Université Pierre et Marie Curie *Friday, January 13, 2017*
Master Mathématiques & Applications
Cours: Variational Approximations of PDE's

Examen

Only the paper documents are authorized

Length: 3 hours

First part — Finite element approximation of a parameter dependent elliptic problem .

Let X be a closed subspace of the Sobolev space $H^1(\Omega)$ over Ω being a bounded convex polygonal domain in $\mathbb{R}^2$ or $\mathbb{R}^3$. Let $\mathcal{D}$ a given set of parameters.

We consider the following problem : given $\mu \in \mathcal{D}$, find $u(\mu) \in X$ such that

$$\forall v \in X, \quad a(u(\mu), v; \mu) = (f, v), \tag{1}$$

where a is a bilinear form, continuous and coercive over X that depends additionally on a parameter $\mu \in \mathcal{D}$ and $f \in L^2(\Omega)$.
Let us assume that the continuity and coercivity (or ellipticity) constants do not depend on μ.

1) Express in a mathematical way the above statements.

2) Provide two simple examples of problems that fit in the above frame, one when $X = H^1_0(\Omega)$ and the other when $X = H^1(\Omega)$.

3) Let be given a family of finite element approximations of the above problems parametrized with h, with the associated discrete spaces X_h. Provide two simple examples of simplicial Lagrange finite elements approaches that provide an error

$$\|u(\mu) - u_h(\mu)\|_X \leq ch \tag{2}$$

for the first method and

$$\|u(\mu) - u_h(\mu)\|_X \leq ch^2 \tag{3}$$

for the second method. Write the discrete problems that the approximation $u_h(\mu)$ solve. Explain under which assumption over the solution $u(\mu)$ such an approximation holds.

4) Under which hypothesis and how is it possible to get an improved estimate

$$\|u(\mu) - u_h(\mu)\|_{L^2(\Omega)} \leq ch^2 \tag{4}$$

for the first method and

$$\|u(\mu) - u_h(\mu)\|_{L^2(\Omega)} \leq ch^4 \tag{5}$$

we expect here that the proof is provided.

Second part — Reduced basis approximation of a parameter dependent elliptic problem .

Let us consider the manifold of discrete solutions

$$\mathcal{M}_h = \{u_h(\mu),\ \mu \in \mathcal{D}\} \tag{6}$$

Let us assume we have been able to select a sequence of parameters $\{\mu_i\}_i$ such that the discrete spaces

$$X_N = \text{Span}\{u_h(\mu_i),\ 1 \leq i \leq N\} \tag{7}$$

is a good approximation of $\mathcal{M}_h$ in the sense that

$$\max_{\mu} \min_{v_N \in X_N} \|u_h(\mu) - v_N\|_X \leq d_N \tag{8}$$

where d_N converges to 0 very fast.

5) Consider the reduced basis approximation : Find $u_N(\mu) \in X_N$ such that

$$\forall v_N \in X_N, \quad a(u_N(\mu), v_N; \mu) = (f, v_N), \tag{9}$$

Provide the numerical analysis of this problem and express the error $\|u_h(\mu) - u_N(\mu)\|_X$ and the error $\|u(\mu) - u_N(\mu)\|_X$.

6) Explicit the construction of the matricial form of the problem and explain why, under the hypothesis that there exists P continuous bilinear forms a_p and P functions $\mu \mapsto g_p(\mu)$ such that

$$a(u, v; \mu) = \sum_{p=1}^{P} g_p(\mu) a_p(u, v) \tag{10}$$

the construction of the reduced basis matrix can be done with a smaller complexity in an offline/online procedure. The above hypothesis is known as the affine decomposition.

Third part — Two grids method.

We assume that we have a second basis set in X_N : $X_N = \text{Span}\{\xi_i,\ 1 \leq i \leq N\}$ composed of functions ξ_i that are

— orthonormal in $L^2(\Omega)$, i.e. $\int_\Omega \xi_i \xi_j = \delta_{ij}$ where δ is the Kronecker symbol,

— orthogonal in $H^1(\Omega)$, i.e. $\int_\Omega \nabla\xi_i \nabla\xi_j = 0$ whenever $i \neq j$.

7) Propose a way to get such a basis in X_N based on the resolution of eigenvalue problem.

We assume now that the above affine decomposition hypothesis is not satisfied. In order to circumvent the complexity of the implementation we assume now that we pick two values of the finite element discretization. Still the same value h as above and a coarser value H that is such that $H^2 \simeq h$ (about the same size, meaning that H is much larger than h). Explain why the statement "The solution $u_H(\mu)$ is much more rapid to evaluate, but the accuracy is much degraded" holds.

8) Let us set, for any $\mu \in \mathcal{D}$ a new approximation

$$u_{h,H,N}(\mu) = \sum_{i=1}^{N} (u_H(\mu), \xi_i)_{L^2(\Omega)} \xi_i. \tag{11}$$

Explain why this approximation depends on h, H, N and state in which discrete space it belongs to (among X_H, X_h). Explain how and offline/online algorithm can provide a complexity of the online process that does not scale with h.

9) By using the above orthogonalities assumed over the ξ_i's and using the improved L^2 estimate proven in question 4) show that

$$\|u(\mu) - u_{h,H,N}(\mu)\|_X \leq c(h + H^2 + d_N) \tag{12}$$

in the case of the first finite element method and

$$\|u(\mu) - u_{h,H,N}(\mu)\|_X \leq c(h^2 + H^4 + d_N) \tag{13}$$

at least if some regularity hypothesis holds. The statement on this hypothesis should be explicitly formulated.

Fourth part — generalization to the Legendre spectral approximation.

10) State the generalization of this two grids method to the case of Legendre spectral approximation based on two possible polynomial degree approximations $m << M$, we can assume here that $\Omega =]-1, 1[^d$.