Université Pierre et Marie Curie
Master Mathématiques & Applications
Cours: Variational Approximations of PDE's

Wednesday, May 11, 2016

Examen

Only the notes corresponding the course notes and associated book are authorized

Length: 3 hours

You may accept the results to be proven in one question in order to skip to another one

First part

Let Ω be an open domain in $I\!R^1$, $I\!R^2$ or $I\!R^3$. We consider the problem to find a solution u to the problem

$$\begin{array}{rcl} -\nabla(a\nabla u) + \lambda u & = & f, \quad \text{in} \quad \Omega \\ u & = & 0 \quad \text{over} \quad \partial\Omega \end{array}$$

where a is a regular given function satisfying $a(x) \geq 1$ and λ is a given real number.

1/ Write the previous problem in variational formulation
2/ Prove that this problem is well posed in $H^1_0(\Omega)$ when $\lambda \geq 0$
3/ By referring to the Poincaré's inequality, prove that the problem is still well posed in $H^1_0(\Omega)$ for $\lambda \geq \gamma$ where $\gamma < 0$ is to be precised. What is the value of γ when $\Omega =]-1,1[$.
4/ Prove that, under the same condition, the problem is also well posed if a is only in L^∞
5/ Let us assume that we are now in one dimension with $\Omega =]-1,1[$. Let us assume that $a = 1$ for $x \leq 0$ and $a = 2$ for $x > 0$, and assume in addition that $\lambda = 0$ and $f = 1$. Give the explicit form for the solution u. Precise its regularity in the class $\mathcal{C}^n$.

In the general case where $\Omega =]-1,1[^d$, and a is constant over $\Omega^- =]-1,0[\times]-1,1[^{d-1}$ and over $\Omega^+ =]0,1[\times]-1,1[^{d-1}$, it will be admitted that for regular right hand sides f the maximal regularity of u is $H^{3/2}(\Omega)$ and is not attained (i.e. u may belong to H^σ with $\sigma < 3/2$). However the solution is more regular, locally, over each subdomain Ω^- and Ω^+.

In what follows, we shall assume that $a = 1$ over Ω^- and $a = 2$ over Ω^+

Spectral discretizations

6/ Propose a spectral discretization in $I\!P_N(\Omega)$ for the previous problem without and with numerical integration.
7/ Prove that the problem without numerical integration is well posed under the same condition over λ as for the continuous problem
8/ Give a sufficient condition over λ for the problem with numerical integration to be well posed
9/ Provide an error estimate for the spectral discretization with numerical integration (assuming the H^σ-regularity of u as above).
10/ We now propose a spectral element approximation over two subdomains : Ω^- and Ω^+. The approximation is thus based on a subspace X_N of

$$Y_N = \{v_N \in L^2(\Omega), \quad v_{N|\Omega^-} \in I\!P_N(\Omega^-), v_{N|\Omega^+} \in I\!P_N(\Omega^+)\}$$

What is the exact definition of X_N.

11/ Provide the discrete formulation with and without numerical integration for this spectral element approximation.

12/ Assuming that $u_{|\Omega_k} \in H^r(\Omega_k)$ for $k = 1, 2$ with $r \geq 2$, provide the numerical analysis of the above discretizations (without and with numerical integration).

Finite element discretization

13/ We are back in 1 dimension and we propose a "triangulation " of Ω with equal segments of size $h = 2/(P+1)$. By recalling the exact solution computed in the first part, prove that the $\mathbb{P}_1$ finite element approximation (best fit) has, in the H^1-norm, a different behavior according to the parity of the number of points used in the triangulation. The precise estimate of the behavior of the convergence rate is expected here.

14/ Following the intuition that the previous question provides, what do you suggest for designing the best mesh in case of dimension 2 and 3. Write the Galerkin $\mathbb{P}_1$ finite element approximation of the problem above.

15/ In this favorable case, perform the numerical analysis of the $\mathbb{P}_1$ finite element approximation of the problem above in dimension equal to 2 ... this analysis will involve Clément-type discretization operator the definition of which should be recalled.