Université Pierre et Marie Curie
Master Mathématiques & Applications
Tuesday, January 12, 2016
Cours: Variational Approximations of PDE's

Examen

Only the notes corresponding the the course and the book "Bernardi-Maday-Rapetti" are authorized

Length: 3 hours

You may accept the results to be proven in one question in order to skip to another one

First part

Let Ω be a bounded convex polygonal domain in $\mathbb{R}^d$, $d = 2$ or 3, Let us consider the biharmonic problem : Find u such that

$$\Delta^2 u = f$$

where f is given in $L^2(\Omega)$. This problem is complemented with the additional boundary conditions

$$u = g_1, \frac{\partial u}{\partial n} = g_2, \text{over } \partial\Omega$$

1) Assume first that $g_1 = g_2 = 0$ show that this problem is equivalent to find $u \in H_0^2(\Omega)$ such that

$$\forall w \in H_0^2(\Omega); \quad \int_\Omega \Delta u \Delta w = \int_\Omega f w$$

where you shall remind what the space H_0^2 stands for.

2) Assume first the lemma holds

Lemma There exists a constant $C > 0$ such that $\forall \varphi \in H_0^2(\Omega)$

$$C\|\varphi\|^2_{H^2(\Omega)} \leq \int_\Omega [\Delta\varphi]^2 \tag{1}$$

show that the above homogeneous harmonic problem is well posed

3) State the precise trace results from $H^2(\Omega)$ into $H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$ for the trace of the functions and the normal derivatives.

4) State and prove the correct result of existence, uniqueness and stability of the full bi-harmonic problem in the non homogeneous case where g_1 and g_2 are not zero anymore.

5) Propose in a precise way a conforming discretization of the above problem, based on a series of discrete spaces denoted as X_δ. The solution is denoted as u_δ

6) prove that

$$\|u - u_\delta\|_{H^2} \leq C \inf_{w_\delta \in X_\delta} \|u - w_\delta\|_{H^2}$$

7) propose a finite element method in the case $d = 2$ that fits to this general framework and state the error estimate

8) propose an improved error estimate for $\|u - u_\delta\|_{L^2}$

9) propose a spectral approximation (with no numerical integration) in $d = 2$ (Ω is a square) or $d = 3$ (Ω is a cube) and indicate (without proof) what is the error between u and $u - N$, where N stands for the degree of the spectral approximation.

Second part

We want here to consider the problem in two steps where we find (u, v) such that

$$-\Delta u = v$$

$$-\Delta v = f$$

$$u = g_1, \frac{\partial u}{\partial n} = g_2, \text{over } \partial\Omega$$

10) We start over with the assumption that $g_1 = g_2 = 0$ that will hold untill question 19). Prove that the above problem is equivalent to find (u, v) in $H_0^1(\Omega) \times H^1(\Omega)$ such that

$$\forall w \in H^1(\Omega), \quad \int_\Omega \nabla u \nabla w - \int_\Omega vw = 0$$

$$\forall w \in H_0^1(\Omega), \quad \int_\Omega \nabla v \nabla w = \int fw$$

11) By denoting $a(v, w) = \int_\Omega vw$ and $b(u, w) = \int_\Omega \nabla u \nabla w$ write the problem as a problem in a mixed formulation

12) What are the hypothesis that should be done on a and b so that this problem is well posed

13) prove the inf-sup condition on the bilinear form b

14) prove the ellipticity condition over a on the kernel of b

15) propose a (mixed) discretization of this formulation

16) state the hypothesis on this discretization so that the discrete problem is well posed

17) what is the interest of the approach in the finite element framework

18) propose an example of a lagrangian simplicial finite element discretization of this problem

19) provide some elements for the numerical analysis leading to an error estimate between u and u_h and v and v_h

Third part

The solution procedure, in particular, in the case where g_1 and g_2 are not zero is a little bit complex so we propose here to replace the initial problem by

$$-\Delta u = v$$

$$-\Delta v = f$$

$$u = g_1 \text{over } \partial\Omega$$

$$v = \gamma_2 \text{over } \partial\Omega$$

20) explain why this problem is more simple to handle

21) propose a spectral discretization of this problem when Ω is a square

22) what do you propose is Ω still in $d = 2$ is not a square

23) Explain the mapping that can be constructed from the data of f, g_1, γ_2 to the value of $\frac{\partial u}{\partial n}$ over $\partial\Omega$ by solving the above problem

24) assume $f = 0, g_1 = 0$ are given, this provides a mapping A that to any γ_2 associates $g_2 = \frac{\partial u}{\partial n}$. Prove that A is invertible

25) propose an iterative scheme that allows to solve the original problem in the second part by solving only problems like in the third part.