Navier-Stokes Equation
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Liste des questions de cours pour l'examen

Théorème 1.2.3 page 10 (énoncé et démonstration)

Théorème 1.2.8 page 15 (énoncé et démonstration)

Chapitre 3 (énoncé du théorème de Leray et structure de la preuve)

Théorème 4.3.1 page 39 (énoncé et démonstration)

Lemme 6.1.1 page 53 (énoncé et démonstration)

Lemme 6.1.2 page 54 (énoncé et démonstration)

Lemme 6.3.1 page 63 (énoncé et démonstration dans le cas où $r = \infty$)

Chapter 1

Some basic results of functionnal analysis

1.1 A short review on ordinary differential equation

Before starting the study of evolution partial differential equation, let us have a look on basic properties of ordinary differential equations.

1.1.1 The linear case

Let E be a Banach space, I an open interval of $\mathbb{R}$ and A a map from I to $\mathcal{L}(E)$, the set of continuous linear maps from E into E. We want to solve the equation

$$(ODE)\left\{\begin{array}{rcl} \dot{u} \stackrel{\text{def}}{=} \dfrac{du}{dt} & = & A(t)u(t) \\ u(0) & = & u_0. \end{array}\right.$$

The proof of the existence and uniqueness of solutions of this equation is very simple. Let λ be a positive real number, let us introduce the space E_λ definied by

$$E_\lambda = \Big\{u \in C(I,E) \,/\, \|u\|_\lambda \stackrel{\text{def}}{=} \sup_{t\in I} \|u(t)\| \exp\Big(-\lambda \int_0^t \|A(t')\|_{\mathcal{L}(E)} dt'\Big) < \infty\Big\}.$$

The solution of (ODE) are the same as the solutions of

$$Lu = u_0 \quad \text{with} \quad Lu(t) \stackrel{\text{def}}{=} u(t) - \int_0^t A(t')u(t')dt'.$$

We have

$$\|(Lu-u)(t)\| \leq \int_0^t \|A(t')\|_{\mathcal{L}(E)} \|u(t')\| dt'.$$

Thus we deduce that

$$\begin{aligned} &\|(Lu-u)(t)\| \exp\Big(-\lambda \int_0^t \|A(t')\|_{\mathcal{L}(E)} dt'\Big) \\ &\leq \int_0^t \exp\Big(-\lambda \int_{t'}^t \|A(t'')\|_{\mathcal{L}(E)} dt''\Big) \|A(t')\|_{\mathcal{L}(E)} \exp\Big(-\lambda \int_0^{t'} \|A(t'')\|_{\mathcal{L}(E)} dt''\Big) \|u(t')\| dt'. \end{aligned}$$

By definition of $\|\cdot\|_\lambda$, we infer that

$$\|(Lu-u)(t)\|\exp\Bigl(-\lambda\int_0^t \|A(t')\|_{\mathcal{L}(E)}dt'\Bigr) \leq \frac{1}{\lambda}\|u\|_\lambda$$

and thus that $\|Lu-u\|_\lambda \leq \lambda^{-1}\|u\|_\lambda$. This implies that, for λ greater than 1, L is invertible in $\mathcal{L}(E)$. Then the proof of the existence and uniqueness of solutions is achieved.

1.1.2 Blow up criteria

The callsical Cauchy-Lipschitz theorem of existence and uniqueness theorem for ordinary differential equations is a local theorem. Let us investigate what can be necessary conditions for a blow up phenomena.

Proposition 1.1.1 *Let F be a function of $\mathbb{R}\times E$ in E satisfying the hypothesis of Theorem* **??** *in any point x_0 of E. Let us assume in addition that a locally bounded function M from $\mathbb{R}^+$ into $\mathbb{R}^+$ and a locally integrable function β from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that*

$$\|F(t,u)\| \leq \beta(t)M(\|u\|).$$

then, if the maximal interval of definition is $]T_\star, T^\star[$, then, if $T^\star$ is finite,

$$\limsup_{t\to T^\star}\|u(t)\| = \infty.$$

Proof. Let us first prove that, if we consider a time $T > T_0$ such that $\|u(t)\|$ is bounded on the interval $[T_0, T[$, then we can define the solution on a larger interval $[T_0, T_1]$ with $T_1 > T$. As the function u is bounded on the interval $[T_0, T[$, the hypothesis on F that, for any t of the interval $[T_0, T[$, we have

$$\|F(t, u(t))\| \leq C\beta(t).$$

The function β being integrable on the interval $[T_0, T]$, we have deduce que, for any ε strictement positive, it exists a positive real number η such that, pour tout t and t' such that $T-t<\eta$ and $T-t'<\eta$,

$$\|u(t)-u(t')\| < \varepsilon.$$

The space E being complete, an element $u_\star$ of E exists such that

$$\lim_{t\to T^\star} u(t) = u_\star.$$

Applying Theorem **??**, we construct solution of (ODE) on some $[T_+^\star, T_1]$ and the continuous function defined by induction on the interval $[T_0, T_1]$ is a solution of the equation (ODE) on the interval $[T_0, T_1]$. □

Corollary 1.1.1 *Under the hypothesis of Proposition 1.1.1, if we have in addition that*

$$\|F(t,u)\| \leq M\|u\|^2,$$

then, if the interval $]T_\star, T^\star[$ is the maximal interval of definition of u and if $T^\star$ is finite, then

$$\int_{t_0}^{T^\star}\|x(t)\|dt = \infty.$$

Proof. The solution satisfies, for any $t \geq t_0$

$$\|x(t)\| \leq \|x(t_0)\| + M\int_{t_0}^t \|x(t')\|^2 dt'. \tag{1.1}$$

Gronwall's Lemma implies that

$$\|x(t)\| \leq \|x_0\| \exp\Big(M\int_0^t \|x(t')\|dt'\Big).$$

A more precise way of proving this result is the following.

Let $T \stackrel{\text{def}}{=} \sup\{t \in [t_0, T^\star[\ /\ \|x(t)\| \leq 2\|x(t_0)\|\}$. For any $t \in [t_0, T^\star[$, we have, using (1.1),

$$\|x(t)\| \leq \|x(t_0)\| + 4M(t-t_0)\|x(t_0)\|^2.$$

Thus we infer

$$\forall t \in \Big[t_0, \min\Big\{T, t_0 + \frac{1}{4M\|x(t_0\|)}\Big\}\Big[\,, \quad \|x(t)\| \leq 2\|x_0\|.$$

Thanks to Proposition 1.1.1, we have

$$T^\star - t_0 \geq \frac{c}{\|x_0\|} \cdot$$

Applying again this result at time $t \in [t_0, T^\star[$, we find that

$$\forall t \in [t_0, T^\star[\,, \quad \|x(t)\| \geq \frac{c}{T^\star - t} \cdot$$

The corollary is proved. □

1.1.3 A compactness theorem : Peano's theorem

The theorem is the following.

Theorem 1.1.1 (Peano) *Let I be an open interval of $\mathbb{R}$. Let us consider a function f from $I \times \mathbb{R}^d$ into $\mathbb{R}^d$ such that*

- *For any compact K of $\mathbb{R}^d$, the function $t \mapsto \|f(t)\|_{L^\infty(K)}$ is locally integrable,*
- *For any t of I, the function $x \mapsto f(t,x)$ is continuous on $\mathbb{R}^d$.*

Then, for any point (t_0, x_0) of $I \times \mathbb{R}^d$, an open interval $J \subset I$ containing t_0 and a continuous function x on J exists such that

$$(ODE) \qquad x(t) = x_0 + \int_{t_0}^t f(t', x(t'))dt'.$$

Proof. The structure of the proof is at least as intesting as the result itself. This proof will be a model for the proof of existence of weak solutions for ther incompressible Navier-Stokes equation we shall study in Chater 3.

There are three steps in the proof

- we regularize the function f and we apply Cauchy-Lipschitz's Theorem to the sequence of regularized functions; Proposition 1.1.1 ensures that the solutions of the regularized problem have a commun interval of definition,

- then, we prove that the sequence of those solutions of the regularized problem are relatively compact in the space $C(J,\mathbb{R}^d)$,
- as a conclusion, we pass to the limit.

Let us procede to a classical regularization; let χ a non negative function of $\mathcal{D}(B(0,1))$ the integral of which is 1. Let us define $\chi_n(x) \stackrel{\text{def}}{=} n^d\chi(nx)$ and $f_n(t) = \chi_n \star f(t)$. We have

$$\|f_n(t)\|_{L^\infty(K)} \leq \|f(t)\|_{L^\infty(K+B(0,n^{-1}))}.$$

Morevoer, we have

$$\|\partial_j f_n(t)\|_{L^\infty(K)} \leq C(n+1)\|f(t)\|_{L^\infty(K+B(0,n^{-1}))}.$$

We can apply Cauchy-Lipschitz's Theorem of to the function f_n. Let J_n the maximal interval of definition of x_n. Let J an interval ouvert such that

$$\int_J \|f(t)\|_{L^\infty(B(x_0,2)}dt \leq 1.$$

Let us define $t_n \stackrel{\text{def}}{=} \sup\Big\{t \in [t_0,\infty[\cap J \cap J_n \,/\, \forall t' \leq t\,,\ x(t') \in B(x_0,1)\Big\}$. For any $t \leq t_n$, we have

$$\begin{aligned}\|x_n(t) - x_0\| &\leq \int_J \|f_n(t)\|_{L^\infty(B(x_0,1))}dt\\ &\leq \int_J \|f(t)\|_{L^\infty(B(x_0,2))}dt\\ &\leq 1.\end{aligned}$$

Thus $t_n \geq \sup J \cap J_n$. working in the same way for the times less to t_0, we find, using Proposition 1.1.1 that, for any n, $J \subset J_n$. This concludes the first part of the proof.

We have

$$\forall t \in J\,,\ X(t) \stackrel{\text{déf}}{=} \{x_n(t),\ n \in \mathbb{N}\} \subset B(x_0,1).$$

As we work on a finite dimensionnal space, $X(t)$ is relatively compact. Moreover, we have

$$\begin{aligned}\|x_n(t) - x_n(t')\| &\leq \Big|\int_t^{t'} \|f_n(t'')\|_{L^\infty(B(x_0,1))}dt''\Big|\\ &\leq \Big|\int_t^{t'} \|f(t'')\|_{L^\infty(B(x_0,2))}dt''\Big|.\end{aligned}$$

Thus, for any positive ϵ, it exists a positive real number α such that

$$\forall (t,t') \in J^2\,,\ |t-t'| < \alpha \Longrightarrow \|x_n(t) - x_n(t')\| < \epsilon.$$

In other words, the family $(x_n)_{n\in\mathbb{N}}$ is equicontinuous on J. Ascoli's Theorem ensures that the set of functions x_n is relatively compact in $C(J;\mathbb{R}^d)$. Thus we can extract a subsequence which converge uniformely on J to a function x of $C(J;\mathbb{R}^d)$. Let omit to note the extraction in the following.

Now let us pass to the limit. For any t of J; we have

$$\|f_n(t,x_n(t)-f(t,x(t))\| \leq \|f_n(t)-f(t)\|_{L^\infty(B(x_0,1))}+\|f(t,x_n(t))-f(t,x(t))\|.$$

Thus for any t of J, we have

$$\lim_{n\to\infty} f_n(t,x_n(t)) = f(t,x(t)).$$

Moreover, $\|f_n(t,x_n(t)\| \leq \|f(t)\|_{L^\infty(B(x_0,2))}$. Lebesgue's Theorem ensures that, for any t, we have

$$\lim_{n\to\infty}\int_{t_0}^t f_n(t',x_n(t'))dt' = \int_{t_0}^t f(t',x(t'))dt'.$$

The theorem is proved. □

1.2 Sobolev spaces

In this course, we shall restrict ourselves to Sobolev spaces modeled on L^2. These spaces definitely play a crucial role in the study of partial differential equations, linear or not. The key tool will be the Fourier transform.

1.2.1 Definition of Sobolev spaces on $\mathbb{R}^d$

Definition 1.2.1 *Let s be a real number, a tempered distribution u belongs to the Sobolev space of index s, denoted $H^s(\mathbb{R}^d)$, or simply H^s if no confusion is possible, if and only if*

$$\widehat{u} \in L^2_{loc}(\mathbb{R}^d) \quad \text{and} \quad \widehat{u}(\xi) \in L^2(\mathbb{R}^d;(1+|\xi|^2)^s d\xi).$$

and we note

$$\|u\|^2_{H^s} = \int_{\mathbb{R}^d}(1+|\xi|^2)^s|\widehat{u}(\xi)|^2 d\xi.$$

Proposition 1.2.1 *For any s real number, the space H^s, equipped with the norm $\|\cdot\|_{H^s}$, is a Hilbert space.*

Proof. The fact that the norm $\|\cdot\|_{H^s}$ comes from the scalar product

$$(u|v)_{H^s} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d}(1+|\xi|^2)^s\widehat{u}(\xi)\overline{\widehat{v}(\xi)}d\xi$$

is obvious. Let us prove that this space is complete. Let $(u_n)_{n\in\mathbb{N}}$ a Cauchy sequence of H^s. By definition of the norm, the sequence $(\widehat{u}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of the space $L^2(\mathbb{R}^d;(1+|\xi|^2)^s d\xi)$. Thus, a function $\widetilde{u}$ exists in the space $L^2(\mathbb{R}^d;(1+|\xi|^2)^s d\xi)$ such that

$$\lim_{n\to\infty}\|\widehat{u}_n-\widetilde{u}\|_{L^2(\mathbb{R}^d;(1+|\xi|^2)^s d\xi)} = 0. \tag{1.2}$$

In particular, the sequence$(\widehat{u}_n)_{n\in\mathbb{N}}$ tends to $\widetilde{u}$ in the space $\mathcal{S}'$ of tempered distribtions. Let $u = \mathcal{F}^{-1}\widetilde{u}$. As the Fourier transform is an isomorphism of $\mathcal{S}'$, the sequence $(u_n)_{n\in\mathbb{N}}$ tends to u in the space $\mathcal{S}'$, and also in H^s thanks to (1.2).

Shortly said, this is nothing more than observing that the Fourier transform is an isometric isomorphism from H^s onto $L^2(\mathbb{R}^d;(1+|\xi|^2)^s d\xi)$. □

Proposition 1.2.2 *Let s be a non negative integer, the space $H^s(\mathbb{R}^d)$ is the space of functions u of L^2 all the derivatives of which of order less or equal to m are distributions which belongs to L^2. Moreover, the space H^m equipped with the norm*

$$\widetilde{\|}u\|_{H^m}^2 \stackrel{\text{def}}{=} \sum_{|\alpha|\leq m} \|\partial^\alpha u\|_{L^2}^2$$

is a Hilbert space and this norm is equivalent to the norme $\|\cdot\|_{H^s}$.

Proof. The fact that

$$\widetilde{\|}u\|_{H^m}^2 = \widetilde{(}u|u)_{H^m} \quad \text{with} \quad \widetilde{(}u|v)_{H^m} \stackrel{\text{déf}}{=} \sum_{|\alpha|\leq m} \int_{\mathbb{R}^d} \partial^\alpha u(x) \overline{\partial^\alpha v(x)} dx.$$

ensures that the norm $\|\cdot\|_{H^m}$ comes from a scalar product. Moreover, a constant C exists such that

$$\forall \xi \in \mathbb{R}^d\,,\ C^{-1}\Big(1 + \sum_{0<|\alpha|\leq m} |\xi|^{2|\alpha|}\Big) \leq (1+|\xi|^2)^s \leq C\Big(1 + \sum_{0<|\alpha|\leq m} |\xi|^{2|\alpha|}\Big). \tag{1.3}$$

As the Fourier transform is, up to a constant, an isometric isomorphism from L^2 onto L^2, we have

$$\partial^\alpha u \in L^2 \Longleftrightarrow \xi^\alpha \widehat{u} \in L^2.$$

Thus, we have deduce that

$$u \in H^m \Longleftrightarrow \forall \alpha \,/\, |\alpha| \leq m\,,\ \partial^\alpha u \in L^2.$$

Inequality (1.3) ensures the equivalence of the two norms using again the fact that the Fourier transform is a isometric isomorphism up to a constant. The proposition is proved. □

Let us prove now an interpolation inequality which will be very useful.

Proposition 1.2.3 *If $s = \theta s_1 + (1-\theta)s_2$ with $\theta \in [0,1]$, then, we have*

$$\|u\|_{H^s} \leq \|u\|_{H^{s_1}}^\theta \|u\|_{H^{s_2}}^{1-\theta}.$$

The proof consits in applying Hölder inequality with the measure $|\widehat{u}(\xi)|^2 d\xi$ and the two functions $(1+|\xi|^2)^{\theta s_1}$ and $(1+|\xi|^2)^{(1-\theta)s_2}$.

Theorem 1.2.1 *For any real number s the space $\mathcal{D}(\mathbb{R}^d)$ is dense in $H^s(\mathbb{R}^d)$.*

Proof. Let us consider a distribution u of H^s such that, for any φ in $\mathcal{D}$, we have $(\varphi|u)_{H^s} = 0$. This means that, for any test function φ, we have

$$\int_{\mathbb{R}^d} \widehat{\varphi}(\xi)(1+|\xi|^2)^s \overline{\widehat{u}(\xi)} d\xi = 0.$$

As $\mathcal{S}$ is continuously included in H^s, as $\mathcal{D}$ is dense in $\mathcal{S}$, and as the Fourier transform an isomorphism of $\mathcal{S}$, we have, for any function f of $\mathcal{S}$,

$$\int_{\mathbb{R}^d} f(\xi)(1+|\xi|^2)^s \overline{\widehat{u}(\xi)} d\xi = 0.$$

As $\mathcal{S}$ is dense in L^2, this implies that $(1+|\xi|^2)^s \widehat{u}(\xi) = 0$, thus $\widehat{u} = 0$ and thus $u = 0$. □

Let us prove a theorem which describes the dual of the space H^s.

Theorem 1.2.2 *The bilinear form B defined by*

$$B\left\{\begin{array}{ccc}\mathcal{S}\times\mathcal{S} & \to & \mathbb{C}\\ (u,\varphi) & \mapsto & \displaystyle\int_{\mathbb{R}^d} u(x)\varphi(x)dx\end{array}\right.$$

can b extended as a bilinear form continuous from $H^{-s}\times H^s$ to $\mathbb{C}$. Moreover, the map δ_B defined by

$$\delta_B\left\{\begin{array}{ccc}H^{-s} & \longrightarrow & (H^s)'\\ u & \longmapsto & \delta_B(u):(\varphi)\mapsto B(u,\varphi)\end{array}\right.$$

is a linear and isometric isomorphism (up to a constant), which means that the bilinear form B identifies the space H^{-s} to the dual space of H^s.

The important point of the proof of this theorem is inverse Fourier formula which ensures that, for any couple (u,φ) of functions of $\mathcal{S}$, we have

$$\begin{aligned} B(u,\varphi) &= \int_{\mathbb{R}^d} u(x)\varphi(x)dx\\ &= \int_{\mathbb{R}^d} u(x)\mathcal{F}(\mathcal{F}^{-1}\varphi)(x)dx\\ &= \int_{\mathbb{R}^d} \widehat{u}(\xi)(\mathcal{F}^{-1}\varphi)(\xi)d\xi\\ &= (2\pi)^{-d}\int_{\mathbb{R}^d} \widehat{u}(\xi)\widehat{\varphi}(-\xi)d\xi. \end{aligned} \tag{1.4}$$

Multiplying and dividing by $(1+|\xi|^2)^{\frac{s}{2}}$, we immediately get thanks to Cauchy-Schwarz inequality,

$$|B(u,\varphi)|\leq (2\pi)^{-d}\|u\|_{H^s}\|\varphi\|_{H^{-s}}.$$

Thus the first point of the theorem. The fact that the map δ_B is one to one comes from the fact that if, for any function $\varphi\in\mathcal{S}$, we have $B(u,\varphi)=0$, then u is 0. We have to prove that the map is onto. In fact, we shall prove that δ_B is one to one and onto. For any real number σ, the Fourier transform is an isometric (up to a constant) isomorphism from H^σ onto $L^2(\mathbb{R}^d,(1+|\xi|^2)^\sigma d\xi)$. Let us now consider the bilinear form $\check{B}$ defined by

$$\check{B}\left\{\begin{array}{ccc}L^2(\mathbb{R}^d,(1+|\xi|^2)^{-s}d\xi)\times L^2(\mathbb{R}^d,(1+|\xi|^2)^{s}d\xi) & \longrightarrow & \mathbb{C}\\ (\phi,f) & \longmapsto & \displaystyle(2\pi)^{-d}\int_{\mathbb{R}^d} f(\xi)\phi(-\xi)d\xi.\end{array}\right.$$

If we prove that

$$\delta_B={}^t\mathcal{F}\delta_{\check{B}}\mathcal{F}, \tag{1.5}$$

then Theorem 1.2.2 will be proved. Indeed, as $\mathcal{F}$ is an isomorphism from H^s onto $L^2(\mathbb{R}^d,(1+|\xi|^2)^s d\xi)$, the map ${}^t\mathcal{F}$ is an isomorphism from $(L^2(\mathbb{R}^d,(1+|\xi|^2)^s d\xi))'$ onto $(H^s)'$. We know that $\delta_{\check{B}}$ is an isomorphism from the space $(L^2(\mathbb{R}^d,(1+|\xi|^2)^s d\xi))'$ onto the space $L^2(\mathbb{R}^d,(1+|\xi|^2)^{-s}d\xi)$.

In order to prove Formula (1.5), let us write that

$$\begin{aligned} \langle {}^t\mathcal{F}\delta_{\check{B}}\mathcal{F}u,\varphi\rangle &= \langle \delta_{\check{B}}\mathcal{F}u,\mathcal{F}\varphi\rangle\\ &= \delta_{\check{B}}(\mathcal{F}u,\mathcal{F}\varphi). \end{aligned}$$

Thanks to Identity (1.4), we have $\langle {}^t\mathcal{F}\delta_{\check{B}}\mathcal{F}u,\varphi\rangle=\langle\delta_B(u),\varphi\rangle$. Thus the theorem is proved. □

1.2.2 Sobolev embeddings

The purpose of this section is the study of embedding properties of Sobolev spaces $H^s(\mathbb{R}^d)$ into L^p spaces . Let us prove the following theorem.

Theorem 1.2.3 *If s is greater than $d/2$, then the space H^s is continuously included in the space of continuous functions which tend to 0 at infinity. If s is a positive real number less than $d/2$, then the space H^s is continuously included in $L^{\frac{2d}{d-2s}}$ and we have*

$$\|f\|_{L^p} \leq C\|f\|_{\dot{H}^s} \quad \textit{with} \quad \|f\|_{\dot{H}^s} \stackrel{\text{def}}{=} \Big(\int_{\mathbb{R}^d} |\xi|^{2s}|\widehat{f}(\xi)|^2 d\xi\Big)^{\frac{1}{2}}.$$

Proof. The first point of this theorem is very easy to prove. Let us use the fact that

$$\|u\|_{L^\infty} \leq (2\pi)^{-d}\|\widehat{u}\|_{L^1} \tag{1.6}$$

Indeed, if s is greater than $d/2$, we have,

$$|\widehat{u}(\xi)| \leq (1+|\xi|^2)^{-s/2}(1+|\xi|^2)^{s/2}|\widehat{u}(\xi)|. \tag{1.7}$$

The fact that s is greater than $d/2$ implies that the function

$$\xi \longmapsto (1+|\xi|^2)^{-s/2}$$

belongs to L^2. Thus, we have

$$\|\widehat{u}\|_{L^1} \leq \Big(\int (1+|\xi|^2)^{-s} d\xi\Big)^{\frac{1}{2}} \|u\|_{H^s}.$$

The first point of the theorem is proved.

The proof of the second point is more delicate. A way to unterstand the index $p = 2d/(d-2s)$ is the use of a scaling argument. Let us consider a function v on $\mathbb{R}^d$ and let us denote by v_λ the function $v_\lambda(x) = v(\lambda x)$. We have

$$\|v_\lambda\|_{L^p} = \lambda^{-\frac{d}{p}}\|v\|_{L^p}$$

and also

$$\begin{aligned} \int |\xi|^{2s}|\widehat{v_\lambda}(\xi)|^2 d\xi &= \lambda^{-2d}\int |\xi|^{2s}|\widehat{v}(\lambda^{-1}\xi)|^2 d\xi \\ &= \lambda^{-d+2s}\|v\|^2_{\dot{H}^s}, \end{aligned}$$

The two quantities $\|\cdot\|_{L^p}$ and $\|\cdot\|_{\dot{H}^s}$ have the same scaling, which means that they have the same behaviour with respect to changes of unit. Thus, it make sense to compare them.

Multiplying f by a positive real number, it is enough to prove the inequality in the case when $\|f\|_{\dot{H}^s} = 1$. On utilise then the fact that for any p de the interval $]1,+\infty[$, we have, for any function measurable f,

$$\|f\|^p_{L^p} = p\int_0^\infty \lambda^{p-1} m(|f|>\lambda)d\lambda.$$

Let us decompose f in a low and in a high frequencies by writing

$$f = f_{1,A} + f_{2,A} \quad \text{with} \quad f_{1,A} = \mathcal{F}^{-1}(\mathbf{1}_{B(0,A)}\widehat{f}) \quad \text{and} \quad f_{2,A} = \mathcal{F}^{-1}(\mathbf{1}_{B^c(0,A)}\widehat{f}). \tag{1.8}$$

As the support of the Fourier transform of $f_{1,A}$ is compact, the function $f_{1,A}$ is bounded and more precisely,

$$
\begin{aligned}
\|f_{1,A}\|_{L^\infty} &\leq (2\pi)^{-d}\|\widehat{f_{1,A}}\|_{L^1} \\
&\leq (2\pi)^{-d}\int_{B(0,A)} |\xi|^{-s}|\xi|^{s}|\widehat{f}(\xi)|d\xi \\
&\leq (2\pi)^{-d}\left(\int_{B(0,A)} |\xi|^{-2s}d\xi\right)^{\frac{1}{2}} \\
&\leq \frac{C}{(d-2s)^{\frac{1}{2}}}A^{\frac{d}{2}-s}. \qquad (1.9)
\end{aligned}
$$

The triangle inequality implies that, for any positive real number A,

$$
(|f|>\lambda)\subset(2|f_{1,A}|>\lambda)\cup(2|f_{2,A}|>\lambda).
$$

Using Inequality (1.9), we have

$$
A=A_\lambda \stackrel{\text{def}}{=} \left(\frac{\lambda(d-2s)^{\frac{1}{2}}}{4C}\right)^{\frac{p}{d}} \Longrightarrow m\left(|f_{1,A}|>\frac{\lambda}{2}\right)=0.
$$

Thus we deduce that

$$
\|f\|_{L^p}^p = p\int_0^\infty \lambda^{p-1}m(2|f_{2,A_\lambda}|>\lambda)d\lambda.
$$

it is well known (this is Bienaimé-Tchebychev inequality) that

$$
\begin{aligned}
m\left(|f_{2,A_\lambda}|>\frac{\lambda}{2}\right) &= \int_{\left(|f_{2,A_\lambda}|>\frac{\lambda}{2}\right)} dx \\
&\leq \int_{\left(|f_{2,A_\lambda}|>\frac{\lambda}{2}\right)} \frac{4|f_{2,A_\lambda}(x)|^2}{\lambda^2}dx \\
&\leq 4\frac{\|f_{2,A_\lambda}\|_{L^2}^2}{\lambda^2}.
\end{aligned}
$$

For such a choice of A, we have

$$
\|f\|_{L^p}^p \leq 4p\int_0^\infty \lambda^{p-3}\|f_{2,A_\lambda}\|_{L^2}^2 d\lambda. \qquad (1.10)
$$

As the Fourier transform is (up to a constant) an isometric isomorphism of L^2, we have

$$
\|f_{2,A_\lambda}\|_{L^2}^2 = (2\pi)^{-d}\int_{(|\xi|\geq A_\lambda)} |\widehat{f}(\xi)|^2 d\xi.
$$

Thanks to Inequality (1.10), we get

$$
\|f\|_{L^p}^p \leq 4p(2\pi)^{-d}\int_{\mathbb{R}_+\times\mathbb{R}^d} \lambda^{p-3}\mathbf{1}_{\{(\lambda,\xi)\,/\ |\xi|\geq A_\lambda\}}(\lambda,\xi)|\widehat{f}(\xi)|^2 d\xi d\lambda.
$$

By definition of A_λ, we have

$$
|\xi|\geq A_\lambda \Longleftrightarrow \lambda\leq C_\xi \stackrel{\text{déf}}{=} \frac{4C}{(d-2s)^{\frac{1}{2}}}|\xi|^{\frac{d}{p}}.
$$

Fubini's theorem implies that

$$\begin{aligned}\|f\|_{L^p}^p &\leq 4p(2\pi)^{-d}\int_{\mathbb{R}^d}\left(\int_0^{C_\xi}\lambda^{p-3}d\lambda\right)|\widehat{f}(\xi)|^2 d\xi\\ &\leq 4\frac{p(2\pi)^d}{p-2}\left(\frac{4C}{(d-2s)^{\frac{1}{2}}}\right)^{p-2}\int_{\mathbb{R}^d}|\xi|^{\frac{d(p-2)}{p}}|\widehat{f}(\xi)|^2 d\xi.\end{aligned}$$

As $2s = \dfrac{d(p-2)}{p}$, the theorem is proved. □

Corollary 1.2.1 *Let p be in $]1,2[$. Then we have*

$$L^p \subset \dot{H}^{-s} \quad \textit{with} \quad s = d\Big(\frac{1}{p}-\frac{1}{2}\Big)\cdot$$

Proof. Let us observe that

$$\|f\|_{\dot{H}^{-s}} \leq C \sup_{\substack{\varphi\in\mathcal{D}\\ \|\varphi\|_{\dot{H}^s}\leq 1}} \langle f,\varphi\rangle.$$

Sobolev embedding implies that

$$\sup_{\substack{\varphi\in\mathcal{D}\\ \|\varphi\|_{\dot{H}^s}\leq 1}} \langle f,\varphi\rangle \leq C \sup_{\substack{\varphi\in\mathcal{D}\\ \|\varphi\|_{L^{p'}}\leq 1}} \langle f,\varphi\rangle \quad \text{with} \quad p' = \frac{2d}{d-2s}\cdot$$

Hölder inequality implies that

$$\|f\|_{\dot{H}^{-s}} \leq C\|f\|_{L^p}$$

and the corollary is proved. □

Let us study interpolation properties.

Proposition 1.2.4 *Let s_1 and s_2 be two reals numbers such that $s_1 < s_2$. We have*

$$\|u\|_{H^s} \leq \|u\|_{H^{s_1}}^{\theta}\|u\|_{H^{s_2}}^{1-\theta} \quad \textit{and} \quad \|u\|_{\dot{H}^s} \leq \|u\|_{\dot{H}^{s_1}}^{\theta}\|u\|_{\dot{H}^{s_2}}^{1-\theta} \quad \textit{with} \quad s=\theta s_1+(1-\theta)s_2.$$

Proof. Applying Hölder inequality for the two functions $(1+|\xi|^2)^{\theta s_1}$ and $(1+|\xi|^2)^{(1-\theta)s_2}$ (or $(|\xi|^{2\theta s_1}$ and $|\xi|2(1-\theta)s_2)$ and the measure $|\widehat{u}(\xi)|^2 d\xi$ gives the result. □

Together with the Sobolev inequality, we have the following corollary.

Corollary 1.2.2

$$\|u\|_{L^p} \leq C\|u\|_{L^2}^{1-\frac{\sigma}{s}}\|u\|_{\dot{H}^s}^{\frac{\sigma}{s}} \quad \textit{with} \quad \sigma = d\Big(\frac{1}{2}-\frac{1}{p}\Big)\cdot$$

1.2.3 The spaces $H_0^1(\Omega)$ and $H^{-1}(\Omega)$

Definition 1.2.2 *Let Ω a domain of $\mathbb{R}^d$, the space $H_0^1(\Omega)$ is defined as the closure of $\mathcal{D}(\Omega)$ in the sense of the norm $H^1(\mathbb{R}^d)$.*

The space $H^{-1}(\Omega)$ is the set of distributions u on Ω such that

$$\|u\|_{H^{-1}(\Omega)} \stackrel{def}{=} \sup_{\substack{f\in\mathcal{D}(\Omega)\\ \|f\|_{H_0^1(\Omega)}\leq 1}} |\langle u,f\rangle| < \infty.$$

Proposition 1.2.5 *The space $H^1_0(\Omega)$ is a Hilbert space equipped with the norm*

$$\left(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$

The proof is an easy exercice left to the reader. The space $H^{-1}(\Omega)$ can be identified to the dual space of $H^1_0(\Omega)$ thanks to the following theorem.

Theorem 1.2.4 *The bilinear map defined by*

$$B\left\{\begin{array}{ccc} H^{-1}(\Omega)\times\mathcal{D}(\Omega) & \longrightarrow & \mathbb{C}\\ (u,\varphi) & \longmapsto & \langle u,\varphi\rangle\end{array}\right.$$

can be extended to a bilinear continuous map from $H^{-1}(\Omega)\times H^1_0(\Omega)$ into $\mathbb{C}$, still denoted by B. Moreover, the map δ_B defined by

$$\delta_B\left\{\begin{array}{ccc} H^{-1}(\Omega) & \longrightarrow & (H^1_0(\Omega))'\\ u & \longmapsto & \delta_B(u)(\varphi)\stackrel{\text{def}}{=}B(u,\varphi)\end{array}\right.$$

is a linear isometric isomorphism between the space $H^{-1}(\Omega)$ and the dual space of $H^1_0(\Omega)$.

The fact that the bilinear map B can be extended because B is uniformely continuous. Let ℓ a linear form continuous on $H^1_0(\Omega)$. Its restriction on $\mathcal{D}(\Omega)$ is a distribution u on Ω such that

$$\forall\varphi\in\mathcal{D}(\Omega)\,,\ \langle u,\varphi\rangle=\langle\ell,\varphi\rangle.$$

By definition of the norm on $(H^1_0(\Omega))'$, the theorem is proved.

Theorem 1.2.5 (Poincaré Inequality) *Let Ω be* **bounded** *open subset of $\mathbb{R}^d$. A constant C exists such that*

$$\forall\varphi\in H^1_0(\Omega)\,,\ \|\varphi\|_{L^2}\leq C\left(\sum_{j=1}^d\|\partial_j\varphi\|^2_{L^2}\right)^{\frac{1}{2}}.$$

Let R a positive real number such that Ω is included in $]-R,R[\times\mathbb{R}^{d-1}$.Then, for any test function φ, we have

$$\varphi(x_1,\cdots,x_d)=\int_{-R}^{x_1}\frac{\partial\varphi}{\partial y_1}(y_1,x_2,\cdots,x_d)dy_1.$$

Cauchy-Schwarz Inequality implies that

$$|\varphi(x_1,\cdots,x_d)|^2\leq 2R\int_{-R}^{x_1}\left|\frac{\partial\varphi}{\partial y_1}(y_1,x_2,\cdots,x_d)\right|^2dy_1.$$

By integration in x_1, we get

$$\int_\Omega|\varphi(x_1,\cdots,x_d)|^2dx_1\leq 2R\int_{\Omega\times]-R,R[}\left|\frac{\partial\varphi}{\partial y_1}(y_1,x_2,\cdots,x_d)\right|^2dy_1.$$

Then, integrating with respect to the other $d-1$ variables, we find

$$\begin{aligned}\int_\Omega|\varphi(x_1,\cdots,x_d)|^2dx &\leq 2R\int_{\Omega\times]-R,R[}\left|\frac{\partial\varphi}{\partial y_1}(y_1,x_2,\cdots,x_d)\right|^2dy_1dx_2\cdots dx_d\\ &\leq 4R^2\sum_{j=1}^d\|\partial_j\varphi\|^2_{L^2}.\end{aligned}$$

As $\mathcal{D}(\Omega)$ is dense in $H^1_0(\Omega)$, the theorem is proved. It obviously implies the following corollary.

Corollary 1.2.3 *Let Ω be a* bounded *domain of $\mathbb{R}^d$. The space $H^1_0(\Omega)$ equipped with the norm*

$$u \longmapsto \Big(\sum_{j=1}^d \|\partial_j u\|^2_{L^2}\Big)^{\frac12} \stackrel{\text{def}}{=} \|\nabla u\|_{L^2}$$

is a Hilbert space and the this norm is equivalent to the previous one.

Let us recall the very important compactness result.

Theorem 1.2.6 *Let Ω be bounded domain on $\mathbb{R}^d$. The space $H^1_0(\Omega)$ is compactly embedded into $L^2(\Omega)$.*

1.2.4 The Dirichlet problem

In this section, Ω denotes a bounded domain of $\mathbb{R}^d$. Let f be an element of $H^{-1}(\Omega)$, let us consider the functionnal F defined par

$$F\left\{\begin{array}{ccc} H^1_0(\Omega) & \longrightarrow & \mathbb{R} \\ u & \longmapsto & \dfrac12\|\nabla u\|^2_{L^2} - \langle f, u\rangle. \end{array}\right.$$

Dirichlet Theorem is the following:

Theorem 1.2.7 *The functionnal F has a unique minimum which is the unique solution in $H^1_0(\Omega)$ of $-\Delta u = f$ in the distribution sense in Ω.*

Proof. Let us observe that the functionnal F bounded from below because

$$\begin{aligned} F(u) &\geq \frac12\|\nabla u\|^2_{L^2} - \|\nabla u\|_{L^2}\|f\|_{H^{-1}(\Omega)} \\ &\geq \frac12\big(\|\nabla u\|_{L^2} - \|f\|_{H^{-1}(\Omega)}\big)^2 - \frac12\|f\|^2_{H^{-1}(\Omega)}. \end{aligned} \tag{1.11}$$

The functionnal F has a lower bound m. Let us consider a minimizing sequence $(u_n)_{n\in\mathbb{N}}$ i.e. a sequence $(u_n)_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to\infty} F(u_n) = m$. Using Inequality (1.11), we have

$$\|\nabla u_n\|_{L^2} \leq \big(2F(u_n) + \|f\|_{H^{-1}(\Omega)}\big)^{\frac12} + \|f\|_{H^{-1}(\Omega)}.$$

The sequence $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence of the space $H^1_0(\Omega)$ which is complete. Thus it exists a function u in $H^1_0(\Omega)$ and a subsequence of $(u_n)_{n\in\mathbb{N}}$ (which we still denote by $(u_n)_{n\in\mathbb{N}}$) such that $(u_n)_{n\in\mathbb{N}}$ tends weakly to u. Moreover, we know that the sequence $(\|\nabla u_n\|_{L^2})_{n\in\mathbb{N}}$ converges to $m + \langle f, u\rangle$. Thanks to the properties of the weak limit we have

$$\lim_{n\to\infty} \|\nabla u_n\|_{L^2} \geq \|\nabla u\|_{L^2}.$$

Let us assume that $\|\nabla u\|_{L^2} < \lim\limits_{n\to\infty} \|\nabla u_n\|_{L^2}$. Then, we have $F(u) < m$ which is in contradiction with the fact that m is the infimum of F. Thus

$$\lim_{n\to\infty} \|\nabla u_n\|_{L^2} = \|\nabla u\|_{L^2}$$

and then the lower bound is a minimum and the sequence $(u_n)_{n\in\mathbb{N}}$ converges strongly to u in $H^1_0(\Omega)$).

Now let us prove that u is a solution of Laplace Equation. The functionnal F is the sum of the quadratic functionnal (the norm to the square) and of of a linear functionnal (both continuous). We have, for any function h of $H^1_0(\Omega)$,

$$F(u+h) = F(u) + 2(\nabla u|\nabla h)_{L^2} - \langle f, h\rangle + \|\nabla h\|^2_{L^2}. \tag{1.12}$$

If u is a minumum, then the differential vanishes at u and thus u is a solution of Laplace Equation. Moreover, Relation (1.12) implies that the minimum is unique and it is characterised by the fact that, for any h in $H^1_0(\Omega)$, we have $(\nabla u|\nabla h)_{L^2} - \langle f, h\rangle = 0$. Thus the theorem is proved. □

Exercice 1.2.1 *Let Ω a bounded domain of $\mathbb{R}^d$ and f a distribution of $H^{-1}(\Omega)$. Prove that a vector field v exists in $L^2(\Omega)$ such that $\operatorname{div} v = f$.*

Let us prove now a result about the spectral structure of the Laplacian in a bounded domain.

Theorem 1.2.8 *It exists a non decreasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ of positive real numbers which thends to infinity and a hilbertian basis of $L^2(\Omega)$ denoted by $(e_k)_{k\in\mathbb{N}}$ such that the sequence $(\lambda_k^{-1}e_k)_{k\in\mathbb{N}}$ is an orthonormal basis of $H^1_0(\Omega)$ such that*

$$-\Delta e_k = \lambda_k e_k.$$

Moreover, if f belongs to $H^{-1}(\Omega)$, then

$$\|f\|^2_{H^{-1}(\Omega)} = \sum_k \lambda_k^{-2}(\langle f, e_k\rangle)^2.$$

Proof. As the space L^2 is continuously included in $H^{-1}(\Omega)$, we can define an operator B as follows:

$$B \begin{cases} L^2 & \longrightarrow \quad H^1_0(\Omega) \subset L^2(\Omega) \\ f & \longmapsto \quad u \end{cases}$$

such that u is the solution in $H^1_0(\Omega)$ of $-\Delta u = f$. The operator B is of course continuous from $L^2(\Omega)$ into $H^1_0(\Omega)$. Thanks to Rellich's theorem, the operator B is compact from $L^2(\Omega)$ into $L^2(\Omega)$. Moreover, the operator B is selfadjoint and positive, i.e. that, for any couple of functions of $L^2(\Omega)$ (f, g), we have

$$(Bf|g)_{L^2} = (f|Bg)_{L^2} \quad \text{and} \quad (Bf|f)_{L^2} > 0 \quad \text{if} \quad f \neq 0.$$

By definition of B, it exists a couple of functions in $H^1_0(\Omega)$ (u, v) such that we have,

$$(Bf|g)_{L^2} = -(Bf|\Delta Bg)_{L^2} = (\nabla Bf|\nabla Bg)_{L^2}.$$

Thus the operator B is compact, selfadjoint and positive. The spectral theorem applied to B implies the existence of a non increasing sequence $(\mu_k)_{k\in\mathbb{N}}$ of positive real numbers which tends to 0 and a hilbertian basis of $L^2(\Omega)$ denoted $(e_k)_{k\in\mathbb{N}}$ such that , for any k, the function e_k belongs to $L^2(\Omega)$ and such that $Be_k = \mu_k e_k$. This implies that $-\Delta e_k = \mu_k^{-1}e_k$. We have,

$$\|f\|_{H^{-1}(\Omega)} = \sup_{(c_k)\in B_f} \langle f, \sum_k \lambda_k^{-1} c_k e_k\rangle$$

where B_f denotes the set of sequences having only a finite number of non zero terms and of ℓ^2 norm less or equal to to 1. Thus

$$\|f\|_{H^{-1}(\Omega)} = \sup_{(c_k)\in B_f} \sum_k \lambda_k^{-1}\langle f, e_k\rangle c_k = \|(\lambda_k^{-1}\langle f, e_k\rangle)_{k\in\mathbb{N}}\|_{\ell^2(\mathbb{N})}.$$

Theorem 1.2.8 is proved. □

Remarks

- To know more about ordinary differential equations and their historical aspect of Osgood's theory, see the book by T.M. Fleet, *Differential analysis*, Cambridge University Press, 1980.

Chapter 2

The Stokes problem

2.1 The stationnary Stokes problem

This problem is analogous to the Dirichlet problem, but we work on the set of divergence free vector field. Nevertheless, the fact that we impose a constrain (even a linear one) of the space on which we search the minimum will introduce an important change. The Laplace eqaution will become the Stokes equation. Let us first define of the space we are going to work with.

Definition 2.1.1 *Let us denote by $\mathcal{V}_\sigma(\Omega)$ the set of divergence free vector fields whose componants are in $H^1_0(\Omega)$ and by $\mathcal{H}(\Omega)$ the closure in $(L^2(\Omega))^d$ de $\mathcal{V}_\sigma(\Omega)$. When no confusion is possible, we omit the mention Ω in the notations.*

Let us state the analogous of Dirichlet theorem in this framework. As in the preceeding section, let us consider a vector field f whose componants are in H^{-1}; then we define the functionnal F

$$F \left\{ \begin{array}{ccl} \mathcal{V}_\sigma & \longrightarrow & \mathbb{R} \\ u & \longmapsto & \dfrac{1}{2}\|\nabla u\|^2_{L^2} - \langle f, u\rangle. \end{array} \right.$$

Theorem 2.1.1 *Let $f \in \mathcal{V}'$. It exists a unique minimum of the functionnal F which is also the unique solution of following equation*

$$-\Delta u - f \in \mathcal{V}_\sigma^\circ$$

which means that, for any vector field v of $\mathcal{V}_\sigma(\Omega)$, we have

$$-\langle \Delta u, v\rangle = \langle f, v\rangle. \tag{2.1}$$

The existence and the uniqueness of a minimum u for the functionnal F can be proved following exactly the same lines as in the case of Dirichlet problem. The fact that the differential of F vanishes at point u implies the relation (2.1).

Remarks

- The fact that a vector field g of H^{-1} belongs the polar set (in the sense of the duality) of H^1_0 implies in particular that, for any function φ of $\mathcal{D}(\Omega)$, we have

  $$\langle g^i, -\partial_j\varphi\rangle + \langle g^j, \partial_i\varphi\rangle = 0$$

  which implies that $\partial_j g^i - \partial_i g^j = 0$, i.e. the curl of g is identically 0.

- Very simple domains exist sucht that it exists a vector field of $H^{-1}(\Omega)$ which are of divergence and of curl identically 0 and which are not gradients of functions.

Let us consider the domain of the plan $\Omega \stackrel{\text{def}}{=} \{x \in \mathbb{R}^2 \;/\; 0 < R_1 < |x| < R_2\}$ and the vector field f defined by $(-\partial_2 \log|x|, \partial_1 \log|x|)$. We have the following lemma.

Proposition 2.1.1 *The vector field f is of curl free, but it is not the gradient of a function.*

The fact that its curl is 0 follows from the fact that that the function $x \mapsto \log|x|$ is harmonic on Ω. Let us assume that f is a gradient of some distribution $-p$. As f is smooth, p is also smooth. Let us consider the flow of $f = -\nabla p$. By definition of f, its trajectories are circles, thus they are periodic. Let us consider a trajectory γ from of a point of Ω such that $f \neq 0$ (here all points are like this). We have

$$\frac{d}{dt}(p \circ \gamma)(t) = \Big(\frac{d\gamma}{dt}\Big|\nabla p(\gamma(t))\Big)_{L^2} = -|\nabla p(\gamma(t))|^2 \leq 0.$$

The fact that the derivative en $t = 0$ is negative is in contradiction with the periodicity of the trajectory γ. Proposition 2.1.1 is proved.

As shown by the following proposition, belonging to the polar space (in the sens of the duality $H^{-1}, H^1_0(\Omega))$ of $(\mathcal{V}_\sigma(\Omega))^\circ$ is stronger than being curl free. Let us admit the following proposition .

Proposition 2.1.2 *Let f in $\mathcal{V}'$. If f belongs à $\mathcal{V}_\sigma^\circ$ i.e. if*

$$\forall v \in \mathcal{V}_\sigma\,,\ \sum_{j=1}^{d} \langle f^j, v^j\rangle = 0,$$

then it exists p in $\mathcal{D}'(\Omega)$ such that $f = -\nabla p$. If the boundary of Ω is a C^1, hypersurface, then $p \in L^2(\Omega)$.

As in the case of Dirichlet problem, let us apply a result of spectral theory on self adjoint compact operators in order to obtain the following theorem.

Theorem 2.1.2 *Let Ω be a bounded domain of $\mathbb{R}^d$. A non decreasing sequence $(\mu_k)_{k\in\mathbb{N}}$ of positive reals number which tends to infinity and a hilbertian basis of $\mathcal{H}$ denoted $(e_k)_{k\in\mathbb{N}}$ such that the sequence $(\mu_k^{-1}e_k)_{k\in\mathbb{N}}$ soit une base hilbertienne of $\mathcal{V}_\sigma$ and such that*

$$-\Delta e_k - \mu_k^2 e_k \in \mathcal{V}_\sigma^\circ.$$

Moreover, if $f \in \mathcal{V}'$, alors

$$\|f\|^2_{\mathcal{V}'_\sigma} = \sum_{k\in N} \mu_k^{-2}(\langle f, e_k\rangle)^2 \quad \text{and} \quad \lim_{n\to\infty}\Big\|f - \sum_{k=0}^{n}\langle f, e_k\rangle e_k\Big\|_{\mathcal{V}'_\sigma} = 0.$$

Proof. The proof is very close to the proof of Theorem 1.2.8. As the space $\mathcal{H}(\Omega)$ is continuously included in $H^{-1}(\Omega)$, we can define the operator B

$$B \left\{ \begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{V}_\sigma \subset \mathcal{H} \\ f & \longmapsto & u \end{array} \right.$$

such that u is the solution in $\mathcal{V}_\sigma$ of $-\Delta u - f \in \mathcal{V}_\sigma^\circ$. The following of the proof is strictement analogous to the one of Dirichlet problem.

Definition 2.1.2 *Let us denote by $\mathbb{P}$ the orthogonal projection of L^2 on $\mathcal{H}$ and by $\mathbb{P}_k$ the orthogonal projection on $\mathbb{E}_k \stackrel{\text{def}}{=} \text{Vect}\langle e_0, \cdots, e_k\rangle$ the vectorial space generated by the first $k+1$ eigenvectors of the Stokes problem.*

Let us observe that, for $u \in \mathcal{H}$,

$$\mathbb{P}_k\, u = \sum_{k=0}^{\infty} \langle u, e_k\rangle e_k \quad \text{and} \quad \mathbb{P}\, u = \sum_{k=0}^{\infty} \langle u, e_k\rangle e_k$$

Thus $\mathbb{P}_k$ can be define on $\mathcal{V}'$. Moreover, we have, for any $f \in \mathcal{V}'$, the sequence $(\mathbb{P}_k\, f)_{k\in\mathbb{N}}$ converge et l'on pose

$$\lim_{k\to\infty} \mathbb{P}_k\, f = \mathbb{P}\, f = \sum_{k=0}^{\infty} \langle f, e_k\rangle e_k.$$

2.2 The time dependent Stokes problem

The evolution Stokes problem reads as follows:

$$(ES)\begin{cases} \partial_t u - \Delta u &= f - \nabla p \\ \operatorname{div} u &= 0 \\ u_{|\partial\Omega} &= 0 \\ u_{|t=0} &= u_0 \in \mathcal{H}. \end{cases}$$

Let us define what a solution of this problem is.

Definition 2.2.1 *Let u_0 be in $\mathcal{H}$ and f in $L^2_{loc}(\mathbb{R}^+; \mathcal{V}')$. We shall say that u is a solution of (ES) with initial data u_0 and external force f if and only if u belongs to the space*

$$C(\mathbb{R}^+; \mathcal{V}'_\sigma) \cap L^\infty_{loc}(\mathbb{R}^+; \mathcal{H}) \cap L^2_{loc}(\mathbb{R}^+; \mathcal{V}_\sigma)$$

and satisfies, for any Ψ in $C^1(\mathbb{R}^+; \mathcal{V}_\sigma)$,

$$\langle u(t), \Psi(t)\rangle + \int_0^t \int_\Omega \Big(\nabla u : \nabla\Psi - u\cdot \partial_t \Psi\Big)(t', x)\, dx dt' = \int_\Omega u_0(x)\cdot \Psi(0, x)\, dx + \int_0^t \langle f(t'), \Psi(t')\rangle\, dt'.$$

The following theorem holds.

Theorem 2.2.1 *The problem (ES) has a unique solution in the sense of the above definition. Moreover this solution belongs to $C(\mathbb{R}^+; \mathcal{H})$ and satisfies*

$$\frac{1}{2}\|u(t)\|^2_{L^2} + \int_0^t \|\nabla u(t')\|^2_{L^2} dt' = \frac{1}{2}\|u_0\|^2_{L^2} + \int_0^t \langle f(t'), u(t')\rangle dt'.$$

Proof. In order to prove uniqueness, let us consider some function u in $C(\mathbb{R}^+; \mathcal{V}'_\sigma) \cap L^2_{loc}(\mathbb{R}^+; \mathcal{V}_\sigma)$ such that, for all Ψ in $C^1(\mathbb{R}^+; \mathcal{V}_\sigma)$,

$$\langle u(t), \Psi(t)\rangle + \int_0^t \int_\Omega \Big(\nabla u : \nabla\Psi - u\cdot \partial_t \Psi\Big)(t', x)\, dx dt' = 0.$$

This is valid in particular for the time independent function $\Psi(t) \equiv e_k$ where the family vector fields $(e_k)_{k\in\mathbb{N}}$ is given by Theorem 2.1.2. This gives

$$\begin{aligned}\langle u(t), e_k\rangle &= -\int_0^t \int_\Omega \nabla u(t',x) : \nabla e_k(x)dxdt'\\ &= \int_0^t \langle u(t'), \Delta\, \mathbb{P}_k\, \Psi\rangle.\end{aligned}$$

Thanks to the spectral Theorem 2.1.2 together with the fact that, for almost every t', $u(t)$ belongs to $\mathcal{V}_\sigma$, we have

$$-\int_\Omega \nabla u(t',x) : \nabla e_k(x)dx = \langle \Delta e_k, u(t')\rangle = \mu_k^2\langle e_k, u(t')\rangle.$$

This gives

$$\langle u(t), e_k\rangle = \int_0^t \mu_k^2\langle e_k, u(t')\rangle dt'.$$

The fact that $\langle u(0), e_k\rangle = 0$ implies that, for any k, $\langle u(t), e_k\rangle = (u|e_k)_{\mathcal{H}} = 0$. Thus $u \equiv 0$.

In order to prove existence, let us consider a sequence $(f_k)_{k\in\mathbb{N}}$ associated with f by Lemma 3.2.1 page 26 and then the approximated problem

$$(ES_k)\left\{\begin{array}{rcl}\partial_t u_k - \mathbb{P}_k\,\Delta u_k &=& \mathbb{P}_k\, f\\ u_{k|t=0} &=& \mathbb{P}_k\, u_0\end{array}\right. \tag{2.1}$$

Again thanks to Theorem 2.1.2 page 18, it is a linear ordinary differential equation on $\mathcal{H}_k$ which has a global solution u_k which is $C^1(\mathbb{R}^+;\mathcal{H}_k)$. By an energy estimate in (ES_k) we get that

$$\frac{1}{2}\frac{d}{dt}\|u_k(t)\|_{L^2}^2 + \|\nabla u_k(t)\|_{L^2}^2 = \langle f_k(t), u_k(t)\rangle.$$

A time integration gives

$$\frac{1}{2}\|u_k(t)\|_{L^2}^2 + \int_0^t \|\nabla u_k(t')\|_{L^2}^2 dt' = \frac{1}{2}\|\,\mathbb{P}_k\, u(0)\|_{L^2}^2 + \int_0^t \langle f_k(t'), u_k(t')\rangle dt'. \tag{2.2}$$

In order to pass to the limit, we write an energy estimate for $u_k - u_{k+\ell}$, which gives

$$\begin{aligned}\delta_{k,\ell}(t) &\stackrel{\text{def}}{=} \frac{1}{2}\|(u_k-u_{k+\ell})(t)\|_{L^2}^2 + \int_0^t \|\nabla(u_k-u_{k+\ell})(t')\|_{L^2}^2 dt'\\ &= \frac{1}{2}\|(\mathbb{P}_k - \mathbb{P}_{k+\ell})u(0)\|_{L^2}^2 + \int_0^t \langle (f_k - f_{k+\ell})(t'), u_k(t')\rangle dt'\\ &\leq \frac{1}{2}\|(\mathbb{P}_k - \mathbb{P}_{k+\ell})u(0)\|_{L^2}^2 + \frac{1}{2}\int_0^t \|(f_k - f_{k+\ell})(t')\|_{\mathcal{V}_\sigma'}^2 dt'\\ &\qquad\qquad + \frac{1}{2}\int_0^t \|\nabla(u_k-u_{k+\ell})(t')\|_{L^2}^2 dt'.\end{aligned}$$

Using Poincaré's inequality, this implies that

$$\begin{aligned}&\|(u_k-u_{k+\ell})(t)\|_{L^2}^2 + \int_0^t \|\nabla(u_k-u_{k+\ell})(t')\|_{L^2}^2 dt'\\ &\qquad\qquad \leq \|(\mathbb{P}_k - \mathbb{P}_{k+\ell})u(0)\|_{L^2}^2 + \int_0^t \|(f_k - f_{k+\ell})(t')\|_{\mathcal{V}_\sigma'}^2 dt'.\end{aligned}$$

This implies immediately that the sequence $(u_k)_{k\in\mathbb{N}}$ is a Cauchy one in the space $C(\mathbb{R}^+;\mathcal{H})\cap L^2_{loc}(\mathbb{R}^+;\mathcal{V}_\sigma)$. Let us denote by u the limit and prove that u is a solution in the sense of Definition 2.2.1. As u_k is a C^1 solution of the ordinary differential equation (ES_k), we have, for a Ψ in $C^1(\mathbb{R}^+;\mathcal{V}_\sigma)$,

$$\frac{d}{dt}\langle u_k(t),\Psi(t)\rangle = \langle \Delta u_k(t),\Psi(t)\rangle + \langle f_k(t),\Psi(t)\rangle + \langle u_k(t),\partial_t\Psi(t)\rangle.$$

By time integration, we get

$$\begin{aligned}\langle u_k(t),\Psi(t)\rangle = -\int_0^t\int_\Omega \nabla u_k(t',x) : \nabla\Psi(t',x)\,dxdt'\\ +\int_0^t\langle f_k(t'),\Psi(t')\rangle dt' + \langle \mathbb{P}_k\, u(0),\Psi(0)\rangle + \int_0^t\langle u_k(t'),\partial_t\Psi(t')\rangle dt'.\end{aligned}$$

Passing to the limit in the above equality and in (2.2) gives the theorem. □

Remark. The solution is given by the explicit formula

$$\begin{aligned} u(t) &= \sum_{j\in\mathbb{N}} U_j(t)e_j \quad\text{with}\\ U_j(t) &\stackrel{\text{def}}{=} e^{-\mu_j^2 t}(u_0|e_j)_{L^2} + \int_0^t e^{-\mu_j^2(t-t')}\langle f(t'),e_j\rangle\,dt'. \end{aligned}\tag{2.3}$$

In the case of the whole space $\mathbb{R}^d$, we have the following analogous formula

$$\begin{aligned} u(t,x) &= (2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\widehat{u}(t,\xi)d\xi \quad\text{with}\\ \widehat{u}(t,\xi) &\stackrel{\text{def}}{=} e^{-t|\xi|^2}\widehat{u}_0(\xi) + \int_0^t e^{-(t-t')|\xi|^2}\widehat{f}(t',\xi)\,dt'. \end{aligned}\tag{2.4}$$

2.3 Some regularity results about the Stokes' problem

Let us first define a family of intermediate spaces between the spaces $\mathcal{V}'_\sigma$ and $\mathcal{V}_\sigma$. This can be done by abstract interpolation theory but we prefer to do it here in an explicit way.

Definition 2.3.1 *Let s be in $[-1,1]$. We shall denote by $\mathcal{V}^s_\sigma$ the space of vector fields u in $\mathcal{V}'$ such that*

$$\|u\|^2_{\mathcal{V}^s_\sigma} \stackrel{\text{def}}{=} \sum_{j\in\mathbb{N}} \mu_j^{2s}\langle u,e_j\rangle^2 < +\infty.$$

Theorem 2.1.2 implies that $\mathcal{V}^0_\sigma = \mathcal{H}$ and $\mathcal{V}^1_\sigma = \mathcal{V}_\sigma$. Moreover, it is obvious that, when s is non negative, $\mathcal{V}^s_\sigma$ endowed with the norm $\|\cdot\|_{\mathcal{V}^s_\sigma}$ is a Hilbert space.

The following proposition will be important in the following two paragraphs.

Proposition 2.3.1 *The space $\mathcal{V}^{\frac12}_\sigma$ is embedded in L^3 and the space $L^{\frac32}$ is embedded in $\mathcal{V}^{-\frac12}_\sigma$.*

Proof. This proposition can be proved using abstract interpolation theory. We prefer to present here a self contained proof in the spirit of the proof of Theorem 1.2.3. Let us consider *a* in $\mathcal{V}_\sigma^{\frac{1}{2}}$. Without any loss of generality, we can assume that $\|a\|_{\mathcal{V}_\sigma^{\frac{1}{2}}} \leq 1$. Let us define, for a positive real number Λ,

$$a_\Lambda \stackrel{\text{def}}{=} \sum_{j\,/\,\mu_j<\Lambda} \langle a, e_j\rangle e_j \quad \text{and} \quad b_\Lambda \stackrel{\text{def}}{=} a - a_\Lambda.$$

Using the fact that $\big\{x\in\Omega\,/\,|a(x)|>\Lambda\big\} \subset \big\{x\in\Omega\,/\,|a_\Lambda(x)|>\Lambda/2\big\} \cup \big\{x\in\Omega\,/\,|b_\Lambda(x)|>\Lambda/2\big\}$, we can write

$$\begin{aligned}
\|a\|_{L^3}^3 &\leq 3\int_0^{+\infty} \Lambda^2 \text{meas}\big(\big\{x\in\Omega\,/\,|a_\Lambda(x)|>\Lambda/2\big\}\big)\,d\Lambda \\
&\qquad\qquad + 3\int_0^{+\infty} \Lambda^2 \text{meas}\big(\big\{x\in\Omega\,/\,|b_\Lambda(x)|>\Lambda/2\big\}\big)\,d\Lambda \\
&\leq 3\times 2^6 \int_0^{+\infty} \Lambda^{-4}\|a_\Lambda\|_{L^6}^6\,d\Lambda + 3\times 2^2\int_0^{+\infty}\|b_\Lambda\|_{L^2}^2\,d\Lambda.
\end{aligned}$$

Thanks to Theorem 1.2.3, we have, by definition of the $\|\cdot\|_{\mathcal{V}_\sigma^s}$ norm,

$$\begin{aligned}
\|a_\Lambda\|_{L^6}^2 &\leq C\|a_\Lambda\|_{\mathcal{V}_\sigma}^2 \\
&\leq C\sum_{j\,/\,\mu_j<\Lambda} \mu_j^2\langle a, e_j\rangle^2 \\
&\leq C\Lambda \sum_{j\,/\,\mu_j<\Lambda} \mu_j\langle a, e_j\rangle^2 \leq C\Lambda.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\|a\|_{L^3}^3 &\leq C\int_0^{+\infty}\Lambda^{-2}\|a_\Lambda\|_{\mathcal{V}_\sigma}^2\,d\Lambda + C\int_0^{+\infty}\|b_\Lambda\|_{L^2}^2\,d\Lambda \\
&\leq C\sum_{j\in\mathbb{N}}\int_{\mu_j}^{+\infty}\Lambda^{-2}\mu_j^2\langle a, e_j\rangle^2\,d\Lambda + C\sum_{j\in\mathbb{N}}\int_0^{\mu_j}\langle a, e_j\rangle^2\,d\Lambda \\
&\leq C\sum_{j\in\mathbb{N}}\mu_j\langle a, e_j\rangle^2 \\
&\leq C.
\end{aligned}$$

This proves the first part of the proposition.

The second part is obtained by a duality argument. By definition, we have, for any a in $\mathcal{V}'$,

$$\begin{aligned}
\|a\|_{\mathcal{V}_\sigma^{-\frac{1}{2}}} &= \|(\mu_j^{-\frac{1}{2}}\langle a, e_j\rangle)_{j\in\mathbb{N}}\|_{\ell^2} \\
&= \sup_{\substack{(\alpha_j)_{j\in\mathbb{N}} \\ \|(\alpha_j)_{j\in\mathbb{N}}\|_{\ell^2}\leq 1}} \sum_{j\in\mathbb{N}}\alpha_j\mu_j^{-\frac{1}{2}}\langle a, e_j\rangle. \qquad (2.5)
\end{aligned}$$

The map L defined by

$$L\left\{\begin{array}{ccc} \ell^2(\mathbb{N}) & \longrightarrow & \mathcal{V}_\sigma^{\frac{1}{2}} \\ (\alpha_j)_{j\in\mathbb{N}} & \longmapsto & \displaystyle\sum_{j\in\mathbb{N}}\alpha_j\mu_j^{-\frac{1}{2}}e_j \end{array}\right.$$

is an onto isometry. Thus, thanks to (2.5), we have

$$\|a\|_{\mathcal{V}_\sigma^{-\frac12}} = \sup_{\|\varphi\|_{\mathcal{V}_\sigma^{\frac12}}\leq 1} \sum_{j\in\mathbb{N}} (L^{-1}\varphi)_j \mu_j^{-\frac12} \langle a, e_j\rangle.$$

For any φ in $\mathcal{V}_\sigma$, we have

$$\sum_{j\in\mathbb{N}} (L^{-1}\varphi)_j \mu_j^{-\frac12} \langle a, e_j\rangle = \langle a, \varphi\rangle.$$

If we assume that a is in $L^{\frac32}$, we have, because φ is in L^3,

$$\langle a, \varphi\rangle = \int_\Omega a(x)\cdot\varphi(x)\,dx.$$

Hölder's inequality and the first part of Proposition 2.3.1 imply that

$$|\langle a, \varphi\rangle| \leq \|a\|_{L^{\frac32}} \|\varphi\|_{L^3} \leq C\|a\|_{L^{\frac32}} \|\varphi\|_{\mathcal{V}_\sigma^{\frac12}}.$$

Thus we have

$$\|a\|_{\mathcal{V}_\sigma^{-\frac12}} \leq \sup_{\|\varphi\|_{\mathcal{V}_\sigma^{\frac12}}\leq 1} \langle a, \varphi\rangle \leq C\|a\|_{L^{\frac32}}.$$

This completes the proof of Proposition 2.3.1. □

We can infer now the following corollary, which will be useful later on.

Corollary 2.3.1 *The bilinear map defined by*

$$\left\{\begin{array}{ccc} L^4([0,T];\mathcal{V}_\sigma)\times L^4([0,T],\mathcal{V}_\sigma) & \longrightarrow & L^2([0,T];\mathcal{V}_\sigma^{-\frac12}) \\ (u,v) & \longmapsto & \operatorname{div}(u\otimes v). \end{array}\right.$$

is continuous.

Lemma 2.3.1 *Let u_0 be $\mathcal{V}_\sigma^{\frac12}$ and f in $L^2([0,T];\mathcal{V}_\sigma^{-\frac12})$. The solution u of the Stokes problem with initial data u_0 and external force f belongs to $C([0,T];\mathcal{V}_\sigma^{\frac12})\cap L^4([0,T];\mathcal{V}_\sigma)$ and satisfies*

$$\max\bigl\{\|u\|_{L^\infty([0,T];\mathcal{V}_\sigma^{\frac12})}, \|u\|_{L^4([0,T];\mathcal{V}_\sigma)}\bigr\} \leq \|u_0\|_{\mathcal{V}_\sigma^{\frac12}} + \|f\|_{L^2([0,T];\mathcal{V}'_\sigma)}.$$

Proof. We known from (2.3) that

$$u(t) = \sum_{j=1}^\infty u_j(t)e_j \quad\text{with}\quad u_j(t) = \langle u_0, e_k\rangle e^{-\mu_j^2 t} + \int_0^t \langle f(t',\cdot), e_j\rangle e^{-\mu_j^2(t-t')}dt'.$$

From Young's inequalities, we get

$$\max\bigl\{\|u_j\|_{L^\infty([0,T])}, \mu_j^{\frac12}\|u_j\|_{L^4([0,T])}\bigr\} \leq |\langle u_0, e_j\rangle| + \|f(t,\cdot), e_j\rangle\|_{L^2([0,T])}.$$

By definition of the spaces $\mathcal{V}_\sigma^s$, we have

$$\sum_j \mu_j\langle u_0, e_j\rangle^2 = \|u_0\|^2_{\mathcal{V}_\sigma^{\frac12}} \quad\text{and}\quad \sum_j \mu_j^{-1}\|\langle f(t,\cdot), e_j\rangle\|^2_{L^2([0,T])} = \|f\|^2_{L^2([0,T];\mathcal{V}_\sigma^{-\frac12})}. \qquad (2.6)$$

We infer that

$$\max\Big\{\sum_{j=0}\mu_j\|u_j\|^2_{L^\infty([0,T])},\sum_{j=0}^{\infty}\mu_j^2\|u_j\|^2_{L^4([0,T];\mathcal{V}_\sigma)}\Big\}\leq\big(\|u_0\|_{\mathcal{V}_\sigma^{\frac12}}+\|f\|_{L^2([0,T];\mathcal{V}_\sigma^{-\frac12})}\big)^2 \tag{2.7}$$

It is obvious that

$$\sup_{t\in[0,T]}\sum_j\mu_ju_j(t)^2\leq\sum_{j=0}^{\infty}\mu_j\|u_j\|^2_{L^\infty([0,T])}.$$

Now let us observe that, thanks to the Cauchy–Schwarz inequality, for any a in $\ell^2(L^4[0,T])$,

$$\begin{aligned}\int_0^T\|a_j(t)\|^4_{\ell^2(\mathbb{N})}dt &= \int_0^T\Big(\sum_{j\in\mathbb{N}}a_j^2(t)\Big)^2dt\\ &= \sum_{j\in\mathbb{N},k\in\mathbb{N}}\int_0^T a_j^2(t)a_k^2(t)dt\\ &\leq \sum_{j\in\mathbb{N},k\in\mathbb{N}}\|a_j\|^2_{L^4([0,T])}\|a_k\|^2_{L^4([0,T])}\\ &\leq \Big\|(\|a_j\|_{L^4([0,T])})_{j\in\mathbb{N}}\Big\|^4_{\ell^2}\end{aligned} \tag{2.8}$$

Let us notice that this is a particular case of the Minkowski inequality. We infer that

$$\|u\|_{L^4([0,T];\mathcal{V}_\sigma)}\leq\|u_0\|_{\mathcal{V}^{\frac12}}+\|f\|_{L^2([0,T];\mathcal{V}_\sigma^{-\frac12})}.$$

In order to prove the continuity, let us consider a positive real number ε. An integer j_0 exists such that

$$\Big(\sum_{j>j_0}\mu_j\|\langle u(t,\cdot),e_j\rangle\|^2_{L^\infty([0,T])}\Big)^{\frac12}<\frac{\varepsilon}{2}\,\cdot$$

Now, it turns out that for all $(t_1,t_2)\in[0,T]^2$, one has

$$\begin{aligned}\|u(t_1)-u(t_2)\|_{\mathcal{V}_\sigma^{\frac12}} &\leq \Big(\sum_{j>j_0}\mu_j\|\langle u(t,\cdot),e_j\rangle\|^2_{L^\infty([0,T])}\Big)^{\frac12}+\Big(\sum_{j\leq j_0}\mu_j\langle u(t_1)-u(t_2),e_j\rangle^2\Big)^{\frac12}\\ &\leq \frac{\varepsilon}{2}+\mu_{j_0}^{\frac12}\|u(t_1)-u(t_2)\|_{L^2}.\end{aligned}$$

Theorem 2.2.1 tells us that u is continuous from $[0,T]$ into $\mathcal{H}$. Thus, the whole lemma is proved. □

Chapter 3

Leray's Theorem on Navier-Stokes equations

3.1 The concept of weak and turbulent solution

Let us state now the weak formulation of the incompressible Navier–Stokes system (NS).

Definition 3.1.1 *Given a domain Ω in $\mathbb{R}^d$, we shall say that u is a weak solution of the Navier–Stokes equations on $\mathbb{R}^+ \times \Omega$ with an initial data u_0 in $\mathcal{H}$ and an external force f in $L^2_{loc}(\mathbb{R}^+; \mathcal{V}')$ if and only if u belongs to the space*

$$C(\mathbb{R}^+; \mathcal{V}'_\sigma) \cap L^\infty_{loc}(\mathbb{R}^+; \mathcal{H}) \cap L^2_{loc}(\mathbb{R}^+; \mathcal{V}_\sigma)$$

and for any function Ψ in $C^1(\mathbb{R}^+; \mathcal{V}_\sigma)$, the vector field u satisfies the following condition:

$$\begin{aligned}\int_\Omega (u \cdot \Psi)(t,x)\,dx &+ \int_0^t \int_\Omega \Big(\nabla u : \nabla\Psi - u \otimes u : \nabla\Psi - u \cdot \partial_t \Psi\Big)(t',x)\,dxdt' \\ &= \int_\Omega u_0(x) \cdot \Psi(0,x)\,dx + \int_0^t \langle f(t'), \Psi(t')\rangle\,dt' \quad \textit{with} \\ \nabla u : \nabla\Psi &= \sum_{j,k=1}^d \partial_j u^k \partial_j \Psi^k \quad \textit{and} \quad u \otimes u : \nabla\Psi = \sum_{j,k=1}^d u^j u^k \partial_j \Psi^k.\end{aligned}$$

Let us remark that the above relation means that the equality in (NS) must be understood as an equality in the sense of $\mathcal{V}'_\sigma$. Now let us state the Leray theorem.

Theorem 3.1.1 *Let Ω be a domain of $\mathbb{R}^d$ and u_0 a vector field in $\mathcal{H}$. Then, there exists a global weak solution u to (NS) in the sense of Definition 3.1.1. Moreover, this solution satisfies the energy inequality for all $t \geq 0$,*

$$\begin{aligned}\frac{1}{2}\int_\Omega |u(t,x)|^2 dx \quad &+ \quad \int_0^t \int_\Omega |\nabla u(t',x)|^2 dxdt' \\ &\leq \quad \frac{1}{2}\int_\Omega |u_0(x)|^2 dx + \int_0^t \langle f(t',\cdot), u(t',\cdot)\rangle dt'. \qquad (3.1)\end{aligned}$$

It is convenient to state the following definition.

Definition 3.1.2 *A solution of (NS) in the sense of the above Definition 3.1.1 which moreover satisfies the energy inequality (3.1) is called a Leray or a turbulent solution of (NS).*

Let us remark that the energy inequality implies a control on the energy.

Proposition 3.1.1 *Any Leray solution u of (NS) satisfies*

$$\|u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(t')\|_{L^2}^2 dt' \leq \|u_0\|_{L^2}^2 + \int_0^t \|f(t')\|_{\mathcal{V}'_\sigma}^2 dt'.$$

Proof. By definition of the norm $\|\cdot\|_{\mathcal{V}'_\sigma}$, we have

$$\langle f(t,\cdot), u(t,\cdot)\rangle \leq \|f(t,\cdot)\|_{\mathcal{V}'_\sigma}\|u(t,\cdot)\|_{\mathcal{V}_\sigma}.$$

Inequality (3.1) becomes

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(t')\|_{L^2}^2 dt' \leq \|u_0\|_{L^2}^2 + \int_0^t \|f(t')\|_{\mathcal{V}'_\sigma}\|u(t',\cdot)\|_{\mathcal{V}}^2\, dt'.$$

As $\|u(t',\cdot)\|_{\mathcal{V}}^2 = \|\nabla u(t',\cdot)\|_{L^2}^2$, we get, using the fact that $2ab \leq a^2 + b^2$,

$$\|u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(t')\|_{L^2}^2 dt' \leq \|u_0\|_{L^2}^2 + \int_0^t \|f(t')\|_{\mathcal{V}'_\sigma}^2\, dt'.$$

Thus the proposition is proved. □

The propose of this chapter is the proof of Theorem 3.1.1. The structure of the proof is the following:

- first approximate solutions are built in spaces with finite frequencies by using simple ordinary differential equations results in L^2–type spaces.
- Next, a compactness result is derived.
- Finally the conclusion is obtained by passing to the limit in the weak formulation, taking especially care of the nonlinear terms.

3.2 Construction of approximate solutions

In this section, we intend to build approximate solutions of the Navier–Stokes equations. We use the projections $\mathbb{P}_k$ defined in (2.1) and denote by $\mathcal{H}_k$ the space $\mathbb{P}_k\,\mathcal{H} = \mathbb{P}_k\,\mathcal{V}'$.

Lemma 3.2.1 *For any external force f in $L^2_{loc}(\mathbb{R}^+;\mathcal{V}')$, a sequence $(f_k)_{k\in\mathbb{N}}$ exists in $C^1(\mathbb{R}^+;\mathcal{V}_\sigma)$ such that for any $k\in\mathbb{N}$ and for any $t>0$, the vector field $f_k(t)$ belongs to $\mathcal{H}_k$, and*

$$\lim_{k\to\infty} \|f_k - f\|_{L^2([0,T];\mathcal{V}'_\sigma)} = 0.$$

Proof. Thanks to Theorem 2.1.2 and to the Lebesgue Theorem, a sequence $(\widetilde{f}_k)_{k\in\mathbb{N}}$ exists in $L^2_{loc}(\mathbb{R}^+;\mathcal{V}_\sigma)$ such that for any positive integer k and for almost all positive t, the vector field $\widetilde{f}_k(t)$ belongs to $\mathcal{H}_k$ and

$$\forall T>0\,,\ \lim_{k\to\infty} \|\widetilde{f}_k - f\|_{L^2([0,T];\mathcal{V}'_\sigma)} = 0.$$

A standard (and omitted) time regularization procedure concludes the proof of the lemma. □

In order to construct the approximate solution, let us study the non linear term.

Definition 3.2.1 *Let us define the bilinear map*

$$Q\left\{\begin{array}{ccc}\mathcal{V}\times\mathcal{V} & \longrightarrow & \mathcal{V}'\\ (u,v) & \longmapsto & -\operatorname{div}(u\otimes v).\end{array}\right.$$

Sobolev embeddings stated in Theorem 1.2.3 ensure that Q is continuous: in the sequel, the following lemma will be useful.

Lemma 3.2.2 *For any u and v in $\mathcal{V}$, the following estimates hold. For d in $\{2,3\}$, a constant C exists such that, for any $\varphi\in\mathcal{V}$,*

$$\langle Q(u,v),\varphi\rangle \leq C\|\nabla u\|_{L^2}^{\frac{d}{4}}\|\nabla v\|_{L^2}^{\frac{d}{4}}\|u\|_{L^2}^{1-\frac{d}{4}}\|v\|_{L^2}^{1-\frac{d}{4}}\|\nabla\varphi\|_{L^2}.$$

Moreover for any u in $\mathcal{V}_\sigma$ and any v in $\mathcal{V}$, $\langle Q(u,v),v\rangle = 0$.

Proof. The first two inequalities follow directly from Gagliardo–Nirenberg's inequality stated in Corollary 1.2.2 page 12, once noticed that

$$\begin{aligned}\langle Q(u,v),\varphi\rangle &\leq \|u\otimes v\|_{L^2}\|\nabla\varphi\|_{L^2}\\ &\leq \|u\|_{L^4}\|v\|_{L^4}\|\nabla\varphi\|_{L^2}.\end{aligned}$$

In order to prove the second assertion, let us assume that u and v are two vector fields the components of which belong to $\mathcal{D}(\Omega)$. Then we deduce from integrations by parts that

$$\begin{aligned}\langle Q(u,v),v\rangle &= -\int_\Omega(\operatorname{div}(u\otimes v)\cdot v)(x)\,dx\\ &= -\sum_{\ell,m=1}^{d}\int_\Omega\partial_m(u^m(x)v^\ell(x))v^\ell(x)\,dx\\ &= \sum_{\ell,m=1}^{d}\int_\Omega u^m(x)v^\ell(x)\partial_m v^\ell(x)\,dx\\ &= -\int_\Omega|v(x)|^2\operatorname{div}u(x)\,dx-\langle Q(u,v),v\rangle.\end{aligned}$$

Thus, we have

$$\langle Q(u,v),v\rangle = -\frac{1}{2}\int_\Omega|v(x)|^2\operatorname{div}u(x)\,dx.$$

The two expressions are continuous on $\mathcal{V}$ and by definition, $\mathcal{D}$ is dense in $\mathcal{V}$. Thus the formula is true for any $(u,v)\in\mathcal{V}_\sigma\times\mathcal{V}$, which completes the proof. □

Thanks to Theorem 2.1.2 and to the above lemma, we can define $F_k(u)\stackrel{\text{def}}{=}\mathbb{P}_k\,Q(u,u)$. Now let us introduce the following ordinary differential equation

$$(NS_k)\quad\left\{\begin{array}{rcl}\dot{u}_k(t) &=& \mathbb{P}_k\,\Delta u_k(t)+F_k(u_k(t))+f_k(t)\\ u_k(0) &=& \mathbb{P}_k\,u_0.\end{array}\right.$$

Theorem 2.1.2 implies that $\mathbb{P}_k\,\Delta$ is a linear map from $\mathcal{H}_k$ into itself. Thus the continuity properties on Q and $\mathbb{P}_k$ allow to apply the Cauchy–Lipschitz theorem. This gives the existence of $T_k \in]0, +\infty]$ and a unique maximal solution u_k of (NS_k) in $C^\infty([0, T_k[; \mathcal{H}_k)$. In order to prove that $T_k = +\infty$, let us observe that, thanks to Lemma 3.2.2 and Theorem 2.1.2

$$\|\dot{u}_k(t)\|_{L^2} \leq \mu_k \|u_k(t)\|_{L^2} + C\mu_k^{\frac{d}{4}} \|u_k(t)\|_{L^2}^2 + \|f_k(t)\|_{L^2}.$$

If $\|u_k(t)\|_{L^2}$ remains bounded on some interval $[0, T[$, so does $\|\dot{u}_k(t)\|_{L^2}$. Thus, for any k, the function u_k satisfies the Cauchy criteria when t tends to T. Thus the solution can be extended beyond T. It follows that a uniform bound on $\|u_k(t)\|_{L^2}$ will imply that $T_k = +\infty$.

3.3 A priori bounds

The purpose of this paragraph is the proof of the following proposition.

Proposition 3.3.1 *The sequence $(u_k)_{k\in\mathbb{N}}$ in bounded in the space*

$$L^\infty_{loc}(\mathbb{R}^+; \mathcal{H}) \cap L^2_{loc}(\mathbb{R}^+; \mathcal{V}_\sigma) \cap L^{\frac{8}{d}}_{loc}(\mathbb{R}^+ L^4(\Omega)).$$

Moreover, the sequence $(\Delta u_k)_{k\in\mathbb{N}}$ is bounded in the space $L^2_{loc}(\mathbb{R}^+; \mathcal{V}'_\sigma)$.

Proof. Let us now estimate the L^2 norm of $u_k(t)$. Taking the L^2 scalar product of equation (NS_k) with $u_k(t)$, we get

$$\frac{1}{2}\frac{d}{dt}\|u_k(t)\|_{L^2}^2 = (\Delta u_k(t)|u_k(t))_{L^2} + (F_k(u_k(t))|u_k(t))_{L^2} + (f_k(t)|u_k(t))_{L^2}.$$

By definition of F_k, Lemma 3.2.2 implies that

$$(F_k(u_k(t))|u_k(t))_{L^2} = \langle Q(u_k(t), u_k(t)), u_k \rangle = 0.$$

Thus we infer that

$$\frac{1}{2}\frac{d}{dt}\|u_k(t)\|_{L^2}^2 + (\nabla u_k(t)|\nabla u_k(t))_{L^2} = (f_k(t)|u_k(t))_{L^2}. \tag{3.2}$$

By time integration, we get the fundamental energy estimate for the approximate Navier–Stokes system: for all $t \in [0, T_k)$

$$\frac{1}{2}\|u_k(t)\|_{L^2}^2 + \int_0^t \|\nabla u_k(t')\|_{L^2}^2 dt' = \frac{1}{2}\|u_k(0)\|_{L^2}^2 + \int_0^t (f_k(t')|u_k(t'))_{L^2} dt'. \tag{3.3}$$

Using the (well known) fact that $2ab \leq a^2 + b^2$, we get

$$\|u_k(t)\|_{L^2}^2 + \int_0^t \|\nabla u_k(t')\|_{L^2}^2 \, dt' \leq \|u_k(0)\|_{L^2}^2 + \int_0^t \|f_k(t')\|_{\mathcal{V}'_\sigma}^2 dt'$$

Gronwall's lemma implies that $(u_k)_{k\in\mathbb{N}}$ remains uniformly bounded in $\mathcal{H}$ for all time, hence that $T_k = +\infty$. In addition, the sequence $(u_k)_{k\in\mathbb{N}}$ is bounded in the space $L^\infty_{loc}(\mathbb{R}^+; \mathcal{H}) \cap L^2_{loc}(\mathbb{R}^+; \mathcal{V}_\sigma)$. Using Gagliardo-Nirenberg inequalities (see Corollary 1.2.2 page 12), we deduce that the sequence $(u_k)_{k\in\mathbb{N}}$ is bounded in the space

$$L^\infty_{loc}(\mathbb{R}^+; \mathcal{H}) \cap L^2_{loc}(\mathbb{R}^+; \mathcal{V}_\sigma) \cap L^{\frac{8}{d}}_{loc}(L^4(\Omega)).$$

Moreover, we have, for any $v \in \mathcal{V}_\sigma$,

$$\langle -\Delta u_k, v \rangle = (\nabla u_k|\nabla v)_{L^2} \leq \|u_k\|_{H^1_0} \|v\|_{\mathcal{V}}.$$

By definition of the norm $\|\cdot\|_{\mathcal{V}'_\sigma}$, we infer that the sequence $(\Delta u_k)_{k\in\mathbb{N}}$ is bounded in $L^2_{loc}(\mathbb{R}^+; \mathcal{V}'_\sigma)$. The whole proposition is proved. □

3.4 Compactness properties

We omit to mention the diagonal process and to note explicitely extraction of subsequences.

Proposition 3.4.1 (Weak compacness) *A vector field u exists in $L^2_{loc}(\mathbb{R}^+;\mathcal{V}_\sigma)$ such that, up to an extraction, we have*

$$\forall f \in L^2_{loc}(\mathbb{R}^+;\mathcal{V}_\sigma)\,,\ \lim_{k\to\infty}\int_0^T \langle f(t),u_k(t)\rangle dt = \int_0^T \langle f(t),u(t)\rangle dt. \tag{3.4}$$

Proof. As $(u_k)_{k\in\mathbb{N}}$ is a bounded seuqence of the Hilbert space $L^2([0,T];\mathcal{V}_\sigma)$, a fonction u exists in $L^2(0,T];\mathcal{V}_\sigma)$ such that

$$\forall \psi \in L^2([0,T];\mathcal{V}_\sigma)\,,\ \int_0^t\int_\Omega \nabla u_k(t,x):\nabla\psi(t,x)dxdt = \int_0^t\int_\Omega \nabla u(t,x):\nabla\psi(t,x)dxdt.$$

Moreover, for almost every t, we have

$$\int_\Omega \nabla u_k(t,x):\nabla\psi(t,x)dx = -\langle\Delta\psi(t),u_k(t)\rangle \quad\text{and}\quad \int_\Omega \nabla u(t,x):\nabla\psi(t,x)dx = -\langle\Delta\psi(t),u_(t)\rangle.$$

Using Theorem 2.1.1 page 17, we get that, for any f in $L^2([0,T];\mathcal{V}_\sigma)$, a function ψ in the space $L^2([0,T];\mathcal{V}_\sigma)$ exists such that, for almost every t, $-\Delta\psi(t)-f(t)$ belongs to $\mathcal{V}_\sigma^\circ$. The proposition is proved. □

Now, let us prove a strong compactnesss result.

Proposition 3.4.2 (Strong compactness) *Up to an extraction, we have*

$$\lim_{k\to\infty}\|u_k-u\|_{L^\infty([0,T];\mathcal{V}'_\sigma)} = 0.$$

Proof. As claimed by Proposition 3.3.1, the sequence $(u_k)_{k\in\mathbb{N}}$ is bounded in $L^{\frac{8}{d}}_{loc}(\mathbb{R}^+;L^4)$. Thus the sequence $(u_k\otimes u_k)_{k\in\mathbb{N}}$ is bounded in $L^{\frac{4}{d}}_{loc}(\mathbb{R}^+;L^2)$. As the sequence $(\Delta u_k)_{k\in\mathbb{N}}$ is bounded in $L^2_{loc}(\mathbb{R}^+;\mathcal{V}'_\sigma)$, we get that a real number M exists such that

$$\forall k\in\mathbb{N}\,,\ \forall t\|\partial_t u_k\|_{L^{\frac{4}{d}}([0,T];\mathcal{V}'_\sigma)} \leq M.$$

Writing that

$$u_k(t)-u_k(t') = \int_{t'}^t \partial_t u_k(t'')dt''$$

gives, using a Hölder inequality, that

$$\|u_k(t)-u_k t')\|_{\mathcal{V}'_\sigma} \leq \int_{t'}^t \|\partial_t u_k(t'')\|_{\mathcal{V}'_\sigma}dt'' \leq |t-t'|^{1-\frac{4}{d}}M.$$

In particular, the sequence $(u_k)_{k\in\mathbb{N}}$ is equicontinuous on $[0,T]$ with values in $\mathcal{V}'_\sigma$. Moreover, Theorem 2.1.2 implies that $\mathcal{H}$ is compactly included in $\mathcal{V}'_\sigma$. As for any t, the sequence $(u_k(t))_{k\in\mathbb{N}}$ is bounded in $\mathcal{H}$, we can apply Ascoli's theorem. Thus, up to an extraction, a function $\widetilde{u}$ exists in $C(\mathbb{R};\mathcal{V}'_\sigma)$ such that, for any T,

$$\lim_{k\to\infty}\|u_k-\widetilde{u}\|_{L^\infty([0,T];\mathcal{V}'_\sigma)} = 0.$$

In particluar, we have, for any ψ in $L^2([0,T];\mathcal{V}_\sigma)$,

$$\lim_{k\to\infty}\int_0^T\langle u_k(t),\psi(t)\rangle dt = \int_0^T\langle u(t),\psi(t)\rangle dt.$$

As $(u_k)_{k\in\mathbb{N}}$ tends weakly to u in $L^2([0,T];\mathcal{V}_\sigma)$, this implies that $\widetilde{u}\equiv u$. This proves the proposition. □

From this proposition, we shall deduce the following result, which is an interpolation result.

Corollary 3.4.1 *We have, for any positive T,*

$$\lim_{k\to\infty} \|u_k - u\|_{L^{\frac{16}{4+d}}([0,T];L^4)} = 0.$$

Proof. Using Corollary 1.2.2 page 12, we get

$$\|u_k(t) - u(t)\|_{L^4} \leq C\|u_k(t) - u(t)\|_{L^2}^{1-\frac{d}{4}} (\|\nabla u_k(t)\|_{L^2} + \|\nabla u(t)\|_{L^2})^{\frac{d}{4}}.$$

As we have $\|v\|_{\mathcal{H}} \leq \|v\|_{\mathcal{V}'_\sigma}^{\frac{1}{2}} \|v\|_{\mathcal{V}_\sigma}^{\frac{1}{2}}$, we deduce that

$$\|u_k(t) - u(t)\|_{L^4} \leq C\|u_k(t) - u(t)\|_{\mathcal{V}'_\sigma}^{\frac{1}{2}-\frac{d}{8}} \big(\|\nabla u_k(t)\|_{L^2} + \|\nabla u(t)\|_{L^2}\big)^{\frac{1}{2}+\frac{d}{8}}.$$

Thus we get

$$\|u_k - u\|_{L^{\frac{16}{4+d}}([0,T];L^4)} \leq C\|u_k - u\|_{L^\infty([0,T];\mathcal{V}'_\sigma)}^{\frac{1}{2}-\frac{d}{8}} \big(\|\nabla u_k\|_{L^2([0,T]\times\Omega)} + \|\nabla u(t)\|_{L^2([0,T]\times\Omega)}\big)^{\frac{1}{2}+\frac{d}{8}}.$$

This gives the corollary. □

3.5 End of the proof of the Leray Theorem

The local strong convergence of $(u_k)_{k\in\mathbb{N}}$ will be crucial to pass to the limit in (NS_k) to obtain solutions of (NS).

According to the definition of a weak solution of (NS), let us consider a test function Ψ in $C^1(\mathbb{R}^+;\mathcal{V}_\sigma)$. Because u_k is a solution of (NS_k), we have

$$\begin{aligned}
\frac{d}{dt}\langle u_k(t), \Psi(t)\rangle &= \langle \dot{u}_k(t), \Psi(t)\rangle + \langle u_k(t), \dot{\Psi}(t)\rangle \\
&= \langle \mathbb{P}_k\, \Delta u_k(t), \Psi(t)\rangle + \langle \mathbb{P}_k\, Q(u_k(t), u_k(t)), \Psi(t)\rangle \\
&\qquad\qquad + \langle f_k(t), \Psi(t)\rangle + \langle u_k(t), \dot{\Psi}(t)\rangle.
\end{aligned}$$

We have after integration by parts

$$\begin{aligned}
\langle \mathbb{P}_k\, \Delta u_k(t), \Psi(t)\rangle &= -(u_k(t)|\, \mathbb{P}_k\, \Psi(t))_{\mathcal{V}_\sigma} = -(u_k(t)|\Psi(t))_{\mathcal{V}_\sigma} \\
\langle \mathbb{P}_k\, Q(u_k(t), u_k(t)), \Psi(t)\rangle &= \int_\Omega u_k(t,x)\otimes u_k(t,x) : \nabla\, \mathbb{P}_k\, \Psi(t,x)\, dx \quad \text{and} \\
\langle u_k(t), \dot{\Psi}(t)\rangle &= \int_\Omega u_k(t,x)\cdot \partial_t\Psi(t,x)\, dx.
\end{aligned}$$

By time integration between 0 and t, we infer that

$$\begin{aligned}
\langle u_k(t), \Psi(t)\rangle &+ \int_0^t \big((\nabla u_k(t')|\nabla\Psi(t'))_{\mathcal{V}_\sigma} - (u_k(t')|\partial_t\Psi(t'))_{\mathcal{H}}\big)dt' \\
&- \int_0^t\int_\Omega \big(u_k\otimes u_k : \nabla\, \mathbb{P}_k\, \Psi\big)dxdt' = \langle u_k(0), \Psi(0)\rangle + \int_0^t \langle f_k(t'), \Psi(t')\rangle\, dt'.
\end{aligned}$$

We now have to pass to the limit. Proposition 3.4.1 implies that, for any t in $[0,T]$,

$$\lim_{k\to\infty}\int_0^t \big(\nabla u_k(t')|\nabla\Psi(t')\big)_{L^2}dt' = \int_0^t \big(\nabla u(t')|\nabla\Psi(t')\big)_{L^2}dt' \tag{3.5}$$

Proposition 3.4.2 implies that, for any t in $[0,T]$,

$$\lim_{k\to\infty} \langle u_k(t), \Psi(t)\rangle = \langle u(t), \Psi(t)\rangle. \tag{3.6}$$

and

$$\lim_{k\to\infty} \int_0^t (u_k(t')|\partial_t \Psi(t'))_{\mathcal{H}} dt' = \int_0^t (u(t')|\partial_t \Psi(t'))_{\mathcal{H}} dt'. \tag{3.7}$$

Thanks to Theorem 2.1.2, we have

$$\lim_{k\to\infty} \int_0^t \langle f_k(t'), \Psi(t')\rangle \, dt' = \int_0^t \langle f(t'), \Psi(t')\rangle \, dt'. \tag{3.8}$$

Now, we have to treat the non linear term. Let us start by proving the following preliminary lemma.

Lemma 3.5.1 *Let* H *be a Hilbert space, and let* $(A_n)_{n\in\mathbb{N}}$ *be a bounded sequence of linear operators on* H *such that*

$$\forall h \in \mathrm{H}, \quad \lim_{n\to\infty} \|A_n h - h\|_{\mathrm{H}} = 0.$$

Then if $\psi \in C([0,T];\mathrm{H})$ *we have* $\lim_{n\to\infty} \sup_{t\in[0,T]} \|A_n\psi(t) - \psi(t)\|_{\mathrm{H}} = 0.$

Proof. The function ψ is continuous in time with values in H, so for all positive ε, the compact $\psi([0,T])$, can be covered by a finite number of balls of radius

$$\frac{\varepsilon}{2(\mathcal{A}+1)} \quad \text{with} \quad \mathcal{A} \stackrel{\text{def}}{=} \sup_n \|A_n\|_{\mathcal{L}(\mathrm{H})}.$$

and center $(\psi(t_\ell))_{0\leq\ell\leq N}$. Then we have, for all t in $[0,T]$ and ℓ in $\{0,\cdots,N\}$,

$$\|A_n\psi(t) - \psi(t)\|_{\mathrm{H}} \leq \|A_n\psi(t) - A_n\psi(t_\ell)\|_{\mathrm{H}} + \|A_n\psi(t_\ell) - \psi(t_\ell)\|_{\mathrm{H}} + \|\psi(t_\ell) - \psi(t)\|_{\mathrm{H}}.$$

The assumption on A_n implies that for any ℓ, the sequence $(A_n\psi(t_\ell))_{n\in\mathbb{N}}$ tends to $\psi(t_\ell)$. Thus, an integer n_N exists such that, if $n \geq n_N$,

$$\forall \ell \in \{0,\cdots N\}\,,\ \|A_n\psi(t_\ell) - \psi(t_\ell)\|_{\mathrm{H}} < \frac{\varepsilon}{2}\cdot$$

We infer that, if $n \geq n_N$, for all $t \in [0,T]$ and all $\ell \in \{0,\cdots,N\}$,

$$\|A_n\psi(t) - \psi(t)\|_{\mathrm{H}} \leq \|A_n\psi(t) - A_n\psi(t_\ell)\|_{\mathrm{H}} + \|\psi(t_\ell) - \psi(t)\|_{\mathrm{H}} + \frac{\varepsilon}{2}\cdot$$

For any t, let us choose ℓ such that

$$\|\psi(t) - \psi(t_\ell)\|_{\mathrm{H}} \leq \frac{\varepsilon}{2(\mathcal{A}+1)}\cdot$$

The lemma is proved. □

Now let us pass to the limit in the non linear term. Lemma 3.5.1 implies

$$\lim_{k\to\infty} \sup_{t\in[0,T]} \|\,\mathbb{P}_k\,\psi(t) - \psi(t)\|_{\mathcal{V}_\sigma} = 0.$$

As the sequence u_k is bounded in $L^2([0,T];L^4(\Omega))$, we have in fact

$$\lim_{k\to\infty} \biggl(\int_0^t \int_\Omega (u_k \otimes u_k : \nabla \mathbb{P}_k \Psi)(t',x)\, dx dt' - \int_0^t \int_\Omega (u_k \otimes u_k : \nabla \Psi)(t',x)\, dx dt' \biggr) = 0.$$

So it is enough to prove that

$$\lim_{k\to\infty}\int_0^t\int_\Omega (u_k\otimes u_k : \nabla\Psi)(t',x)\,dxdt' = \int_0^t\int_\Omega (u\otimes u : \nabla\Psi)(t',x)\,dxdt'.$$

From Corollary 3.4.1, we get that

$$\lim_{k\to\infty}\|u_k - u\|_{L^2([0,T];L^4(\Omega))} = 0.$$

This implies that

$$\lim_{k\to\infty}\|u_k\otimes u_k - u\otimes u\|_{L^1([0,T];L^2(\Omega))} = 0.$$

This concludes the proof of the fact that u is a solution of (NS) in the sense of Definition 3.1.1.

It remains to prove the energy inequality (3.1). Assertion (**??**) of Proposition **??** implies in particular that for any time $t\geq 0$ and any $v\in\mathcal{V}_\sigma$,

$$\lim_{k\to\infty}(u_k(t)|v)_{\mathcal{H}} = \lim_{k\to\infty}\langle u_k(t),v\rangle = \langle u(t),v\rangle = (u(t)|v)_{\mathcal{H}}.$$

As $\mathcal{V}_\sigma$ is dense in $\mathcal{H}$, we get that for any $t\geq 0$, the sequence $(u_k(t))_{k\in\mathbb{N}}$ converges weakly towards $u(t)$ in the Hilbert space $\mathcal{H}$. Hence

$$\|u(t)\|_{L^2}^2 \leq \liminf_{k\to\infty}\|u_k(t)\|_{L^2}^2 \quad \text{for all} \quad t\geq 0.$$

On the other hand, $(u_k)_{k\in\mathbb{N}}$ converges weakly to u in $L^2_{loc}(\mathbb{R}^+;\mathcal{V})$, so that for all non negative t, we have

$$\int_0^t \|\nabla u(t')\|_{L^2}^2\,dt' \leq \liminf_{k\to\infty}\int_0^t \|\nabla u_k(t')\|_{L^2}^2\,dt'.$$

Taking the $\liminf\limits_{k\to\infty}$ in the energy equality for approximate solutions (3.3) yields the energy inequality (3.1).

Remarks

- If you want to know more about the basis of the subject, you can read the seminal paper of J. Leray "Essai on the mouvement of a liquide visqueux emplissant the space, *Acta Matematica*, **63**, 1933, pages 193–248.
- To have a more recent review of results on incompressible Navier-Stokes, you can see te books of P. Constantin and C. Foias *Navier-Stokes equations*, Chigago University Press, 1988 and of P.-G. Lemarié-Rieusset, *Recent developments in the Navier-Stokes problem.* Chapman & Hall/CRC, Research Notes in Mathematics, **431**, 2002.
- If you are interested in developments related to geophysical fluids, you can see the book of J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, *Mathematical Geophysics; an introduction to rotating fluids and Navier-Stokes equations*, Oxford Lecture series in Mathematics and its maps, **32**, Oxford University Press, 2006.

Chapter 4

Stability of Navier-Stokes equations

In this chapter we intend to investigate the stability of the Leray solutions constructed in the previous chapter. It is useful to start by analyzing the linearised version of the Navier-Stokes equations, so the first section of the chapter is devoted to the proof of the wellposedness of the time dependent Stokes system. The study will be applied in Section 4.1 to the two dimensional Navier-Stokes equations, and the more delicate case of three space dimensions will be dealt with in Sections 4.2 to 4.3.

4.1 Stability in dimension two

In a two dimensional domain, the Leray weak solutions are unique and even stable. More precisely, we have the following theorem.

Theorem 4.1.1 *For any data u_0 in $\mathcal{H}$ and f in $L^2_{loc}(\mathbb{R}^+;\mathcal{V}')$, the Leray weak solution is unique. Moreover, it belongs to $C(\mathbb{R}^+;\mathcal{H})$ and satisfies, for any (s,t) such that $0 \leq s \leq t$,*

$$\frac{1}{2}\|u(t)\|^2_{L^2} + \int_s^t \|\nabla u(t')\|^2_{L^2}\, dt' = \frac{1}{2}\|u(s)\|^2_{L^2} + \int_s^t \langle f(t'), u(t')\rangle\, dt'. \tag{4.1}$$

Furthermore, the Leray solutions are stable in the following sense. Let u (resp. v) be the Leray solution associated with u_0 (resp. v_0) in $\mathcal{H}$ and f (resp. g) in the space $L^2_{loc}(\mathbb{R}^+;\mathcal{V}')$ then,

$$\|(u-v)(t)\|^2_{L^2} + \int_0^t \|\nabla(u-v)(t')\|^2_{L^2}\, dt' \leq \Big(\|u_0 - v_0\|^2_{L^2} + \int_0^t \|(f-g)(t')\|^2_{\mathcal{V}'_\sigma}\, dt'\Big)\exp(CE^2(t)) \quad \textit{with}$$

$$E(t) \stackrel{\text{def}}{=} \min\Big\{\|u_0\|^2_{L^2} + \int_0^t \|f(t')\|^2_{\mathcal{V}'_\sigma}\, dt'\,,\, \|v_0\|^2_{L^2} + \int_0^t \|g(t')\|^2_{\mathcal{V}'_\sigma}\, dt'\Big\}.$$

Proof. As u belongs to $L^\infty_{loc}(\mathbb{R}^+;\mathcal{H}) \cap L^2_{loc}(\mathbb{R}^+;\mathcal{V}_\sigma)$, thanks to Lemma 3.2.2 page 27, the non linear term $Q(u,u)$ belongs to $L^2_{loc}(\mathbb{R}^+;\mathcal{V}')$. Thus u is the solution of (ES) with initial data u_0 and external force $f+Q(u,u)$. Theorem 2.2.1 immediately implies that u belongs to $C(\mathbb{R}^+;\mathcal{H})$ and satisfies, for any (s,t) such that $0 \leq s \leq t$,

$$\frac{1}{2}\|u(t)\|^2_{L^2} + \int_s^t \|\nabla u(t')\|^2_{L^2}\, dt' = \frac{1}{2}\|u(s)\|^2_{L^2} + \int_s^t \langle f(t'), u(t')\rangle\, dt' + \int_s^t \langle Q(u(t'), u(t')), u(t')\rangle\, dt'.$$

Using Lemma 3.2.2, we get the energy equality (4.1).

To prove the stability, let us observe that, by difference $w \stackrel{\text{def}}{=} u - v$ is the solution of (ES) with data $u_0 - v_0$ and external force $f - g + Q(u,u) - Q(v,v)$, Theorem 2.2.1 implies that

$$\|w(t)\|_{L^2}^2 + 2\int_0^t \|\nabla w(t')\|_{L^2}^2\,dt' = \|w(0)\|_{L^2}^2 + 2\int_0^t \langle (f-g)(t'), w(t')\rangle\,dt' + 2\int_0^t \langle (Q(u,u)-Q(v,v))(t'), w(t')\rangle\,dt'.$$

The non linear term is estimated thanks to the following lemma.

Lemma 4.1.1 *In two dimensional domains, if a and b belong to $\mathcal{V}_\sigma$, we have*

$$|\langle (Q(a,a)-Q(b,b)), a-b\rangle| \leq C\|\nabla(a-b)\|_{L^2}^{\frac{3}{2}}\|a-b\|_{L^2}^{\frac{1}{2}}\|\nabla a\|_{L^2}^{\frac{1}{2}}\|a\|_{L^2}^{\frac{1}{2}}.$$

Proof. It is an exercise of algebra to deduce from Lemma 3.2.2 that

$$\langle Q(a,a) - Q(b,b), a-b\rangle = \langle Q(a-b,a), a-b\rangle. \tag{4.2}$$

Using again Lemma 3.2.2, we get the result. □

Conclusion of the proof of Theorem 4.1.1. Using that $2ab \leq a^2 + b^2$, we get

$$\|w(t)\|_{L^2}^2 + \frac{3}{2}\int_0^t \|\nabla w(t')\|_{L^2}^2\,dt' \leq \|w(0)\|_{L^2}^2 + 2\int_0^t \|(f-g)(t')\|_{\mathcal{V}_\sigma'}^2\,dt' + C\int_0^t \|\nabla w(t')\|_{L^2}^{\frac{3}{2}}\|w(t')\|_{L^2}^{\frac{1}{2}}\|\nabla u(t')\|_{L^2}^{\frac{1}{2}}\|u(t')\|_{L^2}^{\frac{1}{2}}\,dt'.$$

Using (with $\theta = 1/4$) the convexity inequality

$$ab \leq \theta a^{\frac{1}{\theta}} + (1-\theta)b^{1-\frac{1}{\theta}} \tag{4.3}$$

we infer that

$$\|w(t)\|_{L^2}^2 + \int_0^t \|\nabla w(t')\|_{L^2}^2\,dt' \leq \|w(0)\|_{L^2}^2 + 2\int_0^t \|(f-g)(t')\|_{\mathcal{V}_\sigma'}^2\,dt' + C\int_0^t \|w(t')\|_{L^2}^2\|\nabla u(t')\|_{L^2}^2\|u(t')\|_{L^2}^2\,dt'.$$

Gronwall's lemma implies that

$$\|w(t)\|_{L^2}^2 + \int_0^t \|\nabla w(t')\|_{L^2}^2\,dt' \leq \Big(\|w(0)\|_{L^2}^2 + 2\int_0^t \|(f-g)(t')\|_{\mathcal{V}_\sigma'}^2\,dt'\Big) \times C\sup_{t'\in[0,t]}\|u(t')\|_{L^2}^2\int_0^t \|\nabla u(t')\|_{L^2}^2\,dt'\Big).$$

The energy estimate tells us that

$$\sup_{t'\in[0,t]}\|u(t')\|_{L^2}^2\int_0^t \|\nabla u(t')\|_{L^2}^2\,dt' \leq \Big(\|u_0\|_{L^2}^2 + \int_0^t \|f(t')\|_{\mathcal{V}_\sigma'}^2\,dt'\Big)^2.$$

As u and v play the same role, the theorem is proved. □

4.2 Stability in dimension three

In order to get stability, we need to enforce the time regularity of the Leray solution. The precise stability theorem is the following.

Theorem 4.2.1 *Let u be a Leray solution associated with initial velocity u_0 in $\mathcal{V}_\sigma^{\frac{1}{2}}$ and bulk force f in $L^2([0,T];\mathcal{V}_\sigma^{\frac{1}{2}})$. We assume that u belongs to the space $L^4([0,T];\mathcal{V}_\sigma)$ for some positive T. Then u is unique, belongs to $C([0,T];\mathcal{V}_\sigma^{\frac{1}{2}})$ and satisfies, for any (s,t) such that $0\leq s\leq t\leq T$,*

$$\frac{1}{2}\|u(t)\|_{L^2}^2+\int_s^t\|\nabla u(t')\|_{L^2}^2\,dt'=\frac{1}{2}\|u(s)\|_{L^2}^2+\int_s^t\langle f(t'),u(t')\rangle\,dt'. \tag{4.1}$$

Let v be any turbulent solution associated with v_0 in $\mathcal{H}$ and g in $L^2_{loc}([0,T];\mathcal{V}')$. Then, for all t in $[0,T]$,

$$\begin{aligned}\|(u-v)(t)\|_{L^2}^2&+\int_0^t\|\nabla(u-v)(t')\|_{L^2}^2\,dt'\\ &\leq\Big(\|u_0-v_0\|_{L^2}^2+\int_0^t\|(f-g)(t')\|_{\mathcal{V}'_\sigma}^2\,dt'\Big)\exp\Big(C\int_0^t\|\nabla u(t')\|_{L^2}^4\,dt'\Big).\end{aligned}$$

Proof. Thanks to Corollary 2.3.1 page 23, $-\operatorname{div}(u\otimes u)$ belongs to $L^2([0,T];\mathcal{V}_\sigma^{-\frac{1}{2}})$. Thus, Lemma 2.3.1 page 23 implies that u belongs to $C([0,T];\mathcal{V}_\sigma^{\frac{1}{2}})$. Theorem 2.2.1 page 19 implies that

$$\frac{1}{2}\|u(t)\|_{L^2}^2+\int_s^t\|\nabla u(t')\|_{L^2}^2dt'=\frac{1}{2}\|u(s)\|_{L^2}^2+\int_s^t\big(\langle f(t'),u(t')\rangle-\langle\operatorname{div}(u(t')\otimes u(t'),u(t')\rangle\big)dt'.$$

As $\langle\operatorname{div}(u(t')\otimes u(t'),u(t')\rangle=0$, this gives the energy equality (4.1).

The main issue is now the stability inequality. In order to prove it, let us prove the following approximation lemma, which has its own interest.

Lemma 4.2.1 *Under the hypothesis of Theorem 4.2.1, let us consider the sequence $(u_k)_{k\in\mathbb{N}}$ defined by (NS_k) page 27, then*

$$\lim_{k\to\infty}\|u-u_k\|_{L^\infty([0,T];\mathcal{V}_\sigma^{\frac{1}{2}})}=\lim_{k\to\infty}\|u-u_k\|_{L^4([0,T];\mathcal{V}_\sigma)}=0.$$

Proof. Let us study the equation satisfied by $\delta_k\stackrel{\text{def}}{=}u-u_k$. Straighforward computation implies that δ_k is the solution of the evolution Stokes problem with initial data $u_0-\mathbb{P}_k\,u_0$ and external force $-\mathbb{P}_k\,F(\delta_k)+R_k$ with

$$F(\delta)\stackrel{\text{def}}{=}\operatorname{div}(\delta\otimes u)+\operatorname{div}(u\otimes\delta)-\operatorname{div}(\delta\otimes\delta)\quad\text{and}\quad R_k\stackrel{\text{def}}{=}-(\operatorname{Id}-\mathbb{P}_k)\operatorname{div}(u\otimes u).$$

Let us consider a positive real number ε_0 which will be choosen (small enough) later on and let us define

$$\mathcal{T}_k\stackrel{\text{def}}{=}\sup\big\{t<T\,/\,\|\delta_k\|_{L^4([0,t];\mathcal{V}_\sigma)}\leq\varepsilon_0\big\}. \tag{4.2}$$

Let us decompose the interval $[0,T]=\displaystyle\bigcup_{\ell=0}^{N}[T_\ell,T_{\ell+1}]$ as follows:

$$
T_0=0,\ T_{N+1}=T,\ \int_{T_\ell}^{T_{\ell+1}}\|u(t)\|_{\mathcal{V}_\sigma}^4dt=\varepsilon_0\quad\text{if}\quad \ell<N\quad\text{and}\quad \int_{T_N}^{T}\|u(t)\|_{\mathcal{V}_\sigma}^4dt\leq\varepsilon_0. \tag{4.3}
$$

Let us notice that

$$
N\leq\frac{1}{\varepsilon_0^4}\int_0^T\|u(t)\|_{\mathcal{V}_\sigma}^4dt.
$$

Let us denote by j_k the maximum of indices $\ell<j_k$. Let us apply Lemma 2.3.1 page 23 between T_ℓ and T_{ell+1}. This gives

$$
\begin{aligned}
\max\bigl\{\|\delta_k\|_{L^\infty([T_\ell,T_{\ell+1}];\mathcal{V}_\sigma^{\frac12})},\|\delta_k\|_{L^4([T_\ell,T_{\ell+1}];\mathcal{V}_\sigma)}\bigr\}&\\
\leq\|\delta_k(T_\ell)\|_{\mathcal{V}_\sigma^{\frac12}}+\|F(\delta_k)\|_{L^2([T_\ell,T_{\ell+1}];\mathcal{V}_\sigma^{-\frac12})}&+\|R_k\|_{L^2(0,T];\mathcal{V}_\sigma^{-\frac12})}.
\end{aligned}
$$

Let us notice that we do not pay attention to the length of the interval when we estimate R_k. It is useless here because

$$
\lim_{k\to\infty}\eta_k=0\quad\text{with}\quad \eta_k\stackrel{\text{def}}{=}\|R_k\|_{L^2(0,T];\mathcal{V}_\sigma^{-\frac12})}. \tag{4.4}
$$

Using Corollary 2.3.1, a constant C exists such that, for any $k\in\mathbb{N}$, and any interval $[a,b]$, we have

$$
\|\mathbb{P}_k\,F(\delta)\|_{L^2([a,b];\mathcal{V}_\sigma^{-\frac12}}\leq C\|\delta\|_{L^4([a,b];\mathcal{V}_\sigma)}\bigl(\|\delta\|_{L^4([a,b];\mathcal{V}_\sigma)}+\|u\|_{L^4([a,b];\mathcal{V}_\sigma)}\bigr).
$$

Using (4.2) and (4.3), we get

$$
\|F(\delta_k)\|_{L^2([T_\ell,T_{\ell+1}];\mathcal{V}_\sigma^{-\frac12})}\leq C\varepsilon_0\|\delta_k\|_{L^4([T_\ell,T_{\ell+1}];\mathcal{V}_\sigma)}.
$$

Exactly along the same lines, we get, for any $t<\mathcal{T}_k$,

$$
\|F(\delta_k)\|_{L^2([T_{j_k},t];\mathcal{V}_\sigma^{-\frac12})}\leq C\varepsilon_0\|\delta_k\|_{L^4([T_{j_k},t];\mathcal{V}_\sigma)}.
$$

Now, using Lemma 2.3.1 page 23, we get

$$
\max\bigl\{\|\delta_k\|_{L^\infty([T_\ell,T_{\ell+1}];\mathcal{V}_\sigma^{\frac12})},\|\delta_k\|_{L^4([T_\ell,T_{\ell+1}];\mathcal{V}_\sigma)}\bigr\}\leq\|\delta_k(T_\ell)\|_{\mathcal{V}_\sigma^{\frac12}}+C\varepsilon_0\|\delta_k\|_{L^2([T_\ell,T_{\ell+1}];\mathcal{V}_\sigma^{-\frac12})}+\eta_k \tag{4.5}
$$

and, for any $t\in[T_{j_k},\mathcal{T}_k[$,

$$
\max\bigl\{\|\delta_k\|_{L^\infty([T_{j_k},t];\mathcal{V}_\sigma^{\frac12})},\|\delta_k\|_{L^4([T_{j_k},t];\mathcal{V}_\sigma)}\bigr\}\leq\|\delta_k(T_{j_k})\|_{\mathcal{V}_\sigma^{\frac12}}+C\varepsilon_0\|\delta_k\|_{L^2([T_\ell,t];\mathcal{V}_\sigma^{-\frac12})}+\eta_k \tag{4.6}
$$

Now let us choose ε_0 such that $C\varepsilon_0<1$. This implies that, for any $\ell<j_k$,

$$
\|\delta_k\|_{L^4([T_\ell,T_{\ell+1}];\mathcal{V}_\sigma)}\leq\frac{1}{1-C\varepsilon_0}(\|\delta_k(T_\ell)\|_{\mathcal{V}_\sigma^{\frac12}}+\eta_k) \tag{4.7}
$$

and, for any $t\in[T_{j_k},\mathcal{T}_k[$,

$$
\|\delta_k\|_{L^4([T_{j_k},t];\mathcal{V}_\sigma)}\leq\frac{1}{1-C\varepsilon_0}(\|\delta_k(T_{j_k})\|_{\mathcal{V}_\sigma^{\frac12}}+\eta_k) \tag{4.8}
$$

We deduce from (4.5) and (4.6) that

$$\|\delta_k\|_{L^\infty([T_\ell,T_{\ell+1}];\mathcal{V}_\sigma^{\frac12})} \leq \frac{1+C\varepsilon_0}{1-C\varepsilon_0}(\|\delta_k(T_\ell)\|_{\mathcal{V}_\sigma^{\frac12}}+\eta_k) \tag{4.9}$$

and, for any $t\in[T_{j_k},\mathcal{T}_k[$,

$$\|\delta_k\|_{L^\infty([T_{j_k},t];\mathcal{V}_\sigma^{\frac12})} \leq \frac{1+C\varepsilon_0}{1-C\varepsilon_0}(\|\delta_k(T_{j_k})\|_{\mathcal{V}_\sigma^{\frac12}}+\eta_k) \tag{4.10}$$

Now, let us choose ε_0 such that

$$\frac{1+C\varepsilon_0}{1-C\varepsilon_0}\leq 2.$$

Then (4.7) and (4.8) becomes

$$\begin{aligned}\|\delta_k\|_{L^\infty([T_\ell,T_{\ell+1}];\mathcal{V}_\sigma^{\frac12})} &\leq 2(\|\delta_k(T_\ell)\|_{\mathcal{V}_\sigma^{\frac12}}+\eta_k) \quad\text{and}\\ \|\delta_k\|_{L^\infty([T_{j_k},t];\mathcal{V}_\sigma^{\frac12})} &\leq 2(\|\delta_k(T_{j_k})\|_{\mathcal{V}_\sigma^{\frac12}}+\eta_k)\end{aligned}$$

$$\|\delta_k\|_{L^\infty([T_{j_k},t];\mathcal{V}_\sigma^{\frac12})} \leq 2^{N+1}\|\delta_k(T_{j_k})\|_{\mathcal{V}_\sigma^{\frac12}}+\eta_k). \tag{4.11}$$

Using (4.11), the choice of ε_0 implies that (4.7) and (4.8) becomes

$$\|\delta_k\|_{L^4([T_\ell,T_{\ell+1}];\mathcal{V}_\sigma)} \leq 2^{N+1}(\|\delta_k(0)\|_{\mathcal{V}_\sigma^{\frac12}}+\eta_k) \tag{4.12}$$

and, for any $t\in[T_{j_k},\mathcal{T}_k[$,

$$\|\delta_k\|_{L^4([T_{j_k},t];\mathcal{V}_\sigma)} \leq 2^{N+1}(\|\delta_k(0)\|_{\mathcal{V}_\sigma^{\frac12}}+\eta_k). \tag{4.13}$$

By summation, we get, for any $t<\mathcal{T}_k$

$$\|\delta_k\|^4_{L^4(0,t];\mathcal{V}_\sigma)} \leq N2^{N+1}(\|\delta_k(0)\|_{\mathcal{V}_\sigma^{\frac12}}+\eta_k) \leq e^{N+1}(\|\delta_k(0)\|_{\mathcal{V}_\sigma^{\frac12}}+\eta_k).$$

As we have

$$\lim_{k\to\infty}(\|\delta_k(0)\|_{\mathcal{V}_\sigma^{\frac12}}+\eta_k)=0,$$

we choose k large enough such that

$$(\|\delta_k(0)\|_{\mathcal{V}_\sigma^{\frac12}}+\eta_k) \leq \frac{\varepsilon_0}{100C}\exp\Big(-\frac{1}{\varepsilon_0^4}\int_0^T\|u(t)\|^4_{\mathcal{V}_\sigma}dt\Big).$$

This gives that $t=\mathcal{T}_k$ and conclude the proof of the lemma. □

Continuation of the proof of Theorem 4.2.1. We are going to estimate

$$E_k(t)\stackrel{\text{def}}{=}\frac12\|(u_k-v)(t)\|^2_{L^2}+\int_0^t\|\nabla(u_k-v)(t')\|^2_{L^2}\,dt'$$

Let us write that as v is a Leray solutions,

$$\begin{aligned}E_k(t) &= \frac12|u_k(t)\|^2_{L^2}+\int_0^t\|\nabla u_k(t')\|^2_{L^2}dt'+\|v(t)\|^2_{L^2}+\int_0^t\|\nabla v(t')\|^2_{L^2}\,dt'\\ &\qquad -(u_k(t)|v(t))_{L^2}-2\int_0^t(\nabla u_k(t')|\nabla v(t'))_{L^2}\,dt'\\ &\leq \|\mathbb{P}_k u_0\|^2_{L^2}+\|v_0\|^2_{L^2}-(u_k(t)|v(t))_{L^2}-2\int_0^t(\nabla u_k(t')|\nabla v(t'))_{L^2}\,dt'.\end{aligned} \tag{4.14}$$

Now the problem consists in evaluating the cross-product terms

$$(u_k(t)|v(t))_{L^2} + 2\int_0^t (\nabla u_k(t')|\nabla v(t'))_{L^2}\,dt'.$$

The function u_k belongs to $C^1(\mathbb{R}^+;\mathcal{V}_\sigma)$, thus it can be used as a test function in Definition 3.1.1. As v is a Leray solution, we have

$$\begin{aligned}\mathcal{B}_k(t) &\stackrel{\text{def}}{=} (v(t)|u_k(t))_{L^2} + 2\int_0^t (\nabla u_k(t')|\nabla v(t'))_{L^2}\,dt'\\ &= (v(0)|u_k(0))_{L^2} + \int_0^t (v(t')\otimes v(t')|\nabla u_k(t'))_{L^2}\,dt' + \int_0^t \langle v(t'),\partial_t u_k(t')\rangle\,dt'.\end{aligned}$$

As u_k satisfies (NS_k), we get

$$\begin{aligned}\mathcal{B}_k(t) = (v(0)|u_k(0))_{L^2} &+ \int_0^t (v(t')\otimes v(t')|\nabla u_k(t'))_{L^2}\,dt'\\ &+ \int_0^t \langle \mathbb{P}_k \operatorname{div}(u_k(t')\otimes u_k(t')), v(t')\rangle\,dt'.\end{aligned}$$

As $(u_k)_{k\in\mathbb{N}}$ is bounded in $L^4([0,T];\mathcal{V}_\sigma)$, the sequence $(\operatorname{div}(u_k(t')\otimes u_k)_{k\in\mathbb{N}}$ is bounded in the space $L^2([0,T]);\mathcal{V}_\sigma^{-\frac12})$. Thus

$$\lim_{k\to\infty}\eta_k = 0 \quad\text{with}\quad \eta_k \stackrel{\text{def}}{=} \int_0^T \big|\langle(\mathrm{Id}-\mathbb{P}_k)\operatorname{div}(u_k(t')\otimes u_k(t')), v(t')\rangle\big|\,dt'.$$

Indeed, we have

$$\eta_k \leq \|(\mathrm{Id}-\mathbb{P}_k)\operatorname{div}(u_k\otimes u_k)\|_{L^2([0,T];\mathcal{V}'_\sigma)}\|v\|_{L^2([0,T];\mathcal{V}_\sigma)}$$

Using Theorem 2.1.2 page 18, we get, for any a in $\mathcal{V}_\sigma^{\frac12}$,

$$\|(\mathrm{Id}-\mathbb{P}_k)a\|_{\mathcal{V}'_\sigma} \leq \mu_k^{-\frac12}\|a\|_{\mathcal{V}_\sigma^{-\frac12}}.$$

This gives

$$\eta_k \leq \mu_k^{-\frac12}\sup_k \|u_k\|^2_{L^4([0,T];\mathcal{V}_\sigma)}\|v\|_{L^2([0,T];\mathcal{V}_\sigma)}.$$

Thus,

$$\begin{aligned}\mathcal{B}_k(t) = (v(0)|u_k(0))_{L^2} &+ \int_0^t (v(t')\otimes v(t')|\nabla u_k(t'))_{L^2}\,dt'\\ &+ \int_0^t ((u_k(t')\otimes u_k(t'))|v(t'))_{L^2}\,dt' + \eta_k(t).\end{aligned}$$

where $\eta_k(t)$ tends uniformely to 0 on $[0,T]$.

Now, let us observe that, for any vector field a and b in $\mathcal{V}_\sigma$, we have $(b\otimes b|\nabla a)_{L^2} = \langle Q(b,b),a\rangle$ and thus

$$(b\otimes b|\nabla a)_{L^2} + \langle Q(a,a),b\rangle = \langle Q(b,b),a\rangle + \langle Q(a,a),b\rangle.$$

Using Lemma 3.2.2, we can write

$$\begin{aligned}\langle Q(b,b),a\rangle + \langle Q(a,a),b\rangle &= \langle Q(b,b),a-b\rangle + \langle Q(a,a),b-a\rangle\\ &= \langle Q(b,a),a-b\rangle + \langle Q(a,a),b-a\rangle.\end{aligned}$$

Thus, it turns out that

$$\begin{aligned}\langle Q(b,b),a\rangle + \langle Q(a,a),b\rangle &= \langle Q(a-b,a),b-a\rangle\\ &= ((a-b)\otimes a|\nabla(b-a))_{L^2}.\end{aligned}$$

Using the Gagliardo–Nirenberg inequality (see Corollary 1.2.2), we get for any a and c in $\mathcal{V}_\sigma$,

$$|(c\otimes a|\nabla c)_{L^2}| \leq C\|a\|_{L^6}\|c\|_{L^3}\|\nabla c\|_{L^2} \leq C\|\nabla a\|_{L^2}\|c\|_{L^2}^{\frac{1}{2}}\|\nabla c\|_{L^2}^{\frac{3}{2}}. \tag{4.15}$$

For almost every time t, the vector field $v(t)$ belongs to $\mathcal{V}_\sigma$. It follows that for all $k\in\mathbb{N}$ and $t\geq 0$, taking $a=u_k(t')$ and $b=v(t')$, $t'\in[0,t]$, we have

$$\mathcal{B}_k(t) = (v(0)|u_k(0))_{L^2} + \leq C\int_0^t \|\nabla u_k(t')\|_{L^2}\|(u_k-v)(t')\|_{L^2}^{\frac{1}{2}}\|\nabla(u_k-v)(t')\|_{L^2}^{\frac{3}{2}}\,dt'.$$

We infer that

$$E_k(t) \leq \|\mathbb{P}_k u_0 - v_0\|_{L^2}^2 + C\int_0^t \|\nabla u(t')\|_{L^2}\|\nabla(u-v)(t')\|_{L^2}^{\frac{3}{2}}\|(u-v)(t')\|_{L^2}^{\frac{1}{2}}\,dt'.$$

Using the convexity inequality (4.3) with $\theta=1/4$ and $\theta=1/2$, we obtain

$$\begin{aligned}\|(u_k-v)(t)\|_{L^2}^2 + \int_0^t \|\nabla(u_k-v)(t')\|_{L^2}^2 dt' &\leq \|\mathbb{P}_k u_0 - v_0\|_{L^2}^2\\ &\quad + C\int_0^t \|\nabla u_k(t')\|_{L^2}^4\|(u_k-v)(t')\|_{L^2}^2\,dt'.\end{aligned}$$

Gronwall's Lemma implies that

$$\begin{aligned}&\|(u_k-v)(t)\|_{L^2}^2 + \int_0^t \|\nabla(u_k-v)(t')\|_{L^2}^2 dt'\\ &\qquad\leq \|\mathbb{P}_k u_0 - v_0\|_{L^2}^2 \exp C\Big(\int_0^t \|\nabla u_k(t')\|_{L^2}^4\|(u_k-v)(t')\|_{L^2}^2\,dt'\Big).\end{aligned}$$

The approximation lemma 4.2.1 allows to conclude the proof. □

4.3 Existence of stable solutions in a bounded domain

The aim of this paragraph is the proof of the following existence theorem with data in $\mathcal{V}_\sigma^{\frac{1}{2}}$.

Theorem 4.3.1 *If the initial data u_0 belongs to $\mathcal{V}_\sigma^{\frac{1}{2}}$ and the bulk force f belongs to the space $L^2_{loc}(\mathbb{R}_+;\mathcal{V}_\sigma^{-\frac{1}{2}})$, then a positive time T exists such that a solution u of (NS_ν) exists in $L^4([0,T];\mathcal{V}_\sigma)$. This solution is unique and belongs to $C([0,T];\mathcal{V}_\sigma^{\frac{1}{2}})$.*

Moreover, a constant c exists (which can be chosen independent of the domain Ω) such that, if

$$\|u_0\|_{\mathcal{V}_\sigma^{\frac{1}{2}}} + \|f\|_{L^2(\mathbb{R}^+;\mathcal{V}_\sigma^{-\frac{1}{2}})} \leq c,$$

then the above solution is global.

Proof. Let us explain the basic idea of H. Fujita and T. Kato, known as Kato's scheme. A solution of (NS) can be decompose as

$$u = u_L + w.$$

where u_L is the solution of the Stokes' problem with initial data u_0 and external force 0, and w is the solution of the Stokes problem with initial data 0 and external force $\operatorname{div}(u \otimes u)$ (provided $\operatorname{div}(u \otimes u)$ is regular enough. Let us donote by B the bilinear operator defined by the fact that $B(v, w)$ is the solution of the Stokes problem

$$\begin{cases} \partial_t B - \Delta B &= \frac{1}{2} \operatorname{div}(u \otimes v + v \otimes u) - \nabla p \\ B_{|t=0} &= 0. \end{cases}$$

Lemma 4.3.1 *The bilinear operator B maps continuously $L^4([0,T];\mathcal{V}_\sigma) \times L^4([0,T];\mathcal{V}_\sigma)$ into the space $L^4([0,T];\mathcal{V}_\sigma) \cap C([0,T];\mathcal{V}_\sigma^{\frac{1}{2}})$.*

Proof. For any u and v in $\mathcal{V}_\sigma$, we have $\operatorname{div}(u \otimes v) = u \cdot \nabla v$. Thus, using Hölder estimates we have

$$\|\operatorname{div}(u \otimes v)\|_{L^{\frac{3}{2}}} \leq \|u\|_{L^6} \|\nabla v\|_{L^2} \leq \|u\|_{\mathcal{V}_\sigma} \|v\|_{\mathcal{V}_\sigma}.$$

Proposition 2.3.1 and Lemma 2.3.1 imply that

$$\|\operatorname{div}(u \otimes v)\|_{L^2([0,T];\mathcal{V}_\sigma^{-\frac{1}{2}})} \leq \|u\|_{L^2([0,T];\mathcal{V}_\sigma)} \|\nabla v\|_{L^2([0,T];\mathcal{V}_\sigma)}.$$

The lemma is proved. □

Proof of Theorem 4.3.1 (continued). Let us recall (without proof) Picard's fixed point theorem for bilinear maps.

Lemma 4.3.2 *Let E be a Banach space and $\mathcal{B}$ a bilinear map continuous from $E \times E$ into E and α a positive number such that*

$$\alpha < \frac{1}{4\|\mathcal{B}\|} \quad \textit{with} \quad \|\mathcal{B}\| \stackrel{\text{def}}{=} \sup_{\substack{\|u\| \leq 1 \\ \|v\| \leq 1}} \|\mathcal{B}(u, v)\|.$$

Then for any a in the ball $B(0, \alpha)$ of center 0 and radius α of E, a unique x exists in the ball of radius 2α such that

$$x = a + \mathcal{B}(x, x).$$

Moreover, we have $\|x\| \leq 2\|a\|$.

Now, the question is simply to ensure that $\|u_L\|_{L^4([0,T];\mathcal{V}_\sigma)}$ is small enough. Using (2.3) page 21 and (2.8), we have

$$\|u_L\|_{L^4([0,T];\mathcal{V}_\sigma)} \leq \left(\sum_{j=0}^{\infty} \mu_j^2 (\langle u_0, e_j \rangle)^2 \|e^{-\mu_j^2 t}\|^2_{L^4([0,T])} \right)^{\frac{1}{2}}. \tag{4.16}$$

First of all, let observe that this gives

$$\|u_L\|_{L^4([0,T];\mathcal{V}_\sigma)} \leq \frac{1}{\nu^{\frac{1}{4}}} \|u_0\|_{\mathcal{V}_\sigma}.$$

Moreover, using the definition of $\mathcal{V}_\sigma^s$ norm and the fact that the sequence $(\mu_j)_{j\in\mathbb{N}}$ is non decreasing, we can write that

$$\begin{aligned}\|u_L\|_{L^4([0,T];\mathcal{V}_\sigma)} &\leq \Big(\sum_{j=j_0+1}^{\infty}\mu_j^2\langle u_0,e_j\rangle^2\Big)^{\frac12} + T^{\frac14}\Big(\sum_{j=0}^{j_0}\mu_j\langle u_0,e_j\rangle^2\Big)^{\frac12}\\ &\leq \Big(\sum_{j=j_0+1}^{\infty}\mu_j\langle u_0,e_j\rangle^2\Big)^{\frac12} + \mu_{j_0}T^{\frac14}\|u_0\|_{\mathcal{V}_\sigma}\end{aligned}$$

Then, choosing j_0 large enough and then T small ernough, we claim that, for any $u0$ in $\mathcal{V}_\sigma^{\frac14}$,

$$\lim_{T\to 0}\|u_L\|_{L^4([0,T];\mathcal{V}_\sigma)} = 0.$$

The whole Theorem 4.3.1 is proved. □

Let us conclude this paragraph, we shall assume that the bulk force f is identically 0. We shall establish some results about the maximal existence time of the solution constructed in the preceding paragraph.

Proposition 4.3.1 *Let us assume that the initial data u_0 belongs to $\mathcal{V}_\sigma$. Then the maximal time of existence $T^\star$ of the solution u in the space $C([0,T^\star[;\mathcal{V}_\sigma^{\frac12})\cap L^4_{loc}([0,T^\star[;\mathcal{V}_\sigma)$ satisfies*

$$T^\star \geq \frac{c}{\|\nabla u_0\|_{L^2}^4}\cdot$$

Proof. Let us observe that

$$\Big(\sum_{j=0}^{\infty}\mu_j^2(\langle u_0,e_j\rangle)^2\|e^{-\mu_j^2 t}\|^2_{L^4([0,T])}\Big)^{\frac12} \leq T^{\frac14}\|u_0\|_{\mathcal{V}_\sigma}.$$

Thanks to (4.16), we get the proposition. □

From this proposition, we infer the following corollary.

Corollary 4.3.1 *Let $T^\star$ be the maximal time of existence for a solution u of the system (NS_ν) in the space $C([0,T^\star);\mathcal{V}_\sigma^{\frac12})\cap L^4_{loc}([0,T^\star);\mathcal{V}_\sigma)$. If $T^\star$ is finite, then*

$$\int_0^{T^\star}\|\nabla u(t)\|^4_{L^2}dt = +\infty \quad \textit{and} \quad T^\star \leq C\|u_0\|^4_{L^2}.$$

Proof. For almost every t, $u(t)$ belongs to $\mathcal{V}_\sigma$. Then, thanks to the above proposition, the maximal time of existence of the solution starting at time t, which is of course $T^\star - t$, satisfies

$$T^\star - t \geq c\|\nabla u(t)\|^4_{L^2}\cdot$$

This can be written as

$$\|\nabla u(t)\|^4_{L^2} \geq \frac{c}{T^\star - t}\cdot$$

This gives the first part of the corollary. Taking the square root of the above inequality gives, thanks to the energy estimate,

$$c\int_0^{T^\star}\frac{dt}{(T^\star-t)^{\frac12}} \leq \frac12\|u_0\|^2_{L^2}.$$

The corollary is proved. □

Remarks

- Sections 4.1 and 4.3 must be known.
- Again books of P. Constantin et C. Foias *Navier-Stokes equations*, Chigago University Press, 1988, de P.-G. Lemarié-Rieusset, *Recent developments in the Navier-Stokes problem.* Chapman & Hall/CRC, Research Notes in Mathematics, **431**, 2002 and of J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, *Mathematical Geophysics; an introduction to rotating fluids and Navier-Stokes equations*, Oxford Lecture series in Mathematics and its maps, **32**, Oxford University Press, 2006 give more details.

Chapter 5

Kato's approach in the whole space

In this chapter, we study the incompressible Navier-Stokes equation in the whole space $\mathbb{R}^3$. The reason why is that the space we are going to use are much easier to define and to use in the whole space than in the case of a bounded domain.

5.1 The Navier-Stokes system in the whole space

Let us first recall what the incompressible Navier-Stokes system is. We consider as unknown the speed $u = (u^1, u^2, u^3)$ a time dependant divergence free vector field on $\mathbb{R}^3$ and the pressure p. We consider the system

$$(NS) \quad \begin{cases} \partial_t u + u \cdot \nabla u - \Delta u = -\nabla p + f \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3 \\ \operatorname{div} u = 0 \\ u|_{t=0} = u_0. \end{cases}$$

Again the notion of C^2 solution (i.e. classical solution) is not efficient here. Let us define the notion of weak solution (that we shall denote simply solution in all that follows).

Definition 5.1.1 *A time-dependent vector field u with components in $L^2_{loc}([0,T] \times \mathbb{R}^d)$ is a weak solution (simply a solution in this paper) of (NS_ν) if for any smooth compactly supported divergence free vector field Ψ,*

$$\langle u(t,\cdot), \Psi(t,\cdot)\rangle = \langle u_0, \Psi(0,\cdot)\rangle + \langle f(t',\cdot), \Psi(t',\cdot)\rangle dt' + \int_0^t \big(\langle u, \Delta\Psi\rangle + \langle u\otimes u, \nabla\Psi\rangle + \langle u, \partial_t\Psi\rangle\big)(t')\, dt'.$$

As in Chapter 3, let us define the concept of turbulent (or Leray) solution.

Definition 5.1.2 *A turbulent solution of (NS) is a divergence free vector field u which is a weak solution, has component is $L^\infty_T(L^2) \cap L^2_T(H^1)$ and satisfies in addition the energy inequality*

$$\frac{1}{2}\|u(t)\|^2_{L^2} + \int_0^t \|\nabla u(t')\|^2_{L^2} dt' \le \frac{1}{2}\|u_0\|^2_{L^2} + \int_0^t \int_{\mathbb{R}^3} f(t',x) u(t',x) dt' dx. \tag{5.1}$$

Remark For a turbulent solution, Definition 5.1.1 of a weak solution becomes

$$\int_{\mathbb{R}^d} u(t,x)\cdot\Psi(t,x)\,dx = \int_{\mathbb{R}^d} u_0(x)\cdot\Psi(0,x)\,dx + \int_0^t \langle f(t'),\Psi(t')\rangle dt'$$
$$- \int_0^t\int_{\mathbb{R}^d} \big(\nabla u : \nabla\Psi - u\otimes u : \nabla\Psi - u\cdot\partial_t\Psi\big)(t',x)\,dx\,dt'.$$

Following the lines of Chapter 3 (taking care of infinity in space), we can proof the following global existence theorem.

Theorem 5.1.1 *Let u_0 be a divergence free vector field in $L^2(\mathbb{R}^d)$. Then a turbulent solution u exists on $\mathbb{R}^+\times\mathbb{R}^3$.*

Another fundamental fact about Navier-Stokes equation (also ignored by Definition 5.1.1) is the scaling invariance. Let us observe that if u is a solution of (NS) on $[0,T]\times\mathbb{R}^3$, then for any positive λ, the vector field $u_\lambda(t,x) \stackrel{\text{def}}{=} \lambda u(\lambda^2 t,\lambda x)$ is also a solution of (NS) on $[0,\lambda^{-2}T]\times\mathbb{R}^3$.

Let us define the notion of scaling invariant spaces.

Definition 5.1.3 *Let X be a Banach space of functions on $\mathbb{R}^+\times\mathbb{R}^3$. It is scaling invariant if, for any positive λ,*

$$u\in X \Longleftrightarrow u_\lambda\in X \quad\textit{and}\quad \|u\|_X\sim\|u_\lambda\|_X.$$

Let $(X_T)_{T>0}$ be a family of Banach spaces of distributions on $[0,T]\times\mathbb{R}^3$. It is a scaling invariant family if, for any positive λ and any positive T,

$$u\in X_T \Longleftrightarrow u_\lambda\in X_{\lambda^{-2}T} \quad\textit{and}\quad \|u\|_{X_T}\sim\|u_\lambda\|_{X_{\lambda^{-2}T}}.$$

Let us give some classical examples of scaling invariant families:

- The spaces $L^4([0,T];\dot H^1)$ define a scaling family. More generaly, the spaces

  $$L^p([0,T];\dot H^s)\quad\text{with}\quad s=\frac12+\frac2p$$

  define a scaling family.

- The spaces $L^4([0,T];L^6)$ or more generally the spaces

  $$L^r([0,T];L^p)\quad\text{with}\quad \frac3p+\frac2r=1$$

  define a scaling family.

- The space of functions u on $[0,T]\times\mathbb{R}^3$ such that $\sup\limits_{t\in[0,T]} t^{\frac14}\|u(t)\|_{L^6}$ or more generally the space of functions u on $[0,T]\times\mathbb{R}^3$ such that

  $$\sup_{t\in[0,T]} t^{\frac1r}\|u(t)\|_{L^p}\quad\text{with}\quad \frac3p+\frac2r=1$$

  are scaling invariant families.

All the solutions constructed following Kato's approach can be included in the following concept of scaled solution. The idea is that the high frequency part belongs to a scaling invariant space.

Definition 5.1.4 *A global solution scaled of (NS) is a global solution which belongs to a scaling invariant space.*

A scaled solution of (NS) on $[0,T]\times\mathbb{R}^3$ is a solution u such that, if for some smooth compactly supported function χ on $\mathbb{R}^3$ with value 1 near 0,

$$(\mathrm{Id}-\chi(D))u \stackrel{\text{def}}{=} \mathcal{F}^{-1}\big(1-\chi(\xi))\widehat{u}\big) \in X_T$$

where X_T belongs to a scaling invariant family in the sense of Definition 5.1.3.

Now, let us compute the pressure. This is possible only because there is no boundary. As seen in the preceeding chapter, it is not possible for domain with boundary. This simplifies a lot the technicalities. Applying the divergence operator to the incompresible Naver-Stokes equation gives

$$-\Delta p = \sum_{1\leq j,k\leq d} \partial_j\partial_k(v^j v^k).$$

So formally, we have that

$$p = -\sum_{1\leq j,k\leq d} \Delta^{-1}\partial_j\partial_k(v^j v^k) \quad \text{with} \quad \Delta^{-1}\partial_j\partial_k a \stackrel{\text{def}}{=} \mathcal{F}^{-1}(|\xi|^{-2}\xi_j\xi_k\widehat{a}). \tag{5.2}$$

In all this chapter, we shall denote by Q any bilinear map of the form

$$Q^j(u,v) \stackrel{\text{def}}{=} \sum_{k,\ell,m} q^{j,m}_{k,\ell}\partial_m(u^k v^\ell),$$

where $q^{j,m}_{k,\ell}$ are Fourier multipliers of the form

$$q^{j,m}_{k,\ell}a \stackrel{\text{def}}{=} \sum_{n,p} \alpha^{j,m,n,p}_{k,\ell}\mathcal{F}^{-1}\Big(\frac{\xi_n\xi_p}{|\xi|^2}\widehat{a}(\xi)\Big) \quad \text{with} \quad \alpha^{j,m,n,p}_{k,\ell} \in \mathbb{R}\,.$$

We shall denote by Q_{NS} the particular one related to Navier-Stokes equation, namely

$$Q^j_{NS}(u,v) \stackrel{\text{def}}{=} \frac{1}{2}\big(\mathrm{div}(v^j u) + \mathrm{div}(u^j v)\big) - \sum_{1\leq k,\ell\leq d} \partial_j\Delta^{-1}\partial_k\partial_\ell(u^k v^\ell).$$

Now the incompressible Navier-Stokes system appears as a particular case of

$$(GNS)\left\{\begin{array}{rcl} \partial_t v - \Delta v + Q(v,v) & = & 0\\ v_{|t=0} & = & v_0. \end{array}\right.$$

with the quadratic operator Q define above. Let us define $B(u,v)$ (resp. $B_{NS}(u,v)$) by

$$\left\{\begin{array}{c} \partial_t B(u,v) - \Delta B(u,v) = Q(u,v) \text{ (resp. } Q_{NS}(u,v))\\ B(u,v)_{|t=0} = 0. \end{array}\right.$$

Now solving (GNS) (resp. (NS)) can be seen as finding a fixed point for the map

$$u \longmapsto e^{t\Delta}u_0 + B(u,u) \text{ (resp. } B_{NS}(u,u)).$$

In all this chapter, we shall solve (GNS) or (NS) using a contraction argument in a well choosen Banach space. It is based on the following classical lemma, that we recall for the reader's convenience.

Lemma 5.1.1 *Let E be a Banach space and $\mathcal{B}$ a bilinear map continuous from $E\times E$ into E and α a positive number such that*

$$\alpha < \frac{1}{4\|\mathcal{B}\|} \quad \textit{with} \quad \|\mathcal{B}\| \stackrel{\text{def}}{=} \sup_{\substack{\|u\|\leq 1\\ \|v\|\leq 1}} \|\mathcal{B}(u,v)\|.$$

Then for any a in the ball $B(0,\alpha)$ of center 0 and radius α of E, a unique x exists in the ball of radius 2α such that

$$x = a + \mathcal{B}(x,x).$$

Moreover, we have $\|x\| \leq 2\|a\|$.

Proof. It consists in applying the classical iterative scheme define by

$$x_0 = a \quad \text{and} \quad x_{n+1} = a + \mathcal{B}(x_n, x_n).$$

Let us first prove by induction that $\|x_n\| \leq 2\alpha$. Using this hypothesis on α, we get, by definition of x_{n+1},

$$\|x_{n+1}\| \leq \alpha(1+4\alpha\|\mathcal{B}\|) \leq 2\alpha.$$

Thus the sequence remains in the ball $B(0,2\alpha)$. Then

$$\begin{aligned} x_{n+1} - x_n &= \mathcal{B}(x_n,x_n) - \mathcal{B}(x_{n-1},x_{n-1})\\ &\leq \mathcal{B}(x_n - x_{n-1}, x_n) + \mathcal{B}(x_{n-1}, x_n - x_{n-1}). \end{aligned}$$

Then we have

$$\|x_{n+1} - x_n\| \leq 4\alpha\|\mathcal{B}\|\,\|x_n - x_{n-1}\|.$$

The hypothesis $4\alpha\|\mathcal{B}\| < 1$, makes that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in E the limit of which is a fixed point of $x \mapsto a + \mathcal{B}(x,x)$ in the ball $B(0,2\alpha)$. This fixed point is unique because if x and y are two such fixed points, then

$$\|x-y\| \leq \|\mathcal{B}(x-y,y) + \mathcal{B}(x,x-y)\| \leq 4\alpha\|\mathcal{B}\|\,\|x-y\|.$$

The lemma is proved. □

The Kato' scheme consists in finding a scaled family of spaces $(X_T)_{T>0}$ such that the bilinear operator B maps $X_T \times X_T$ into X_T continously. This will produce automatically local or global wellposesness result.

5.2 An elementary L^p approach

Let us define the space we are going to work with.

Definition 5.2.1 *If p is in $[1,\infty]\setminus\{3\}$ and T in $]0,\infty]$, let us define $K_p(T)$ by*

$$K_p(T) \stackrel{\text{def}}{=} \Big\{ u \in C(]0,T]; L^p) \,/\, \|u\|_{K_p(T)} \stackrel{\text{def}}{=} \sup_{t\in[0,T]} t^{\frac12\left(1-\frac3p\right)} \|u(t)\|_{L^p} < \infty \Big\}.$$

We shall denote by $K_3(T)$ the space of bounded continuous functions from $]0,T]$ with value L^3 equipped with the norm $\|\cdot\|_{L^\infty(]0,T];L^3)}$.

Theorem 5.2.1 *For any p in $]3,\infty[$, a constant c exists which satisfies the following properties. Let u_0 be an initial data in $\mathcal{S}'$ such that, for some positive T,*

$$\|e^{t\Delta}u_0\|_{K_p(T)} \leq c. \tag{5.3}$$

Then a unique solution u of (GNS) exists in the ball of center 0 and radius $2c\nu$ in the Banach space $K_p(T)$.

Proof. Thank to Lemma 5.1.1, the proof reduces to the proof of the following lemma. □

Lemma 5.2.1 *For any p, q and r such that*

$$0 < \frac{1}{p} + \frac{1}{q} \leq 1 \quad \text{and} \quad \frac{1}{r} \leq \frac{1}{p} + \frac{1}{q} < \frac{1}{3} + \frac{1}{r}\cdot$$

Then, for any positive T, the bilinear map B maps $K_p(T) \times K_q(T)$ into $K_r(T)$. Moreover, a constant C (independant of T) exists such that

$$\|B(u,v)\|_{K_r(T)} \leq C\|u\|_{K_p(T)}\|v\|_{K_q(T)}.$$

Proof. The method consists in computing B as a convolution operator. More precisely, we have the following proposition.

Proposition 5.2.1 *We have*

$$B^j(u,v)(t,x) = \sum_{k,\ell} \int_0^t \Gamma^j_{k,\ell}(t-t',\cdot) \star \big(u^j(t',\cdot)v^\ell(t',\cdot)\big)dt'$$

where the functions $\Gamma^j_{k,\ell}$ belongs to $C(]0,\infty[;L^s)$ for any s in $[1,\infty[$ and satisfies, for any j, k and ℓ,

$$\|\Gamma^j_{k,\ell}(t,\cdot)\|_{L^s} \leq \frac{C}{t^{2-\frac{3}{2s}}}\cdot$$

Proof. In Fourier space, we have

$$\mathcal{F}B^j(u,v)(t,\xi) = i\int_0^t e^{-(t-t')|\xi|^2} \sum_{k,\ell} \alpha_{j,k\ell}\xi_j\xi_k\xi_\ell|\xi|^{-2}\mathcal{F}Q(u(t'),v(t'))(\xi)dt'.$$

In order to write this operator as a convolution operator, it is enough to compute the inverse Fourier transform of $\xi_j\xi_k\xi_\ell|\xi|^{-2}e^{-t|\xi|^2}$. Using the fact that

$$e^{-t|\xi|^2}|\xi|^{-2} = \int_t^\infty e^{-t'|\xi|^2}dt',$$

we get that

$$\begin{aligned} \Gamma^j_{k,\ell}(t,x) &= (2\pi)^{-d} i \int_t^\infty \int_{\mathbb{R}^3} \xi_j\xi_k\xi_\ell e^{i(x|\xi)-t'|\xi|^2}dt'd\xi \\ &= (2\pi)^{-d}\partial_j\partial_k\partial_\ell \int_t^\infty \int_{\mathbb{R}^3} e^{i(x|\xi)-t'|\xi|^2}dt'd\xi. \end{aligned}$$

Using the formula about the Fourier transform of the Gaussian functions, we get

$$\begin{aligned}\Gamma^j_{k,\ell}(t,x) &= \partial_j\partial_k\partial_\ell \int_t^\infty \frac{1}{(4\pi t')^{\frac{3}{2}}} e^{-\frac{|x|^2}{4t'}}\,dt' \\ &= \pi^{-\frac{3}{2}} \int_t^\infty \frac{1}{(4t')^3}\Psi^j_{k,\ell}\Big(\frac{x}{\sqrt{4t'}}\Big)dt' \quad \text{with} \\ \Psi^j_{k,\ell}(z) &\stackrel{\text{def}}{=} \partial_j\partial_k\partial_\ell e^{-|z|^2}.\end{aligned}$$

Changing variable $r=(4t')^{-1}|x|^2$ gives

$$|\Gamma^j_{k,\ell}(t,x)| \leq \frac{1}{\pi^{\frac{3}{2}}}\frac{1}{|x|^4}\int_0^{\frac{|x|^2}{4t}} r\Psi^j_{k,\ell}\Big(\frac{x}{|x|}r\Big)dr.$$

This implies that

$$|\Gamma^j_{k,\ell}(t,x)| \leq c\min\Big\{\frac{1}{t^2},\frac{1}{|x|^4}\Big\} \quad \text{and thus} \quad \|\Gamma^j_{k,\ell}(t,\cdot)\|_{L^s} \leq \frac{C}{t^{2-\frac{3}{2s}}}\cdot$$

In order to prove the continuity, let us observe that, for $0\leq c\leq t_1\leq t_2$, we have

$$|\Gamma^j_{k,\ell}(t_2,x)-\Gamma^j_{k,\ell}(t_1,x)| \leq \frac{C}{|x|^4}\int_{\frac{|x|^2}{4t_2}}^{\frac{|x|^2}{4t_1}} re^{-r}dr.$$

This implies that

$$|\Gamma^j_{k,\ell}(t_2,x)-\Gamma^j_{k,\ell}(t_1,x)| \leq C\min\Big\{\frac{t_2^2-t_1^2}{(t_1t_2)^2},\frac{1}{|x|^4}\Big\}.$$

The proposition is proved. □

Proof of Lemma 5.2.1 (continued) Thanks to Young's and Hölder inequality and the condition

$$\frac{1}{r}\leq\frac{1}{p}+\frac{1}{q}\leq 1,$$

we have, using Proposition 5.2.1 with s defined by $1+\dfrac{1}{r}=\dfrac{1}{s}+\dfrac{1}{p}+\dfrac{1}{q}$,

$$\|B(u,v)(t)\|_{L^r} \leq C\int_0^t \frac{1}{\sqrt{(t-t')}^{4-3\left(1+\frac{1}{r}-\frac{1}{p}-\frac{1}{q}\right)}}\|u(t')\|_{L^p}\|v(t')\|_{L^q}dt'.$$

By definition of the $K_p(T)$ norms, we get that

$$\begin{aligned}\|B(u,v)(t)\|_{L^r} &\leq \|u\|_{K_p(T)}\|v\|_{K_q(T)}\int_0^t \frac{1}{\sqrt{(t-t')}^{1-3\left(\frac{1}{r}-\frac{1}{p}-\frac{1}{q}\right)}}\frac{1}{\sqrt{t'}^{2-3\left(\frac{1}{p}+\frac{1}{q}\right)}}dt' \\ &\leq C\frac{1}{t^{\frac{1}{2}\left(1-\frac{3}{r}\right)}}\|u\|_{K_p(T)}\|v\|_{K_q(T)}.\end{aligned}$$

Lemma 5.2.1 is proved. □

5.3 Besov spaces of negative index

In this section, we interpret Theorem 5.2.1 in term of Besov spaces. Let us introduce the following spaces.

Definition 5.3.1 *Let s be a positive real number and $p \in [1, \infty]$. We define $\dot{B}_p^{-s}$ as the space of temperated distributions such that*

$$\|u\|_{\dot{B}_p^{-s}} \stackrel{def}{=} \sup_{t>0} t^{\frac{s}{2}} \|e^{t\Delta} u\|_{L^p} < \infty.$$

Let us point out that as, for $q \geq p$, $e^{t\Delta} u = e^{\frac{t}{2}\Delta} e^{\frac{t}{2}\Delta} u$ and

$$\|e^{\tau\Delta}\|_{\mathcal{L}(L^p;L^q)} \leq C \tau^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}. \tag{5.4}$$

Thus we get that

$$\|u\|_{\dot{B}_q^{-s+d\left(\frac{1}{p}-\frac{1}{q}\right)}} \leq C \|u\|_{\dot{B}_p^{-s}}. \tag{5.5}$$

The part of Theorem 5.2.1 relative of small initial data can be stated as follows.

Theorem 5.3.1 *Let p be in $]3, \infty[$. A constant c_p exists such that, if $\|u_0\|_{\dot{B}_p^{-1+\frac{3}{p}}}$ is small enough, then a unique global solution of (NS) exists in $K_p(\infty)$.*

Now let us give some examples of functions on $\dot{B}_p^{-1+\frac{3}{p}}$. The first one is very simple : indeed, if a function f belongs to L^3, then (5.4) implies that

$$\|e^{t\Delta} f\|_{L^p} \leq t^{-\frac{3}{2}\left(\frac{1}{3}-\frac{1}{p}\right)} \|f\|_{L^3}.$$

Thus L^3 is included in any $\dot{B}^{-1+\frac{3}{p}}$ for p in $]3, \infty]$.

Homogeneous functions of negative degree will also belong to some Besov spaces. We have the following proposition.

Proposition 5.3.1 *Let p be in $]1, \infty]$ and α in $]d/p, d[$. Then a homogeneous function f of degree α of $\mathbb{R}^d \setminus \{0\}$ which is bounded on the sphere of $\mathbb{R}^d$ belongs to $\dot{B}_p^{-\alpha+\frac{d}{p}}$.*

Proof. We have, by changing of variable and because homogeneïty of f that

$$(e^{t\Delta} f)(x) = t^{-\frac{\alpha}{2}} (e^{\Delta} f)\Big(\frac{x}{\sqrt{t}}\Big)\cdot$$

As f is bounded on the sphere of $\mathbb{R}^d$, we have

$$|f(x)| \leq C|x|^{-\alpha}$$

As α belongs to $]d/p, d[$, the function f belongs to $L^1 + L^p$. Thus $e^{\Delta} f$ belongs to L^p. After a change of variable, we get the result. □

From this, we can infer the following theorem about so called "self similar solutions" of (NS).

Corollary 5.3.1 *Let u_0 be a smooth divergence free vector field on $\mathbb{R}^3 \setminus \{0\}$ homogeneous of degree -1. Then, if u_0 is small enough in $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$, then there exists a unique solution of (NS) which is self-similar in the sense that it satisfies*

$$u(t,x) = \frac{1}{\sqrt{t}} U\Big(\frac{x}{\sqrt{t}}\Big) \quad \textit{with} \quad U(x) = u(1,x).$$

Proof. Using the scaling invariance of the Navier-Stokes equation, we have that, for any positive λ, $\lambda u(\lambda^2 t, \lambda x)$ is the global solution with initial data $\lambda u_0(\lambda x)$ which is equal $u_0(x)$ because of the homogeneity. Thus, for any positive λ, we have

$$u(t,x) = \lambda u(\lambda^2 t, \lambda x)$$

Choosing $\lambda = (\sqrt{t})^{-1}$ gives the result. □

Another examples about Besov norm of negative index is the following proposition.

Proposition 5.3.2 *Let us consider a function ϕ of $\mathcal{S}(\mathbb{R}^d)$ and ω a unit vector of $\mathbb{R}^d$. Let us define for ε positive,*

$$\phi_\varepsilon(x) = e^{i\frac{(x|\omega)}{\varepsilon}}\phi(x).$$

Then if $0 < s \leq d(1-1/p)$, we have, for $\varepsilon \leq 1$,

$$\|\phi_\varepsilon\|_{\dot{B}^{-s}_p} \leq C_\phi \varepsilon^s.$$

Proof. Using (5.4), we get that

$$t^{\frac{s}{2}}\|e^{t\Delta}f\|_{L^p} \leq C_\phi t^{\frac{s}{2}}.$$

$$t^{\frac{1}{2}} \leq \varepsilon \Rightarrow t^{\frac{s}{2}}\|e^{t\Delta}f\|_{L^p} \leq C_\phi \varepsilon^s. \tag{5.6}$$

Now, let us assume that $t^{\frac{1}{2}} \geq \varepsilon$. We can assume without loss of geneality Using that

$$(-i\varepsilon)^k e^{i\frac{x|\omega)}{\varepsilon}} = \partial_1^k e^{i\frac{x|\omega)}{\varepsilon}}.$$

We get after k integrations by parts and Leibnitz formula,

$$t^{\frac{s}{2}}e^{t\Delta}\phi_\varepsilon = (-i\varepsilon)^k t^{\frac{s}{2}} \sum_{\ell=0}^{k} C_k^\ell \frac{1}{t^{\frac{d}{2}+\frac{\ell}{2}}} f_\ell\Big(\frac{\cdot}{\sqrt{t}}\Big) \star (e^{\frac{i(x|\omega)}{\varepsilon}}\phi_{k-\ell})$$

where f_ℓ and $\phi_{k-\ell}$ are functions of $\mathcal{S}(\mathbb{R}^d)$. Using the convolution inequality, we infer that

$$\|t^{\frac{s}{2}}e^{t\Delta}\phi_\varepsilon\|_{L^p} \leq C_{\phi,k} \sum_{\ell=0}^{k} \min\Big\{\Big(\frac{1}{\sqrt{t}}\Big)^{\ell-\sigma}, \Big(\frac{1}{\sqrt{t}}\Big)^{\ell-s+\frac{d}{p'}}\Big\}.$$

As $t^{\frac{1}{2}} \geq \varepsilon$, if $\ell \geq s$, then we have

$$\Big(\frac{1}{\sqrt{t}}\Big)^{\ell-\sigma} \leq \varepsilon^{s-\ell} \leq \varepsilon^{s-k}$$

and if $\ell < s \leq d/p'$, we have

$$\Big(\frac{1}{\sqrt{t}}\Big)^{\ell-s+\frac{d}{p'}} \leq \varepsilon^{s-\frac{d}{p'}-\ell} \leq \varepsilon^{-\frac{d}{p'}}.$$

Thus, we can

$$\|t^{\frac{s}{2}}e^{t\Delta}\phi_\varepsilon\|_{L^p} \leq C_{\phi,k}\max\big\{\varepsilon^s, \varepsilon^{k-\frac{d}{p'}}\big\}.$$

Choosing $k \geq s + \dfrac{d}{p'}$ gives the result. $\square$

Translated in term of Navier-Stokes equation, this givers the following corollary.

Corollary 5.3.2 *Let α be in $]0,1[[$ and ϕ in $\mathcal{S}(\mathbb{R}^3)$. For ε in $\mathbb{R}^+_\star$, let us define*

$$v_{\lambda,\varepsilon}(x) = \frac{\lambda}{\varepsilon^\alpha}\cos\Big(\frac{x_3}{\varepsilon}\Big)(-\partial_2\phi, \partial_1\phi, 0)$$

Then, for any α, a positive real number ε_0 exist such that if $\varepsilon \leq \varepsilon_0$, then a unique global solution of (NS) exists in $K_p(\infty)$ with $p > \dfrac{3}{1-\alpha}\cdot$

Chapter 6

Littlewood-Paley theory

Introduction

In this chapter, we introduce the Littelwood-Paley theory. The basic idea of this theory is that the description of the regularity of functions can be much more precise if we consider a function as a countable sum of smooth functions the Fourier transform of wich is compactly supported in a ball or an annulus. The Littlewood-Paley theory provides such a decomposition.

The first section is dedicated to the description of the basic property of functions with compactly supported Fourier transforms and to the introduction of the dyadic decomposition of tempered distribution.

The second section is devoted to the definition and the basic study of general Besov spaces using Littlewood-Paley theory.

The third section investigate the special case of Besov spaces of negative index. An equivalent of definition in term of the heat flow is established. This equivalent definition will be very important is the context of imcompressible Navier-Stokes system.

6.1 Localization in frequency space and dyadic decomposition

The very basic idea of this theory consists in a localization procedure in the frequency space. The interest of this method is that the derivatives (or more generally Fourier multipliers) act in a very special way on distributions the Fourier transform of which is supported in a ball or a ring. More precisely, we have the following lemma.

Lemma 6.1.1 *Let $\mathcal{C}$ be a ring, B a ball. A constant C exists so that, for any non negative integer k, any smooth homogeneous function σ of degree m, any couple of real (a,b) so that $b \geq a \geq 1$ and any function u of L^a, we have*

$$\text{Supp } \widehat{u} \subset \lambda B \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leq C^{k+1} \lambda^{k+d(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a};$$

$$\text{Supp } \widehat{u} \subset \lambda \mathcal{C} \Rightarrow C^{-k-1} \lambda^k \|u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^a} \leq C^{k+1} \lambda^k \|u\|_{L^a};$$

$$\text{Supp } \widehat{u} \subset \lambda \mathcal{C} \Rightarrow \|\sigma(D) u\|_{L^b} \leq C_{\sigma,m} \lambda^{m+d(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a}.$$

Proof. Using a dilation of size λ, we can assume all along the proof that $\lambda = 1$. Let ϕ be a function of $\mathcal{D}(\mathbb{R}^d)$, the value of which is 1 near B. As $\widehat{u}(\xi) = \phi(\xi)\widehat{u}(\xi)$, we can write, if g denotes the inverse fourier transform of ϕ,

$$\partial^\alpha u = \partial^\alpha g \star u.$$

Applying Young inequalities the result follows through

$$\begin{aligned}\|\partial^\alpha g\|_{L^c} &\leq \|\partial^\alpha g\|_{L^\infty} + \|\partial^\alpha g\|_{L^1}\\ &\leq 2\|(1+|\cdot|^2)^d \partial^\alpha g\|_{L^\infty}\\ &\leq 2\|(\mathrm{Id}-\Delta)^d\big((\cdot)^\alpha \phi\big)\|_{L^1}\\ &\leq C^{k+1}.\end{aligned}$$

To prove the second assertion, let us consider a function $\widetilde{\phi}$ which belongs to $\mathcal{D}(\mathbb{R}^d \setminus \{0\})$ the value of which is identically 1 near the ring $\mathcal{C}$. Using the algebraic identity

$$\begin{aligned}|\xi|^{2k} &= \sum_{1\leq j_1,\cdots,j_k\leq d} \xi_{j_1}^2 \cdots \xi_{j_k}^2\\ &= \sum_{|\alpha|=k} (i\xi)^\alpha(-i\xi)^\alpha, \end{aligned}\tag{6.1}$$

and stating $g_\alpha \stackrel{\text{def}}{=} \mathcal{F}^{-1}(i\xi_j)^\alpha|\xi|^{-2k}\widetilde{\phi}(\xi)$, we can write, as $\widehat{u} = \widetilde{\phi}\widehat{u}$ that

$$\widehat{u} = \sum_{|\alpha|=k} (-i\xi)^\alpha \widehat{g}_\alpha \widehat{u},$$

which implies that

$$u = \sum_{|\alpha|=k} g_\alpha \star \partial^\alpha u \tag{6.2}$$

and then the result. In order to prove the third assertion, let us observe that the function $\widetilde{\phi}\sigma$ is smooth and compactly supported. Thus stating $g_\sigma \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\widetilde{\phi}\sigma)$, we have that $\sigma(D)u = g_\sigma \star u$ and then

$$\|\sigma(D)u\|_{L^b} \leq C\|u\|_{L^b} \leq C\|u\|_{L^a}.$$

This proves the whole lemma. □

The following lemma is in the same spirit. It describes the action of the semi-group of the heat equation on distributions the Fourier transform of which is supported in a ring.

Lemma 6.1.2 *Let $\mathcal{C}$ be a ring. Two positive constants c and C exist such that, for any real a greater than 1, any couple (t,λ) of positive real numbers, we have*

$$\mathrm{Supp}\ \widehat{u} \subset \lambda\mathcal{C} \Rightarrow \|e^{t\Delta}u\|_{L^a} \leq Ce^{-ct\lambda^2}\|u\|_{L^a}.$$

Proof. Again, let us consider a function ϕ of $\mathcal{D}(\mathbb{R}^d \setminus \{0\})$, the value of which is identically 1 near the ring $\mathcal{C}$. Let us also assume that $\lambda = 1$. Then, we have

$$\begin{aligned}e^{t\Delta}u &= \phi(D)e^{t\Delta}u\\ &= \mathcal{F}^{-1}\left(\phi(\xi)e^{-t|\xi|^2}\widehat{u}(\xi)\right)\\ &= g(t,\cdot)\star u \quad \text{with}\\ g(t,x) &\stackrel{\text{def}}{=} (2\pi)^{-d}\int e^{i(x|\xi)}\phi(\xi)e^{-t|\xi|^2}d\xi. \end{aligned}\tag{6.3}$$

If we prove that two strictly positive real numbers c and C exist such that, for all strictly positive t, we have

$$\|g(t,\cdot)\|_{L^1} \leq Ce^{-ct}, \tag{6.4}$$

then the lemma is proved. Let us do integrations by part in (6.3) . We get

$$\begin{aligned} g(t,x) &= (1+|x|^2)^{-d}\int (1+|x|^2)^d e^{i(x|\xi)}\phi(\xi)e^{-t|\xi|^2}d\xi \\ &= (1+|x|^2)^{-d}\int \Big((\mathrm{Id}-\Delta_\xi)^d e^{i(x|\xi)}\Big)\,\phi(\xi)e^{-t|\xi|^2}d\xi \\ &= (1+|x|^2)^{-d}\int_{\mathbb{R}^d} e^{i(x|\xi)}(\mathrm{Id}-\Delta_\xi)^d\Big(\phi(\xi)e^{-t|\xi|^2}\Big)d\xi. \end{aligned}$$

Through Leibnitz's formula, we obtain

$$(\mathrm{Id}-\Delta)^d\left(\phi(\xi)e^{-t|\xi|^2}\right) = \sum_{\beta\leq|\alpha|\leq 2d} C_\beta^\alpha\left(\partial^{(\alpha-\beta)}\phi(\xi)\right)\left(\partial^\beta e^{-t|\xi|^2}\right).$$

The Faà-di-Bruno's formula tells us that

$$e^{t|\xi|^2}\partial^\beta(e^{-t|\xi|^2}) = \sum_{\substack{\beta_1+\cdots+\beta_m=\beta\\ |\beta_j|\geq 1}} (-t)^m\prod_{j=1}^m \partial^{\beta_j}(|\xi|^2).$$

As the support of ϕ is included in a ring, it turns out that it exists a couple (c,C) of strictly positive real numbers such that, for any ξ in the support of ϕ,

$$\begin{aligned} \left|\left(\partial^{(\alpha-\beta)}\phi(\xi)\right)\left(\partial^\beta e^{-t|\xi|^2}\right)\right| &\leq C(1+t)^{|\beta|}e^{-t|\xi|^2} \\ &\leq C(1+t)^{|\beta|}e^{-ct}. \end{aligned}$$

Thus we have proved that $|g(t,x))|\leq(1+|x|^2)^{-d}e^{-ct}$, which proves Inequality (6.4). □

Using Lemmas 6.1.1 and 6.1.2 together with Duhamel's formula, we infer immediately the following corollary.

Corollary 6.1.1 *Let $\mathcal{C}$ be a ring. Two positive constants c and C exist such that, for any real a greater than 1, any positive λ and any f satisfying, for any $t\in[0,T]$, $\operatorname{Supp}\widehat{f}(t)\subset\lambda\mathcal{C}$, we have for u the solution of*

$$\partial_t u-\Delta u = f \quad\text{and}\quad u_{|t=0}=0.$$

and for any $(a,b,p,q)\in[1,\infty]^4$ such that $b\geq a$ and $q\geq p$

$$\|u\|_{L^q([0,T];L^b)}\leq C(\nu\lambda^2)^{-1+\left(\frac{1}{p}-\frac{1}{q}\right)}\lambda^{d\left(\frac{1}{a}-\frac{1}{b}\right)}\|f\|_{L^p([0,T];L^a)}.$$

Now, let us define a dyadic partition of unity. We shall use it all along this text.

Proposition 6.1.1 *Let us define by $\mathcal{C}$ the ring of center 0, of small radius 3/4 and great radius 8/3. It exists two radial functions χ and φ the values of which are in the interval $[0,1]$, belonging respectively to $\mathcal{D}(B(0,4/3))$ and to $\mathcal{D}(\mathcal{C})$ such that*

$$\forall\xi\in\mathbb{R}^d\,,\ \chi(\xi)+\sum_{j\geq 0}\varphi(2^{-j}\xi)=1, \tag{6.5}$$

$$\forall\xi\in\mathbb{R}^d\backslash\{0\}\,,\ \sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi)=1, \tag{6.6}$$

$$|j-j'|\geq 2 \Rightarrow \text{Supp }\ \varphi(2^{-j}\cdot)\cap \text{Supp }\ \varphi(2^{-j'}\cdot)=\emptyset, \tag{6.7}$$

$$j\geq 1 \Rightarrow \text{Supp }\ \chi\cap \text{Supp }\ \varphi(2^{-j}\cdot)=\emptyset, \tag{6.8}$$

If $\widetilde{\mathcal{C}}=B(0,2/3)+\mathcal{C}$, then $\widetilde{\mathcal{C}}$ is a ring and we have

$$|j-j'|\geq 5 \Rightarrow 2^{j'}\widetilde{\mathcal{C}}\cap 2^j\mathcal{C}=\emptyset, \tag{6.9}$$

$$\forall \xi\in\mathbb{R}^d\,,\ \frac{1}{3}\leq \chi^2(\xi)+\sum_{j\geq 0}\varphi^2(2^{-j}\xi)\leq 1, \tag{6.10}$$

$$\forall \xi\in\mathbb{R}^d\setminus\{0\}\,,\ \frac{1}{2}\leq \sum_{j\in\mathbb{Z}}\varphi^2(2^{-j}\xi)\leq 1. \tag{6.11}$$

Proof. Let us choose α in the interval $]1,4/3[$ let us denote by $\mathcal{C}'$ the ring of small radius α^{-1} and big radius 2α. Let us choose a smooth function θ, radial with value in $[0,1]$, supported in $\mathcal{C}$ with value 1 in the neighbourhood of $\mathcal{C}'$. The important point is the following. For any couple of integers (p,q) we have

$$|j-j'|\geq 2 \Rightarrow 2^j\mathcal{C}\cap 2^{j'}\mathcal{C}=\emptyset. \tag{6.12}$$

Let us suppose that $2^{j'}\mathcal{C}\cap 2^j\mathcal{C}\neq\emptyset$ and that $j'\geq j$. It turns out that $2^{j'}\times 3/4\leq 4\times 2^{j+1}/3$, which implies that $j'-j\leq 1$. Now let us state

$$S(\xi)=\sum_{j\in\mathbf{Z}}\theta(2^{-j}\xi).$$

Thanks to (6.12), this sum is locally finite on the space $\mathbb{R}^d\setminus\{0\}$. Thus the function S is smooth on this space. As α is greater than 1,

$$\bigcup_{j\in\mathbf{Z}}2^j\mathcal{C}'=\mathbb{R}^d\setminus\{0\}.$$

As the function θ is non negative and has value 1 near $\mathcal{C}'$, it comes from the above covering property that the above function is positive. Then let us state

$$\varphi=\frac{\theta}{S}\cdot \tag{6.13}$$

Let us check that φ fits. It is obvious that $\varphi\in\mathcal{D}(\mathcal{C})$. The function $1-\sum_{j\geq 0}\varphi(2^{-j}\xi)$ is smooth thanks to (6.12). As the support of θ is included in $\mathcal{C}$, we have

$$|\xi|\geq\frac{4}{3}\Rightarrow\sum_{j\geq 0}\varphi(2^{-j}\xi)=1. \tag{6.14}$$

Thus stating

$$\chi(\xi)=1-\sum_{j\geq 0}\varphi(2^{-j}\xi), \tag{6.15}$$

we get Identites (6.5)and (6.7). Identity (6.8) is a obvious consequence of (6.12) and of (6.14). Now let us prove (6.9) which will be useful in Section **??**. It is clear that the ring $\widetilde{\mathcal{C}}$ is the ring of center 0, of small radius 1/12 and of big radius 10/3. Then it turns out that

$$2^p\widetilde{\mathcal{C}}\cap 2^j\mathcal{C}\neq\emptyset\Rightarrow\Big(\frac{3}{4}\times 2^j\leq 2^p\times\frac{10}{3}\quad\text{or}\quad\frac{1}{12}\times 2^p\leq 2^j\frac{8}{3}\Big),$$

and (6.9) is proved. Now let us prove (6.10). As χ and φ have their values in $[0,1]$, it is clear that

$$\chi^2(\xi) + \sum_{j\geq 0} \varphi^2(2^{-j}\xi) \leq 1. \tag{6.16}$$

Let us bound from below the sum of squares. The notation $a \equiv b(2)$ means that $a-b$ is even. So we have

$$1 = (\chi(\xi) + \Sigma_0(\xi) + \Sigma_1(\xi))^2 \quad \text{with}$$
$$\Sigma_0(\xi) = \sum_{j\equiv 0(2), j\geq 0} \varphi(2^{-j}\xi) \quad \text{and} \quad \Sigma_1(\xi) = \sum_{j\equiv 1(2), q\geq 0} \varphi(2^{-j}\xi).$$

From this it comes that $1 \leq 3(\chi^2(\xi) + \Sigma_0^2(\xi) + \Sigma_1^2(\xi))$. But thanks to (6.7), we get

$$\Sigma_i^2(\xi) = \sum_{j\geq 0, q\equiv i(2)} \varphi^2(2^{-j}\xi)$$

and the proposition is proved. □

We shall consider all along this text two fixed functions χ and φ satisfying the assertions (6.5)–(6.10). Now let us fix the notations that will be used in all the following of this text.

Notations

$$\begin{gathered}
h = \mathcal{F}^{-1}\varphi \quad \text{and} \quad \widetilde{h} = \mathcal{F}^{-1}\chi,\\
\Delta_{-1}u = \chi(D)u = \mathcal{F}^{-1}(\chi(\xi)\widehat{u}(\xi)),\\
\text{if} \quad j \geq 0\,,\ \Delta_j u = \varphi(2^{-j}D)u = 2^{jd}\int_{\mathbb{R}^d} h(2^j y)u(x-y)dy,\\
\text{if} \quad j \leq -2\,,\ \Delta_j u = 0,\\
S_j u = \sum_{j'\leq j-1} \Delta_p u = \chi(2^{-j}D)u = 2^{jd}\int_{\mathbb{R}^d} \widetilde{h}(2^j y)u(x-y)dy,\\
\text{if} \quad j \in \mathbb{Z}\,,\ \dot{\Delta}_j u = \varphi(2^{-j}D)u = 2^{jd}\int_{\mathbb{R}^d} h(2^j y)u(x-y)dy,\\
\text{if} \quad j \in \mathbb{Z}\,,\ \dot{S}_j u = \sum_{j'\leq j-1} \dot{\Delta}_p u.
\end{gathered}$$

Remark Let us point that all the above operators Δ_j and S_j maps L^p into L^p with norms which do not depend on j. This fact will be used all along this book.

Now let us have a look of the case when we may write

$$\mathrm{Id} = \sum_j \Delta_j \quad \text{or} \quad \mathrm{Id} = \sum_j \dot{\Delta}_j.$$

This is described by the following proposition, the proof of which is left as an exercise.

Proposition 6.1.2 *Let u be in $\mathcal{S}'(\mathbb{R}^d)$. Then, we have, in the sense of the convergence in the space $\mathcal{S}'(\mathbb{R}^d)$,*

$$u = \lim_{j\to\infty} S_j u.$$

The following proposition tells us that the condition of convergence in $\mathcal{S}'$ is somehow weak for series, the Fourier transform of which is supported in dyadic rings.

Proposition 6.1.3 *Let $(u_j)_{j\in\mathbb{N}}$ be a sequence of bounded functions such that the Fourier transform of u_j is supported in $2^j\widetilde{\mathcal{C}}$ where $\widetilde{\mathcal{C}}$ is a given ring. Let us assume that*

$$\|u_j\|_{L^\infty} \leq C2^{jN}.$$

Then the series $(u_j)_{j\in\mathbb{N}}$ is convergent in $\mathcal{S}'$.

Proof. Let us use the relation (6.2). After rescaling it can be written

$$u_j = 2^{-jk}\sum_{|\alpha|=k} 2^{jd}g_\alpha(2^j\cdot)\star\partial^\alpha u_j.$$

Then for any test function ϕ in $\mathcal{S}$, let us write that

$$\begin{aligned}\langle u_j,\phi\rangle &= 2^{-jk}\sum_{|\alpha|=k}\langle u_j, 2^{jd}\check{g}_\alpha(2^j\cdot)\star(-\partial)^\alpha\phi\rangle \qquad (6.17)\\ &\leq C2^{-jk}\sum_{|\alpha|=k}2^{jN}\|\partial^\alpha\phi\|_{L^1}.\end{aligned}$$

Let us choose $k>N$. Then $(\langle u_j,\phi\rangle)_{j\in\mathbb{N}}$ is a convergent series, the sum of which is less than $C\|\phi\|_{M,\mathcal{S}}$ for some integer M. Thus the formula

$$\langle u,\phi\rangle \stackrel{\text{def}}{=} \lim_{j\to\infty}\sum_{j'\leq j}\langle\Delta_{j'}u,\phi\rangle$$

defines a tempered distribution.

For the case of the operators $\dot\Delta_j$, the problem is a little bit more delicate. Obviously, it is not true for $u=\mathbf{1}$ because, for any integer j, we have $\dot\Delta_j\mathbf{1}=0$. This leads to the following definition.

Definition 6.1.1 *Let us denote by $\mathcal{S}'_h$ the space of tempered distribution such that*

$$\lim_{j\to-\infty}\dot S_j u = 0 \quad \textit{in} \quad \mathcal{S}'.$$

Examples

- If a tempered distribution u is such that its Fourier transform $\widehat{u}$ is locally integrable near 0, then u belongs to $\mathcal{S}'_h$.
- If u is a tempered distribution such that for some function θ in $\mathcal{D}(\mathbb{R}^d)$ with value 1 near the origin, we have $\theta(D)u$ in L^p for some $p\in[1,+\infty[$, then u belongs to $\mathcal{S}'_h$.
- A non zero constant function u does not belong to $\mathcal{S}'_h$ because $\dot S_j u = u$ for any j in $\mathbb{Z}$.

Remarks

- The space $\mathcal{S}'_h$ is exactly the space of tempered distributions for which we may write
$$u=\sum_j\dot\Delta_j u.$$
- The fact that u belongs to $\mathcal{S}'_h$ or not is an information about low frequencies.
- The space $\mathcal{S}'_h$ is not a closed subspace of $\mathcal{S}'$ for the topology of weak convergence.
- It is an exercice left to the reader to prove that u belongs to $\mathcal{S}'_h$ if and only if, for any θ in $\mathcal{D}(\mathbb{R}^d)$ with value 1 near the origin, we have $\lim_{\lambda\to\infty}\theta(\lambda D)u=0$ in $\mathcal{S}'$.

6.2 Homogeneous Besov spaces

Definition 6.2.1 *Let u be a tempered distribution, s a real number, and $(p,r) \in [1,+\infty]^2$. The space $\dot{B}^s_{p,r}$ is the space of distribution in $\mathcal{S}'_h$ such that*

$$\|u\|_{\dot{B}^s_{p,r}} \overset{def}{=} \Big(\sum_{j\in\mathbb{Z}} 2^{rqs}\|\dot{\Delta}_j u\|^r_{L^p}\Big)^{\frac{1}{r}}.$$

There are two important facts to point out. The first one is about the homogeneity. If u is a tempered distribution, then let us consider for any integer N, the tempered distribution u_N defined by $u_N \overset{\text{def}}{=} u(2^N\cdot)$. We have the following proposition.

Proposition 6.2.1 *If $\|u\|_{\dot{B}^s_{p,r}}$ is finite, so it is for u_N and we have*

$$\|u_N\|_{\dot{B}^s_{p,r}} = 2^{N(s-\frac{d}{p})}\|u\|_{\dot{B}^s_{p,r}}.$$

Proof. We go back to the definition of the operator $\dot{\Delta}_j$. This gives

$$\begin{aligned}\dot{\Delta}_j u_N(x) &= 2^{jd}\int h(2^j(x-y)u_N(y)dy\\ &= 2^{jd}\int h(2^j(x-y))u(2^N y)dy.\end{aligned}$$

By the change of variables $z = 2^N y$, we get that

$$\begin{aligned}\dot{\Delta}_j u_N(x) &= 2^{(j-N)d}\int h(2^{j-N}(2^N x - z))u(z)dz\\ &= (\dot{\Delta}_{j-N}u)(2^N x).\end{aligned}$$

So it turns out that $\|\dot{\Delta}_j u_N\|_{L^p} = 2^{-N\frac{d}{p}}\|\dot{\Delta}_{j-N}u\|_{L^p}$. We deduce from this that

$$\|2^{js}\dot{\Delta}_j u_N\|_{L^p} = 2^{N(s-\frac{d}{p})}2^{(j-N)s}\|\dot{\Delta}_{j-N}u\|_{L^p}.$$

And the proposition follows immediately by summation. □

Theorem 6.2.1 *The space $(\dot{B}^s_{p,r}, \|\cdot\|_{\dot{B}^s_{p,r}})$ is a normed space. Moreover, if $s < \dfrac{d}{p}$, then $(\dot{B}^s_{p,r}, \|\cdot\|_{\dot{B}^s_{p,r}})$ is a Banach space.*

Proof. It is obvious that $\|\cdot\|_{\dot{B}^s_{p,r}}$ is a semi-norm. Let us assume that $\|u\|_{\dot{B}^s_{p,r}} = 0$ for some u in $\mathcal{S}'_h$. This implies that the support of $\widehat{u}$ is included in $\{0\}$ and thus that $\dot{S}_j u = u$ for any j in $\mathbb{Z}$. As u belongs to $\mathcal{S}'_h$, this implies that $u = 0$.

Let us prove the second part of the theorem. First let us prove that those spaces are continuously embedded in $\mathcal{S}'$. Thanks to Lemma 6.1.1, we have

$$\|\dot{\Delta}_j u\|_{L^\infty} \leq C2^{j\frac{d}{p}}\|\dot{\Delta}_j u\|_{L^p}. \tag{6.18}$$

As $s < d/p$, let us write that, for negative j and for large enough M,

$$\begin{aligned} |\langle \dot\Delta_j u, \phi\rangle| &\leq \|\dot\Delta_j u\|_{L^\infty} \|\phi\|_{L^1} \\ &\leq 2^{j\frac{d}{p}} \|\dot\Delta_j u\|_{L^p} \|\phi\|_{L^1} \\ &\leq C 2^{j\left(\frac{d}{p}-s\right)} \|u\|_{\dot B^s_{p,r}} \|\phi\|_{M,\mathcal{S}}. \end{aligned} \tag{6.19}$$

For non negative j, Formula (6.2) applied with $u = \dot\Delta_j u$ gives (after a dilation by 2^j)

$$\dot\Delta_j u = 2^{-jk} \sum_{|\alpha|=k} \partial^\alpha \big(2^{jd} g_\alpha(2^j \cdot) \star \dot\Delta_j u\big) \quad \text{with} \quad g_\alpha = \mathcal{F}^{-1} (i\xi)^\alpha |\xi|^{-2k} \widetilde{\phi}(\xi).$$

Thus we infer that

$$\begin{aligned} \langle \dot\Delta_j u, \phi\rangle &= 2^{-jk} \sum_{|\alpha|=k} \langle \partial^\alpha \big(2^{jd} g_\alpha(2^j \cdot) \star \dot\Delta_j\big), \phi\rangle \\ &= 2^{-jk} \sum_{|\alpha|=k} \langle \dot\Delta_j, 2^{jd} \check g_\alpha(2^j \cdot) \star (-\partial)^\alpha \phi\rangle \\ &\leq \|\dot\Delta_j u\|_{L^\infty} 2^{-jk} \|\phi\|_{M_k,\mathcal{S}} \end{aligned}$$

for large enough M_k. By definition of $\dot B^s_{p,r}$, this gives $\langle \dot\Delta_j u, \phi\rangle \leq C 2^{j\left(s-\frac{d}{p}-k\right)} \|u\|_{\dot B^s_{p,r}} \|\phi\|_{M_k,\mathcal{S}}$. Choosing k greater than $s - \dfrac{d}{p}$ and then M_k large enough, gives, using the fact that u is in $\mathcal{S}'_h$ and the inequality (6.19),

$$\langle u, \phi\rangle \leq C \|u\|_{\dot B^s_{p,r}} \|\phi\|_{M_k,\mathcal{S}}. \tag{6.20}$$

Let us consider a Cauchy sequence $(u_n)_{n\in\mathbb{N}}$ in $\dot B^s_{p,r}$. Using (6.20), this implies that a tempered distribution u exists such that the sequence $(u_n)_{n\in\mathbb{N}}$ converges to u in $\mathcal{S}'$. The main point of the proof consists in proving that this distribution u belongs to $\mathcal{S}'_h$. As $s < d/p$, we have, thanks to (6.20),

$$\forall j \in \mathbb{Z}\,,\ \forall n \in \mathbb{N}\,,\ |\langle \dot S_j u_n, \phi\rangle| \leq C_s 2^{j\left(\frac{d}{p}-s\right)} \sup_n \|u_n\|_{\dot B^s_{p,r}} \|\phi\|_{M,\mathcal{S}}.$$

As the sequence $(u_n)_{n\in\mathbb{N}}$ tends to u in $\mathcal{S}'$, we have

$$\forall j \in \mathbb{Z}\,,\ |\langle \dot S_j u, \phi\rangle| \leq C_s 2^{j\left(\frac{d}{p}-s\right)} \sup_n \|u_n\|_{\dot B^s_{p,r}} \|\phi\|_{M,\mathcal{S}}.$$

Thus u belongs to $\mathcal{S}'_h$.

By definition of the norm of $\dot B^s_{p,r}$ the sequence $(\dot\Delta_j u^{(n)})_{n\in\mathbb{N}}$ is a Cauchy one in L^p for any j. Thus an element u_j of L^p exists such that $(\dot\Delta_j u^{(n)})_{n\in\mathbb{N}}$ converges to u_j in L^p. As $(u^{(n)})_{n\in\mathbb{N}}$ converges to u in $\mathcal{S}'$ we have $\dot\Delta_j u = u_j$. Let us define

$$a_j^{(n)} = 2^{js} \|\dot\Delta_j u^{(n)}\|_{L^p} \quad \text{and} \quad a_j = 2^{js} \|\dot\Delta_j u\|_{L^p}.$$

For any j, $\lim_{n\to\infty} a_j^{(n)} = a_j$. As $(a_j^{(n)})_{n\in\mathbb{N}}$ is a bounded sequence of $\ell^r(\mathbb{Z})$, $a \stackrel{\text{def}}{=} (a_j)_{j\in\mathbb{Z}}$ is in $\ell^r(\mathbb{Z})$ and thus $u \in \dot B^s_{p,r}$. As $(u^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\dot B^s_{p,r}$, we have,

$$\forall \varepsilon > 0\,,\ \exists n_0 \,/\, \forall n \geq n_0\,,\ \forall m\ \|a^{(n+m)} - a^{(n)}\|_{\ell^r(\mathbb{Z})} \leq \varepsilon.$$

As $(a^{(n)})$ tends weakly to a in $\ell^r(\mathbb{Z})$, we get, passing to the limit in m in the above inequality that $\|u^{(n)} - u\|_{\dot B^s_{p,r}} = \|a - a^{(n)}\|_{\ell^r(\mathbb{Z})} \leq \varepsilon$. This ends the proof of the theorem. □

Let us give the first example for Besov space, the Sobolev spaces $\dot{H}^s$.

Proposition 6.2.2 *The two spaces $\dot{H}^s$ and $\dot{B}^s_{2,2}$ are equal and the two norms satisfies*

$$\frac{1}{C^{|s|+1}}\|u\|_{\dot{B}^s_{2,2}} \leq \|u\|_{\dot{H}^s} \leq C^{|s|+1}\|u\|_{\dot{B}^s_{2,2}}.$$

Proof. As the support of the Fourier transform of $\dot{\Delta}_j u$ is included in the ring $2^j\mathcal{C}$, it is clear, as $j \geq 0$, that a constant C exists such that, for any real s and any u such that $\widehat{u}$ belongs to L^2_{loc},

$$\frac{1}{C^{|s|+1}}2^{2js}\|\dot{\Delta}_j u\|^2_{L^2} \leq \|\dot{\Delta}_j u\|^2_{\dot{H}^s} \leq C^{|s|+1}2^{2js}\|\dot{\Delta}_j u\|^2_{L^2}. \tag{6.21}$$

Using Identity (6.11), we get

$$\frac{1}{2}\|u\|^2_{\dot{H}^s} \leq \sum_{j\in\mathbb{Z}}\int \varphi^2(2^{-j}\xi)|\xi|^{2s}|\widehat{u}(\xi)|^2 d\xi \leq \|u\|^2_{\dot{H}^s}$$

which proves the proposition. □

Let us give an example of a function which belongs to a large class of Besov spaces. Let us give an example of an L^1_{loc} function which belongs to $\dot{B}^s_{p,r}$.

Proposition 6.2.3 *Let σ be in $]0,d[$. Let us consider f a smooth function on $\mathbb{R}^d\setminus\{0\}$ which is homogeneous of degree $-\sigma$. Then f belongs to $\dot{B}^{d-\sigma}_{1,\infty}$.*

Proof. As f canbe written as a sum of an L^1 and an L^q function with q gretar than d/σ, f is in $\mathcal{S}'_h$. Now let us compute $\dot{\Delta}_j|\cdot|^{-\sigma}$. By definition of the operator $\dot{\Delta}_j$, usinf the homogenneity of f, we have

$$(\dot{\Delta}_j f)(x) = 2^{j\sigma}(\dot{\Delta}_0 f)(2^j x)$$

Let us consider ϕ in $\mathcal{D}(\mathbb{R}^d)$ with value 1 near 0. The function ϕf belongs to L^1 ans so is $\Delta_0(\phi f)$. Using 6.2 page 54, we have that, for any k,

$$\Delta_0(1-\phi)f = \sum_{|\alpha|=k} g_\alpha \partial^\alpha((1-\phi)f)$$

If k is gretar than $d-\sigma$, the function $g_\alpha\partial^\alpha((1-\phi)f)$ belongs to L^1 and the proposition is proved. □

Lemma 6.2.1 *Let $\mathcal{C}'$ be a ring in $\mathbb{R}^d$; let (s,p,r) be as in Theorem 6.2.1. Let $(u_j)_{j\in\mathbb{Z}}$ be a sequence of smooth functions such that*

$$\text{Supp }\ \widehat{u}_j \subset 2^j\mathcal{C}' \quad \text{and} \quad \left\|(2^{js}\|u_j\|_{L^p})_{j\in\mathbb{Z}}\right\|_{\ell^r} < +\infty.$$

Then we have $u = \displaystyle\sum_{j\in\mathbb{Z}} u_j \in \dot{B}^s_{p,r}$ *and* $\|u\|_{\dot{B}^s_{p,r}} \leq C_s\left\|(2^{js}\|u_j\|_{L^p})_{j\in\mathbb{Z}}\right\|_{\ell^r}.$

This immediately implies the following corollary.

Corollary 6.2.1 *Let (s,p,r) be as above; then the space $\dot{B}^s_{p,r}$ does not depend on the choice of the functions χ and φ used in the Definition 6.2.1.*

Proof of Lemma 6.2.1 Let us first observe that, using Lemma 6.1.1, we have $(u_j)_{j\leq 0}$ is a convergent series in L^∞. Let us denote by u^- its limit. It is obvious that u^- belongs to $\mathcal{S}'_h$. Using again Lemma 6.1.1, we get that $\|u_j\|_{L^\infty} \leq C2^{j\left(\frac{d}{p}-s\right)}$. Proposition 6.1.3 implies that $(u_j)_{j>0}$ is a convergente series in $\mathcal{S}'$. Let us denote by u^+ its limit. The support of the Fourier transform of u^+ does not contain the origin. Thus u^+ is in $\mathcal{S}'_h$. So does $u \stackrel{\text{def}}{=} u^- + u^+$. Then, let us study $\Delta_{j'}u$. As $\mathcal{C}$ and $\mathcal{C}'$ are two rings, an integer N_0 exists so that $|j'-j|\geq N_0$ then $2^j\mathcal{C}\cap 2^{j'}\mathcal{C}' = \emptyset$. Here $\mathcal{C}$ is the ring defined in the Proposition 6.1.1. Now, it is clear that if $|j'-j|\geq N_0$, then $\Delta_{j'}u_j = 0$. Then we can write that

$$\begin{aligned}\|\dot\Delta_{j'}u\|_{L^p} &= \|\sum_{|j-j'|<N_0} \dot\Delta_{j'}u_j\|_{L^p}\\ &\leq C\sum_{|j-j'|<N_0} \|u_j\|_{L^p}.\end{aligned}$$

So, we obtain that

$$\begin{aligned}2^{j's}\|\dot\Delta_{j'}u\|_{L^p} &\leq C\sum_{\substack{j'\geq -1\\ |j'-j|\leq N_0}} 2^{j's}\|u_j\|_{L^p}\\ &\leq C\sum_{\substack{j'\geq -1\\ |j-j'|\leq N_0}} 2^{js}\|u_j\|_{L^p}.\end{aligned}$$

We deduce from this that

$$2^{j's}\|\dot\Delta_{j'}u\|_{L^p} \leq (c_k)_{k\in\mathbb{Z}} \star (d_\ell)_{\ell\in\mathbb{Z}} \quad\text{with}\quad c_k = \mathbf{1}_{[-N_0,N_0]}(k) \quad\text{and}\quad d_\ell = \mathbf{1}_{\mathbb{N}}(\ell)2^{\ell s}\|u_\ell\|_{L^p}.$$

The classical property of convolution between $\ell^1(\mathbb{Z})$ and $\ell^r(\mathbb{Z})$ gives that

$$\|u\|_{\dot B^s_{p,r}} \leq C\left\|(2^{js}\|u_j\|_{L^p})_{j\in\mathbb{Z}}\right\|_{\ell^r}.$$

This proves the lemma. □

The following theorem is the equivalent of Sobolev embedding.

Theorem 6.2.2 *Let $1\leq p_1\leq p_2\leq\infty$ and $1\leq r_1\leq r_2\leq\infty$. Then for any real number s the space $\dot B^s_{p_1,r_1}$ is continuously embedded in $\dot B^{s-d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}_{p_2,r_2}$.*

Proof. In order to prove this result, we simply apply Lemma 6.1.1 which

$$\|\dot\Delta_j u\|_{L^{p_2}} \leq C2^{jd\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|\dot\Delta_j u\|_{L^{p_1}}.$$

Considering that $\ell^{r_1}(\mathbb{Z})\subset\ell^{r_2}(\mathbb{Z})$, the theorem is proved. □

Now let us study the way Fourier multipliers acts of Besov spaces.

Proposition 6.2.4 *Let σ be a smooth function on $\mathbb{R}^d$ which is homogeneous of degree m. Then for any $(s,p,r)\in\mathbb{R}\times[1,+\infty]^2$ such that $\dot B^{s-m}_{p,r}$ is a Banach space, the operator $\sigma(D)$ maps continuously $\dot B^s_{p,r}$ into $\dot B^{s-m}_{p,r}$.*

Proof. Lemma 6.1.1 tells us that $\|\sigma(D)\dot\Delta_j u\|_{L^p} \leq C2^{jm}\|\dot\Delta_j u\|_{L^p}$. Then Lemma 6.2.1 implies the proposition. □

Remark Let us point out that this proof is very simple compared with the similar result on L^p spaces when p belongs to $]1,+\infty[$. Moreover, as we shall see in the next section, Fourier multipliers does not map L^∞ into L^∞ in general. From this point of view Besov spaces are much easier to use that classical L^p spaces or Sobolev spaces modeled on L^p.

6.3 The special case of Besov spaces with negative index

We shall give equivalent definitions of the Besov norm. These definitions does not use the localisation in frequency space. The first one concerns negative indices and uses the heat flow.

Theorem 6.3.1 *Let s be a positive real number and $(p,r) \in [1,\infty]^2$. A constant C exists which satisfies the following property. For u in $\mathcal{S}'_h$, we have*

$$C^{-1}\|u\|_{\dot{B}^{-2s}_{p,r}} \leq \Big\| \|t^s e^{t\Delta}u\|_{L^p}\Big\|_{L^r(\mathbb{R}^+,\frac{dt}{t})} \leq C\|u\|_{\dot{B}^{-2s}_{p,r}}.$$

Proof. The proof relies on Lemma 6.1.2. Let us estimate $\|t^s\dot{\Delta}_j e^{t\Delta}u\|_{L^p}$. Using Lemma 6.1.2, we can write

$$\|t^s\dot{\Delta}_j e^{t\Delta}u\|_{L^p} \leq Ct^s 2^{2js}e^{-ct2^{2j}}2^{-2js}\|\dot{\Delta}_j u\|_{L^p}.$$

Using that u belongs to $\mathcal{S}'_h$ and the definition of the homogeneous Besov (semi) norm, we have

$$\begin{aligned}\|t^s e^{t\Delta}u\|_{L^p} &\leq \sum_{j\in\mathbb{Z}} \|t^s\dot{\Delta}_j e^{t\Delta}u\|_{L^p}\\ &\leq C\|u\|_{\dot{B}^{-2s}_{p,r}}\sum_{j\in\mathbb{Z}} t^s 2^{2js}e^{-ct2^{2j}}c_{r,j}\end{aligned}$$

where $(c_{r,j})_{j\in\mathbb{Z}}$ denotes, as in all this proof, a generic element of the unit sphere of $\ell^r(\mathbb{Z})$. If $r=\infty$, the inequality comes immedialtely from the following lemma, the proof of which is an exercice left to the reader.

Lemma 6.3.1 *For any positive s, we have*

$$\sup_{t>0}\sum_{j\in\mathbb{Z}} t^s 2^{2js}e^{-ct2^{2j}} < \infty.$$

If $r<\infty$, using the Hölder inequality with the weight $2^{2js}e^{-ct2^{2j}}$, the above lemma and Fubini's theorem, we obtain

$$\begin{aligned}\int_0^\infty t^{rs}\|e^{t\Delta}u\|^r_{L^p}\frac{dt}{t} &\leq C\|u\|^r_{\dot{B}^{-2s}_{p,r}}\int_0^\infty\Big(\sum_{j\in\mathbb{Z}} t^s 2^{2js}e^{-ct2^{2j}}c_{r,j}\Big)^r\frac{dt}{t}\\ &\leq C\|u\|^r_{\dot{B}^{-2s}_{p,r}}\int_0^\infty\Big(\sum_{j\in\mathbb{Z}} t^s 2^{2js}e^{-ct2^{2j}}\Big)^{r-1}\Big(\sum_{j\in\mathbb{Z}} t^s 2^{2js}e^{-ct2^{2j}}c^r_{r,j}\Big)\frac{dt}{t}\\ &\leq C\|u\|^r_{\dot{B}^{-2s}_{p,r}}\int_0^\infty\sum_{j\in\mathbb{Z}} t^s 2^{2js}e^{-ct2^{2j}}c^r_{r,j}\frac{dt}{t}\\ &\leq C\|u\|^r_{\dot{B}^{-2s}_{p,r}}\sum_{j\in\mathbb{Z}}c^r_{r,j}\int_0^\infty t^s 2^{2js}e^{-ct2^{2j}}\frac{dt}{t}\\ &\leq C_s\|u\|^r_{\dot{B}^{-2s}_{p,r}}\quad\text{with}\quad C_s\stackrel{\text{def}}{=}\int_0^\infty t^{s-1}e^{-t}dt.\end{aligned}$$

To prove the other inequality, by definition of C_s, we have

$$\dot{\Delta}_j u = C^{-1}_{s+1}\int_0^\infty t^s(-\Delta)^{s+1}e^{t\Delta}\dot{\Delta}_j u dt.$$

As $e^{t\Delta}u = e^{\frac{t}{2}\Delta}e^{\frac{t}{2}\Delta}u$, we can write, using Lemmas 6.1.1 and 6.1.2,

$$\begin{aligned}\|\dot\Delta_j u\|_{L^p} &\leq C\int_0^\infty t^s 2^{2j(s+1)}e^{-ct2^{2j}}\|\dot\Delta_j e^{\frac{t}{2}\Delta}u\|_{L^p}dt\\ &\leq C\int_0^\infty t^s 2^{2j(s+1)}e^{-ct2^{2j}}\|e^{t\Delta}u\|_{L^p}dt.\end{aligned}$$

If $r=\infty$, we have

$$\begin{aligned}\|\dot\Delta_j u\|_{L^p} &\leq C\Big(\sup_{t>0} t^s\|e^{t\Delta}u\|_{L^p}\Big)\int_0^\infty 2^{2j(s+1)}e^{-ct2^{2j}}dt\\ &\leq C2^{2js}\Big(\sup_{t>0} t^s\|e^{t\Delta}u\|_{L^p}\Big).\end{aligned}$$

If $r<\infty$, let us write that

$$\sum_j 2^{-2jsr}\|\dot\Delta_j u\|_{L^p}^r \leq C\sum_{j\in\mathbb{Z}} 2^{2jr}\bigg(\int_0^\infty t^s e^{-ct2^{2j}}\|e^{t\Delta}u\|_{L^p}dt\bigg)^r.$$

Hölder inequality with the weight $e^{-ct2^{2j}}$ implies that

$$\begin{aligned}\bigg(\int_0^\infty t^s e^{-ct2^{2j}}\|e^{t\Delta}u\|_{L^p}dt\bigg)^r &\leq \bigg(\int_0^\infty e^{-ct2^{2j}}dt\bigg)^{r-1}\int_0^\infty t^{rs}e^{-ct2^{2j}}\|e^{t\Delta}u\|_{L^p}^r dt\\ &\leq C2^{-2j(r-1)}\int_0^\infty t^{rs}e^{-ct2^{2j}}\|e^{t\Delta}u\|_{L^p}^r dt.\end{aligned}$$

Thanks to Lemma 6.3.1 and Fubini's theorem, we get

$$\begin{aligned}\sum_j 2^{-2jsr}\|\dot\Delta_j u\|_{L^p}^r &\leq C\sum_{j\in\mathbb{Z}} 2^{2j}\int_0^\infty t^{rs}e^{-ct2^{2j}}\|e^{t\Delta}u\|_{L^p}^r dt\\ &\leq C\int_0^\infty\bigg(\sum_{j\in\mathbb{Z}} t2^{2j}e^{-ct2^{2j}}\bigg)t^{rs}\|e^{t\Delta}u\|_{L^p}^r\frac{dt}{t}\\ &\leq C\int_0^\infty t^{rs}\|e^{t\Delta}u\|_{L^p}^r\frac{dt}{t}\cdot\end{aligned}$$

The theorem is proved. □

One of the main facts about Besov spaces of negative index is that the norm is small when the function is strongly oscillating. More precisely, we have the following proposition.

Proposition 6.3.1 *Let ϕ be a function of $\mathcal{S}(\mathbb{R}^d)$, p in $[1,\infty]$ and s be in $\Big]0,\dfrac{d}{p'}\Big[$. Let us define*

$$\phi_\varepsilon(x) \stackrel{\text{def}}{=} e^{i\frac{x_1}{\varepsilon}}\phi(x).$$

Then, we have $\|\phi_\varepsilon\|_{\dot B^{-s}_{p,1}} \leq C_\phi\varepsilon^s$.

Proof. Let us first observe that $\|\Delta_j\phi_\varepsilon\|_{L^p}\leq C\|\phi\|_{L^p}$. The fact that s is positive implies that

$$\sum_{j\geq -\log_2\varepsilon} 2^{-js}\|\Delta_j\varphi_\varepsilon\|_{L^p}\leq C_s\varepsilon^s. \tag{6.22}$$

Now, let us study the cas when $2^j < \varepsilon^{-1}$. Let us recal that by definition of φ_ε, we have

$$\Delta_j\phi_\varepsilon(x) = 2^{jd}\int_{\mathbb{R}^d} h(2^j(x-y)\phi(y)e^{i\frac{y_1}{\varepsilon}}dy.$$

We use now the non stationnary phase method in the above integral. More precisely, using that

$$(-i\varepsilon\partial_1)^k e^{i\frac{y_1}{\varepsilon}} = e^{i\frac{y_1}{\varepsilon}},$$

we have, thanks repeated integration by parts and Leibnitz formula,

$$\Delta_j\phi_\varepsilon(x) = (i\varepsilon)^k\sum_{\ell\leq k} C_k^\ell\int_{\mathbb{R}^d} 2^{j(d+k-\ell)}e^{i\frac{y_1}{\varepsilon}}(\partial_1^{k-\ell}h)(2^j(x-y))\partial_1^\ell\phi(y)dy.$$

As we have

$$2^{j(k-\ell)} \leq \max\{2^{jk},1\} \quad\text{and}\quad \|(\partial_1^{k-\ell}h)(2^j\cdot)\star\partial_1^\ell\phi\|_{L^p} \leq C_\phi\min\{2^{-jd},2^{-j\frac{d}{p}}\},$$

we infer that

$$2^{-js}\|\Delta_j\phi_\varepsilon\|_{L^p} \leq C_\phi\varepsilon^k 2^{j(d-s)}\max\Big\{2^{j(k-d)},2^{-j\frac{d}{p}}\Big\}.$$

As $\varepsilon 2^j$ is less than 1, we get

$$2^{-js}\|\Delta_j\phi_\varepsilon\|_{L^p} \leq C_\phi\max\Big\{2^{-js},\varepsilon^k 2^{j(d-\frac{d}{p}-s)}\Big\}.$$

As s is less than $d(1-1/p)$, we infer, choosing k large enough, that

$$\sum_{j<-\log_2\varepsilon} 2^{-js}\|\Delta_j\phi_\varepsilon\|_{L^p} \leq C\varepsilon^s\max\Big\{1,\varepsilon^{k-d\left(1-\frac{1}{p}\right)}\Big\} \leq C\varepsilon^s.$$

Together with (6.22), this gives the proposition. □

Proposition 6.3.2 *Let ϕ be a function of $\mathcal{S}(\mathbb{R}^d)$, p in $]1,\infty[$ and s in $\Big]0,\dfrac{d}{p'}\Big[$. For (ε,Λ) in $]0,1]\times[1,\infty[$, let us define*

$$\phi_{\varepsilon,\Lambda}(x) \stackrel{\text{def}}{=} e^{i\frac{x_1}{\varepsilon}}\phi(x_1,\Lambda x_2,x_3).$$

Then, we have

$$\|\phi_{\varepsilon,\Lambda}\|_{\dot{B}^{-s}_{p,1}} \leq C_\phi\varepsilon^s\Lambda^{-\frac{1}{p}}.$$

Proof. Let us first observe the right inside inequality. We have to bound

$$\sum_{j\in\mathbb{Z}} 2^{-j\sigma}\|\Delta_j\phi_{\varepsilon,\Lambda}\|_{L^p}.$$

We shall start by estimating the high frequencies, i.e. frequencies that are larger than $-\log_2\varepsilon$. The fact that s is positive implies that

$$\begin{aligned}\sum_{j\geq J} 2^{-js}\|\Delta_j\phi_{\varepsilon,\Lambda}\|_{L^p} &\leq C_s\varepsilon^s\|\phi_{\varepsilon,\Lambda}\|_{L^p}\\ &\leq C_s\varepsilon^s\Lambda^{-\frac{1}{p}}\|\phi\|_{L^p} \qquad (6.23)\end{aligned}$$

Now, let us study the low frequency case i.e. the case when $j < -\log_2 \varepsilon$. Let us recall that by definition of ϕ_ε, we have

$$\Delta_j \phi_{\varepsilon,\Lambda}(x) = 2^{3j} \int_{\mathbb{R}^3} e^{i\frac{y_1}{\varepsilon}} h(2^j(x-y)\phi(y_1, \Lambda y_2, y_3)dy.$$

As in the proof of Propostion 6.3.1 page 64, let us use that

$$(-i\varepsilon\partial_1)^k e^{i\frac{y_1}{\varepsilon}} = e^{i\frac{y_1}{\varepsilon}}.$$

This gives, using repeated integration by parts and Leibnitz formula,

$$\Delta_j \varphi_{\varepsilon,\Lambda}(x) = (i\varepsilon)^k \sum_{\ell\leq k} C_k^\ell \int_{\mathbb{R}^d} 2^{j(3+k-\ell)} e^{i\frac{y_1}{\varepsilon}} (\partial_1^{k-\ell} h)(2^j(x-y))(\partial_1^\ell \phi)(y_1, \Lambda y_2, y_3)dy.$$

Using that

$$\varepsilon^k 2^{-js} \Big\|(\partial_1^{k-\ell} h)(2^j\cdot) \star (\partial_1^\ell)\phi(\cdot, \Lambda\cdot, \cdot)\Big\|_{L^p} \leq C_\phi \varepsilon^k \min\big\{2^{j\left(\ell-s+\frac{3}{p'}\right)}\Lambda^{-1}, 2^{j(\ell-s)}\Lambda^{-\frac{1}{p}}\big\},$$

we infer that, if $\ell \leq s$

$$\varepsilon^k 2^{-js} \sum_{j\leq -\log_2\varepsilon} \Big\|(\partial_1^{k-\ell} h)(2^j\cdot)\star(\partial_1^\ell)\phi(\cdot, \Lambda\cdot, \cdot)\Big\|_{L^p} \leq C_\phi \Lambda^{-1}\varepsilon^k \sum_{j\leq -\log_2\varepsilon} 2^{j\left(\ell-s\frac{3}{p'}\right)} \leq C_\phi \varepsilon^s \Lambda^{-1} \varepsilon^{k-s-\frac{3}{p'}}$$

and, if $\ell > s$,

$$\varepsilon^k 2^{-js} \sum_{j\leq -\log_2\varepsilon} \Big\|(\partial_1^{k-\ell} h)(2^j\cdot) \star (\partial_1^\ell)\phi(\cdot, \Lambda\cdot, \cdot)\Big\|_{L^p} \leq C_\phi \Lambda^{-\frac{1}{p}}\varepsilon^k \sum_{j\leq -\log_2\varepsilon} 2^{j(\ell-s)} \leq C_\phi \varepsilon^s \Lambda^{-\frac{1}{p}} \varepsilon^{k-\ell}.$$

Thus, we get, for $\Lambda \geq 1$,

$$\sum_{j\leq -\log_2\varepsilon} \|\Delta_j \phi_{\varepsilon,\Lambda}\|_{L^p} \leq C_\phi \Lambda^{-1}\varepsilon^s \max\big\{1, \varepsilon^{k-s-\frac{3}{p'}}\big\}.$$

If k is choosen large enough, using (6.23), we get the result. □

Now, let us prove a bound from below.

Proposition 6.3.3 *Let ϕ be a function of $\mathcal{S}(\mathbb{R}^3)$ and s in $]0,3[$. For (ε,Λ) in $]0,1]\times[1,\infty[$, let us define*

$$\phi_{\varepsilon,\Lambda}(x) \stackrel{\text{def}}{=} e^{i\frac{x_1}{\varepsilon}}\phi(x_1, \Lambda x_2, x_3).$$

Then, if $\Lambda\varepsilon$ is small enough, we have

$$\|\phi_{\varepsilon,\Lambda}\|_{\dot{B}^{-s}_{\infty,\infty}} \geq C_\phi \varepsilon^s.$$

Proof. Let us first observe that, as the space of smooth compactly supported functions is dense in $\mathcal{S}$ and the Fourier transform is continuous on $\mathcal{S}$. Thus, for any positive η, a function φ exists, the Fourier transform of which is smooth and compactly supported such that, denoting as before $\theta_\varepsilon(x) = e^{i\frac{x_3}{\varepsilon}}\theta(x_1, \Lambda x_2, x_3)$,

$$\|\phi_\varepsilon - \theta_\varepsilon\|_{\dot{B}^{-\sigma}_{\infty,\infty}} \leq \eta\varepsilon^\sigma \quad \text{and} \quad \|\phi-\theta\|_{L^\infty} \leq \eta. \tag{6.24}$$

As the support of the Fourier transform of θ is included in the ball $B(0,R)$ for some positive R, that of $\theta(x_1,\Lambda x_2,x_3)$ is included in the ball $B(0,R\Lambda)$. Then the support of $\mathcal{F}\theta_{\varepsilon,\Lambda}$ is included in the ball $B(\varepsilon^{-1}(0,0,1),\Lambda R)$ which can be written as

$$\frac{1}{\varepsilon}B((0,0,1),\Lambda\varepsilon R)$$

If $\lambda\varepsilon$ is small enough, we can assume that this set is included in $\varepsilon^{-1}\mathcal{C}$ where $\mathcal{C}$ denotes a fixed ring. Now we use the heat flow through Theorem 6.3.1. Let us write that

$$\begin{aligned}\|\theta_{\varepsilon,\Lambda}\|_{\dot{B}^{-s}_{\infty,\infty}} &\sim \sup_{t>0} t^{\frac{s}{2}}\|e^{t\Delta}\theta_{\varepsilon,\Lambda}\|_{L^\infty}\\ &\geq C\varepsilon^{\sigma}\|e^{\varepsilon^2\Delta}\theta_{\varepsilon,\Lambda}\|_{L^\infty}.\end{aligned}$$

For any function h such that the support of $\widehat{h}$ is included in $\varepsilon^{-1}\mathcal{C}$, we have

$$\|\mathcal{F}^{-1}(e^{\varepsilon^2|\xi|^2}h)\|_{L^\infty}\leq C\|h\|_{L^\infty}.$$

Applied with $h=e^{\varepsilon^2\Delta}\theta_{\varepsilon,\Lambda}$, this inequality gives

$$\|\theta_{\varepsilon,\Lambda}\|_{L^\infty}\leq C\|e^{\varepsilon^2\Delta}\theta_{\varepsilon,\Lambda}\|_{L^\infty}\quad\text{and thus}\quad \|\theta_{\varepsilon,\Lambda}\|_{\dot{B}^{-s}_{\infty,\infty}}\geq C^{-1}\varepsilon^{\sigma}\|\theta_{\varepsilon,\Lambda}\|_{L^\infty}=C^{-1}\varepsilon^{s}\|g\|_{L^\infty}.$$

Now let us write that

$$\begin{aligned}\|\phi_\varepsilon\|_{\dot{B}^{-s}_{\infty,\infty}} &\geq \|\theta_{\varepsilon,\Lambda}\|_{\dot{B}^{-s}_{\infty,\infty}}-\eta\varepsilon^{\sigma}\\ &\geq C^{-1}\varepsilon^{s}(\|\phi\|_{L^\infty}-2\eta).\end{aligned}$$

Together with (6.24), this gives the proposition. □

Chapter 7

Large oscillating data: an example of the use of the structure of the equation

The purpose of this chapter is to provide examples of large initial data which generates global smooth solutions. First of all, we need to understand what is large data is. In fact this implies to prove a end point theorem in the following sense. The spaces that appear on Theorem 5.2.1 page 47 are increasing with p. A theorem can contains all this ones is proved in the first section and a notion of large data is defined.

In the second

7.1 The endpoint space for Picard's scheme

According to Theorem 5.2.1, the generalized Navier-Stokes system (GNS) is globally well-posed whenever the initial data u_0 is small the homogeneous Besov space $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$ with p in $]3,\infty[$. In this section, we aim at finding the largest space for solving (GNS) by an iterative scheme. Since the spaces $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$ are increasing with p, a good candidate would be the space $\dot{B}^{-1}_{\infty,\infty}$. As a matter of fact, the following proposition guarantees that it is hopeless to go beyond that space:

Proposition 7.1.1 *Let E be a Banach space continuously embedded in the set $\mathcal{S}'(\mathbb{R}^3)$. Assume that, for any (λ, a) in $\mathbb{R}^+_\star \times \mathbb{R}^3$,*

$$\|f(\lambda(\cdot - a))\|_E = \lambda^{-1}\|f\|_E.$$

Then E is continuously embedded in $\dot{B}^{-1}_{\infty,\infty}$.

Proof. As B is continuously included in $\mathcal{S}'$, we have that $|\langle f, e^{-|\cdot|^2}\rangle| \leq C\|f\|_B$. Then by dilation and translation, we deduce that

$$\|f\|_{\dot{B}^{-1}_{\infty,\infty}} = \sup_{t>0} t^{\frac{1}{2}}\|e^{t\Delta}f\|_{L^\infty} \leq C\|f\|_B.$$

This proves the proposition. □

It turns out however that $\dot{B}^{-1}_{\infty,\infty}$ is too large a space. The may reason why is that if we want to solve the problem using an iterative scheme then we need that $e^{t\Delta}u_0$ belongs to $L^2_{loc}(\mathbb{R}^+ \times \mathbb{R}^3)$ so that $B(e^{t\Delta}u_0, e^{t\Delta}u_0)$ makes sense. Taking into consideration the scaling and the translation invariance thus leads to the following definition.

Definition 7.1.1 *We denote by X_0 the space of tempered distributions u such that*

$$\|u\|_{X_0} \stackrel{def}{=} \|u\|_{\dot{B}^{-1}_{\infty,\infty}} + \sup_{\substack{x\in\mathbb{R}^3\\ R>0}} R^{-\frac{3}{2}} \Big(\int_{P(x,R)} |e^{t\Delta}u(y)|^2\, dy\, dt \Big)^{\frac{1}{2}} < \infty$$

where $P(x,R) = [0,R^2] \times B(x,R)$ and $B(x,R)$ denotes the ball of $\mathbb{R}^3$ of center x and radius R.

We denote by X be the space of functions f on $\mathbb{R}^+_\star \times \mathbb{R}^3$ such that

$$\|f\|_X \stackrel{def}{=} \sup_{t>0} \Big(t^{\frac{1}{2}} \|f(t)\|_{L^\infty} + \sup_{\substack{x\in\mathbb{R}^3\\ R>0}} R^{-\frac{3}{2}} \Big(\int_{P(x,R)} |f(t,y)|^2\, dy\, dt \Big)^{\frac{1}{2}} \Big) < \infty$$

We denote by Y the space of functions on $\mathbb{R}^+_\star \times \mathbb{R}^3$ such that

$$\|f\|_Y \stackrel{def}{=} \sup_{t>0} t\|f(t)\|_{L^\infty} + \sup_{\substack{x\in\mathbb{R}^3\\ R>0}} R^{-3} \int_{P(x,R)} |f(t,y)|\, dy\, dt < \infty.$$

Proposition 7.1.2 *For any p in $]3,\infty[$, the space $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$ is continuously embedded in X_0.*

Proof. Let us notice that for any $x \in \mathbb{R}^3$ and $R > 0$, we have

$$\int_0^{R^2} \int_{B(x,R)} \big|e^{t\Delta}u_0(y)\big|^2\, dy\, dt \leq \mu(B(x,R))^{1-\frac{2}{p}} \int_0^{R^2} \Big(\int_{B(x,R)} \big|e^{t\Delta}u_0(y)\big|^p\, dy \Big)^{\frac{2}{p}} dt.$$

By definition of the space $\dot{B}_p^{-1+\frac{3}{p}}$, we have

$$\int_0^{R^2} \int_{B(x,R)} \big|e^{t\Delta}u_0(y)\big|^2\, dy\, dt \leq \|u_0\|_{\dot{B}_p^{-1+\frac{3}{p}}} \mu(B(x,R))^{1-\frac{2}{p}} \int_0^{R^2} t^{-1+\frac{3}{p}} dt$$

which obviously entails the announced embedding. □

Proposition 7.1.3 *The space $\dot{B}^{-1}_{\infty,2}$ is included in X_0.*

Proof. As $\dot{B}^{-1}_{\infty,2}$ is included in $B^{-1}_{\infty,\infty}$, we have

$$\sup_{t>0} t^{\frac{1}{2}} \|e^{t\Delta}u\|_{L^\infty} \leq C\|u\|_{B^{-1}_{\infty,2}}. \tag{7.1}$$

Moreover, we have

$$\begin{aligned} \frac{1}{R^3} \int_0^{R^2} \int_{B(x,R)} |(e^{t\Delta}f)(y)|^2 dy dt &\leq \int_0^\infty \|e^{t\Delta}f\|^2_{L^\infty} dt \\ &\leq C\|u\|_{\dot{B}^{-1}_{\infty,2}}. \end{aligned}$$

Together with (7.1), this proves the proposition. □

The following theorem tells us that space X is suitable for solving the generalized Navier-Stokes system.

Theorem 7.1.1 *A constant c exists such that, if u_0 is in X_0 and $\|u_0\|_{X_0} \leq c$, then (GNS) has a unique solution u in X such that $\|u\|_X \leq 2\|u_0\|_{X_0}$.*

According to Lemma 5.1.1, it suffices to prove that there exists some constant C such that

$$\|B(u,v)\|_X \leq C\|u\|_X\|v\|_X. \tag{7.2}$$

Once noticed that $\|fg\|_Y \leq \|f\|_X\|g\|_X$, we see that the above inequality stems from the following lemma.

Theorem 7.1.2 *Let us define the operators $(L_j)_{1\leq j\leq N}$ by*

$$\begin{cases} \partial_t L_j f - \Delta L_j f = \partial_j f - \nabla p \\ \operatorname{div} v = 0 \quad \text{and} \quad L_{j|t=0} = 0. \end{cases}$$

The operators L_j maps continuously Y into X.

Proof. Using commutations of the proof of Lemma 5.2.1 page 47, we get that

$$(L_j f)^k(t,x) = \sum_{\ell=1}^{3} \Gamma^k_{j,\ell}(t-t', x-y) f^\ell(t',y)\, dt'\, dy$$

with for all positive real number R,

$$|\Gamma^k_{j,\ell}(\tau,\zeta)| \leq \frac{C}{(\sqrt{\tau}+|\zeta|)^4} \leq C'\big(\Gamma^{(1)}_R(\tau,\zeta)+\Gamma^{(2)}_R(\tau,\zeta)\big)$$

with $\Gamma^{(1)}_R(\tau,\zeta) \stackrel{\text{def}}{=} \mathbf{1}_{|\zeta|\geq R}\dfrac{1}{|\zeta|^4}$ and $\Gamma^{(2)}_R(\tau,\zeta) \stackrel{\text{def}}{=} \mathbf{1}_{|\zeta|\leq R}\dfrac{1}{(\sqrt{\tau}+|\zeta|)^4}\cdot$

The operators of convolution with functions $\Gamma^{(1)}_R$ and $\Gamma^{(2)}_R$ may be bounded according to the following proposition.

Lemma 7.1.1 *There exists a constant C such that, for any $R>0$,*

$$\|\Gamma^{(1)}_R \star f\|_{L^\infty([0,R^2]\times\mathbb{R}^3)} \leq \frac{C}{R}\|f\|_Y, \tag{7.3}$$

$$\|\Gamma^{(2)}_R \star f\|_{L^\infty([R^2,\infty[\times\mathbb{R}^3)} \leq \frac{C}{R}\|f\|_Y. \tag{7.4}$$

Proof. Let us decompose $\Gamma^{(1)}_R \star f(t,x)$ as a sum of integrals on annulus:

$$\begin{aligned} |\Gamma^{(1)}_R \star f(t,x)| &\leq \sum_{p=0}^{\infty}\int_0^t \int_{B(0,2^{p+1}R)\setminus B(0,2^pR)} \frac{1}{|y|^4}|f(t',x-y)|\,dy\,dt' \\ &\leq \frac{1}{R}\sum_{p=0}^{\infty} 2^{-p+3}(2^{p+1}R)^{-3}\int_0^t\int_{B(0,2^{p+1}R)} |f(t',x-y)|\,dy\,dt'. \end{aligned}$$

As p is nonnegative, we have for $t \leq R^2$,

$$
\begin{aligned}
|\Gamma_R^{(1)} \star f(t,x)| &\leq \frac{C}{R} \sum_{p=0}^{\infty} 2^{-p} (2^{p+1}R)^{-3} \int_{P(x,2^{p+1}R)} |f(t,z)|\, dt\, dz \\
&\leq \frac{C}{R} \sum_{p=0}^{\infty} 2^{-p} \sup_{R'>0} \frac{1}{R'^3} \int_{P(x,R')} |f(t,z)|\, dt\, dz.
\end{aligned}
$$

By definition of $\|\cdot\|_Y$, Inequality (7.3) is proved.

In order to prove the second inequality, let us observe that for all $x \in \mathbb{R}^3$ and $t \geq R^2$, we have

$$
\begin{aligned}
|(\Gamma_R^{(2)} \star f)(t,x)| &\leq \Gamma_R^{(21)}(t,x) + \Gamma_R^{(22)}(t,x) \quad \text{with} \\
\Gamma_R^{(21)}(t,x) &\stackrel{\text{def}}{=} \int_0^{\min(R^2,\frac{t}{2})} \int_{B(0,R)} \frac{1}{(\sqrt{t-t'}+|y|)^4} |f(t',x-y)|\, dy\, dt', \\
\Gamma_R^{(22)}(t,x) &\stackrel{\text{def}}{=} \int_{\min(R^2,\frac{t}{2})}^{t} \int_{B(0,R)} \frac{1}{(\sqrt{t-t'}+|y|)^4} |f(t',x-y)|\, dy\, dt'.
\end{aligned}
$$

For bounding $\Gamma_R^{(21)}(t,x)$, we use that $t \leq 2(t-t')$. We get

$$
\Gamma_R^{(21)}(t,x) \leq C \frac{R^3}{t^2} \biggl(\frac{1}{R^3} \int_0^{R^2} \int_{B(0,R)} |f(t',x-y)|\, dt'\, dy \biggr)
$$

so that, for any $t \geq R^2$ and x in $\mathbb{R}^3$,

$$
\Gamma_R^{(21)}(t,x) \leq \frac{C}{t^{\frac{1}{2}}} \|f\|_Y. \tag{7.5}
$$

In order to estimate $\Gamma_R^{(22)}$, let us use that $t \leq 2t'$ and that, for any $a > 0$,

$$
\int_{B(0,R)} \frac{dy}{(a+|y|)^4} \leq \frac{1}{a} \int_{\mathbb{R}^3} \frac{dz}{(1+|z|)^4}.
$$

This enables us to write that

$$
\begin{aligned}
\Gamma_R^{(22)}(t,x) &\leq \int_{\min(R^2,\frac{t}{2})}^{t} \int_{B(0,R)} \frac{1}{(\sqrt{t-t'}+|y|)^4} \|f(t',\cdot)\|_{L^\infty}\, dy\, dt', \\
&\leq C\|f\|_Y \biggl(\int_{t/2}^{t} \frac{1}{\sqrt{t-t'}} \frac{dt'}{t'} + \int_{R^2}^{t} \frac{\mu(B(0,R))}{t^2} \frac{dt'}{t'} \biggr), \\
&\leq C\|f\|_Y \biggl(\frac{1}{t^{\frac{1}{2}}} + \frac{1}{R^2} \frac{tR^3}{t^2} \biggr).
\end{aligned}
$$

As $R \leq \sqrt{t}$, this concludes the proof of the lemma. □

Proof of Lemma 7.1.2 (continued). Note that applying the above proposition with $R = \sqrt{t}$ yields

$$
\|(L_j f)(t,\cdot)\|_{L^\infty} \leq \frac{C}{t^{\frac{1}{2}}} \|f\|_Y. \tag{7.6}
$$

Hence, it suffices to estimate $\|L_j f\|_{L^2(P(x,R))}$ for an arbitrary $x \in \mathbb{R}^3$. Using translations and dilations, we can assume that $x = 0$ and $R = 1$. Let us write

$$L_j f = L_j(\mathbf{1}_{^cB(0,2)} f) + L_j(\mathbf{1}_{B(0,2)} f).$$

Observing that, for any $y \in B(0,1)$, we have

$$|L_j(\mathbf{1}_{^cB(0,2)} f)(t,y)| \leq C\Gamma_1^{(1)} \star (\mathbf{1}_{^cB(0,2)}|f|)(t,y),$$

and using Inequality (7.3), we get

$$\|L_j(\mathbf{1}_{^cB(0,2)} f)\|_{L^\infty(P(0,1))} \leq C\|f\|_Y.$$

As the volume of $P(0,1)$ is finite, we infer that

$$\|L_j(\mathbf{1}_{^cB(0,2)} f)\|_{L^2(P(0,1))} \leq C\|f\|_Y. \tag{7.7}$$

Now the proof of Lemma 7.1.2 is reduced to the proof the following Lemma.

Lemma 7.1.2 *For any function $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ such that $f(t,\cdot)$ be supported in $B(0,2)$ for all $t \in [0,1]$, we have*

$$\|(L_j f)(t,\cdot)\|_{L^2([0,1]\times\mathbb{R}^3)} \leq C\|f\|_Y$$

Proof. Let us point out that, for any t, $L_j f(t) = \mathbb{P}\widetilde{L}_j f(t)$ where $\widetilde{L}_j$ is the solution of

$$\partial_t \widetilde{L}_j f - \Delta \widetilde{L}_j f = \partial_j f \quad \text{and} \quad \widetilde{L}_j f_{|t=0} = 0.$$

As $\mathbb{P}$ is an orthogonal projection in L^2, we get

$$\|L_j f(t,\cdot)\|_{L^2} \leq \|\widetilde{L}_j f(t,\cdot)\|_{L^2}. \tag{7.8}$$

Let us decompose f into low and high frequencies in the sense of the heat flow:

$$f = f^\flat + f^\sharp \quad \text{with} \quad f^\flat(t,\cdot) \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\widehat{\theta}(t^{\frac{1}{2}}\xi)\widehat{f}(t,\xi))$$

where θ denotes a function such that $\widehat{\theta}$ be compactly supported and with value 1 near the origin. Let us write that

$$\begin{aligned}
\|f^\sharp\|^2_{L^2([0,1];\dot{H}^{-1})} &= (2\pi)^{-3} \int_{[0,1]\times\mathbb{R}^3} \frac{|1-\widehat{\theta}(t^{\frac{1}{2}}\xi)|^2}{t|\xi|^2}\, t|\widehat{f}(t,\xi)|^2\, dt\, d\xi \\
&\leq C \int_{[0,1]\times\mathbb{R}^3} t\|f(t,\cdot)\|^2_{L^2}\, dt \\
&\leq C\|f\|_{L^1([0,1]\times\mathbb{R}^3)} \sup_{t>0} t\|f(t,\cdot)\|_{L^\infty}.
\end{aligned}$$

So using the energy estimate on the heat equation and (7.8), we end up with

$$\|L_j f^\sharp\|_{L^2([0,1]\times\mathbb{R}^3)} \leq C\|f\|_Y. \tag{7.9}$$

Now let us estimate $\|\widetilde{L}_j f^\flat\|_{L^2([0,1]\times\mathbb{R}^3)}$. Let us first observe that, by definition of $\widetilde{L}_j$ and $f^\flat$, we have

$$\begin{aligned}
\mathcal{F}\widetilde{L}_j f^\flat(t,\xi) &= i\xi_j \int_0^t e^{(-t-t')|\xi|^2} \widehat{f^\flat}(t',\xi)\, dt' \\
&= i\xi_j e^{-t|\xi|^2} \int_0^t \mathcal{F}(\widetilde{f^\flat})(t',\xi)\, dt' \quad \text{with} \quad \mathcal{F}\widetilde{f^\flat}(t',\xi) \stackrel{\text{def}}{=} e^{t'|\xi|^2}\widehat{\theta}(t'|\xi|^2)\widehat{f}(t,\xi).
\end{aligned}$$

Let us notice that, by definition of θ, we have that

$$\widetilde{f}^\flat(t,\cdot)=t^{-\frac{3}{2}}\widetilde{\theta}\Big(\frac{\cdot}{\sqrt{t}}\Big)\star f(t,\cdot)\quad\text{with}\quad \widetilde{\theta}\in\mathcal{S}(\mathbb{R}^3). \tag{7.10}$$

Thus, using (7.8), we get

$$\sum_{j=1}^3\|L_jf^\flat\|^2_{L^2([0,1]\times\mathbb{R}^3)}\leq\mathcal{N}(f)\quad\text{with}\quad\mathcal{N}(f)\stackrel{\text{def}}{=}\int_0^1\Big\|\nabla e^{t\Delta}\int_0^t\widetilde{f}^\flat(t')\,dt'\Big\|^2_{L^2}\,dt.$$

By symmetry, we can write

$$\begin{aligned}\mathcal{N}(f) &= 2\int_A\big(\nabla e^{t\Delta}\widetilde{f}^\flat(t'')\big|\nabla e^{t\Delta}\widetilde{f}^\flat(t')\big)_{L^2}\,dt''\,dt'\,dt\quad\text{with}\\ A &\stackrel{\text{def}}{=} \{(t'',t',t)\in[0,1]^3\,/\,t''\leq t'\leq t\}.\end{aligned}$$

By integration by parts and because $e^{t\Delta}$ is self-adjoint on L^2, we get

$$\big(\nabla e^{t\Delta}\widetilde{f}^\flat(t'')\big|\nabla e^{t\Delta}\widetilde{f}^\flat(t')\big)_{L^2}=-\langle\Delta e^{2t\Delta}\widetilde{f}^\flat(t''),\widetilde{f}^\flat(t')\rangle.$$

For any positive t' and t'' such that $t''\leq t'$, the function $\widetilde{f}^\flat(t',\cdot)$ and $\widetilde{f}^\flat(t'',\cdot)$ have Fourier transform with compact support in a ball of radius $C/\sqrt{t''}$. In this space (denoted in $\mathcal{F}L^2_{t''}$), we have, in the sense of $\mathcal{L}(\mathcal{F}L^2_{t''})$,

$$2\Delta e^{t\Delta}=-\frac{d}{dt}e^{2t\Delta}\,.$$

We infer that

$$\forall(t'',t')\in]0,1]^2\,,\quad\big(\nabla e^{t\Delta}\widetilde{f}^\flat(t'')\big|\nabla e^{t\Delta}\widetilde{f}^\flat(t')\big)_{L^2}=-\frac{1}{2}\frac{d}{dt}\big(e^{2t\Delta}\widetilde{f}^\flat(t'')|\widetilde{f}^\flat(t')\big)_{L^2}.$$

By integration, we deduce that

$$\begin{aligned}\int_{t'}^1\big(\nabla e^{t\Delta}\widetilde{f}^\flat(t'')\big|\nabla e^{t\Delta}\widetilde{f}^\flat(t')\big)_{L^2}dt &= -\int_{t'}^1\frac{d}{dt}\big(e^{2t\Delta}\widetilde{f}^\flat(t'')\big|\widetilde{f}^\flat(t')\big)_{L^2}dt\\ &= \big((e^{2t'\Delta}-e^{2\Delta})\widetilde{f}^\flat(t'')\big|\widetilde{f}^\flat(t')\big)_{L^2}\end{aligned}$$

Thanks to Fubini' theorem, we deduce that

$$\mathcal{N}(f)=\int_0^1\Big((e^{2t'\Delta}-e^{2\Delta})\int_0^{t'}\widetilde{f}^\flat(t'')\,dt''\Big|\widetilde{f}^\flat(t')\Big)_{L^2}dt'$$

By definition of the L^2 inner product, we infer that

$$\mathcal{N}(f)\leq\|\widetilde{f}^\flat\|_{L^1([0,1]\times\mathbb{R}^3)}\sup_{t'\in[0,1]}\Big\|(e^{2t'\Delta}-e^{2\Delta})\int_0^{t'}\widetilde{f}^\flat(t'')\,dt''\Big\|_{L^\infty}.$$

Using (7.10), we infer that

$$\mathcal{N}(f)\leq\|f\|_{L^1([0,1]\times\mathbb{R}^3)}\sup_{t'\in[0,1]}\Big\|(e^{2t'\Delta}-e^{2\Delta})\int_0^{t'}\widetilde{f}^\flat(t'')\,dt''\Big\|_{L^\infty}. \tag{7.11}$$

First of all, let us notice that, using (7.10) and the fact that operator $e^{2\Delta}$ maps $L^1(\mathbb{R}^3)$ in $L^\infty(\mathbb{R}^3)$, we have

$$\Big\|e^{2\Delta}\int_0^{t'}\widetilde{f}^\flat(t'')\,dt''\Big\|_{L^\infty}\leq C\|f^\flat\|_{L^1([0,1]\times\mathbb{R}^3)}.$$

Thanks to (7.10), we have, as f is supported in the ball $B(0,2)$,

$$\forall t\in[0,1]\,,\,,\;\|f^\flat(t,\cdot)\|_{L^1(\mathbb{R}^3)}\leq C\|f\|_Y. \tag{7.12}$$

Thus, we get

$$\Big\|e^{2\Delta}\int_0^{t'}\widetilde{f}^\flat(t'')\,dt''\Big\|_{L^\infty}\leq C\|f\|_Y. \tag{7.13}$$

Let us admit for a while the following two lemmas.

Lemma 7.1.3 *Let θ in $\mathcal{S}(\mathbb{R}^3)$. Let us define*

$$\widetilde{f}_\theta(t,\cdot)\stackrel{\text{def}}{=}t^{-\frac32}\theta(t^{-\frac12}\cdot)\star f(t,\cdot)$$

We have

$$\|\widetilde{f}_\theta\|_Y\leq C\|f\|_Y.$$

Lemma 7.1.4 *Let us define*

$$Ef(t)=e^{t\Delta}\int_0^t f(t')dt'.$$

Then a constant C exists such that for any function f in Y and supported in $[0,1]\times B(0,2)$, we have

$$\|Ef\|_{L^\infty([0,1]\times\mathbb{R}^3)}\leq C\|f\|_Y$$

These two lemmas imply that

$$\sup_{t'\in[0,1]}\Big\|e^{2t'\Delta}\int_0^{t'}\widetilde{f}^\flat(t'')\,dt''\Big\|_{L^\infty}\leq C\|f\|_Y.$$

Inequalities (7.6), (7.9) and (7.13) allows to conclude the proof of the theorem. □

Proof of Lemma 7.1.3 Let us first obverse that, for any t, we have

$$\|\widetilde{f}(t,\cdot)\|_{L^\infty}\leq\|\theta\|_{L^1}\|f(t,\cdot)\|_{L^\infty}.$$

Thus we have

$$t\|\widetilde{f}(t,\cdot)\|_{L^\infty}\leq\|\theta\|_{L^1}t\|f(t,\cdot)\|_{L^\infty}. \tag{7.14}$$

Now, let us write that, for any x in the ball of center 0 and radius R, we have

$$\begin{aligned}|\widetilde{f}_\theta(t,x)| &\leq t^{-\frac32}\int_{\mathbb{R}^3}\Big|\theta\Big(\frac{x-y}{\sqrt t}\Big)\Big|\mathbf{1}_{B(0,2R)}(y)\,|f(t,y)|\,dy\\ &\qquad\qquad\qquad +Ct^{-\frac32}\int_{\mathbb{R}^3}\frac{1}{\Big(1+\frac{|x-y|}{\sqrt t}\Big)^4}\frac{t}{R^2}|f(t,y)|\,dy\\ &\leq t^{-\frac32}\Big(|\theta(t^{-\frac12}\cdot)|\star\mathbf{1}_{B(0,2R)}|f(t,\cdot)|\Big)(x)+\frac{C}{R^2}\sup_{t>0}t\|f(t,\cdot)\|_{L^\infty}.\end{aligned}$$

Thus, we infer

$$\frac{1}{R^3}\|\widetilde{f}_\theta\|_{L^1(P(0,R))}\leq\frac{C}{R^3}\int_{P(0,R)}|f(t,y)|\,dt\,dy+C\sup_{t>0}t\|f(t,\cdot)\|_{L^\infty}.$$

This proves Lemma 7.1.3. □

Proof of Lemma 7.1.4 Let us write that

$$\Big|e^{2t\Delta}\int_0^t f(t',x)\,dt'\Big| \leq \sum_{n\in\mathbb{Z}^3}\frac{1}{(4\pi t')^{\frac{3}{2}}}\int_{\mathbb{R}^3}\!\int_0^{t'} e^{-\frac{|x-y|^2}{4t'}}\mathbf{1}_{B_{n,t'}}(y)f(t',y)dt'dy$$

where $B_{n,t'}$ denotes the ball of center $n\sqrt{t'}$ and radius $\sqrt{t'}$. Using the translation invariance, it is enough to estimate the above integral at point $x=0$. We write, thanks to Proposition 7.1.3,

$$\begin{aligned}\Big|\Big(e^{2t'\Delta}\int_0^{t'}\widetilde{f}^\flat(t'')\,dt''\Big)(0)\Big| &\leq \sum_{|n|>2} e^{-\frac{|n|^2}{4}}\Big(\frac{1}{|n|^3}\int_{P(n,t')}|\widetilde{f}^\flat(t'',y)|\,dt''dy\Big)\\ &+\sum_{|n|\leq 2}\frac{1}{(4\pi t')^{\frac{3}{2}}}\int_{\mathbb{R}^3}\!\int_0^{t'} e^{-\frac{|x-y|^2}{4t'}}\mathbf{1}_{B_{n,t'}}(y)\big|\widetilde{f}^\flat(t'',y)\big|\,dt''dy\\ &\leq C\|f\|_Y.\end{aligned}$$

Thanks to Inequality (7.9), this concludes the proof of Lemma 7.1.4. □

7.2 An abstract non linear smallness condition

Let us define a space which is a good space for being an external force that Theorem 7.1.1.

Definition 7.2.1 *We shall denote by E the space of functions f in $L^1(\mathbb{R}^+;\dot{B}^{-1}_{\infty,1})$ such that*

$$\sum_{j\in\mathbb{Z}} 2^{-j}\Big\|\|\Delta_j f(t)\|_{L^\infty}\Big\|_{L^2(\mathbb{R}^+;tdt)}<\infty$$

equipped with the norm

$$\|f\|_E \stackrel{\text{def}}{=} \|f\|_{L^1(\mathbb{R}^+;\dot{B}^{-1}_{\infty,1})}+\sum_{j\in\mathbb{Z}} 2^{-j}\Big\|\|\Delta_j f(t)\|_{L^\infty}\Big\|_{L^2(\mathbb{R}^+;tdt)}.$$

Let us remark that, for any homogeneous function σ of order 0 smooth outside 0, we have

$$\forall p\in[1,\infty]\,,\ \|\sigma(D)\Delta_j f\|_{L^p}\leq C\|\Delta_j f\|_{L^p}.$$

Thus the Leray projection $\mathbb{P}$ on divergence free vectors fields maps continuously E into E.

Theorem 7.2.1 *There is a constant C_0 such that the following result holds. Let $u_0\in\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ be a divergence free vector field. Suppose that*

$$\big\|\mathbb{P}\big(e^{t\Delta}u_0\cdot\nabla e^{t\Delta}u_0\big)\big\|_E\leq C_0^{-1}\exp\big(-C_0U_0^4\ \big)\quad\textit{with}\quad U_0\stackrel{\text{def}}{=}\Big(\|u_0\|^2_{\dot{B}^{-1}_{\infty,2}}+\|u_0\|^4_{\dot{B}^{-1}_{\infty,4}}\Big)^{\frac{1}{4}}\quad(7.15)$$

Let us define

$$U(t)=\Big(\|e^{t\Delta}u_0\|^2_{L^\infty}+t\|e^{t\Delta}u_0\|^4_{L^\infty}\Big)^{\frac{1}{4}}.$$

Then there is a unique global solution to (NS) such that

$$\Big\|\exp\Big(-C_0\int_0^t U^4(t')dt'\Big)\big(u(t)-e^{t\Delta}u_0\big)\Big\|_X\leq C_0^{-1}\exp(-CU_0^4).$$

The proof of this theorem is the purpose of all this section.

7.2.1 Main steps of the proof

Let us start by remarking that in the case when u_0 is small then there is nothing to be proved, so in the following we shall suppose that $\|u_0\|_{\dot{B}^{-1}_{\infty,2}}$ is not small, say $\|u_0\|_{\dot{B}^{-1}_{\infty,2}} \geq 1$.

We search for the solution u under the form

$$u_L + R \quad \text{where} \quad u_L(t) \stackrel{\text{def}}{=} e^{t\Delta} u_0.$$

Then R is the solution of

$$(MNS) \qquad R = B(u_L, u_L) + 2B(u_L, R) + B(R, R).$$

To prove the global existence of u, we are reduced to prove that (MNS) has a global solution. We use the following easy lemma, the proof of which is omitted.

Lemma 7.2.1 *Let X be a Banach space, let L be a continuous linear map from X to X, and let B be a bilinear map from $X \times X$ to X. Let us define*

$$\|L\|_{\mathcal{L}(X)} \stackrel{\text{def}}{=} \sup_{\|x\|=1} \|Lx\| \quad \textit{and} \quad \|B\|_{\mathcal{B}(X)} \stackrel{\text{def}}{=} \sup_{\|x\|=\|y\|=1} \|B(x,y)\|.$$

If $\|L\|_{\mathcal{L}(X)} < 1$, then for any x_0 in X such that

$$\|x_0\|_X < \frac{(1 - \|L\|_{\mathcal{L}(X)})^2}{4\|B\|_{\mathcal{B}(X)}},$$

the equation

$$x = x_0 + Lx + B(x,x)$$

has a unique solution in the ball of center 0 *and radius* $\dfrac{1 - \|L\|_{\mathcal{L}(X)}}{2\|B\|_{\mathcal{B}(X)}}$.

Let us introduce the functional space for which we shall apply the above lemma. We define the quantity

$$U(t) \stackrel{\text{def}}{=} \|u_L(t)\|^2_{L^\infty} + t\|u_L(t)\|^4_{L^\infty},$$

which satisfies, thanks to Theorem 6.3.1 page 63,

$$\begin{aligned}\int_0^\infty U(t)dt &\leq C\|u_0\|^2_{\dot{B}^{-1}_{\infty,2}} + C\|u_0\|^4_{\dot{B}^{-1}_{\infty,4}} \\ &\leq C\|u_0\|^4_{\dot{B}^{-1}_{\infty,2}} \end{aligned} \tag{7.16}$$

recalling that we have supposed that $\|u_0\|_{\dot{B}^{-1}_{\infty,2}} \geq 1$ to simplify the proof.

For all $\lambda \geq 0$, let us denote by X_λ the set of functions on $\mathbb{R}^+ \times \mathbb{R}^3$ such that

$$\|v\|_\lambda \stackrel{\text{def}}{=} \sup_{t>0} \biggl(t^{\frac12} \|v_\lambda(t)\|_{L^\infty} + \sup_{\substack{x\in\mathbb{R}^3 \\ R>0}} R^{-\frac32} \Bigl(\int_{P(x,R)} |v_\lambda(t,y)|^2 dy \Bigr)^{\frac12} \biggr) < \infty, \tag{7.17}$$

where

$$v_\lambda(t,x) \stackrel{\text{def}}{=} v(t,x) \exp\Bigl(-\lambda \int_0^t U(t')dt' \Bigr)$$

while $P(x,R)=[0,R^2]\times B(x,R)$ and $B(x,R)$ denotes the ball of $\mathbb{R}^3$ of center x and radius R. Let us point out that, in the case when $\lambda=0$, this is exactly the space of Definition 7.1.1 page 70. For any non negative λ and that for any $\lambda\geq 0$ we have due to (7.16),

$$\|v\|_\lambda \leq \|v\|_0 \leq C\|v\|_\lambda \exp\Big(C\lambda\|u_0\|^4_{\dot B^{-1}_{\infty,2}}\Big). \tag{7.18}$$

From (7.2) page 71 together with the above equivalence of norms, we infer that

$$\|B(v,w)\|_\lambda \leq C\|v\|_\lambda\|w\|_\lambda \exp\big(C\lambda\|u_0\|^4_{\dot B^{-1}_{\infty,2}}\big). \tag{7.19}$$

Theorem 7.2.1 follows from the following two lemmas we admit for a while.

Lemma 7.2.2 *There is a constant $C>0$ such that the following holds. For any non negative λ, for any $t\geq 0$ and any $f\in E$, we have*

$$\Big\|\int_0^t e^{(t-t')\Delta} f(t')dt'\Big\|_\lambda \leq C\|f\|_E.$$

Lemma 7.2.3 *Let $u_0\in \dot B^{-1}_{\infty,2}$ be given, and define $u_L(t)=e^{t\Delta}u_0$. There is a constant $C>0$ such that the following holds. For any $\lambda\geq 1$, for any $t\geq 0$ and any $v\in X_\lambda$, we have*

$$\|B(u_L,v)(t)\|_\lambda \leq \frac{C}{\lambda^{\frac14}}\|v\|_\lambda.$$

Conclusion of the proof of Theorem 7.2.1 Let us apply Lemma 7.2.1 to Equation (MNS) satisfied by R, in a space X_λ. We choose λ so that according to Lemma 7.2.3,

$$\|B(u_L,\cdot)(t)\|_{\mathcal{L}(X_\lambda)} \leq \frac14\cdot$$

Then according to Lemma 7.2.1, there is a unique solution R to (MNS) in X_λ as soon as $B(u_L,u_L)$ satisfies

$$\|B(u_L,u_L)\|_{X_\lambda} \leq \frac{1}{16\|B\|_{\mathcal{B}(X_\lambda)}}\cdot$$

But (7.19) guarantees that

$$\|B\|_{\mathcal{B}(X_\lambda)} \leq \exp\Big(\lambda\|u_0\|^4_{\dot B^{-1}_{\infty,2}}\Big),$$

So, if

$$\|B(u_L,u_L)\|_{X_\lambda} \leq C^{-1}\exp\Big(-\lambda\|u_0\|^4_{\dot B^{-1}_{\infty,2}}\Big),$$

by Lemma 7.2.1, there is a unique solution of (MNS). The above condition is exactly re is precisely condition (7.15) of Theorem 7.2.1, so under assumption (7.15), there is a unique small (in the sense of $\|\cdot\|_\lambda$) solution R to (MNS).

7.2.2 Proof of Lemma 7.2.2

Thanks to (7.18), it is enough to prove Lemma 7.2.2 for $\lambda=0$.

Let us start by proving that $\displaystyle\int_0^t e^{(t-t')\Delta}f(t')dt'$ belongs to $L^2(\mathbb{R}^+;L^\infty)$; that will give in particular the boundedness of the second norm entering in the definition of X_λ.

Using Lemma 6.1.2 page 54, we get

$$\Big\|\int_0^t \Delta_j e^{(t-t')\Delta} f(t')dt'\Big\|_{L^\infty} \leq C\int_0^t e^{-C^{-1}2^{2j}(t-t')}\|\Delta_j f(t')\|_{L^\infty}dt'.$$

Young's inequality then gives

$$\Big\|\int_0^t \Delta_j e^{(t-t')\Delta} f(t')dt'\Big\|_{L^2(\mathbb{R}^+;L^\infty)} \leq C2^{-j}\|\Delta_j f\|_{L^1(\mathbb{R}^+;L^\infty)},$$

thus the series $\Big(\Delta_j \displaystyle\int_0^t e^{(t-t')\Delta} f(t')dt'\Big)_{j\in\mathbb{Z}}$ converges in $L^2(\mathbb{R}^+;L^\infty)$, and

$$\Big\|\int_0^t e^{(t-t')\Delta} f(t')dt'\Big\|_{L^2(\mathbb{R}^+;L^\infty)} \leq C\|f\|_{L^1(\mathbb{R}^+;\dot{B}^{-1}_{\infty,1})}.$$

This implies in particular that

$$\sup_{\substack{x\in\mathbb{R}^3\\ R>0}} R^{-\frac{3}{2}}\bigg(\int_{P(x,R)}\Big|\int_0^t (e^{(t-t')\Delta}f(t'))(y)dt'\Big|^2 dy\bigg)^{\frac{1}{2}} \leq C\|f\|_{L^1(\mathbb{R}^+;\dot{B}^{-1}_{\infty,1})}. \tag{7.20}$$

The second part of the norm defining $\|\cdot\|_{X_\lambda}$ in (7.17) is therefore controlled by the norm of f in E.

To estimate the first part of that norm, let us write that for any $t\geq 0$ and any $j\in\mathbf{Z}$,

$$\begin{aligned} t^{\frac{1}{2}}\Delta_j\int_0^t e^{(t-t')\Delta}f(t')dt' &= G_j^{(1)}(t)+G_j^{(2)}(t) \quad \text{with}\\ G_j^{(1)}(t) &\stackrel{\text{def}}{=} t^{\frac{1}{2}}\int_0^{\frac{t}{2}} e^{(t-t')\Delta}\Delta_j f(t')dt' \quad \text{and}\\ G_j^{(2)}(t) &\stackrel{\text{def}}{=} t^{\frac{1}{2}}\int_{\frac{t}{2}}^t e^{(t-t')\Delta}\Delta_j f(t')dt'. \end{aligned}$$

Using again Lemma 6.1.2 page 54, we have, since $t\leq 2(t-t')$,

$$\begin{aligned} \|G_j^{(1)}(t)\|_{L^\infty} &\leq C\int_0^{\frac{t}{2}}(t-t')^{\frac{1}{2}}2^j e^{-C^{-1}2^{2j}(t-t')}2^{-j}\|\Delta_j f(t')\|_{L^\infty}dt'\\ &\leq 2^{-j}\|\Delta_j f\|_{L^1(\mathbb{R}^+;L^\infty)}. \end{aligned}$$

In order to estimate $\|G_j^{(2)}(t)\|_{L^\infty}$, let us write, since $t\leq 2t'$,

$$\|G_j^{(2)}(t)\|_{L^\infty} \leq C\int_0^t e^{-C^{-1}2^{2j}(t-t')}t'^{\frac{1}{2}}\|\Delta_j f(t')\|_{L^\infty}dt'.$$

Using the Cauchy-Schwarz inequality, we get

$$\|G_j^{(2)}(t)\|_{L^\infty} \leq C2^{-j}\|t^{\frac{1}{2}}\Delta_j f(t)\|_{L^2(\mathbb{R}^+;L^\infty)}.$$

Then using (7.20) and summing over $j\in\mathbb{Z}$ concludes the proof of Lemma 7.2.2. □

7.2.3 Proof of Lemma 7.2.3

From Proposition 5.2.1 page 47, we have

$$\begin{aligned} B(v,w)(t,x) &= \int_0^t \int_{\mathbb{R}^3} k(t-t',y)v(t',x-y)w(t',x-y)dydt' \\ &= k\star(vw)(t,x) \quad \text{with} \quad |k(\tau,\zeta)| \leq \frac{C}{(\sqrt{\tau}+|\zeta|)^4}\cdot \end{aligned}$$

The proof relies now mainly on the following proposition.

Proposition 7.2.1 *Let $u_0 \in \dot{B}^{-1}_{\infty,2}$ be given, and define $u_L(t) = e^{t\Delta}u_0$. There is a constant C such that the following holds. Consider, for any positive R and for $(\tau,\zeta) \in \mathbb{R}^+ \times \mathbb{R}^3$, the following functions:*

$$K_R^{(1)}(\tau,\zeta) \stackrel{\text{def}}{=} \mathbf{1}_{|\zeta|\geq R}\frac{1}{|\zeta|^4} \quad \text{and} \quad K_R^{(2)}(\tau,\zeta) \stackrel{\text{def}}{=} \mathbf{1}_{|\zeta|\leq R}\frac{1}{(\sqrt{\tau}+|\zeta|)^4}\cdot$$

Then for any $\lambda \geq 1$ and any $R > 0$,

$$\Big\|e^{-\lambda\int_0^t U(t')dt'}K_R^{(1)}\star(u_Lv)\Big\|_{L^\infty([0,R^2]\times\mathbb{R}^3)} \leq \frac{C}{\lambda^{\frac12}R}\|v\|_\lambda. \tag{7.21}$$

Moreover, for any $\lambda \geq 1$ and any $R > 0$,

$$\Big\|e^{-\lambda\int_0^t U(t')dt'}K_R^{(2)}\star(u_Lv)\Big\|_{L^\infty([R^2,2R^2]\times\mathbb{R}^3)} \leq \frac{C}{\lambda^{\frac14}R}\|v\|_\lambda. \tag{7.22}$$

Proof. Let us write that

$$\begin{aligned} V_\lambda^{(1)}(t,x) &\stackrel{\text{def}}{=} e^{-\lambda\int_0^t U(t')dt'}|K_R^{(1)}\star(u_Lv)(t,x)| \\ &\leq \int_0^t\int_{^cB(0,R)} \frac{1}{|y|^4}e^{-\lambda\int_{t'}^t U(t'')dt''}\|u_L(t',\cdot)\|_{L^\infty}|v_\lambda(t',x-y)|dt'dy. \end{aligned}$$

By the Cauchy-Schwarz inequality and by definition of U, we infer that

$$\begin{aligned} V_\lambda^{(1)}(t,x) &\leq \bigg(\int_0^t\int_{^cB(0,R)} \frac{1}{|y|^4}e^{-2\lambda\int_{t'}^t U(t'')dt''}\|u_L(t',\cdot)\|^2_{L^\infty}dt'dy\bigg)^{\frac12} \\ &\qquad\qquad \times\bigg(\int_0^t\int_{^cB(0,R)}\frac{1}{|y|^4}|v_\lambda(t',x-y)|^2dt'dy\bigg)^{\frac12} \\ &\leq \Big(\frac{C}{\lambda R}\Big)^{\frac12}\bigg(\int_0^t\int_{^cB(0,R)}\frac{1}{|y|^4}|v_\lambda(t',x-y)|^2dt'dy\bigg)^{\frac12}. \end{aligned} \tag{7.23}$$

Now let us decompose the integral on the right on rings; this gives

$$\begin{aligned} \int_0^t\int_{^cB(0,R)}\frac{1}{|y|^4}|v_\lambda(t',x-y)|^2dt'dy &= \sum_{p=0}^\infty\int_0^t\int_{B(0,2^{p+1}R)\setminus B(0,2^pR)}\frac{1}{|y|^4}|v_\lambda(t',x-y)|^2dt'dy \\ &\leq \frac1R\sum_{p=0}^\infty 2^{-p+3}(2^{p+1}R)^{-3} \\ &\qquad \times\int_0^t\int_{B(0,2^{p+1}R)}|v_\lambda(t,x-y)|^2dtdy. \end{aligned}$$

As $t \leq R^2$ and p is non negative, we have

$$\begin{aligned}
\int_0^t \int_{{}^cB(0,R)} \frac{1}{|y|^4} |v_\lambda(t', x-y)|^2 dt' dy &\leq \frac{C}{R} \sum_{p=0}^{\infty} 2^{-p} (2^{p+1}R)^{-3} \int_{P(x, 2^{p+1}R)} |v_\lambda(t,z)|^2 dt dz \\
&\leq \frac{C}{R} \sum_{p=0}^{\infty} 2^{-p} \sup_{R'>0} \frac{1}{R'^3} \int_{P(x,R')} |v_\lambda(t,z)|^2 dt dz.
\end{aligned}$$

By definition of $\|\cdot\|_\lambda$, we infer that

$$\int_0^t \int_{{}^cB(0,R)} \frac{1}{|y|^4} |v_\lambda(t', x-y)|^2 dt' dy \leq \frac{C}{R} \|v\|_\lambda^2.$$

Then, using (7.23), we conclude the proof of (7.21).

In order to prove the second inequality, let us observe that

$$\begin{aligned}
e^{-\lambda \int_0^t U(t')dt'} |(K_R^{(2)} \star (u_L v))(t,x)| &\leq \mathcal{K}_R^{(21)}(t,x) + \mathcal{K}_R^{(21)}(t,x) \quad \text{with} \\
\mathcal{K}_R^{(22)}(t,x) &\stackrel{\text{def}}{=} \int_0^{\frac{t}{2}} \int_{B(0,R)} \frac{1}{(\sqrt{t-t'} + |y|)^4} e^{-\lambda \int_{t'}^t U(t'')dt''} \|u_L(t',\cdot)\|_{L^\infty} |v_\lambda(t', x-y)| dt' dy\,, \\
\mathcal{K}_R^{(22)}(t,x) &\stackrel{\text{def}}{=} \int_{\frac{t}{2}}^t \int_{B(0,R)} \frac{1}{(\sqrt{t-t'} + |y|)^4} e^{-\lambda \int_{t'}^t U(t'')dt''} \|u_L(t',\cdot)\|_{L^\infty} |v_\lambda(t', x-y)| dt' dy.
\end{aligned}$$

Using the Cauchy-Schwarz inequality, as $t \in [R^2, 2R^2]$ and $t \leq 2(t-t')$, we infer that

$$\begin{aligned}
\mathcal{K}_R^{(21)}(t,x) &\leq \biggl(\int_0^{\frac{t}{2}} \int_{B(0,R)} \frac{1}{(\sqrt{t-t'} + |y|)^8} e^{-2\lambda \int_{t'}^t U(t'')dt''} \|u_L(t',\cdot)\|_{L^\infty}^2 dt' dy \biggr)^{\frac{1}{2}} \\
&\qquad\qquad \times \biggl(\int_0^{\frac{t}{2}} \int_{B(0,R)} |v_\lambda(t', x-y)|^2 dt' dy \biggr)^{\frac{1}{2}} \\
&\leq \frac{C}{\lambda^{\frac{1}{2}}} \biggl(\int_{B(0,R)} \frac{dy}{(R+|y|)^8} \biggr)^{\frac{1}{2}} \biggl(\int_0^{\frac{t}{2}} \int_{B(0,R)} |v_\lambda(t', x-y)|^2 dt' dy \biggr)^{\frac{1}{2}} \\
&\leq \frac{C}{(t\lambda)^{\frac{1}{2}}} R^{-\frac{3}{2}} \biggl(\int_0^{R^2} \int_{B(0,R)} |v_\lambda(t', x-y)|^2 dt' dy \biggr)^{\frac{1}{2}},
\end{aligned}$$

so that

$$K_R^{(21)}(t,x) \leq \frac{C}{(t\lambda)^{\frac{1}{2}}} \|v\|_\lambda. \tag{7.24}$$

In order to estimate $\mathcal{K}_R^{(22)}$, let us write that

$$\begin{aligned}
\mathcal{K}_R^{(22)}(t,x) &\leq \int_{\frac{t}{2}}^t \int_{\mathbb{R}^3} \frac{1}{(\sqrt{t-t'} + |y|)^4} e^{-\lambda \int_{t'}^t U(t'')dt''} \|u_L(t',\cdot)\|_{L^\infty} \|v_\lambda(t',\cdot)\|_{L^\infty} dt' dy \\
&\leq C\|v\|_\lambda \int_{\frac{t}{2}}^t \frac{1}{\sqrt{t-t'}} e^{-\lambda \int_{t'}^t U(t'')dt''} \frac{\|u_L(t',\cdot)\|_{L^\infty}}{t'^{\frac{1}{2}}} dt'.
\end{aligned}$$

By definition of U and using the fact that $t \leq 2t'$, Hölder's inequality implies that

$$\begin{aligned}\mathcal{K}_R^{(22)}(t,x) &\leq \frac{C}{t^{\frac12}}\|v\|_\lambda\Big(\int_0^t e^{-4\lambda\int_{t'}^t U(t'')dt''} t'\|u_L(t',\cdot)\|_{L^\infty}^4 dt'\Big)^{\frac14}\\ &\leq \frac{C}{\lambda^{\frac14}t^{\frac12}}\|v\|_\lambda.\end{aligned}$$

Together with (7.24), this concludes the proof of the proposition. □

From this proposition, we infer immediately the following corollary. This corollary proves directly one half of Lemma 7.2.3, as it gives a control of $B(u_L, v)$ in the first norm out of the two entering in the definition of X_λ.

Corollary 7.2.1 *Under the assumptions of Proposition 7.2.1, we have*

$$t^{\frac12} e^{-\lambda\int_0^t U(t')dt'}\|B(u_L,v)(t,\cdot)\|_{L^\infty} \leq \frac{C}{\lambda^{\frac14}}\|v\|_\lambda.$$

Proof. Let us write that

$$k\star(u_L v)(t,x) = k\star(u_L \mathbf{1}_{^cB(x,2\sqrt{t})}v)(t,x) + k\star(u_L\mathbf{1}_{B(x,2\sqrt{t})}v)(t,x).$$

From Proposition 7.2.1, we infer that

$$\begin{aligned} e^{-\lambda\int_0^t U(t')dt'}|k\star(u_L\mathbf{1}_{^cB(x,2\sqrt{t})}v)(t,x)| &\leq e^{-\lambda\int_0^t U(t')dt'} K^{(1)}_{2\sqrt{t}}\star(|u_L\mathbf{1}_{B(x,2\sqrt{t})}v|)(t,x)\\ &\leq \frac{C}{(t\lambda)^{\frac12}}\|v\|_\lambda.\end{aligned}$$

Moreover, thanks to Proposition 7.2.1, we have also

$$\begin{aligned} e^{-\lambda\int_0^t U(t')dt'}|k\star(u_L\mathbf{1}_{B(x,2\sqrt{t})}v)(t,x)| &\leq e^{-\lambda\int_0^t U(t')dt'} K^{(2)}_{2\sqrt{t}}\star(|u_L|\mathbf{1}_{B(x,2\sqrt{t})}|v|)(t,x)\\ &\leq \frac{C}{\lambda^{\frac14}t^{\frac12}}\|v\|_\lambda.\end{aligned}$$

This proves the corollary. □

Conclusion the proof of Lemma 7.2.3 Let us estimate $\|k\star(u_L v)\|_{L^2(P(x,R))}$, for an arbitrary x in $\mathbb{R}^3$. Let us write that

$$k\star(u_L v) = k\star(u_L\mathbf{1}_{^cB(x,2R)}v) + k\star(u_L\mathbf{1}_{B(x,2R)}v).$$

Observing that, for any $y \in B(x,R)$, we have

$$|k\star(u_L\mathbf{1}_{^cB(x,2R)}v)(t,y)| \leq CK_R^{(1)}\star(|u_L|\mathbf{1}_{^cB(x,2R)}|v|)(t,y),$$

and using Inequality (7.3) of Proposition 7.2.1, we get

$$e^{-\lambda\int_0^t U(t')dt'}\|k\star(u_L\mathbf{1}_{^cB(x,2R)}v\|_{L^\infty(P(x,R))} \leq \frac{C}{\lambda^{\frac12}R}\|v\|_\lambda.$$

As the volume of $P(x,R)$ is proportional to R^5, we infer that

$$\|k\star(u_L v_\lambda)\|_{L^2(P(x,R))} \leq \frac{C}{\lambda^{\frac12}}R^{\frac32}\|v\|_\lambda.$$

The following inequality is easy and classical, so its proof is omitted.

$$\left\| e^{-\lambda\int_0^t U(t')dt'} B(u_L,v)(t)\right\|_{L^2([0,T]\times\mathbb{R}^3)} \leq \frac{C}{\lambda^{\frac12}}\|v_\lambda\|_{L^2([0,T]\times\mathbb{R}^3)}.$$

We deduce that

$$\begin{aligned}
\left\| e^{-\lambda\int_0^t U(t')dt'} k\star(u_L\mathbf{1}_{B(x,2R)})v\right\|_{L^2(P(x,R))} &\leq \left\| e^{-\lambda\int_0^t U(t')dt'} k\star(u_L\mathbf{1}_{B(x,2R)}v)\right\|_{L^2([0,R^2]\times\mathbb{R}^3)}\\
&\leq \frac{C}{\lambda^{\frac12}}\|\mathbf{1}_{B(x,2R)}v_\lambda\|_{L^2([0,R^2]\times\mathbb{R}^3)}\\
&\leq \frac{C}{\lambda^{\frac12}}\|v_\lambda\|_{L^2(P(x,2R))}.
\end{aligned}$$

This concludes the proof of Lemma 7.2.3 □

7.3 A particular case of large oscillating data

It is not obvious that Theorem 7.2.1 is not empty. Of course, the non linear smallness condition is satisfied in the case when $\|u_0\|_{\dot B^{-1}_{\infty,2}}$ is small. Let us first state the theorem that presents a class of large oscillating initial data satisfying hypotheses of Theorem 7.2.1.

Proposition 7.3.1 *Let φ be a function in $\mathcal{S}(\mathbb{R}^3)$. Let us consider $P=(\varepsilon,\Lambda)$ in $]0,1]\times[1,\infty[$ such that $\varepsilon\Lambda$ is small enough. Let us define*

$$\varphi_\varepsilon(x)=\cos\Big(\frac{x_3}{\varepsilon}\Big)\varphi(x_1,\Lambda x_2,x_3).$$

The divergence free vector field

$$u_0(x)=\frac{A}{\varepsilon\Lambda}(-\partial_2\varphi_P(x),-\partial_1\varphi_P(x),0)$$

satisfies

$$\|u_0\|_{\dot B^{-1}_{\infty,2}}\geq c_0A \tag{7.25}$$

and the hypotheses of Theorem 7.2.1.

Proof. The fact that (7.25) is satisfied is an obvious consequence of Proposition 6.3.3.

As $\mathbb{P}$ is a Fourier multiplier operator of order 0, we have

$$\big\|\mathbb{P}\big(e^{t\Delta}u_0\cdot\nabla e^{t\Delta}u_0\big)\big\|_E\leq C\big\|e^{t\Delta}u_0\cdot\nabla e^{t\Delta}u_0\big\|_E.$$

Let us observe that

$$\big(e^{t\Delta}u_0\big)^1\partial_1\big(e^{t\Delta}u_0\big)^1+\big(e^{t\Delta}u_0\big)^2\partial_2\big(e^{t\Delta}u_0\big)^1 = \Big(\frac{A}{\varepsilon}\Big)^2 e^{t\Delta}f_Pe^{t\Delta}g_P \quad\text{and} \tag{7.26}$$

$$\big(e^{t\Delta}u_0\big)^1\partial_1\big(e^{t\Delta}u_0\big)^2+\big(e^{t\Delta}u_0\big)^2\partial_2\big(e^{t\Delta}u_0\big)^2 = \Big(\frac{A}{\varepsilon}\Big)^2\frac{1}{\Lambda}e^{t\Delta}\widetilde f_Pe^{t\Delta}\widetilde g_P \tag{7.27}$$

where f, g, $\widetilde f$ and $\widetilde g$ are fuction in $\mathcal{S}(\mathbb{R}^3)$. Now, the main lemma is the following.

Lemma 7.3.1 *There is a constant C such that the following result holds. Let f and g be in $\dot B^{-1}_{\infty,2}\cap \dot H^{-1}$. Then we have*

$$\|(e^{t\Delta}fe^{t\Delta}g)\|_E \leq C\Big(\|f\|_{\dot B^{-1}_{\infty,2}}\|g\|_{\dot B^{-1}_{\infty,2}}\Big)^{\frac23}\Big(\|f\|_{\dot B^{-1}_{2,2}}\|g\|_{\dot B^{-1}_{2,2}}\Big)^{\frac13}$$

Proof. As the Leray projection $\mathbb{P}$ is continuous on E, it is enough to prove the lemma without $\mathbb{P}$. Using Bernstein's estimate, we get that

$$\|\Delta_j(e^{t\Delta}fe^{t\Delta}g)\|_{L^\infty} \leq C2^{3j}\|e^{t\Delta}fe^{t\Delta}g\|_{L^1}.$$

Then, using the Cauchy-Schwarz inequality, we infer that

$$\begin{aligned} E_j &\stackrel{\text{def}}{=} \|\Delta_j(e^{t\Delta}fe^{t\Delta}g)\|_{L^1(\mathbb{R}^+;L^\infty)} + \|t^{\frac12}\Delta_j(e^{t\Delta}fe^{t\Delta}g)\|_{L^2(\mathbb{R}^+;L^\infty)}\\ &\leq C2^{3j}\Big(\|e^{t\Delta}f\|_{L^2(\mathbb{R}^+;L^2)} + \|t^{\frac12}e^{t\Delta}f\|_{L^\infty(\mathbb{R}^+;L^2)}\Big)\|e^{t\Delta}g\|_{L^2(\mathbb{R}^+;L^2)}.\end{aligned}$$

We deduce that

$$E_j \leq C2^{3j}\|f\|_{\dot B^{-1}_{2,2}}\|g\|_{\dot B^{-1}_{2,2}}. \tag{7.28}$$

Let us observe that we also have

$$\begin{aligned} E_j &\leq C\Big(\|e^{t\Delta}f\|_{L^2(\mathbb{R}^+;L^\infty)} + \|t^{\frac12}e^{t\Delta}f\|_{L^\infty(\mathbb{R}^+;L^\infty)}\Big)\|e^{t\Delta}g\|_{L^2(\mathbb{R}^+;L^\infty)}\\ &\leq C\|f\|_{\dot B^{-1}_{\infty,2}}\|g\|_{\dot B^{-1}_{\infty,2}}.\end{aligned}$$

Using this estimate for high frequencies and (7.28) for low frequencies, we get, for any j_0 in $\mathbb{Z}$,

$$\begin{aligned} \|e^{t\Delta}fe^{t\Delta}g\|_E &= \sum_j 2^{-j}E_j\\ &\leq C\Big(\|f\|_{\dot B^{-1}_{2,2}}\|g\|_{\dot B^{-1}_{2,2}}\sum_{j\leq j_0}2^{2j} + \|f\|_{\dot B^{-1}_{\infty,2}}\|g\|_{\dot B^{-1}_{\infty,2}}\sum_{j\geq j_0}2^{-j}\Big)\\ &\leq C\Big(\|f\|_{\dot B^{-1}_{2,2}}\|g\|_{\dot B^{-1}_{2,2}}2^{2j_0} + \|f\|_{\dot B^{-1}_{\infty,2}}\|g\|_{\dot B^{-1}_{\infty,2}}2^{-j_0}\Big).\end{aligned}$$

Choosing j_0 such that

$$2^{3j_0} \sim \frac{\|f\|_{\dot B^{-1}_{\infty,2}}\|g\|_{\dot B^{-1}_{\infty,2}}}{\|f\|_{\dot B^{-1}_{2,2}}\|g\|_{\dot B^{-1}_{2,2}}}$$

gives the result. □

Conclusion of the proof of Proposition 7.3.1 Then, using (7.26) and (7.27), we infer that

$$\begin{aligned} \big\|\mathbb{P}(e^{t\Delta}u_0\cdot\nabla e^{t\Delta}u_0)\big\|_E &\leq \Big(\frac{A}{\varepsilon}\Big)^2\Big(\|f_P\|_{\dot B^{-1}_{\infty,2}}\|g_P\|_{\dot B^{-1}_{\infty,2}}\Big)^{\frac23}\Big(\|f_P\|_{\dot B^{-1}_{2,2}}\|g_P\|_{\dot B^{-1}_{2,2}}\Big)^{\frac13}\\ &\leq \Big(\frac{A}{\varepsilon}\Big)^2\varepsilon^{\frac43}\big(\varepsilon^2\Lambda^{-1}\big)^{\frac13}\\ &\leq A^2\Lambda^{-\frac13}.\end{aligned}$$

Thus if

$$\Lambda^{-\frac13} \leq C_0^{-1} \leq A^{-2}C_0^{-1}\exp(-C_0A^4)$$

the hypotheses of Theorem 7.2.1 □

Remark Let us point out the importance of the algebraic structure of a non linear term. A term like

$$\big(e^{t\Delta}u_0\big)^1\partial_2\big(e^{t\Delta}u_0\big)^1$$

will produce terms we can bound only in E by $A^2\Lambda^{\frac23}$.

7.4 The role of the special structure: a Navier-Stokes type equation which blows up

The incompressible Navier-Stokes system has three important features: the scaling invariance, the incompressibilty condition and the very special structure of the non linear term. This structure leads to energy estimate, but also appears in relations likes (7.26) and (7.27) which has been crucial for proving the global well posedness result of Theorem 7.3.1. The purpose of this section is to study a modifed system which has the first two properties (scaling invariance and divergence free condition) and with a different structure of the non linear term which will lead to blow up at finite time for some initial data that satisfies the hypotheses of Theorem 7.3.1,

$$\mathcal{E} \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^3\,,\ \xi_1\xi_2 < 0\,,\ \xi_2\xi_3 < 0\,|\xi_1| < \min\{|\xi_2|, |\xi_3|\}\} \tag{7.29}$$

and the matrix $q(\xi)$

$$q(\xi) \begin{pmatrix} \xi_2^2+\xi_3^2-\xi_1\xi_2-\xi_1\xi_3 & \xi_2^2+\xi_3^2-\xi_1\xi_2-\xi_1\xi_3 & \xi_2^2+\xi_3^2-\xi_1\xi_2-\xi_1\xi_3 \\ \xi_1^2+\xi_3^2-\xi_1\xi_2-\xi_2\xi_3 & \xi_1^2+\xi_3^2-\xi_1\xi_2-\xi_2\xi_3 & \xi_1^2+\xi_3^2-\xi_1\xi_2-\xi_2\xi_3 \\ \xi_1^2+\xi_2^2-\xi_1\xi_3-\xi_2\xi_3 & \xi_1^2+\xi_2^2-\xi_1\xi_3-\xi_2\xi_3 & \xi_1^2+\xi_2^2-\xi_1\xi_3-\xi_2\xi_3 \end{pmatrix} \tag{7.30}$$

Let us observe that

$$q(\xi) = |\xi| q(\xi)\, \mathbb{P}(\xi) \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

if $\mathbb{P}(\xi)$ denotes the matrix of the Leray projection of divergence free vector field in the Fourier space. Let us consider the following modified incompressible Navier-Stokes system.

$$(MNS) \left\{ \begin{array}{c} \partial_t - \Delta u = Q(u,u) \\ \operatorname{div} u = 0\,,\ u_{|t=0} = u_0 \end{array} \right. \qquad \text{with} \quad Q(u,u) \stackrel{\text{def}}{=} \sum_{k=1}^{3} q_{j,k}(D)(u^j u^k).$$

The main point of this modified Navier-Stokes system is the following property which plays a key role in the proof of blow up for finite time.

Proposition 7.4.1 *The coefficients of the matrix $q(\xi)$ are non negative.*

Proof. Let us first notice that on $\mathcal{E}$, $\xi_1 xi_3$ is positive.The components of the first line may be written

$$\xi_2^2+\xi_3^2-\xi_1\xi_2-\xi_1\xi_3 = \xi_2^2-\xi_1\xi_2+\xi_3(\xi_3-\xi_1)$$

which is also positive since either ξ_1 and ξ_3 are both positive, in which case $\xi_3 > \xi_1$, or they are both negative in which case $\xi_3 < \xi_1$. Thus the first line of the above matrix is clearly made of positive scalars. The fact that the terms of the second line are non negative is obvious due to the sign condition imposed on the components of ξ.

Similarly one has

$$\xi_1^2+\xi_2^2-\xi_1\xi_3-\xi_2\xi_3 = \xi_1^2+\xi_2^2-\xi_3(\xi_1+\xi_2)$$

and either $\xi_1 < 0$, $\xi_2 > 0$, $\xi_3 < 0$ and $\xi_1+\xi_2 > 0$, or $\xi_1 > 0$, $\xi_2 < 0$, $\xi_3 > 0$ and $\xi_1+\xi_2 < 0$. So the third line is also made of positive real numbers. □

Theorem 7.4.1 *Let us consider an initial data u_0 such that, for any $j \in \{1,2,3\}$, the component $\widehat{u}_0^j$ is a non negative function. Let us assume that for some j_0, we have*

$$u_0^{j_0}(\xi) \geq \widehat{v}_0(\xi) \geq 0 \quad \textit{with} \quad \operatorname{Supp}\ \widehat{v}_0 \subset \mathcal{C}_{r,R}. \tag{7.31}$$

where $\mathcal{C}_{r,R}$ is some ring of $\mathbb{R}^3$. Let us assume a positive real number m exists such that, for any ξ in the union of the iterated sum of Supp $\widehat{v}_0$,

$$q_{j_0,j_0}(\xi) \geq m|\xi|. \tag{7.32}$$

If the quantity

$$m\frac{r}{R}\|\widehat{v}_0\|_{L^1}$$

is large enough, then the unique solution to (MNS) associated with the initial date u_0 blows up for finite time.

Proof. As the positivity of the components of $\widehat{u}$ is preserved by the flow of (MSN), we have that

$$\begin{aligned} u^{j_0}(\xi) &= e^{-t|\xi|^2}u_0^{j_0}(\xi) + \sum_{k=1}^{3}\int_0^t e^{-t-t')|\xi|^2} q_{j_0,k}(\xi)\big(\widetilde{u}^{j_0}(t'\cdot) \star \widehat{u}^k(t',\cdot)\big)(\xi)dt' \\ &\geq e^{-t|\xi|^2}u_0^{j_0}(\xi) + \sum_{k=1}^{3}\int_0^t e^{-t-t')|\xi|^2} q_{j_0,j_0}(\xi)\big(\widetilde{u}^{j_0}(t'\cdot) \star \widehat{u}^k(t',\cdot)\big)(\xi)dt'. \end{aligned}$$

As $q_{j_0,j_0}(\xi)$ is non negative, we get that, for any t, $u^{j_0}(\xi) \geq \widehat{v}(\xi)$ where v is the solution of

$$\partial_t v - \Delta v = q_{j_0,j_0}(v^2) \quad \text{with} \quad v_{|t=0} = v_0.$$

We give here a variation of the proof of [50]. Let us define the sequence $(t_k)_{k\in\mathbb{N}}$ by

$$t_0 = 0 \quad \text{and} \quad t_k \stackrel{\text{def}}{=} \frac{1}{R^2}\sum_{\ell=1}^{k} 2^{-2\ell}.$$

We use denote by $\underline{T}$ its limit (which is $4/(3R^2)$). Let us define the sequence $(v_0^{(\ell)})_{\ell\in\mathbb{N}}$ by

$$v_0^{(1)} \stackrel{\text{def}}{=} \widehat{v}_0 \quad \text{and} \quad v^\ell \stackrel{\text{def}}{=} v_0^{(1)} \star v_0^{\ell-1}.$$

Let us notice that

$$\operatorname{Supp}\ v^{(\ell)} \subset \ell\mathcal{C}_{r,R} \quad \text{and} \quad \forall \xi \in \operatorname{Supp}\ v_0^{(\ell)}\,,\,,\ q(\xi) \geq mr\ell.$$

Let us make the following induction hypothesis for some sequence $(A_k)_{k\in\mathbb{N}}$ which will be choosen later on:

$$(H_k) \quad \forall t \in [2\underline{T}, t_k]\,,\ \widehat{v}(t,\xi) \geq A_k v_0^{(2^k)}(\xi).$$

Using the hypothesis on the support ofn $\widehat{v}_0$, we have that, for any t in $[0, 2T]$,

$$\widehat{v}(t,\xi) \geq e^{-t\xi|^2} \geq e^{-2\underline{T}R^2}\widehat{v}_0(\xi).$$

Thus, if we choose $A_0 = e^{2\underline{T}R^2}$, (H_0) is satisfied. Now let us assume (H_k). As q_{j_0,j_0} and $\widehat{v}(t,\cdot)$ are non negative functions, we have, for any $t \geq t_{k+1}$,

$$\begin{aligned}\widehat{v}(t,\xi) &= e^{-t|\xi|^2}\widehat{v}_0(\xi) + \int_0^t e^{-(t-t')|\xi|^2} q_{j_0,j_0}(\xi)\big(\widehat{v}(t',\cdot)\star\widehat{v}(t',\cdot)\big)(\xi)dt' \\ &\geq \Big(\int_t^{t_k} e^{-(t-t')|\xi|^2}dt'\Big)A_k^2 q_{j_0,j_0}(\xi)\big(v_0^{(2^k)}\star v_0^{(2^k)}\big)(\xi).\end{aligned}$$

By definition of the sequence $(v_0^{(\ell)})_{\ell\in\mathbb{N}}$ we get using (7.32),

$$\widehat{v}(t,\xi) \leq \Big(\int_{t_k}^t e^{-(t-t')2^{2k+2}R^2}dt'\Big)mr2^k v_0^{(2^{k+1})}.$$

As $t \geq t_{k+1}$, we have $t - \dfrac{1}{2^{2k+2}R^2} \geq t_k$. Thus we get

$$\begin{aligned}\widehat{v}(t,\xi) &\geq \Big(\int_{t-\frac{1}{2^{2k+2}R^2}}^t e^{-(t-t')2^{2k+2}R^2}dt'\Big)mr2^k v_0^{(2^{k+1})}(\xi) \\ &\geq 4e^{-1}\frac{r}{R^2}2^{-k}A_k^2 v_0^{(2^{k+1})}(\xi).\end{aligned}$$

Choosing

$$A_{k+1} \stackrel{\text{def}}{=} m_0 2^{-k}k^2 \quad \text{with} \quad \stackrel{\text{def}}{=} 4e^{-1}m\frac{r}{R^2}$$

gives (H_{k+1}). Let us compute A_k. By iteration, we find that

$$A_{k+1} = m_0^{2k+1}2^{-\sum_{\ell=0}^k \ell 2^{k-\ell}}A_0^{2^{k+1}}.$$

As $\displaystyle\sum_{\ell=0}^k \ell 2^{k-\ell} = 2^k$, we get that $A_{k+1} = 4\big(m_0 e^{-\frac{8}{3}}\big)^{2^{k+1}}$. As $\widehat{v}_0^{(\ell)}\|_{L^1} = \|\widehat{v}_0\|_{L^1}$, we infer that, for any t in $[\underline{T};2\underline{T}]$, we have

$$\widehat{v}(t,\xi) \geq 4\big(m_0 e^{-\frac{8}{3}}\|\widehat{v}_0\|_{L^1}\big)^{2^{k+1}}$$

Thus, if $m_0 e^{-\frac{8}{3}}\|\widehat{v}_0\|_{L^1}$ is large enough, then $\lim\limits_{k\to\infty} A_k = +\infty$ and thus $\|\widehat{u}(t,\cdot)\|_{L^1}$ blows up for finite time and the theorem is proved. □

The purpose of this section is the proof of the following theorem.

Theorem 7.4.2 *Let ϕ be a function in $\mathcal{S}(\mathbb{R}^3)$ such that its Fourier transform is non negative, even and has its support in the region $\mathcal{E}$. Let $P = (\varepsilon,\Lambda)$ be in $]0,1]\times[1,\infty[$ such that $\varepsilon\Lambda$ is small enough,andy A a positive real number. Let us consider the initial data*

$$u_0(x) \stackrel{\text{def}}{=} A(\partial_2\varphi_P(x), -\partial_1\varphi_P(x), 0) \quad \textit{with} \quad \varphi_P(x) \stackrel{\text{def}}{=} \frac{1}{\varepsilon\Lambda}\cos\Big(\frac{x_3}{\varepsilon}\Big)(\partial_1\phi)(x_1,\Lambda x_2,x_3).$$

If

$$\Lambda^{\frac{4}{3}} \leq C_0^{-1}A^{-2}\exp(-C_0A^4) \quad \textit{and} \quad \frac{A}{\varepsilon} \quad \textit{large enough,}$$

then u_0 satisfies the hypothesis of Theorem 7.2.1 and the local solution to (MNS) blows up for finite time.

Proof. We have

$$u_0(x) = \frac{A}{\varepsilon}\cos\Big(\frac{x_3}{\varepsilon}\Big)\Big((\partial_1\partial_2\varphi)(x_1,\Lambda x_2,x_3), -\frac{1}{\Lambda}(\partial_1^2\varphi)(x_1,\Lambda x_2,x_3)\Big).$$

First of all, let us check that $\widehat{u}_0^j$ are non negative functions for $j \in \{1,2,3\}$ and that their support intersects the set $|\xi_j| \geq 1/2$. Indeed we have

$$\widehat{u}_0(\xi) = \frac{A}{2\varepsilon\Lambda^2}\Big(\sum_{\pm}(-\xi_1\xi_2)\varphi\Big(\xi_1,\frac{\xi_2}{\Lambda},\xi_3 \pm \frac{1}{\varepsilon}\Big), \sum_{\pm} -\xi_1^2\varphi\Big(\xi_1,\frac{\xi_2}{\Lambda},\xi_3 \pm \frac{1}{\varepsilon}\Big)\Big)$$

which gives the non negativity of the Fourier transform of the components of u_0. Then let us consider a point ω_0 such that

$$-\omega_0^1\omega_2\varphi(\omega_0) \geq 2c_0$$

and a real number ε_0 such that

$$\forall \xi \in B(\omega_0,\varepsilon_0), -\xi_1\xi_2\varphi(\xi) \geq c_0.$$

Let us define v_0 by

$$\widehat{v}_0(\xi \stackrel{\text{def}}{=} \frac{A}{2\varepsilon\Lambda^2}w_0\Big(\xi_1,\frac{\xi_2}{\Lambda},\xi_3 \pm \frac{1}{\varepsilon}\Big) \quad \text{with} \quad w_0(\eta) \stackrel{\text{def}}{=} \mathbf{1}_{B(\omega_0,\varepsilon_0)}(\eta)\eta_1\eta_2\widehat{\varphi}(\eta).$$

As we have

$$\begin{aligned}\|\widehat{v}_0\|_{L^1} &\geq c_0\frac{A}{\varepsilon\Lambda}\mu\Big\{\xi \in \mathbb{R}^3 \;/\; \Big(\xi_1,\frac{\xi_2}{\Lambda},\xi_3\Big) \in B(\omega_0,\varepsilon_0)\Big\}\\ &\geq \mu(B(\omega_0,\varepsilon_0)c_0\frac{A}{\varepsilon},\end{aligned}$$

we infer that if A/ε is large enough, the hypotheses of Theorem 7.4.1 are satisfied and thus the theorem is proved. □

Examen du 25 janvier 2006
M2"Théorie des équations d'évolution"

On utilise les notations standard des notes de cours
Cet examen en comporte pas de questions de cours

Exercice

Soit $\alpha \in]0, d[$, *on pose, pour* $x \in \mathbb{R}^d$, $r_\alpha(x) = |x|^{-\alpha}$.

1) En écrivant, pour une fonction χ de $\mathcal{D}(\mathbb{R}^d)$ valant 1 près de 0, que

$$r_\alpha = \chi r_\alpha + (1 - \chi) r_\alpha,$$

démontrer que $\lim_{j \to -\infty} \dot{S}_j r_\alpha = 0$ dans $\mathcal{S}'$ et que $\Delta_0 r_\alpha \in L^1$.

2) En déduire que $r_\alpha \in \dot{B}^{d-\alpha}_{1,\infty}$.

3) Démontrer que l'application

$$\begin{cases} \mathcal{S}(\mathbb{R}^d) & \to & \mathbb{R} \\ \phi & \mapsto & \displaystyle\int_{\mathbb{R}^d} \frac{1}{|x|^\alpha} \phi(x) dx \end{cases}$$

se prolonge en une forme linéaire continue sur $\dot{B}^{\alpha-d}_{\infty,1}$.

4) En déduire que, pour tout $s \in]0, d/2[$, il existe C telle que

$$\forall u \in \dot{H}^s\,, \ \int_{\mathbb{R}^d} \frac{u^2(x)}{|x|^{2s}} dx \leq C \|u\|^2_{\dot{H}^s}.$$

5) L'inégalité ci-dessus est-elle vraie pour $s = d/2$?

TSVP

Problème

Soit $d \geq 2$, on considère $\mathbf{P}$ la projection orthogonale de $(L^2(\mathbb{R}^d))^d$ sur $\mathcal{H}$ l'ensemble des chams de vecteurs à diverngence nulle de $(L^2(\mathbb{R}^d))^d$.

1) Calculer $\mathbf{P}$ en terme de transformée de Fourier et démontrer que c'est un multiplicateur de Fourier homogène de degré 0.

2) Démontrer que, pour s positif ou nul, $\mathbf{P}_{|(H^s(\mathbb{R}^d))^d}$ est la projection sur $\mathcal{H}^s \stackrel{\text{déf}}{=} (H^s(\mathbb{R}^d))^d \cap \mathcal{H}$.

3) Démontrer que, pour s négatif, $\mathbf{P}$ s'étend à$(H^s(\mathbb{R}^d))^d$ et que cette extension est la projection orthogonale de $(H^s(\mathbb{R}^d))^d$ sur l'ensemble des chams de vecteurs à divergnece nulle de $(H^s(\mathbb{R}^d))^d$.

On suppose dorénavant que $d = 3$.

4) Soit v une solution de $L^2([0,T];H^1) \cap C([0,T];\mathcal{S}')$ de

$$\begin{cases} \partial_t v - \Delta v + \mathbf{P}(v \cdot \nabla v) = 0 \\ v_{|t=0} = v_0 \\ \operatorname{div} v_0 = 0. \end{cases}$$

Démontrer que v appartient à $L^1([0,T];B^{\frac{3}{2}}_{2,1})$.

5) En déduire que, pour tout $x_0 \in \mathbb{R}^3$, il existe une fonction x continue de $[0,T]$ à valeurs dans $\mathbb{R}^3$ telle que

$$x(t) = x_0 + \int_0^t v(t', x(t'))dt'.$$

On veillera à expliquer pourquoi $\displaystyle\int_0^t v(t', x(t'))dt'$ a un sens.

6) Interpréter le résultat en terme de mécanique des fluides.

EXAMEN DU 8 JANVIER 2007
"THÉORIE DES ÉQUATIONS D'ÉVOLUTION"

Première partie : Questions de cours

1) Énoncer et démontrer le théorème d'inclusion de Sobolev (Théorème 2.2.1 des notes).

2) Soit S_j l'opérateur défini au chaiptre 7 en théorie de Littlewood-Paley, c'est-à-dire $\mathcal{F}S_j a = \chi(2^{-j}\xi)\widehat{a}$ où χ est une fonction de $\mathcal{D}(\mathbb{R}^3)$ valan 1 près de 0. Démontrer en détail que, pour toute fonction ϕ de $\mathcal{S}(\mathbb{R}^d)$, on a

$$\lim_{j\to\infty} S_j\phi = \phi \quad \text{dans l'espace } \mathcal{S}(\mathbb{R}^d).$$

Avertissement : Cette partie doit être traitée en une heure. Les copies relatives à cette partie seront ramassées à la fin de cette heure et la seconde partie du sujet distribuée.

EXAMEN DU 8 JANVIER 2007
"THÉORIE DES ÉQUATIONS D'ÉVOLUTION"

Seconde partie

EXERCICE I

Dans tout cet exercice, nous considérerons une partition de l'unité dyadique, c'est-à-dire une fonction $\varphi \in \mathcal{D}(\mathbb{R}^d \setminus \{0\})$ *telle que l'on ait*

$$\forall \xi \in \mathbb{R}^d \setminus \{0\}, \ \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1.$$

On notera $\dot{\Delta}_j$ *l'opérateur défini sur* $\mathcal{S}'(\mathbb{R}^d)$ *par*

$$\dot{\Delta}_j u = \mathcal{F}^{-1}(\varphi(2^{-j}\cdot)\widehat{u}).$$

On définit alors

$$\|a\|_{\dot{B}^s_{2,1}} = \sum_{j \in \mathbb{Z}} 2^{js} \|\dot{\Delta}_j a\|_{L^2}.$$

1) Démontrer qu'il existe une constante C telle que, pour toute fonction f de $H^1(\mathbb{R}^d)$, on ait

$$\|f\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \leq C\|f\|_{L^2}^{\frac{1}{2}} \|\nabla f\|_{L^2}^{\frac{1}{2}},$$

Indication : On distinguera le cas des hautes fréquences de celui des basses fréquences.

On se place dans le cas où $d = 1$. *On considère deux réels strictement positifs* R *et* λ *tels que* $R \leq \dfrac{1}{\lambda} \cdot$ *Soit* $\chi_{\lambda,R}$ *la fonction paire de* $\mathbb{R}$ *dans* $\mathbb{R}$ *définie par* $\chi_{\lambda,R} \equiv 0$ *pour* $x \geq R + 1/\lambda$ *et* $\chi_{\lambda,R} \equiv 1$ *sur* $[0, R]$ *et*

$$\chi_{\lambda,R}(x) = -\lambda\left(x - R - \frac{1}{\lambda}\right) \quad \text{sur} \quad \left[R, R + \frac{1}{\lambda}\right] \cdot$$

2) Calculer $\|\chi_{\lambda,R}\|_{L^2}$ et $\|\nabla \chi_{\lambda,R}\|_{L^2}$.

3) En déduire l'existence d'une constante C strictement positive telle que, pour tout λ et R tels que $\lambda R \leq 1$, on ait

$$\|\chi_{\lambda,R}\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \leq C.$$

EXERCICE II

Soit a une fonction appartenant à $H^s(\mathbb{R}^d)$ avec $s > d/2$. On rappelle que $B^0_{\infty,\infty}$ est l'espace des distributions tempérées u telles que

$$\|u\|_{B^0_{\infty,\infty}} \stackrel{\text{déf}}{=} \sup_j \|\Delta_j u\|_{L^\infty} < \infty$$

1) Démontrer que L^∞ est inclus dans $B^0_{\infty,\infty}$ et qu'il existe une constante C telle que

$$\|u\|_{B^0_{\infty,\infty}} \leq C\|u\|_{L^\infty}$$

2) Démontrer que, pour tout entier positif j, on a

$$\|a\|_{L^\infty} \leq (j+1) \sup_{j\geq -1} \|\Delta_j a\|_{L^\infty} + C2^{-j\left(s-\frac{d}{2}\right)}\|a\|_{H^s}.$$

3) En déduire que

$$\|a\|_{L^\infty} \leq C\|a\|_{B^0_{\infty,\infty}} \log\left(e + \frac{\|a\|_{H^s}}{\|a\|_{B^0_{\infty,\infty}}}\right).$$

4) Démontrer que, pour tout nombre réel strictement positif α, la fonction

$$x \longmapsto x \log\left(e + \frac{\alpha}{x}\right)$$

est croissante sur $]0, \infty[$. En déduire que

$$\|a\|_{L^\infty} \leq C\left(\|a\|_{B^0_{\infty,\infty}} + 1\right) \log\left(e + \|a\|_{H^s}\right).$$

EXERCICE III

Soit un système quasi-linéaire symétrique

$$(S)\begin{cases} \partial_t U + \displaystyle\sum_{k=1}^d A_k(U)\partial_k U &= 0 \\ U_{|t=0} &= U_0 \end{cases}$$

avec

$$A_k(U) = A_k^{(0)} + \sum_{j=1}^N A_k^j U_j.$$

On considère une solution U associée à une donnée initale U_0 appartenant à $H^s(\mathbb{R}^d)$ avec $s > d/2 + 1$ fournie par le Théorème 8.2.1. On suppose que le temps maximal d'existence $T^\star$ de cette solution est fini.

1) Démontrer que l'on a, pour tout $t < T^\star$,

$$\|\nabla U(t)\|_{L^\infty} \leq C(\|\nabla U(t)\|_{B^0_{\infty,\infty}} + 1) \log\left((e + \|U_0\|_{H^s}) \int_0^t \|\nabla U(t')\|_{L^\infty} dt'\right).$$

2) Démontrer que $\displaystyle\int_0^{T^\star} \|\nabla U(t)\|_{B^0_{\infty,\infty}} dt = \infty.$

EXAMEN DU 7 JANVIER 2008
"THÉORIE DES ÉQUATIONS D'ÉVOLUTION"

Première partie : Questions de cours

1) Énoncer et démontrer les inégalités de Sobolev (Théorème 2.1.1 des notes de cours)

2) Énoncer et démontrer le théorème spectral pour le Laplacien (Théorème 3.1.2 des notes de cours).

Avertissement : Cette partie doit être traitée en une heure. Les copies relatives à cette partie seront ramassées à la fin de cette heure et la seconde partie du sujet distribuée.

EXAMEN DU 7 JANVIER 2008
"THÉORIE DES ÉQUATIONS D'ÉVOLUTION"

Seconde partie

EXERCICE I

On rappelle les notations du cours. Dans tout cet exercice, nous considérerons une partition de l'unité dyadique, c'est-à-dire une fonction $\varphi \in \mathcal{D}(\mathbb{R}^d \setminus \{0\})$ et une fonction χ de $\mathcal{D}(\mathbb{R}^d \setminus \{0\})$ telle que l'on ait

$$\forall \xi \in \mathbb{R}^d\,,\ \chi(\xi) + \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1.$$

On notera Δ_j l'opérateur défini pour$j \geq 0$ sur $\mathcal{S}'(\mathbb{R}^d)$ par

$$\Delta_j u = \mathcal{F}^{-1}(\varphi(2^{-j}\cdot)\widehat{u}).$$

On posera $\Delta_{-1}u = \mathcal{F}^{-1}(\chi\widehat{u})$ et $\Delta_j = 0$ pour $j \leq -2$.

1) Démontrer l'existence d'un entier positif N_0 tel que $2^{N_0}\,\mathrm{Supp}\ \chi + \mathrm{Supp}\ \varphi$ soit une couronne.

Dans la suite de l'exercice, on fixe un tel entier N_0 et on pose

$$T_a b = \sum_j S_{j-N_0}\Delta_j u.$$

2) Soit ρ appartenant à l'intervalle $]0,1[$. Démontrer qu'il existe une constante C telle que, l'on ait, pour toute fonction b dans $B^\rho_{\infty,\infty}$, tout entier j, et toute fonction u de L^2,

$$\|\Delta_j(S_{j-N_0}bu) - S_{j-N_0}b\Delta_j u\|_{L^2} \leq C2^{-j\rho}\|b\|_{B^\rho_{\infty,\infty}}\|u\|_{L^2}.$$

3) Soit s un réel tel que $s+\rho$ soit positif. Démontrer l'existence d'une constante C que pour toute fonction a de L^∞ et toute fonction b de $B^\rho_{\infty,\infty}$, on ait

$$\Big\|T_aT_bu - \sum_{j,j'} S_{j-N_0}aS_{j'-N_0}b\Delta_j\Delta_{j'}u\|_{H^{s+\rho}} \leq C\|a\|_{L^\infty}\|b\|_{B^\rho_{\infty,\infty}}\|u\|_{H^s}.$$

4) En déduire que si s est un réel tel que $s+\rho$ soit positif, alors il existe une constante C tel que

$$\|T_aT_bu - T_bT_au\|_{H^{s+\rho}} \leq C\Big(\|a\|_{L^\infty}\|b\|_{B^\rho_{\infty,\infty}} + \|a\|_{B^\rho_{\infty,\infty}}\|b\|_{L^\infty}\Big)\|u\|_{H^s}.$$

EXERCICE II

On considère une solution u de Kato de l'équation de Navier-Stokes

$$\left\{\begin{array}{rcl} \partial_t u + v\cdot\nabla u - \nu\Delta u & = & -\nabla p \\ \operatorname{div} u & = & 0 \\ u_{|t=0} & = & u_0 \in \mathcal{V}_\sigma^{\frac12} \\ u_{|\partial\Omega} & = & 0. \end{array}\right.$$

donné par le Théorème 5.3.1 des notes de cours. On suppose que u_0 appartient à $\mathcal{V}_\sigma$.

1) Démontrer que l'application bilinéaire

$$(v,w) \longmapsto \mathbf{P}(v\cdot\nabla w)$$

envoie continûment $\mathcal{V}_\sigma \times \mathcal{V}_\sigma$ dans $\mathcal{V}_\sigma^{-\frac12}$.

2) Soit $(e_j)_{j\in\mathbb{N}}$ la suite des vecteurs propres de l'opérateur de Stokes. Démontrer l'existence d'une constante C, pour tout couple (v,w) de $\mathcal{V}_\sigma \times \mathcal{V}_\sigma$, il existe une suite $(c_j)_{j\in\mathbb{N}}$ de nombre positifs telle que

$$\sum_j c_j^2 = 1 \quad \text{et} \quad |\langle v\cdot\nabla w, e_j\rangle| \leq C c_j \lambda_j^{\frac12} \|v\|_{\mathcal{V}_\sigma} \|w\|_{\mathcal{V}_\sigma}.$$

3) Démontrer que l'application bilinéaire, qui à un couple (v,w) de $L^\infty([0,T];\mathcal{V}_\sigma) \times L^\infty([0,T];\mathcal{V}_\sigma)$ associe le solution de problème de Stokes d'évolution

$$\left\{\begin{array}{rcl} \partial_t B - \Delta B & = & v\cdot\nabla w - \nabla p \\ B_{|t=0} & = & 0 \end{array}\right.$$

envoie continûment $L^\infty([0,T];\mathcal{V}_\sigma) \times L^\infty([0,T];\mathcal{V}_\sigma)$ dans $L^\infty([0,T];\mathcal{V}_\sigma)$ et que

$$\|B\|_{L^\infty([0,T];\mathcal{V}_\sigma)} \leq C T^{\frac14} \|v\|_{L^\infty([0,T];\mathcal{V}_\sigma)} \|w\|_{L^\infty([0,T];\mathcal{V}_\sigma)}.$$

4) Soit $T^\star$ le temps maximal d'existence de la solution de Kato. Démontrer l'existence d'un temps maximal $\widetilde{T}^\star \leq T^\star$ d'existence d'une solution bornée en temps à valeurs $\mathcal{V}_\sigma$.

5) Démontrer que $\widetilde{T}^\star = T^\star$.