for some n. Lemma 8.2.1 allows us to write that

$$\partial_t U_j^{(n+1)} + \sum_{k=1}^{d}(S_{j-1}A_k(U^{(n)}))\partial_k U_j^{(n+1)} = R_j(U^{(n)}, U^{(n+1)}).$$

The L^2 energy estimate (6.2) implies that

$$\frac{1}{2}\frac{d}{dt}\|U_j^{(n+1)}(t)\|_{L^2}^2 \leq C\|\nabla U^{(n)}(t)\|_{L^\infty}\|U_j^{(n+1)}(t)\|_{L^2}^2 + C\|R_j(U^{(n)}, U^{(n+1)})\|_{L^2}\|U_j^{(n+1)}(t)\|_{L^2}.$$

Using Lemma 8.2.1, the fact that as $s - 1 > \dfrac{d}{2}$, the space H^{s-1} is continuously embedded in L^∞ and the induction hypothesis, we get, for any $t \leq T$,

$$\frac{d}{dt}\|U_j^{(n+1)}(t)\|_{L^2}^2 \leq C\|U_0\|_{H^s}\|U_j^{(n+1)}(t)\|_{L^2}\Big(\|U_j^{(n+1)}(t)\|_{L^2} + c_j(t)2^{-js}\|U_j^{(n+1)}(t)\|_{L^2}\Big).$$

By definition of the Sobolev norm, we get

$$\frac{d}{dt}\|U_j^{(n+1)}(t)\|_{L^2}^2 \leq C\|U_0\|_{H^s}c_j^2(t)2^{-2js}\|U^{(n+1)}(t)\|_{H^s}^2.$$

By time integration, we obtain that

$$\|U_j^{(n+1)}\|_{L^\infty([0,T];L^2)}^2 \leq \|\Delta_j U_0\|_{H^s}^2 + 2C\|U_0\|_{H^s}\|U^{(n+1)}\|_{L^\infty([0,T];H^s)}^2 2^{-2js}\int_0^T c_j^2(t)dt.$$

Let us remine that for any t, we have $\displaystyle\sum_j c_j^2(t) = 1$. Multiplying by 2^{2js} and taking the sum over j gives,

$$\begin{aligned}\sum_j 2^{2js}\|U_j^{(n+1)}\|_{L^\infty([0,T];L^2)}^2 &\leq \|U_0\|_{H^s}^2 + 2C\|U_0\|_{H^s}\|U^{(n+1)}\|_{L^\infty([0,T];H^s)}^2\sum_j\int_0^T c_j^2(t)dt\\ &\leq \|U_0\|_{H^s}^2 + 2C\|U_0\|_{H^s}T\|U^{(n+1)}\|_{L^\infty([0,T];H^s)}^2. \qquad (8.7)\end{aligned}$$

Let us observe that

$$\|V\|_{L^\infty([0,T];H^s)}^2 \leq \sum_j 2^{2js}\|\Delta_j V\|_{L^\infty([0,T];L^2)}^2$$

Then choosing $C_0 \geq 4C$ where C is the constant that appears in the above inequality we get that

$$\|U^{(n+1)}\|_{L^\infty([0,T];H^s)} \leq 2\|U_0\|_{H^s}. \tag{8.8}$$

This is the conclusion of the first step of the proof.

Remark Let us point out that we proved a little bit more than what we announced. In fact plugging (8.8) into (8.7) gives

$$\sum_j 2^{2js}\|U_j^{(n+1)}\|_{L^\infty([0,T];L^2)}^2 \leq 2\|U_0\|_{H^s}. \tag{8.9}$$

This will be important in the proof of the continuity of the solution with value in H^s.

The second step consists mainely in the proof of the fact that $(U^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty(([0,T];L^2)$. Let us assume this. Then using interpolation Inequality and remembering (8.8), we get for any s' in $]0,s[$,

$$\|U^{(n+p)}-U^{(n)}\|_{L^\infty([0,T];H^{s'})}\leq \|U^{(n+p)}-U^{(n)}\|_{L^\infty([0,T];L^2)}^{1-\frac{s'}{s}}(2\|U_0\|_{H^s})^{\frac{s'}{s}}.$$

By difference, we get

$$\partial_t(U^{(n+p)}(t)-U^{(n)}(t))+\sum_{k=1}^d A_k(U^{(n+p-1)})\partial_k(U^{(n+p)}(t)-U^{(n)}(t))$$
$$=-\sum_{k=1}^d\Big(A_k(U^{(n-1+p)})-A_k(U^{(n-1)}\Big)\partial_k U^{(n)}.$$

Then using the energy estimate (6.2), we get

$$\frac{d}{dt}\|U^{(n+p)}(t)-U^{(n)}(t)\|_{L^2}^2\leq C\big(\|\nabla U^{(n)}\|_{L^\infty}\|+\|\nabla U^{(n+p-1)}\|_{L^\infty}\big)\|U^{(n+p)}(t)-U^{(n)}(t)\|_{L^2}$$
$$\times\Big(\|U^{(n+p)}(t)-U^{(n)}(t)\|_{L^2}+\|U^{(n-1)}(t)-U^{(n+p-1)}(t)\|_{L^2}\Big).$$

using (8.8) , we infer that by time integration that

$$\frac{d}{dt}\|U^{(n+p)}(t)-U^{(n)}(t)\|_{L^2}^2\leq C\|U_0\|_{H^s}\|U^{(n+p)}(t)-U^{(n)}(t)\|_{L^2}$$
$$\times\Big(\|U^{(n+p)}(t)-U^{(n)}(t)\|_{L^2}+\|U^{(n-1)}(t)-U^{(n+p-1)}(t)\|_{L^2}\Big).$$

Stating $\rho_n(T)\stackrel{\text{def}}{=}\sup_p\|U^{(n+p)}(t)-U^{(n)}\|_{L^\infty([0,T];L^2)}$ we get by time integration that

$$\begin{aligned}\rho_n^2(T)&\leq 2^{-2ns}\|U_0\|_{H^s}^2+C\|U_0\|_{H^s}T\big(\rho_n^2(T)+\rho_n(T)\rho_{n-1}(T)\big)\\ &\leq 2^{-2ns}\|U_0\|_{H^s}^2+C\|U_0\|_{H^s}T\big(\rho_n^2(T)+\rho_{n-1}^2(T)\big)\end{aligned}$$

Choosing T small enough gives

$$\rho_n^2(T)\leq 2^{-2ns}\|U_0\|_{H^s}^2+\frac{1}{2}\rho_{n-1}^2(T).$$

By iteration of this inequality, we get

$$\rho_n^2(T)\leq 2^{-2ns}\|U_0\|_{H^s}^2\sum_{j=1}^n 2^{-j}.$$

This implies that $(U^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty([0,T];L^2)$. As this implies that it is a Cauchy sequence in $L^\infty(([0,T];H^{s'})$, using the fact that the product maps continuously $H^{s'}\times H^{s'-1}$ into $H^{s'-1}$ when s' is greater than $d/2$, we get that the limit U of $(U^{(n)})_{n\in\mathbb{N}}$ is a solution of (S).

To sum up, the whole existence part of Theorem 8.2.1 is proved except the fact that U is continuous in time with value in H^s. The last step consists in proving it. Unfortunately, the improvement given in (8.9) cannot be used directly. But an analogous inequality will give us the result.

Let us consider a solution U of (S) belonging to

$$L^\infty([0,T];H^s)\cap C([0,T];H^1)\cap C^1([0,T];L^2).$$

We shall prove that U is continuous in time with value in H^s. Let us apply Lemma 8.2.1. We get that $\Delta_j U$ satisfies

$$\begin{cases} \partial_t \Delta_j U + \displaystyle\sum_{k=1}^d (S_{j-1}A_k(U))\partial_k \Delta_j U &= R_j(U,U) \\ \Delta_j U_{|t=0} &= \Delta_j U_0 \end{cases}$$

with

$$\|R_j(U,U)(t)\|_{L^2} \leq Cc_j(t)2^{-js}\|\nabla U(t)\|_{L^\infty}\|U(t)\|_{H^s}.$$

By L^2 energy estimate we get that

$$\frac{d}{dt}\|\Delta_j U(t)\|_{L^2}^2 \leq Cc_j(t)2^{-2js}\|\nabla U(t)\|_{L^\infty}\|U(t)\|_{H^s}^2. \tag{8.10}$$

By integration, we get

$$\|\Delta_j U(t)\|_{L^2}^2 \leq \|\Delta_j U(0)\|_{L^2}^2 + C2^{-2js}\int_0^t c_j^2(t')\|\nabla U(t')\|_{L^\infty}\|U(t')\|_{H^s}^2 dt'. \tag{8.11}$$

Using the fact that U belongs to $L^\infty([0,T];H^s)$, we get that

$$\|\Delta_j U\|_{L^\infty([0,T];L^2)}^2 \leq \|\Delta_j U(0)\|_{L^2}^2 + C2^{-2js}\|U\|_{L^\infty([0,T];H^s)}^3\Big(\int_0^T c_j^2(t)dt\Big).$$

After multiplication by 2^{2js} and summation in j, it turns out that

$$\sum_j 2^{2js}\|\Delta_j U\|_{L^\infty([0,T];L^2)}^2 \leq \|U_0\|_{H^s}^2 + CT\|U\|_{L^\infty([0,T];H^s)}^3.$$

Now let us consider any positive ε. The above inequality implies that an integer N exists such that

$$\sum_{j\geq N} 2^{2js}\|\Delta_j U\|_{L^\infty([0,T];L^2)}^2 \leq \frac{\varepsilon^2}{4}\cdot$$

Thus we have

$$\begin{aligned} \|U(t)-U(t')\|_{H^s}^2 &\leq \sum_{j<N} 2^{2js}\|\Delta_j(U(t)-U(t'))\|_{L^2}^2 \\ &\qquad\qquad + 2\sum_{j\geq N} 2^{2js}\|\Delta_j U\|_{L^\infty([0,T];L^2)}^2 \\ &\leq \sum_{j<N} 2^{2js}\|\Delta_j(U(t)-U(t'))\|_{L^2}^2 + \frac{\varepsilon^2}{2} \\ &\leq C2^{Ns}\|U(t)-U(t')\|_{L^2}^2 + \frac{\varepsilon^2}{2}\cdot \end{aligned}$$

Thus $U \in C([0,T];H^s)$.

The uniqueness is an obvious consequence of the following proposition.

Proposition 8.2.1 *Let U and V be two solutions of (S) in $C([0,T];H^1)\cap C^1(([0,T];L^2)$ the gradient of which is continuous and bounded on $[0,T]\times\mathbb{R}^d$. Then we have*

$$\|U(t)-V(t)\|_{L^2}\leq \|U_0-V_0\|_{L^2}\exp\Big(C\int_0^t(\|\nabla U(t')\|_{L^\infty}+\|\nabla V(t')\|_{L^\infty})dt'\Big)$$

By difference, we get

$$\partial_t(U(t)-V(t))+\sum_{k=1}^d A_k(U)\partial_k(U-V)=\sum_{k=1}^d A_k(V-U)\partial_k V\cdot$$

Using (6.3) which is valid under the assumptions of the proposition, we get the result.

In order to prove the blow up condition, let us first observe that the maximal time of existence $T^\star$ satisfies

$$T^\star\geq c\|U_0\|_{H^s}.$$

Of course, the maximal time of existence of the solution of (S) with initial data $U(t)$ is $T^\star-t$; Thus we have

$$T^\star-t\geq c\|U(t)\|_{H^s},$$

which can be written

$$\|U(t)\|_{H^s}\geq\frac{C}{(T^\star-t)}\cdot \tag{8.12}$$

This implies that $\|U(t)\|_{H^s}$ does not remain bounded when t tends to $T^\star$. Now let us apply (8.11) and multiply this inequality by 2^{2js}. After summation, we get

$$\|U(t)\|_{H^s}^2\leq\|U(0)\|_{H^s}^2+C\int_0^t\|\nabla U(t')\|_{L^\infty}\|U(t')\|_{H^s}^2dt'. \tag{8.13}$$

Gronwall's Lemma implies that, for any $t<T^\star$,

$$\|U(t)\|_{H^s}^2\leq\|U(0)\|_{H^s}^2\exp\Big(C\int_0^t\|\nabla U(t')\|_{L^\infty}dt'\Big).$$

Thanks to Inequality (8.12), the whole Theorem 8.2.1 is proved.

Chapter 9

The Strichartz inequality for the Schrödinger operator

9.1 The Schrödinger equation and the Strichartz estimate

The Schrödinger equation is the following:

$$(LS) \quad \begin{cases} i\partial_t u + \Delta u = f \\ u_{|t=0} = u_0 \end{cases}$$

is the whole space $\mathbb{R}^d$. Let us first explain what a solution (in the sense of distributions) is.

Definition 9.1.1 *Let u be a continuous function from $\mathbb{R}$ with value in $\mathcal{S}'(\mathbb{R}^d)$, which means exactly that for any ϕ in $\mathcal{S}(\mathbb{R}^d)$, that maps defined by*

$$t \longmapsto \langle u(t), \phi \rangle$$

is continuous. Let f be a locally integrale map from $\mathbb{R}$ into $\mathcal{S}'(\mathbb{R}^d)$, which means exactly thatfor any ϕ in $\mathcal{S}(\mathbb{R}^d)$, that maps defined by

$$t \longmapsto \langle f(t), \phi \rangle$$

is locally integrable on $\mathbb{R}$. Let u_0 be in $\mathcal{S}'(\mathbb{R}^d)$. We say that u is a solution of (LS) if and only if, for anyfunction ϕ of $\mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$, we have, for any t in $\mathbb{R}$,

$$\langle u(t), \phi(t) \rangle - \langle u_0, \phi(0) \rangle = i \int_0^t \big\langle u(t'), i\Delta\phi(t') + \partial_t \phi(t') \big\rangle dt' - i \int_0^t \langle f(t'), \phi(t') \rangle dt'.$$

Proposition 9.1.1 *If u_0 belongs to $\mathcal{S}'(\mathbb{R}^d)$ and f is locally integrable form $\mathbb{R}$ into $\mathcal{S}'(\mathbb{R}^d)$, there is a unique solution of (LS) with a given by*

$$u(t) = \mathcal{F}^{-1}\bigg(e^{-it|\xi|^2} \widehat{u}_0(\xi) + \int_0^t e^{-i(t-t')|\xi|^2} \widehat{f}(t', \xi) dt' \bigg).$$

The purpose of this chapter is to prove the following theorem

Theorem 9.1.1 *Let u be a solution of (LS) with u_0 in $L^2(\mathbb{R}^d)$ and f in $L^1(\mathbb{R}; L^2(\mathbb{R}^d))$. Then, for any (p, q)in $[2, \infty]$ such that*

$$\frac{2}{p} + \frac{d}{r} = \frac{d}{2} \quad \textit{with} \quad (p, r) \neq (2, \infty). \tag{9.1}$$

a constant C exists such that

$$\|u\|_{L^p(\mathbb{R};L^q(\mathbb{R}^d))} \leq C\big(\|u\|_{L^2} + \|f\|_{L^1(\mathbb{R};L^2(\mathbb{R}^d))}\big).$$

The basic idea is that the solution of (LS) with $f = 0$ has some decay because it belongs (globally) to some L^p space in time. Even if it does not give the result immediately, this idea is illustrated by the following estimate.

Proposition 9.1.2 *If the denote by $e^{it\Delta}u_0$ the solution of (LS) with $f \equiv 0$, we have*

$$\|e^{it\Delta}u_0\|_{L^\infty} \leq \Big(\frac{1}{4\pi|t|}\Big)^{\frac{d}{2}} \|u_0\|_{L^1}$$

Proof. It is based on the fact that if z be a nonzero complex number with nonnegative real part, then

$$\mathcal{F}\Big(e^{-z|\cdot|^2}\Big)(\xi) = \Big(\frac{\pi}{z}\Big)^{\frac{d}{2}} e^{-\frac{|\xi|^2}{4z}} \tag{9.2}$$

with $z^{-\frac{d}{2}} \stackrel{\text{def}}{=} |z|^{-\frac{d}{2}} e^{-i\frac{d}{2}\theta}$ if $z = |z|e^{i\theta}$ with θ in $[-\pi/2, \pi/2]$.

In order to prove this formula, let us remark that, for any ξ in $\mathbb{R}^d$, the two functions

$$z \longmapsto \int_{\mathbb{R}^d} e^{-i\langle\xi,x\rangle} e^{-z|x|^2}\,dx \quad \text{and} \quad z \longmapsto \Big(\frac{\pi}{z}\Big)^{\frac{d}{2}} e^{-\frac{|\xi|^2}{4z}} \tag{9.3}$$

are holomorphic on the domain D of complex numbers with positive real part. As we have for any positive real number a, we have

$$\int_{\mathbb{R}^d} e^{-i\langle\xi,x\rangle} e^{-a|x|^2}\,dx = \Big(\frac{\pi}{a}\Big)^{\frac{d}{2}} e^{-\frac{|\xi|^2}{4a}}.$$

This claims that the two functions of (9.3) coincide on the intersection of the real line with D. Thus they also coincide on the whole domain D. Now let $(z_n)_{n\in\mathbb{N}}$ be a sequence of elements of D which converges to it for $t \neq 0$. For any function ϕ in $\mathcal{S}$, we have by virtue of Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} e^{-z_n|x|^2}\phi(x)\,dx = \int_{\mathbb{R}^d} e^{-it|x|^2}\phi(x)\,dx \quad \text{and}$$
$$\lim_{n\to\infty} \int_{\mathbb{R}^d} e^{-\frac{|\xi|^2}{4z_n}}\phi(\xi)\,d\xi = \int_{\mathbb{R}^d} e^{-\frac{|\xi|^2}{4it}}\phi(\xi)\,d\xi.$$

As we have

$$\mathcal{F}\left(e^{-z_n|\cdot|^2}\right) = \Big(\frac{\pi}{z_n}\Big)^{\frac{d}{2}} e^{-\frac{|\xi|^2}{4z_n}},$$

passing to the limit in $\mathcal{S}'(\mathbb{R}^d)$ when n tends to 0 gives (9.2).

Then using the Proposition 9.2 page 100, we infer from Proposition 9.1.1 that

$$e^{it\Delta}u_0 = \frac{1}{(4\pi it)^{\frac{d}{2}}} e^{i\frac{|\cdot|^2}{4t}} \star u_0.$$

The Young inequality implies the result. $\square$

9.2 The complex interpolation method in L^p space

We present here the theory of complex interpolation in the particular case of L^p space. It allows to extend inequalities between L^p space The basic theorem is the following.

Theorem 9.2.1 *Let us consider* $(X_k,\mu_k)_{1\leq k\leq 2}$ *two measured spaces and* $(p_j,q_j)_{j\in\{0,1\}}$ *two elements of* $[1,\infty]^2$. *Let us consider an operator* A *which maps continuously the space* $L^{p_j}(X_1)$ *into the space* $L^{q_j}(X_2)$ *for* j *in* $\{0,1\}$. *For any* θ *in* $[0,1]$, *if*

$$\left(\frac{1}{p_\theta},\frac{1}{q_\theta}\right)\stackrel{\text{def}}{=}(1-\theta)\left(\frac{1}{p_0},\frac{1}{q_0}\right)+\theta\left(\frac{1}{p_1},\frac{1}{q_1}\right),$$

then A *maps continuously* $L^{p_\theta}(X_1)$ *into* $L^{q_\theta}(X_2)$ *and*

$$\|A\|_{\mathcal{L}(L^{p_\theta}(X_1);L^{q_\theta}(X_2))}\leq\mathcal{A}_\theta\quad\textit{with}\quad\mathcal{A}_\theta\stackrel{\text{def}}{=}\|A\|^{1-\theta}_{\mathcal{L}(L^{p_0}(X_1);L^{q_0}(X_2))}\|A\|^{\theta}_{\mathcal{L}(L^{p_1}(X_1);L^{q_1}(X_2))}.$$

Proof. Let us consider (f,φ) in $L^{p_\theta}(X_1)\times L^{q'_\theta}(X_2)$ such that $\|f\|_{L^{p_\theta}(X_1)}\|=\|\varphi\|_{L^{q'_\theta}(X_2)}=1$. It is enough to prove that

$$\int_{X_2}(Af)(x_2)\varphi(x_2)d\mu_2(x_2)\leq\mathcal{A}_\theta. \tag{9.4}$$

Let us consider a complex number z in the strip S of complex numbers the real part of which is between 0 and 1. Let us define

$$\begin{aligned}f_z(x_1)&\stackrel{\text{def}}{=}\frac{f(x_1)}{|f(x_1)|}|f(x_1)|^{p_\theta\left(\frac{1-z}{p_0}+\frac{z}{p_1}\right)}\quad\text{and}\\ \varphi_z(x_2)&\stackrel{\text{def}}{=}\frac{\varphi(x_2)}{|\varphi(x_2)|}|\varphi(x_2)|^{q'_\theta\left(\frac{1-z}{q'_0}+\frac{z}{q'_1}\right)}\end{aligned} \tag{9.5}$$

Obviously, we have $f_\theta=f$ and $\varphi_\theta=\varphi$. Moreover, for any t in $\mathbb{R}$, we have, for j in $\{0,1\}$,

$$|f_{j+it}(x_1)|=|f(x_1)|^{\frac{p_\theta}{p_j}}\quad\text{and}\quad|\varphi_{j+it}(x_2)|=|\varphi(x_2)|^{\frac{q'_\theta}{q'_j}}. \tag{9.6}$$

It can be checked that the function defined by

$$F(z)\stackrel{\text{def}}{=}\int_{X_2}(Af_z)(x_2)\varphi_z(x_2)d\mu_2(x_2)$$

is holomorphic and bounded on S and continuous on the closure of S. Using Phragmen-Lindelhof principle, we infer that

$$F(\theta)\leq M_0^{1-\theta}M_1^{\theta}\quad\text{with}\quad M_j\stackrel{\text{def}}{=}\sup_{t\in\mathbb{R}}|F(j+it)|. \tag{9.7}$$

Using (9.6), we observe that that f_{j+it} belongs to L^{p_j} and $\|f_{j+it}\|_{L^{p_j}(L^{q_j})}=1$. We infer that

$$M_j\leq\|A\|_{\mathcal{L}(L^{p_j}(X_1);L^{q_j}(X_2))}.$$

and the lemma is proved. □

Let us give some immediate applications of this result.

Corollary 9.2.1 *For any* p *in* $[1,2]$, *the Fourier transform maps* $L^p(\mathbb{R}^d)$ *into* $L^{p'}(\mathbb{R}^d)$ *with norm less or equal to* $(2\pi)^{\frac{d}{p'}}$.

Corollary 9.2.2 *In* $\mathbb{R}^d$, *we have, for any* p *in* $[1,2]$,

$$\|e^{it\Delta}\|_{\mathcal{L}(L^p;L^{p'})}\leq\left(\frac{1}{4\pi|t|}\right)^{d\left(\frac{1}{p}-\frac{1}{2}\right)}.$$

9.3 The duality method and the $TT^\star$ argument

This section describes the so-called $TT^\star$ argument which is the standard method for converting the dispersive estimates (presented in the previous section) into inequalities involving suitable space-time Lebesgue norms of the solution.

In all this section, we denote by $\|\cdot\|_{L^p(L^q)}$ the norm in $L^q(\mathbb{R};L^r(\mathbb{R}^d))$. Let us now state the "abstract" Strichartz estimates.

Theorem 9.3.1 *Let $(U(t))_{t\in\mathbb{R}}$ be a bounded family of continuous operators on $L^2(\mathbb{R}^d)$ such that, for some positive real numbers σ and C_0, we have*

$$\|U(t)U^\star(t')f\|_{L^\infty} \leq \frac{C_0}{|t-t'|^\sigma}\|f\|_{L^1}. \tag{9.8}$$

Then, for any (p,q) in $[2,\infty]^2$ such that

$$\frac{2}{p}+\frac{2\sigma}{r}=\sigma \quad \text{and} \quad (p,r,\sigma)\neq(2,\infty,1)\,, \tag{9.9}$$

we have for some positive constant C

$$\|U(t)u_0\|_{L^p(L^r)} \leq C\|u_0\|_{L^2}.$$

Proof. It is based on a duality argument together with the Hardy-Littlewood-Sobolev inequality stated in Theorem 9.5.2. Let us first notice that

$$\begin{aligned}\|U(t)u_0\|_{L^p(L^q)} &= \sup_{\varphi\in\mathcal{B}_{p,r}}\left|\int_{\mathbb{R}\times\mathbb{R}^d} U(t)u_0(x)\varphi(t,x)\,dt\,dx\right|\\ &= \sup_{\varphi\in\mathcal{B}_{p,r}}\left|\int_{\mathbb{R}}(U(t)u_0|\varphi(t))_{L^2}\,dt\right|\end{aligned}$$

where

$$\mathcal{B}_{p,r} \stackrel{\text{def}}{=} \big\{\phi\in\mathcal{D}(\mathbb{R}^{1+d};\mathbb{C})\,/\,\|\phi\|_{L^{p'}(L^{r'})}\leq 1\big\}.$$

By definition of the adjoint operator, we have

$$\|U(t)u_0\|_{L^p(L^q)} = \sup_{\varphi\in\mathcal{B}_{p,r}}\left|\Big(u_0\Big|\int_{\mathbb{R}}U^\star(t)\varphi(t)\,dt\Big)_{L^2}\right|.$$

By virtue of the Cauchy-Schwarz inequality, we deduce that

$$\|U(t)u_0\|_{L^p(L^q)} \leq \|u_0\|_{L^2}\sup_{\varphi\in\mathcal{B}_{p,r}}\Big\|\int_{\mathbb{R}}U^\star(t)\varphi(t)\,dt\Big\|_{L^2}. \tag{9.10}$$

Let us write that

$$\begin{aligned}\Big\|\int_{\mathbb{R}}U^\star(t)\varphi(t)\,dt\Big\|_{L^2}^2 &= \int_{\mathbb{R}^2}\big(U^\star(t')\varphi(t')\big|U^\star(t)\varphi(t)\big)_{L^2}\,dt'\,dt\\ &= \int_{\mathbb{R}^2}\big(U(t)U^\star(t')\varphi(t')\big|\varphi(t)\big)_{L^2}\,dt'\,dt\\ &= \int_{\mathbb{R}^2}\big\langle U(t)U^\star(t')\varphi(t'),\overline{\varphi}(t)\big\rangle\,dt'\,dt.\end{aligned} \tag{9.11}$$

Now, let us observe that, using Theorem 9.2.1, we infer that

$$\forall r \in [2,\infty]\,,\ \|U(t)U^\star(t')f\|_{L^r} \leq \frac{C}{|t-t'|^{\sigma\left(1-\frac{2}{r}\right)}}\|f\|_{L^{r'}}. \tag{9.12}$$

Thanks to (9.9) we infer that

$$\Big\|\int_{\mathbb{R}} U^\star(t)\varphi(t)\,dt\Big\|_{L^2}^2 \leq C\int_{\mathbb{R}^2}\frac{1}{|t-t'|^{\frac{2}{p}}}\|\varphi(t')\|_{L^{p'}}\|\varphi(t)\|_{L^{p'}}\,dt'\,dt.$$

Because $p>2$, the Hardy-Littlewood-Sobolev inequality page 105 gives

$$\Big\|\int_{\mathbb{R}} U^\star(t)\varphi(t)\,dt\Big\|_{L^2}^2 \leq C\|\varphi\|_{L^{p'}(L^{q'})}^2.$$

Thanks to (9.10), the theorem is proved. □

Proof of theorem 9.1.1 It is a clear consequence of Theorem 9.3.1 that

$$\|e^{it\Delta}u_0\|_{L^p(L^r)} \leq C\|u_0\|_{L^2}. \tag{9.13}$$

Now, we can to prove the theorem in the case when $u_0=0$. In this case, let us write the solution as

$$u(t)=\int_0^t e^{i(t-t')\Delta}f(t')dt'.$$

We have, for any t,

$$\begin{aligned}\|u(t)\|_{L^r(\mathbb{R}^d)} &\leq C\int_0^t \|e^{i(t-t')\Delta}f(t')\|_{L^r(\mathbb{R}^d)}dt'\\ &\leq C\int_{\mathbb{R}}\|e^{i(t-t')\Delta}f_+(t')\|_{L^r(\mathbb{R}^d)}dt'\end{aligned}$$

with $f_+(t)\stackrel{\text{def}}{=}\mathbf{1}_{\mathbb{R}^+}(t)f(t)$. Taking the L^p norm in time in the above inequality, using (9.13) and the translation invariance of the Lebesgue measure on $\mathbb{R}$ gives

$$\begin{aligned}\|u\|_{L^p(L^r(\mathbb{R}^d))} &\leq C\int_{\mathbb{R}}\|f_+(t')\|_{L^2(\mathbb{R}^d)}dt'\\ &\leq C\|f\|_{L^1(\mathbb{R};L^2(\mathbb{R}^d))}.\end{aligned}$$

This concluded the proof of Theorem 9.1.1. □

9.4 An example of application

As an application of the results of the previous section, we here solve the initial boundary value problem for the *cubic semilinear Schrödinger equation* in $\mathbb{R}^2$:

$$(NLS_3)\ \begin{cases} i\partial_t u-\dfrac{1}{2}\Delta u = P_3(u,\overline{u})\\ u_{|t=0}=u_0\end{cases}$$

where P_3 is some given homogeneous polynomial of degree 3.

Theorem 9.4.1 *There exists a constant c such that, for any initial data u_0 in $L^2(\mathbb{R}^2)$ satisfying $\|u_0\|_{L^2} \leq c$, Equation (NLS_3) has a unique solution u in the space $L^3(\mathbb{R}; L^6(\mathbb{R}^2))$ which in addition belongs to $L^\infty(\mathbb{R}; L^2(\mathbb{R}^2))$.*

Remark Let us first have a look on the scaling properties of Equation (NLS_3). If u is a solution of (NLS_3), then $u_\lambda(t,x) \stackrel{\text{def}}{=} \lambda u(\lambda^2 t, \lambda x)$ is also a solution of the same equation. In the scale of Sobolev spaces, $L^2(\mathbb{R}^2)$ is the only invariant space.

Proof of Theorem 9.4.1. Let Q be the nonlinear functional defined by

$$\begin{cases} i\partial_t Q(u) - \dfrac{1}{2}\Delta Q(u) = P_3(u, \overline{u}) \\ Q(u)_{|t=0} = 0. \end{cases}$$

The functional Q maps continuously the space $L^3(\mathbb{R}; L^6(\mathbb{R}^2))$ into the space $L^\infty(\mathbb{R}; L^2(\mathbb{R}^2)) \cap L^3(\mathbb{R}; L^6(\mathbb{R}^2))$. Indeed Theorem 9.1.1 leads to

$$\begin{aligned} \|Q(u)\|_{L^3(\mathbb{R};L^6(\mathbb{R}^2))} &\leq C\|P_3(u,\overline{u})\|_{L^1(\mathbb{R};L^2(\mathbb{R}^2))} \\ &\leq C\|u\|^3_{L^3(\mathbb{R};L^6(\mathbb{R}^2))}. \end{aligned}$$

As $Q(u) - Q(v)$ satisfies

$$\Big(i\partial_t + \frac{1}{2}\Delta\Big)(Q(u) - Q(v)) = P_3(u,\overline{u}) - P_3(v,\overline{v}),$$

we get, using Theorem 9.1.1 again,

$$\begin{aligned} \|Q(u) - Q(v)\|_{L^\infty(\mathbb{R};L^2(\mathbb{R}^2))\cap L^3(\mathbb{R};L^6(\mathbb{R}^2))} &\leq C\|u - v\|_{L^3(\mathbb{R};L^6(\mathbb{R}^2))} \\ &\quad \times \Big(\|u\|^2_{L^3(\mathbb{R};L^6(\mathbb{R}^2))} + \|v\|^2_{L^3(\mathbb{R};L^6(\mathbb{R}^2))}\Big). \end{aligned} \tag{9.14}$$

Now, it is obvious that u is a solution of (NLS_3) if and only if u is a fixed point of the map

$$F(u) \stackrel{\text{def}}{=} U(t)u_0 + Q(u).$$

Applying Theorem 9.1.1 and Estimate (9.14) with $v = 0$, we get that

$$\|F(u)\|_{L^3(\mathbb{R};L^6)} \lesssim \|u_0\|_{L^2} + \|u\|^3_{L^3(\mathbb{R};L^6)}.$$

Thus, if $8C^2\|u_0\|^2_{L^2} \leq 1$, then the ball $B(0, 2C\|u_0\|_{L^2})$ of center 0 and radius $2C\|u_0\|_{L^2}$ of the Banach space $L^3(\mathbb{R}; L^6(\mathbb{R}^2))$ is invariant by the map F. Using again Inequality (9.14), we get, for any u and v in $B(0, 2C\|u_0\|_{L^2})$,

$$\|F(u) - F(v)\|_{L^3(\mathbb{R};L^6)} \leq 8C^3\|u_0\|^2_{L^2}\|u - v\|_{L^3(\mathbb{R};L^6)}.$$

Thus, if in addition

$$8C^3\|u_0\|^2_{L^2} \leq \frac{1}{2},$$

then Picard's fixed point theorem implies that a unique solution u exists in some neighborhood of 0 in $L^3(\mathbb{R}; L^6)$. Clearly, Inequality (9.14) implies that uniqueness holds true in $L^3(\mathbb{R}; L^6)$ without any smallness condition.

Finally, the energy estimate entails that this solution belongs to $L^\infty(\mathbb{R}; L^2)$. Indeed, multiplying Equation (NLS_3) by $\overline{u}$, integrating over $\mathbb{R}^2$ then taking the real part, we discover that

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 = \Im m \int \overline{u} P_3(u, \overline{u})\, dx,$$

whence, for all t in $\mathbb{R}$,

$$\|u(t)\|_{L^2} \leq \|u_0\|_{L^2} + C\left|\int_0^t \|u(t')\|_{L^6}^3\, dt'\right|.$$

This completes the proof of the theorem. □

9.5 Refined convolution inequalities

The purpose of this section is to improve the classical Young inequalities which clains that, if

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r} \tag{9.15}$$

then $\|f \star g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}$. In order to state these refined inequalities, let us introduce the so caled weak L^p spaces.

Definition 9.5.1 *For p in $[1, \infty[$, we denote by $L^p_w(X, \mu))$ space the set of measure functions such that*

$$\|f\|^p_{L^p_w(X,\mu)} \stackrel{\text{def}}{=} \sup_{\lambda > 0} \lambda^p \mu(|f| > \lambda) < \infty,$$

Let us notice that we have

$$\mu(|f| > \lambda) \leq \int_{(|f\|\lambda)} \Big(\frac{|f(x)|}{\lambda}\Big)^p d\mu(x) \leq \frac{1}{\lambda^p}\|f\|^p_{L^p}.$$

Thus L^p is a subset of L^p_w. Typical examples of weak L^p functions. The function $|\cdot|^{-\alpha}$ belongs for $L_w^{\frac{d}{\alpha}}(\mathbb{R}^d, dx)$.

Theorem 9.5.1 *Let (p, q, r) be in $]1, \infty[^3$ and satisfy (9.15). A constant C exists such that*

$$\|f \star g\|_{L^r} \leq C\|f\|_{L^p}\|g\|_{L^q_w}$$

Let us notice that the above theorem implies the well-known *Hardy-Littlewood-Sobolev* inequalities on $\mathbb{R}^d$.

Theorem 9.5.2 (Hardy-Littlewood-Sobolev inequality) *Let α be in $]0, d[$ and (p, r) in $]1, \infty[^2$ satisfy*

$$\frac{1}{p} + \frac{\alpha}{d} = 1 + \frac{1}{r}\cdot \tag{9.16}$$

Then a constant C exists such that

$$\|\,|\cdot|^{-\alpha} \star f\|_{L^r(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)}.$$

The proof of Theorem 9.5.1 relies on the atomic decomposition which is decribe in the following proposition.

Proposition 9.5.1 *Let (X,μ) be a measure space and p be in $[1,\infty[$. Let f be a nonnegative function in L^p. Then a sequence of positive real numbers $(c_k)_{k\in\mathbb{Z}}$ and a sequence of nonnegative functions $(f_k)_{k\in\mathbb{Z}}$ (the atoms)Atom exist such that*

$$f = \sum_{k\in\mathbb{Z}} c_k f_k$$

where the support of the functions f_k are pairwise disjoint and

$$\mu(\text{Supp } f_k) \leq 2^{k+1}, \tag{9.17}$$

$$\|f_k\|_{L^\infty} \leq 2^{-\frac{k}{p}}, \tag{9.18}$$

$$\frac{1}{2}\|f\|_{L^p}^p \leq \sum_{k\in\mathbb{Z}} c_k^p \leq 2\|f\|_{L^p}^p. \tag{9.19}$$

Remark As inferred by the definition given below, the sequence $(c_k f_k)_{k\in\mathbb{Z}}$ is independent of p and depends only on f.

Proof of Proposition 9.5.1. If $\mu(f>0)$ is not finite, let us define, for any k in $\mathbb{Z}$,

$$\lambda_k \stackrel{\text{def}}{=} \inf\left\{\lambda \;/\mu(f>\lambda) < 2^k\right\}.$$

If $\mu(f>0)$ is finite, let us define by k_0 the smallest integer such that

$$\mu(f>0) \leq 2^{k_0}$$

Then, for $k \leq k_0$, let us define

$$\lambda_k \stackrel{\text{def}}{=} \inf\left\{\lambda \;/\mu(f>\lambda) < 2^k\right\}.$$

In all that follows in this proof, we shall implicitely consider that, if $\mu(f>0)$ is finite, then, all the sequence define are 0 for $k \geq k_0$. Let us notice that $(\lambda_k)_{k\in\mathbb{Z}}$ is a decreasing sequence. The monotonic convergence theorem and the definition of $(\lambda_k)_{k\in\mathbb{Z}}$, we have that

$$\mu(f>\lambda_k) \leq 2^k \quad \text{and} \quad \forall \lambda < \lambda_k\,,\ \mu(f>\lambda) \geq 2^k \tag{9.20}$$

Then, let us define

$$c_k \stackrel{\text{def}}{=} 2^{\frac{k}{p}}\lambda_k \quad \text{and} \quad f_k \stackrel{\text{def}}{=} c_k^{-1}\mathbf{1}_{(\lambda_{k+1}<f\leq\lambda_k)}f.$$

It is obvious that $\|f_k\|_{L^\infty} \leq 2^{-\frac{k}{p}}$. The point is now the proof of (9.19). As the support of the functions $(f_k)_{k\in\mathbb{Z}}$ are pairwise disjoint, one may write

$$\|f\|_{L^p}^p = \sum_{k\in\mathbb{Z}} c_k^p \|f_k\|_{L^p}^p.$$

Taking advantage of Inequalities (9.17) and (9.18), we claim that

$$\|f_k\|_{L^p}^p \leq 2 \quad \text{for all} \;\; k\in\mathbb{Z}\,.$$

which claims exactly that $\displaystyle\sum_{k\in\mathbb{Z}} c_k^p \leq 2\|f\|_{L^p}^p$.

Moreoverwhich, owing to the fact that f is a nonnegative function of L^p, converges to 0 when k tends to $+\infty$.

Thanks to (9.20), we have $\mu(\text{Supp } f_k) \leq 2^{k+1}$. This gives

$$\begin{aligned}\sum_{k\in\mathbb{Z}} c_k^p &= \sum_{k\in\mathbb{Z}} 2^k \lambda_k^p \\ &= p\sum_{k\in\mathbb{Z}} \int_0^\infty 2^k \mathbf{1}_{]0,\lambda_k[}(\lambda)\lambda^{p-1}\,d\lambda.\end{aligned}$$

Using Fubini's theorem, we get

$$\sum_{k\in\mathbb{Z}} c_k^p = p\int_0^\infty \lambda^{p-1}\Big(\sum_{k\,/\,\lambda_k>\lambda} 2^k\Big)\,d\lambda.$$

By definition of sequence $(\lambda_k)_{k\in\mathbb{Z}}$, having $\lambda < \lambda_k$ implies that $\mu(f > \lambda) \geq 2^k$. Thus we infer that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} c_k^p &\leq p\int_0^\infty \lambda^{p-1}\Big(\sum_{k\,/\,2^k\leq\mu(f>\lambda)} 2^k\Big)d\lambda \\ &\leq 2p\int_0^\infty \lambda^{p-1}\mu(f>\lambda)\,d\lambda.\end{aligned}$$

Now, the right inequality in (9.19) follows from the fact that, owing to Fubini's theorem, we have

$$\|f\|_{L^p}^p = p\int_0^\infty \lambda^{p-1}\mu(|f|>\lambda)\,d\lambda \tag{9.21}$$

which implies the desired inequality. □

Proof of Theorem 9.5.1 Let f and g be two nonnegative measurable functions on $\mathbb{R}^d$. Let us consider a nonnegative function h in $L^{r'}$ and let us define

$$I(f,g,h) \stackrel{\text{def}}{=} \int_{\mathbb{R}^{2d}} f(y)g(x-y)h(x)\,d\mu(x)\,d\mu(y).$$

Arguing by homogeneity, one can assume that $\|f\|_{L^p} = \|g\|_{L^q_w} = \|h\|_{L^{r'}} = 1$. Defining

$$C_j \stackrel{\text{def}}{=} \{y\in\mathbb{R}^d\,,\ 2^j \leq g(y) < 2^{j+1}\},$$

we can write

$$\begin{aligned}I(f,g,h) &\leq 2\sum_{j\in\mathbb{Z}} 2^j I_j(f,h) \quad \text{with} \\ I_j(f,h) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^{2d}} f(y)h(x)\mathbf{1}_{C_j}(x-y)\,d\mu(x)\,d\mu(y).\end{aligned}$$

Because $\|g\|_{L^q_w} = 1$, we have $\|\mathbf{1}_{C_j}\|_{L^s} \leq 2^{-j\frac{q}{s}}$ for all s in $[1,\infty]$. Thus if we directly apply Young's inequality with p, q and r, we find that $I_j(f,h) \leq 2^{-j}$ so that series $\sum 2^{j+1} I_j(f,h)$ has no reason to converge. In order to bypass this difficulty, one may introduce the atomic decomposition for f and h given by Proposition 9.5.1. Then we write

$$I_j(f,h) = \sum_{k,\ell} c_k d_\ell I_j(f_k,h_\ell).$$

Using Young's inequalities, we get for any (a,b) in $[1,\infty]^2$ such that $b\leq a'$ and for any $(\widetilde f,\widetilde h)$ in $L^a\times L^b$,

$$I_j(\widetilde f,\widetilde h)\leq \|\widetilde f\|_{L^a}\|\mathbf{1}_{C_j}\|_{L^b}\|\widetilde h\|_{L^{c'}}\quad\text{with}\quad \frac{1}{a}+\frac{1}{b}=1+\frac{1}{c}\,\cdot$$

This gives

$$I_j(\widetilde f,\widetilde h)\leq 2^{-\frac{jq}{b}}\|\widetilde f\|_{L^a}\|\widetilde h\|_{L^{c'}}.$$

Now applying this for f_k and h_ℓ and using Proposition 9.5.1 yields

$$2^jI_j(f_k,h_\ell)\leq 2^{jq\left(\frac{1}{q}-\frac{1}{b}\right)}2^{k\left(\frac{1}{a}-\frac{1}{p}\right)}2^{\ell\left(\frac{1}{c'}-\frac{1}{r'}\right)}=2^{jq\left(\frac{1}{q}-\frac{1}{b}\right)}2^{k\left(\frac{1}{a}-\frac{1}{p}\right)}2^{\ell\left(\frac{1}{r}-\frac{1}{c}\right)}.$$

Using the condition (9.15) on (p,q,r) and (a,b,c) implies

$$\begin{aligned}2^jI_j(f_k,h_\ell)&\leq 2^{jq\left(\frac{1}{q}-\frac{1}{b}\right)}2^{k\left(\frac{1}{a}-\frac{1}{p}\right)}2^{\ell\left(\frac{1}{p}+\frac{1}{q}-\frac{1}{a}-\frac{1}{b}\right)}\\&\leq 2^{(jq-\ell)\left(\frac{1}{b}-\frac{1}{q}\right)}2^{(k-\ell)\left(\frac{1}{a}-\frac{1}{p}\right)}.\end{aligned}\tag{9.22}$$

As (p,q,r) is in $]1,\infty[^2$, a positive real number ε exists, if

$$\frac{1}{a}\stackrel{\text{def}}{=}\frac{1}{p}-\varepsilon\,\mathrm{sg}(k-\ell)\quad\text{and}\quad\frac{1}{b}\stackrel{\text{def}}{=}\frac{1}{q}-\varepsilon\,\mathrm{sg}(jq+\ell)$$

then (a,b) is in $[1,\infty]^2$ and $\dfrac{1}{a}+\dfrac{1}{b}\geq 1$. With this choice of a and b, (9.22) becomes

$$2^jI_j(f_k,h_\ell)\leq 2^{-\varepsilon|jq-\ell|-\varepsilon|k-\ell|}.$$

Now, using Young's inequality for $\mathbb{Z}$ equipped with the counting measure, we deduce that

$$\begin{aligned}I(f,g,h)&\leq C\sum_{j,k,\ell}c_kd_\ell 2^{-\varepsilon|jq+k|-\varepsilon|jq+\ell|-\varepsilon|k-\ell|}\\&\leq \frac{C}{\varepsilon}\sum_{k,\ell}c_kd_\ell 2^{-\varepsilon|k-\ell|}\\&\leq \frac{C}{\varepsilon^2}\|(c_k)\|_{\ell^p}\|\|(d_\ell)\|_{\ell^{p'}}.\end{aligned}$$

Condition (9.15) implies that $r'\leq p'$ and thus

$$I(f,g,h)\leq\frac{C}{\varepsilon^2}\|(c_k)\|_{\ell^p}\|\|(d_\ell)\|_{\ell^{r'}}.$$

The theorem is proved. □

Chapter 10

The wave equation

The purpose of ths chapter is to study the wave equation and in aprticlar its dispersive properties. As an mapplication of this properties, we shall prove a existence theorem ofor some semi linear wave equation where the doispersive effect allows to go below the regularity given by Sobolev embeddings.

10.1 Some basic properties of the linear wave equation

The wave equation is a simplified model for the propagation of waves in a physical medium. In the present subsection, we shall only consider the case of an isotropic medium so that the corresponding system reduces (after suitable normalization) to

$$(W) \qquad \begin{cases} \Box u = 0 \\ (u, \partial_t u)_{|t=0} = (u_0, u_1). \end{cases}$$

Above, $\Box$ denotes the wave operator $\partial_t^2 - \Delta$. The unknown function $u = u(t,x)$ is real valued and depends only on (t,x) in $\mathbb{R} \times \mathbb{R}^d$.

In the one-dimensional case $d = 1$, it may be easily shown that the solution to (W) is given (in the smooth case) by *d'Alembert's formula*:

$$u(t,x) = \frac{1}{2}\Big(u_0(x+t) + u_0(x-t) + \int_{x-t}^{x+t} u_1(y)\,dy\Big)$$

so that one cannot expect the wave operator to have any (global) dispersive property or smoothing effect.

From now on, we shall assume that the dimension d is greater than or equal to 2. In this case the situation turns out to be very different. Easy computations in Fourier variables shows that:

Proposition 10.1.1 *If u_0 and u_1 are tempered distribution, then the unique solution of the linear wave equation (W) in $\mathcal{C}(\mathbb{R};\mathcal{S}')$ reads*

$$u(t) = U^+(t)\gamma_+ + U^-(t)\gamma_- \quad \text{with}$$
$$\mathcal{F}\big(U^\pm(t)f\big)(\xi) \stackrel{\text{def}}{=} e^{\pm it|\xi|}\widehat{f}(\xi) \quad \text{and} \quad \widehat{\gamma}_\pm(\xi) \stackrel{\text{def}}{=} \frac{1}{2}\Big(\widehat{u}_0(\xi) \pm \frac{1}{i|\xi|}\widehat{u}_1(\xi)\Big).$$

In the rest of this chapter, we adopt the notations

$$\partial \stackrel{\text{def}}{=} (\partial_{x_1}, \cdots, \partial_{x_d}), \quad \nabla \stackrel{\text{def}}{=} (\partial_t, \partial_{x_1}, \cdots, \partial_{x_d}) \text{ and } \partial_0 \stackrel{\text{def}}{=} \partial_t,$$

Another key point about the wave equation is the following.

Proposition 10.1.2 *Let us consider a solution u of the wave equation*

$$(W) \qquad \begin{cases} \Box u = f \\ (u, \partial_t u)_{|t=0} = (u_0, u_1). \end{cases}$$

in the space $C^2(\mathbb{R}; L^2)\, \mathbb{C}^1(\mathbb{R}; H^1)$. Then, if $f \equiv 0$, we have

$$\|\nabla u(t)\|_{L^2}^2 = \|u_1\|_{L^2}^2 + \|\partial u_0\|_{L^2}^2.$$

For any f, we have

$$\|\nabla u(t)\|_{L^2}^2 \leq \|u_1\|_{L^2}^2 + \|\partial u_0\|_{L^2}^2 + 2\Big(\int_{[0,t]} \|f(t')\|_{L^2} dt'\Big)^2.$$

Proof. Taking the L^2 scalar product of (W) with $\partial_t u$ gives

$$\frac{1}{2}\frac{d}{dt}\|\partial_t u(t)\|_{L^2}^2 - \int_{\mathbb{R}^d} \Delta u(t,x)\partial_t u(t,x)dx = \int_{\mathbb{R}^d} f(t,x)\partial_t u(t,x)dx.$$

An integration by parts gives

$$\begin{aligned} \int_{\mathbb{R}^d} \Delta u(t,x)\partial_t u(t,x)dx &= -\sum_{j=1}^d \int_{\mathbb{R}^d} \partial_j u(t,x)\partial_t\partial_j u(t,x)dx \\ &= -\frac{d}{dt}\sum_{j=1}^d \int_{\mathbb{R}^d} (\partial_j u(t,x))^2 dx. \end{aligned}$$

Thus we infer that

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 = \int_{\mathbb{R}^d} f(t,x)\partial_t u(t,x)dx.$$

If $f \equiv 0$, the result is proved. For any external force f, if $M(t) \stackrel{\text{def}}{=} \sup_{t' \in [0,t]} \|\nabla u(t')\|_{L^2}$, we get by integration that,

$$\frac{1}{2}M^2(t) \leq \|\nabla u(0)\|_{L^2}^2 + M(t)\int_{[0,t]} \|f(t')\|_{L^2} dt'.$$

This implies that

$$\frac{1}{2}\Big(M(t) - \int_{[0,t]} \|f(t')\|_{L^2} dt)\Big)^2 \leq \frac{1}{2}\|\nabla u(0)\|_{L^2}^2 + \frac{1}{2}\Big(\int_{[0,t]} \|f(t')\|_{L^2} dt)\Big)^2$$

which gives the result. □

10.2 The dispersive estimate for the wave equation

The purpose of this section is the proof of the following theorem.

Theorem 10.2.1 *Assume that $d \geq 2$. Let $\mathcal{C} \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^d \;\; r \leq |\xi| \leq R\}$ for some positive r and R such that $r < R$. Then a constant C exists such that, if the Fourier transform of u_0 and u_1 are supported in the annulus $\mathcal{C}$, then u the associate solution to the wave equation (W) satisfies,*

$$\|u(t)\|_{L^\infty} \leq \frac{C}{|t|^{\frac{d-1}{2}}} \left(\|u_0\|_{L^1} + \|u_1\|_{L^1}\right) \quad \text{for all} \quad t \neq 0.$$

Remark As the support of the Fourier transform is preserved by the flow of the constant coefficients wave equation (a property which is no longer true in the case of variable coefficients) the Fourier transform of the solution u is at each time t supported in the annulus $\mathcal{C}$.

The proof of this theorem relies of estimates on oscillatory integrals which have their own interest. The purpose is the proof of estimatrs of integrals of the form

$$I_\psi(\tau) = \int_{\mathbb{R}^d} e^{i\tau\Phi(\xi)}\psi(\xi)\, d\xi$$

where τ must be understood as a large parameter.

Notation. In all this section, ψ will denote a function of $\mathcal{D}(\mathbb{R}^d)$ and Φ a *real valued* smooth function on a neighborhood of the support of ψ. Moreover, the constants which will appear will be generically denoted by C and depend on a finite number of derivatives of ψ and on a finite number of derivatives of order greater than or equal to 2 of the phase function Φ.

We shall distinguish the case when $\nabla\Phi$ does not vanish (the *nonstationary phase* case) from the case when it may vanish (the *stationary phase* case). The non stationnary phase case will produce fast decay with respect to the parameter τ.

Theorem 10.2.2 *Let us consider a compact set K of $\mathbb{R}^d$ and assume that a constant c_0 exists such that*

$$\forall \xi \in K\,, \;\; |\nabla\Phi(\xi)| \geq c_0.$$

Then, for any integer N, for any function ψ in the set $\mathcal{D}_K$ smooth functions supported in K, a constant C exists such that

$$\left|I_\psi(\tau)\right| \leq \frac{C_N}{(c_0\tau)^N} \cdot$$

Proof. Note that changing Φ in Φ/c_0 and τ in $c_0\tau$ reduces the proof to the case $c_0 = 1$. So let us assume that $c_0 = 1$. Then it is only a matter of using the oscillations to produce decay. This will be achieved by doing suitable integrations by parts. Indeed, let us consider the following first order differential operator, defined for any function a in $\mathcal{D}_K$, by

$$\mathcal{L}a \stackrel{\text{def}}{=} -i \sum_{j=1}^{d} \frac{\partial_j\Phi}{|\nabla\Phi|^2} \partial_j a.$$

This operator obviously satisfies

$$\mathcal{L}e^{i\tau\Phi} = \tau e^{i\tau\Phi},$$

hence, by repeated integrations by parts, we get that

$$I_\psi(\tau) = \frac{1}{\tau^N} \int_{\mathbb{R}^d} e^{i\tau\Phi}\big(({}^t\mathcal{L})^N\psi\big)(\xi)\, d\xi.$$

Let us compute ${}^t\mathcal{L}$ for $a \in \mathcal{D}_K$. We have

$$ {}^t\mathcal{L}a = -\mathcal{L}a + i\frac{\Delta\Phi}{|\nabla\Phi|^2}a - 2i\sum_{1\leq j,k\leq d}\frac{\partial_j\Phi\,\partial_k\Phi\,\partial_j\partial_k\Phi}{|\nabla\Phi|^4}a. $$

Thus, it is obvious that

$$ ({}^t\mathcal{L}\psi)(\xi) = f_{1,1}(\xi,\nabla\Phi(\xi)) + f_{1,2}(\xi,\nabla\Phi(\xi)), $$

where the function $f_{1,j}(\xi,\theta)$ belongs to $\mathcal{D}(K\times\mathbb{R}^d\backslash\{0\})$, is homogeneous of degree $-j$ with respect to θ and satisfies

$$ \forall(\alpha,\beta)\in(\mathbb{N}^d)^2\,,\quad \sup_{(\xi,\theta)\in K\times\mathbb{S}^{d-1}}\left|\partial_\xi^\alpha\partial_\theta^\beta f_{1,j}(\xi,\theta)\right| \leq C_{j,\alpha,\beta}. $$

As the coefficients of the differential operator $\mathcal{L}$ and all their derivatives are bounded on K, an obvious (and omitted) induction implies that

$$ ({}^t\mathcal{L})^N\psi(\xi) = \sum_{j=0}^{N} f_{N,j}(\xi,\nabla\Phi(\xi)), $$

where the function $f_{N,j}(\xi,\theta)$ belongs to $\mathcal{D}(K\times\mathbb{R}^d\backslash\{0\})$, is homogeneous of degree $-N-j$ in θ and satisfies

$$ \forall(\alpha,\beta)\in(\mathbb{N}^d)^2\,,\quad \sup_{(\xi,\theta)\in K\times\mathbb{S}^{d-1}}\left|\partial_\xi^\alpha\partial_\theta^\beta f_{N,j}(\xi,\theta)\right| \leq C_{N,j,\alpha,\beta}. $$

This proves the theorem. □

Now let us consider the case when the gradient of the phase function may vanish. The decay is of a different nature.

Theorem 10.2.3 *Let us consider a compact K of $\mathbb{R}^d$ and assume that a constant c_0 exists such that*

$$ \forall\xi\in K\,,\ |\nabla\Phi(\xi)|\leq c_0. $$

Then, for any integer N and any function ψ in $\mathcal{D}_K$ there exists a constant C_N such that

$$ \left|I_\psi(\tau)\right| \leq C_N\int_K \frac{d\xi}{(1+c_0\tau|\nabla\Phi(\xi)|^2)^N}\cdot $$

Proof. As in the preceding theorem, it suffices to consider the case $c_0 = 1$ and one may perform suitable integrations by parts to pinpoint the decay with respect to τ. Let us consider the following first order differential operator

$$ \mathcal{L}_\tau \stackrel{\text{def}}{=} \frac{1}{1+\tau|\nabla\Phi|^2}\,(\mathrm{Id} - i\nabla\Phi\cdot\partial)\quad\text{with}\quad \nabla\Phi\cdot\partial = \sum_{j=1}^d \partial_j\Phi\,\partial_j. \tag{10.1} $$

This operator obviously satisfies

$$ \mathcal{L}_\tau e^{i\tau\Phi} = e^{i\tau\Phi}. $$

Now, by integration by parts, we get that

$$ I_\psi(\tau) = \int_{\mathbb{R}^d} e^{i\tau\Phi}({}^t\mathcal{L}_\tau)^N\psi(\xi)\,d\xi. $$

Hence, to complete the proof of the theorem, it suffices to state that for any integer N, a constant C exists such that

$$\left|({}^t\mathcal{L}_\tau)^N\psi(\xi)\right| \leq \frac{C}{(1+\tau|\nabla\Phi(\xi)|^2)^N}. \tag{10.2}$$

In order to do so, let us define the following class of functions. □

Definition 10.2.1 *Let N be in $\mathbb{Z}$, we denote by S^N the set of smooth functions on $K\times\mathbb{R}^d$ such that*

$$\forall(\alpha,\beta)\in\mathbb{N}^d\times\mathbb{N}^d\,,\ \exists C\ /\ \forall(\xi,\theta)\in K\times\mathbb{R}^d\,,\ |\partial_\xi^\alpha\partial_\theta^\beta f(\xi,\theta)|\leq C(1+|\theta|^2)^{\frac{N-|\beta|}{2}}.$$

It is obvious that the space S^N is increasing with N and that the product of a function of S^{N_1} by a function of S^{N_2} is a function of $S^{N_1+N_2}$. Moreover, we have $\partial_\theta^\beta(S^N)\subset S^{N-|\beta|}$.

It is clear that the following lemma implies Inequality (10.2).

Lemma 10.2.1 *For any N in $\mathbb{N}$, a function f_N exists in S^{-2N} such that*

$$({}^t\mathcal{L}_\tau)^N\psi(\xi) = f_N\left(\xi,\tau^{\frac12}\nabla\Phi(\xi)\right) \quad \textit{for all}\quad \xi\in K.$$

Proof. By noticing that S^0 contains the space $\mathcal{D}_K$, and by an immediate induction, it is enough to prove that, if f belongs to S^M, then

$${}^t\mathcal{L}_\tau\Big(f(\xi,\tau^{\frac12}\nabla\Phi(\xi))\Big) = g(\xi,\tau^{\frac12}\nabla\Phi(\xi)) \quad\text{with}\quad g\in S^{M-2}. \tag{10.3}$$

We have, for any $a\in\mathcal{D}_K$,

$$\begin{aligned} {}^t\mathcal{L}_\tau a(\xi) &= i\frac{\nabla\Phi(\xi)\cdot\nabla a(\xi)}{1+\tau|\nabla\Phi(\xi)|^2} + \sigma\big(\xi,\tau^{\frac12}\nabla\Phi(\xi)\big)\,a(\xi) \\ &\qquad \text{with}\quad \sigma(\xi,\theta) = \frac{i\Delta\Phi(\xi)+1}{1+|\theta|^2} - 2i\frac{D^2\Phi(\theta,\theta)}{(1+|\theta|^2)^2} \end{aligned} \tag{10.4}$$

where, from now on, we agree that

$$D^2\Phi(\theta_1,\theta_2) \stackrel{\text{def}}{=} \sum_{j,k}\theta_1^j\theta_2^k\partial_{jk}^2\Phi.$$

It is obvious that $\sigma\in S^{-2}$. By using the chain rule, we get

$$\nabla\Phi\cdot\nabla f(\xi,\tau^{\frac12}\nabla\Phi(\xi)) = \Big(\nabla\Phi\cdot\nabla_\xi f + D^2\Phi(\theta,\nabla_\theta f)\Big)(\xi,\tau^{\frac12}\nabla\Phi(\xi)).$$

Thus we have the relation (10.3) with

$$g(\xi,\theta) = \frac{i}{1+|\theta|^2}\Big(\nabla\Phi(\xi)\cdot\nabla_\xi f(\xi,\theta) + D^2\Phi(\theta,\nabla_\theta f(\xi,\theta))\Big) + (\sigma f)(\xi,\theta). \tag{10.5}$$

The lemma is proved and thus Theorem 10.2.3 too. □

Putting together the above two theorems, one gets the following statement.

Theorem 10.2.4 *Let ψ be in $\mathcal{D}(\mathbb{R}^d)$ and Φ be a* real valued *smooth function defined on a neighborhood of the support of ψ. Fix some positive real number c_0. Then, for any couple (N,N') of positive real numbers, there exist two constants C_N and $C_{N'}$ such that*

$$\big|I_\psi(\tau)\big| \leq \frac{C_N}{(c_0\tau)^N} + C_{N'}\int \frac{\mathbf{1}_{\{\xi\in\mathbb{R}^d\,,\,|\nabla\Phi(\xi)|\leq c_0\}}}{(1+c_0\tau|\nabla\Phi|^2)^{N'}}\,d\xi.$$

Besides, the constants C_N and $C_{N'}$ depend only on N, N', on a finite number of derivatives of ψ and on a finite number of derivatives of order greater than or equal to 2 of Φ.

Proof. Let χ be a smooth function supported in the unit ball and with value 1 for $|x|\leq 1/2$. One may write

$$I_\psi(\tau) = I_1(\tau)+I_2(\tau) \quad\text{with}\quad \left\{\begin{array}{rcl} I_1(\tau) &=& \displaystyle\int e^{i\tau\Phi(\xi)}\Big(1-\chi\Big(\frac{\nabla\Phi(\xi)}{c_0}\Big)\Big)\psi(\xi)\,d\xi,\\ I_2(\tau) &=& \displaystyle\int \;e^{i\tau\Phi(\xi)}\chi\Big(\frac{\nabla\Phi(\xi)}{c_0}\Big)\psi(\xi)\,d\xi.\end{array}\right.$$

Applying Theorem 10.2.2 to I_1 and Theorem 10.2.3 to I_2 gives the result. □

Now we go back to the wave equation.

Proof of Theorem 10.2.1. Due to the time reversibility of the wave equation, it suffices to prove the result for non negative times.

Let φ be a function of $\mathcal{D}(\mathbb{R}^d\backslash\{0\})$ with value 1 near $\mathcal{C}$. Then according to Proposition 10.1.1, we have

$$u(t) = K^+(t,\cdot)\star\widetilde{\gamma}^+ + K^-(t,\cdot)\star\widetilde{\gamma}^- \quad\text{with}$$
$$\widetilde{\gamma}^\pm \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\varphi\widehat{\gamma}^\pm) \quad\text{and}\quad K^\pm(t,x) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} e^{i(x|\xi)}e^{\pm it|\xi|}\varphi(\xi)\,d\xi.$$

Let us admit for a while the inequality

$$\|K^\pm(t,\cdot)\|_{L^\infty} \leq \frac{C}{t^{\frac{d-1}{2}}} \quad\text{for}\quad t>0. \tag{10.6}$$

Then, we immediately get

$$\|u(t)\|_{L^\infty} \leq \frac{C}{t^{\frac{d-1}{2}}}\big(\|\widetilde{\gamma}^+\|_{L^1}+\|\widetilde{\gamma}^-\|_{L^1}\big).$$

Now, because

$$\widetilde{\gamma}^\pm = \frac{1}{2}(u_0 \mp ih\star u_1)$$

where the L^1 function h stands for the inverse Fourier transform of $|\cdot|^{-1}\varphi$, one gets the desired inequality for $\|u(t)\|_{L^\infty}$.

In order to complete the proof, let us establish Inequality (10.6). As the L^∞ norm is invariant under dilation, it suffices to estimate $\|K(t,t\,\cdot)\|_{L^\infty}$. Now, Theorem 10.2.4 implies that

$$\begin{array}{rcl}|K^\pm(t,tx)| &\leq& \displaystyle\frac{C}{t^{\frac{d-1}{2}}} + C\int_{\mathcal{C}_x}\Big(1+t\Big|x\pm\frac{\xi}{|\xi|}\Big|^2\Big)^{-d}\,d\xi \quad\text{where}\\ \mathcal{C}_x &\stackrel{\text{def}}{=}& \displaystyle\Big\{\xi\in\mathcal{C}\,/\,\Big|x\pm\frac{\xi}{|\xi|}\Big|\leq\frac{1}{2}\Big\}.\end{array}$$

If $\mathcal{C}_x$ is not empty then $x \neq 0$. Hence one can write for any ξ in $\mathcal{C}_x$ the following orthogonal decomposition:

$$\xi = \zeta_1 + \zeta' \quad \text{with} \quad \zeta_1 = \Big(\xi\Big|\frac{x}{|x|}\Big)\frac{x}{|x|} \quad \text{and} \quad \zeta' = \xi - \Big(\xi\Big|\frac{x}{|x|}\Big)\frac{x}{|x|}\cdot$$

Knowing that ζ' is orthogonal to the vector x, we infer that

$$\left|x \pm \frac{\xi}{|\xi|}\right| \geq \frac{|\zeta'|}{|\xi|}\cdot$$

Therefore, using the fact that $r \leq |\xi| \leq R$ for any $\xi \in \mathcal{C}$, we get

$$|K^{\pm}(t, tx)| \leq \frac{C}{t^{\frac{d-1}{2}}} + C\int_{\mathcal{C}} \frac{1}{(1 + t|\zeta'|^2)^d}\, d\zeta'\, d\zeta_1.$$

The change of variables $\widetilde{\zeta} = t^{\frac{1}{2}}\zeta'$ gives (10.6). This completes the proof of the proposition.

10.3 Strichartz estimates for the wave equation

Let us first introduce the following definition.

Definition 10.3.1 *We shall say that a pair (p, q) in $[2, \infty]^2$ is* wave admissible *if there exists $\widetilde{q}$ in $[2, q]$ such that*

$$\frac{2}{p} + \frac{d-1}{\widetilde{q}} = \frac{d-1}{2} \quad \textit{with} \quad \Big(p, \widetilde{q}, \frac{d-1}{2}\Big) \neq (2, \infty, 1). \tag{10.7}$$

The basic result of this subsection is the following one.

Theorem 10.3.1 *Let us assume that the space dimension d is greater than or equal to 2. For any wave admissible pair (p, q) , a constant C exists such that, for any j in $\mathbb{Z}$,*

$$\|\nabla\dot{\Delta}_j u\|_{L^p_T(L^q)} \leq C2^{j\mu}\|\dot{\Delta}_j \nabla u(0)\|_{L^2} + C2^{j\mu}\|\dot{\Delta}_j \Box u\|_{L^1_T(L^2)} \tag{10.8}$$

with

$$\mu \stackrel{\text{def}}{=} d\Big(\frac{1}{2} - \frac{1}{q}\Big) - \frac{1}{p}\cdot \tag{10.9}$$

Proof. The solution u to the linear Cauchy problem (W) writes $u = v + w$, where v is the solution to the homogeneous wave equation

$$\begin{cases} \partial_t^2 v - \Delta v = 0 \\ (v, \partial_t v)_{|t=0} = (u_0, u_1) \end{cases}$$

and w is the solution to the nonhomogeneous wave equation

$$\begin{cases} \partial_t^2 w - \Delta w = f \stackrel{\text{def}}{=} \Box u, \\ (w, \partial_t w)_{|t=0} = (0, 0). \end{cases}$$

Using the notation introduced in Proposition 10.1.1 and Duhamel principle, one can write for all $t \in [0, T]$,

$$v(t) = U^+(t)\gamma_+ + U^-(t)\gamma_-,$$
$$w(t) = \int_0^t \Big(U^+(t-t')f_+(t') + U^-(t-t')f_-(t')\Big)\, dt'$$

with $\widehat{f}_{\pm}(t', \xi) = \pm\frac{1}{2i|\xi|}\widehat{f}(t', \xi)$.

Owing to Bernstein inequalities, Proposition 10.2.1 and Theorem 9.3.1 page 102, we infer that, for any couple $(p,\widetilde{r})$ satisfying (10.7), we have

$$\|\dot\Delta_0\nabla u\|_{L^p_T(L^{\widetilde{r}})}\leq C\Big(\|\dot\Delta_0\nabla u(0)\|_{L^2}+\|\dot\Delta_0 f\|_{L^1_T(L^2)}\Big).$$

Since $r\geq\widetilde{r}$ we deduce, using Bernstein inequalities, that

$$\|\dot\Delta_0\nabla u\|_{L^p_T(L^{r})}\leq C\Big(\|\dot\Delta_0\nabla u(0)\|_{L^2}+\|\dot\Delta_0 f\|_{L^1_T(L^2)}\Big).$$

This gives the result for $j=0$. The result for all j in $\mathbb{Z}$ follows by an obvious rescaling. □

The two easy corollaries that we are going to state now will prove to be very useful in the next sections.

Corollary 10.3.1 *For a wave admissible pair (p,q) and any real σ, a constant C exists such that with the notation of Theorem 10.3.1,*

$$\|\nabla u\|_{L^p_T(\dot B^{\sigma}_{q,2})}\leq C\Big(\|\nabla u(0)\|_{\dot H^{\sigma+\mu}}+\|\Box u\|_{L^1_T(\dot H^{\sigma+\mu-1})}\Big).\tag{10.10}$$

Proof. Thanks to Theorem 10.3.1, we have for any j in $\mathbb{Z}$,

$$2^{j\sigma}\|\dot\Delta_j\nabla u\|_{L^p_T(L^r)}\leq C2^{j(\sigma+\mu_1)}\|\dot\Delta_j\nabla u(0)\|_{L^2}+C2^{j(\sigma+\mu_{12})}\|\dot\Delta_j\Box u\|_{L^1_T(L^2)}.$$

Taking the $\ell^2(\mathbb{Z})$ norm of both sides, we get

$$\Big(\sum_{j\in\mathbb{Z}}2^{2j\sigma}\|\dot\Delta_j\nabla u\|^2_{L^p_T(L^r)}\Big)^{\frac12}\leq C\bigg(\|\nabla u(0)\|_{\dot H^{\sigma+\mu_1}}+\Big(\sum_{j\in\mathbb{Z}}2^{2j(\sigma+\mu_{12})}\|\dot\Delta_j\Box u\|^2_{L^1_T(L^2)}\Big)^{\frac12}\bigg).$$

As $p\geq 2$ the Minkowski inequality implies the theorem. □

The second corollary is very often used.

Corollary 10.3.2 *For any wave admissible pairs (q,r), a constant C exists such that*

$$\|u\|_{L^p_T(L^q)}\leq C\left(\|\nabla u(0)\|_{\dot H^{\mu-1}}+\|\Box u\|_{L^1_T(\dot H^{\mu-1})}\right)\quad \textit{with}\quad \mu=d\Big(\frac12-\frac1q\Big)-\frac1p\,\cdot$$

Proof. Applying Corollary 10.3.1 with $\sigma=-1$, we get

$$\|u\|_{L^p_T(\dot B^0_{q,2})}\leq C\Big(\|\nabla u(0)\|_{\dot H^{\mu-1}}+\|\Box u\|_{L^1_T(\dot H^{\mu-1})}\Big).$$

In order to conclude the proof, we need the following result of harmonic analysis.

Theorem 10.3.2 *For any r in $[2,\infty[$, $\dot B^0_{r,2}$ is continuously included in L^r.*

Proof. Therefore, denoting $F_r(x)=|x|^r$, we can rewrite $\|u\|^r_{L^r}$ as a telescopic series

$$\|u\|^r_{L^r}=\sum_{j\in\mathbb{Z}}F_p(\dot S_{j+1}u)-F_p(\dot S_ju)\quad\text{whence}$$

$$\|u\|^r_{L^r}=\sum_j\langle\dot\Delta_ju,m_j\rangle\quad\text{with}\quad m_j(x)\stackrel{\text{def}}{=}\int_0^1F'_r\left(\dot S_ju(x)+t\dot\Delta_ju(x)\right)dt.$$

Using Plancherel formula, and stating $\widetilde{\Delta}_j$ the convolution operator by the inverse Fourier transform of $\widetilde{\varphi}(2^{-j}\cdot)$ where $\widetilde{\varphi}$ is in $\mathcal{D}(\mathbb{R}^d\backslash\{0\})$ with value 1 near the support of φ, we can write that

$$\langle\dot{\Delta}_j u, m_j\rangle = \langle\dot{\Delta}_j u, \widetilde{\Delta}_j m_j\rangle.$$

By Lemma 7.1.1, we infer that

$$\|\widetilde{\Delta}_j m_j\|_{L^{r'}} \leq C2^{-j} \sup_{1\leq \ell\leq d} \|\partial_\ell m_j\|_{L^{r'}}. \tag{10.11}$$

The chain rule and the Hölder's inequality imply that

$$\begin{aligned}\|\partial_\ell m_j\|_{L^{r'}} &\leq \int_0^1 \left\|\partial_\ell(\dot{S}_j u + t\dot{\Delta}_j u)F_r''(\dot{S}_j + t\Delta_j u)\right\|_{L^{r'}} dt\\ &\leq \int_0^1 \|\partial_\ell(\dot{S}_j u + t\dot{\Delta}_j u)\|_{L^r}\|F_r''(\dot{S}_j u + t\dot{\Delta}_j u)\|_{L^{\frac{r}{r-2}}} dt.\end{aligned}$$

As $F_r''(x) = r(r-1)|x|^{r-2}$, we get immediately that

$$\forall t\in[0,1]\,,\ \|F_r''(\dot{S}_j u + t\dot{\Delta}_j u)\|_{L^{\frac{r}{r-2}}} \leq r(r-1)\|\dot{S}_j u + t\dot{\Delta}_j u\|_{L^r}^{r-2}.$$

Using Lemma 7.1.1, we infer that for all t in $[0,1]$,

$$\|F_r''(S_j u + t\Delta_j u)\|_{L^{\frac{r}{r-2}}} \leq C^r r(r-1)\|u\|_{L^r}^{r-2}. \tag{10.12}$$

Now, by definition of $\dot{S}_j$, Lemma 7.1.1 and Young's inequality on series, we get

$$\begin{aligned}\|\partial_\ell(\dot{S}_j u + t\dot{\Delta}_j u)\|_{L^r} &\leq \sum_{k\leq j}\|\partial_\ell\dot{\Delta}_k u\|_{L^r}\\ &\leq 2^j\sum_{k\leq j}2^{k-j}\|\dot{\Delta}_k u\|_{L^r}\\ &\leq Cc_j 2^j\|u\|_{\dot{B}^0_{r,2}} \quad\text{with}\quad \sum_j c_j^2 = 1.\end{aligned}$$

Combining (10.11) and (10.12), we deduce that

$$\|\widetilde{\Delta}_j m_j\|_{L^{r'}} \leq C^r r(r-1)c_j\|u\|_{L^r}^{r-2}\|u\|_{\dot{B}^0_{r,2}} \quad\text{with}\quad \sum_j c_j^2 = 1.$$

As we have $\|u\|_{L^r}^r = \sum_j\langle\dot{\Delta}_j u, \tilde{\Delta}_j m_j\rangle$, we infer that

$$\|u\|_{L^r}^2 \leq C^r r(r-1)\|u\|_{\dot{B}^0_{r,2}}\sum_j c_j\|\dot{\Delta}_j u\|_{L^p} \leq C^r r(r-1)\|u\|_{\dot{B}^0_{r,2}}^2. \tag{10.13}$$

This concludes the proof that $\dot{B}^0_{r,2}\hookrightarrow L^r$. □

Conclusion of the proof of Corollary 10.3.2 It is enough to apply the above theorem. □

Remark The term $1/p$ which is in the definition of the index μ may be interpreted as a gain of $1/p$ derivative compared with the Sobolev embedding.

10.4 The quintic wave equation in $\mathbb{R}^3$

In this section, we investigate the quintic wave equation in $\mathbb{R}^3$:

$$(W_5^\pm) \qquad \begin{cases} \Box u \pm u^5 = 0, \\ (u, \partial_t u)_{|t=0} = (u_0, u_1). \end{cases}$$

In this section, we shall prove that Equation $(W_5^\pm)$ is locally well-posed in the scaling invariant space $\mathcal{C}([0,T];L^2) \cap L^5([0,T];L^{10})$.

Theorem 10.4.1 *If $\gamma \stackrel{\text{def}}{=} \nabla u_{|t=0}$ belongs to L^2, then a positive time T exists such that the Cauchy problem (W_5) has a unique solution u in*

$$E_T \stackrel{\text{def}}{=} \Big\{ u \in L^5([0,T];L^{10}) \;/\; \nabla u \in \mathcal{C}([0,T];L^2) \Big\}.$$

In addition, u satisfies the following continuation criterion: if $T^\star$ denotes the maximal time of existence of u in E_T then

- *there exists a constant c such that, if $\|\gamma\|_{L^2} \le c$, then $T^\star = +\infty$ and the solution belongs to*
$$L^\infty(\mathbb{R}^+;\dot H^1) \cap L^5(\mathbb{R}^+;L^{10}),$$
- *if $T^\star$ is finite, then*
$$\int_0^{T^\star} \|u(t)\|_{L^{10}}^5 \, dt = +\infty.$$

Proof. Let us denote by $B(u_1, \cdots, u_5)$ the solution of the wave equation

$$\begin{cases} \Box B(u_1, \cdots, u_5) = -\prod_{j=1}^5 u_j, \\ B(u_1, \cdots, u_5)_{|t=0} = \partial_t B(u_1, \cdots, u_5)_{|t=0} = 0, \end{cases}$$

and by u_F the solution of the free wave equation $\Box u = 0$ satisfying $u_F(0) = u_0$ and $\partial_t u_F(0) = u_1$. A solution of (W_5) is a fixed point of the map

$$u \longmapsto u_F + B(u, \cdots, u).$$

Energy inequality, Corollary 10.3.2 and Hölder inequality imply that, for any T,

$$\|\nabla B(u_1, \cdots, u_5)\|_{L^\infty_T(L^2)} + \|B(u_1, \cdots, u_5)\|_{L^5_T(L^{10})} \le C \prod_{j=1}^5 \|u_j\|_{L^5_T(L^{10})}.$$

Provided $\|u_F\|_{L^5_T(L^{10})}$ is small enough, it is an exercise left to the reader to prove existence of a solution on the interval $[0,T]$ with the desired properties.

More precisely, in the case where $\|\gamma\|_{L^2}$, is *small*, one readily gets *global* existence because, owing to Corollary 10.3.2,

$$\|u_F\|_{L^5(L^{10})} \le C \|\gamma\|_{L^2}.$$

Now, if $\|\gamma\|_{L^2}$ is not small, one may decompose γ into its high frequency part and its low frequency part as follows:

$$\gamma = S_J \gamma + (\mathrm{Id} - S_J)\gamma.$$

Let us denote respectively by $u^\ell_{F,J}$ and $u^h_{F,J}$ the solutions of the free wave equation $\Box u = 0$ associated with $S_J\gamma$ and $(\mathrm{Id} - S_J)\gamma$. As we know that

$$\lim_{j\to\infty} \|(\mathrm{Id} - S_j)\gamma\|_{L^2} = 0,$$

according to Corollary 10.3.2, for all positive ε, there exists some J in $\mathbb{Z}$ such that

$$\|u^h_{F,J}\|_{L^5(L^{10})} \leq \varepsilon. \tag{10.14}$$

For the low frequency part, we use Hölder and Bernstein inequalities which imply that

$$\begin{aligned}\|u^\ell_{F,J}\|_{L^5_T(L^{10})} &\leq T^{\frac{1}{5}}\|u^\ell_{F,J}\|_{L^\infty_T(L^{10})}\\ &\leq CT^{\frac{1}{5}}2^{\frac{J}{5}}\|u^\ell_{F,J}\|_{L^\infty_T(L^6)}.\end{aligned}$$

Using Sobolev inequality and energy identity thus yields that

$$\|u^\ell_{F,J}\|_{L^5_T(L^{10})} \leq C2^{\frac{J}{5}}T^{\frac{1}{5}}\|\gamma\|_{L^2}.$$

Together with (10.14), this gives that

$$\lim_{T\to 0}\|u_F\|_{L^5_T(L^{10})} = 0$$

which leads to local well-posedness for any data in L^2.

Let us finally prove the blow-up criterion. Let us consider a solution u to $(W_5^\pm)$ on the interval $[0,T[$ such that

$$\int_0^T \|u(t)\|^5_{L^{10}}\,dt < \infty.$$

Using the energy estimate between t' and t (with $t' \leq t$) gives

$$\|\nabla u(t) - \nabla u(t')\|_{L^2} \leq \int_{t'}^t \|u(t'')\|^5_{L^{10}}\,dt''.$$

Thus a function u_T exists in $\dot H^1$ such that

$$\lim_{t\to T} u(t) = u_T \quad \text{in} \quad \dot H^1.$$

Then the local well-posedness part of the theorem implies that u can be continued beyond T. This ends the proof of Theorem 10.4.1. □