Stock Price Pinning near Option Expiration Dates

Marco Avellaneda, New York University
Gennady Kasyan, New York University
Michael D. Lipkin, Katama Trading & Columbia University

George Papanicolaou Conference
Paris, December 1st 2011
Summary

Empirical evidence of stock pinning (Ni-Pearson-Poteshman, 2003)

Linear price-impact model (Avellaneda & Lipkin 2002)

Non-linear price-impact model (Ave., Kasyan & Lipkin, 2007)

Numerical simulation of pinning for different price-impact functions

Phase transition at $p=1/2$

Rigorous mathematical proofs for the different regimes.

Bibliography
Pinning on Option Expiration Dates

Stock B pinned
Stock A did not

Option expiration
Friday, (3rd Friday of
the month).
KO: Sep 18 to Oct 17 2003

10/17 Close=$45.05
Statistical Evidence of Pinning

Stock Price Clustering on Option Expiration Dates, Preprint, June 26, 2003

Authors: Sophie Xiaoyan Ni, Neil Pearson and Allen M. Posheshman

(U. of Illinois Urbana-Champaign)

Data 1. Ivy DB (OptionMetrics)
Jan 1996, Sep 2002: All stocks traded in US exchanges
All options traded in US exchanges
End of day bid-ask quotes, volume, open interest

2. CBOE Statistics
Open interest and trading volume, Jan 1996 to Dec 2001
4 Investor Categories: Market Makers, Firm Prop Traders,
Large Firm Clients, Discount Firm Clients
The U. Illinois Urbana study

- At least 80 expiration dates
- 4,395 optionable stocks on at least one date
- 184,449 optionable stock-expiration pairs
- 12,001 non-optionable stocks on at least one date
- 417,007 non-optionable stock-expiration pairs
Percentage of non-optionable stocks closing within $0.25 of an integer multiple of $5

Relative Trading Date from Option Expiration Date

(Courtesy: Ni, Pearson & Poteshman)
Percentage of **optionable** stocks closing within $0.25 of an integer multiple of $5

Relative Trading Date from Option Expiration Date

(Courtesy: Ni, Pearson & Poteshman)
Percentage of optionable stocks closing within $0.25 of a strike price

Relative Trading Date from Option Expiration Date

(Courtesy: Ni, Pearson & Poteshman)
Percentage of non-optionable stocks closing within $0.125 of an integer multiple of $5

(Courtesy: Ni, Pearson & Poteshman)
Percentage of **optional** stocks closing within $0.125 of an integer multiple of $5

**Relative Trading Date from Option Expiration Date**

(Courtesy: Ni, Pearson & Poteshman)
Percentage of optionable stocks closing within $0.125 of a strike price

Relative Trading Date from Option Expiration Date

(Courtesy: Ni, Pearson & Poteshman)
Percentage of non-optionable stocks closing within $0.125 of an integer multiple of $5

(Courtesy: Ni, Pearson & Poteshman)
In search for an explanation…

Large sale of options on this day
Average traded vol in stocks = 1MM shares

Notional number of shares corresponding to OI = 5.6 MM shares
Our Model: Feedback Due to Demand for Deltas

Assumption 1. Open Interest is unusually large

Assumption 2. Market-makers – professional delta-hedgers – are net very long options

Proposed mechanism for pinning:

Hedgers are net long options, hence long Gamma. They sell stock when it rises and buy stock when it falls.

Since the aggregate amount of stock required is large compared to typical daily trading volume, this drives the stock to the strike price
Taking into account demand for stock: Price-Impact Functions

\[
\frac{dS}{S} \propto E\left(\frac{D}{<V>}\right)^p \quad \frac{D}{<V>} \gg 1
\]

- \(p=0.22\) Farmer, Lillo, Mantegna
- \(p=0.5\) X. Gabaix
- \(p=1\) linear model, (A. & Lipkin)
- \(p=1.5\) convex model (Bouchaud, …)

Choice of \(p\) is a fundamental question in Econophysics.
Linear Model (A & Lipkin, 2002)

Price-Demand Elasticity Eq.

\[ \frac{\Delta S}{S} \propto E \cdot \frac{D}{\langle |D| \rangle} \]

Price-response due to demand for deltas

\[ \frac{\Delta S}{S} \propto \frac{E.OI}{\langle |D| \rangle} \Delta \delta \]

\[ \delta = B. - S. \text{ Delta for one option} \]
Estimating the Demand for Deltas using Black-Scholes

\[ \Delta \delta = \frac{\partial \delta}{\partial t} \, dt, \quad \tau = T - t \]

\[ \delta = 2N(d_1), \quad d_1 = \frac{1}{\sigma \sqrt{\tau}} \left( \ln\left( \frac{S}{K} \right) + \left( \mu + \frac{\sigma^2}{2} \right) \frac{\sqrt{\tau}}{2\sigma} \right) \]

From Black-Scholes

\[ y = \ln\left( \frac{S}{K} \right), \quad a = \mu + \frac{\sigma^2}{2}, \]

\[ \frac{\partial \delta}{\partial t} = -\frac{1}{\sqrt{2\pi}} \frac{y - a \tau}{\sigma \tau^{3/2}} e^{-\frac{(y+a\tau)^2}{2\sigma^2\tau}} \]
Dynamics for Stock Price

\[ \frac{dS}{S} = \frac{E.OI}{\langle |D| \rangle} \frac{\partial \delta}{\partial t} dt + \sigma dW \]

\[ y = \ln \left( \frac{S}{K} \right) \]

\[ dy = -\frac{E.OI}{\langle |D| \rangle \sqrt{2\pi}} \cdot \frac{y - a(T-t)}{\sigma(T-t)^{3/2}} e^{-\frac{(y+a(T-t))^2}{2\sigma^2(T-t)}} dt + \sigma dW \]

`coupling constant'  restoring force  bounded support  noise
Monte Carlo Simulation

Time scale = -log(T - t)
Cumulative PDF for price at expiration date ($\text{Beta}=0.1$)
Dimensionless Variables

\[ z = \frac{y}{\sigma \sqrt{T}}, \quad s = \frac{t}{T}, \quad z_0 = \frac{y_0}{\sigma \sqrt{T}} = \frac{1}{\sigma \sqrt{T}} \ln \left( \frac{S_0}{K} \right) \]

\[ \alpha = \frac{a \sqrt{T}}{\sigma}, \quad \beta = \frac{E.OI}{\langle |D| \rangle \sqrt{2\pi \sigma^2 T}} \]

\[ dz = -\frac{\beta(z - \alpha(1-s))}{(1-s)^{3/2}} e^{-(z+\alpha(1-s))^2 / 2(1-s)} ds + d\overline{W} \]
Dimensionless Model (alpha=0) for Linear Price-Impact Function

\[ dz = -\frac{\beta \cdot z}{(1-s)^{3/2}} e^{-\frac{z^2}{2(1-s)}} ds + \overline{dW} \]

Linear restoring force with increasing coupling with time and compact support.
The Potential Well

Price experiences a force that becomes stronger, more localized near expiration

\[ z = \ln(S/K)/(\sigma \sqrt{\tau}) \]

\[ \frac{dz}{ds} = -\frac{z}{(1-s)^{3/2}} e^{\frac{z^2}{2(1-s)}} \]  \hspace{1cm} (\alpha = 0, \beta = 1)
Solving the linear response model (p=1)

Assume $\alpha = 0$

Forward Fokker-Planck equation:

$$\frac{\partial F}{\partial s} + \frac{1}{2} \frac{\partial^2 F}{\partial z^2} - \frac{\beta z}{\tau^{3/2}} e^{\frac{-z^2}{2\tau}} \frac{\partial F}{\partial z} = 0, \quad \tau = 1-s$$

Look for solution of the form:

$$F(z, s) = \exp\left(\frac{1}{\sqrt{\tau}} \phi\left(\frac{z}{\sqrt{\tau}}\right)\right), \quad \phi(\zeta) \text{ unknown, } \quad \zeta = \frac{z}{\sqrt{\tau}}$$
ODE for the `Phase Function' (WKB)

\[
\frac{\phi + \zeta \phi' + \phi''}{2 \tau^{3/2}} + \frac{(\phi')^2 - 2 \beta \zeta \phi' e^{-\frac{\zeta^2}{2}}}{2 \tau^2} = 0
\]

\[
O(\tau^{-2}) \quad (\phi')^2 - 2 \beta \zeta \phi' e^{-\frac{\zeta^2}{2}} = 0 \quad \text{Eikonal Equation}
\]

\[
\therefore \quad \phi(\zeta) = -2 \beta e^{-\frac{\zeta^2}{2}} + c
\]

\[
O(\tau^{-3/2}) \quad \phi + \zeta \phi' + \phi'' = c \quad c = 0
\]

\[
F(z, s) = \exp \left[ -\frac{2 \beta}{\sqrt{1-s}} e^{-\frac{z^2}{2(1-s)}} \right] \quad \text{Exact solution of the FFP Equation!}
\]
A Formula for the Pinning Probability

\[ P(z, s) = 1 - \exp\left[-\frac{2\beta}{\sqrt{1-s}} e^{-\frac{z^2}{2(1-s)}}\right] \]

Satisfies:

\[
\begin{align*}
\lim_{s \to 1^+} P(z, s) &= 0 \\
\lim_{s \to 1^+} P(0, s) &= 1
\end{align*}
\]

\[
\text{Prob}(z(1) = 0 \mid z(0) = z_0) = 1 - e^{-2\beta e^{-\frac{z_0^2}{2}}}
\]
Pinning Probability: Dependence on Beta

\[ \beta = \frac{E.OI}{\langle |D| \rangle \sqrt{2\pi\sigma^2T}} \]

- Increases with OI
- Decreases with volat, expiration
- Decreases with the distance to strike
Pinning probability: dependence on $z_0$

\[z = \ln(S/K)/(\alpha \sqrt{T})\]

Alpha = 0
Beta = 0.1
OK, but does this story explain stock pinning?

- We know that stock pinning at option expiration exists for optionable stocks

- Our model makes two assumptions to justify pinning
  - Large number of deltas relative to total volume
  - Market-makers are long options

Use CBOE data to calculate pinning statistics conditional on market-makers position near expiration
Observations with market-makers net long (~$0.125)

percentage of stocks close within strike price +/- 0.125, marketMaker net long 44024
Market-makers + firm proprietary traders net long
Market-makers net short

percentage of stocks close within strike price +/- 0.125, marketMaker net short 26706

relative trading date from option expiration date
Market-makers + firm proprietary traders net short

percentage of stocks close within strike price +/- 1.25, market & Firm net short 27401

relative trading date from option expiration date
Lipkin & Stanton: Validation of the Dependence of Volatility/OI

\[ P_{pin} \propto \frac{OI}{\langle V \rangle} \]

\[ P_{pin} \propto \frac{OI}{\sigma} \]

Source of Data: IVY/OptionMetrics
Pinning Probability vs. Open Interest

Mike Lipkin, Sasha Stanton, Columbia empirical finance course
Dimensionless Model for Power-Law Price-Impact Function ($p>0$)

$$\frac{dS}{S} \propto \text{const.} \left| \frac{\partial \delta}{\partial t} \right|^p \text{sign} \left( \frac{\partial \delta}{\partial t} \right) dt + \sigma dW$$

$$dz = -\beta \cdot |z|^p \text{sign}(z) e^{-\frac{pz^2}{2(1-s)}} ds + d\bar{W}$$

Dimensionless eq. without irrelevant drift terms ($\alpha=0$).
Pinning under non-linear price-impact models

(i) If $p \leq 1/2$, there is no pinning, i.e. $P[z(1)=0|z(0)=z]=0$, for all $z$.

(ii) If $p > 1/2$ pinning occurs with finite probability ($<1$) and

$$
\ln P(z(1)=0|z(0)=z) \propto -\frac{C(\beta,z)}{2p-1}
$$

$$
P_{pin} \propto e^{-\frac{C}{2p-1}}, \quad p > 1/2
$$
Calculation of Pinning Probabilities by MC Simulation (Gennady Kasyan)

Smooth fit near $p=0.5$
Absence of Pinning for $p<1/2$ (I)

$$dz = -\frac{\beta}{\tau^p} \Omega\left(\frac{z}{\sqrt{\tau}}\right) dt + dW$$

$$\Omega(r) = |r|^p \text{sg}(r) e^{-\frac{pr^2}{2}}$$

Novikov condition: If we have

$$E^W \left\{ \int_0^1 (\text{drift})^2 dt \right\} < \infty$$

the measure induced by the $z$-process in path-space is absolutely continuous with respect to Wiener Measure.
Absence of Pinning for $p < 1/2$ (II)

Verify that Novikov condition holds:

\[
E^W \left\{ e^0 \int_0^1 (\text{drift})^2 \, dt \right\} = E^W \left\{ e^0 \int_0^1 \frac{\beta^2}{\tau^{2p}} \Omega^2 (\zeta_t) \, dt \right\}
\]

\[
< E^W \left\{ e \left\| \Omega \right\|_{L_\infty}^2 \beta^2 \int_0^1 \frac{dt}{(1-t)^2 p} \right\}
\]

\[
< \infty, \quad \text{if } p < 1/2
\]
Self-similarity approach (RG)

Kasyan (2007) : exploit the self-similarity properties of the model

\[ z' = a z \]
\[ \tau' = a^2 \tau \quad \therefore \quad t' = (1 - a^2) + a^2 t \]

\[ dz' = -\frac{\beta a^{2p-1}}{(\tau')^p} \Omega \left( \frac{z'}{\sqrt{\tau'}} \right) dt' + dW \]

Parabolic re-scaling transformation

Re-scaling give the same model with a different coupling constant (RG)

\[ \beta' = \beta a^{2p-1} \]
Absence of pinning at p=1/2

1. Chapman-Kolmogorov

\[ P^\beta \left[z(1) = 0 \mid z(0) = 0 \right] = \int \pi(z', 0.5) P^\beta \left[z(1) = 0 \mid z(0.5) = z' \right] dz' \]

2. Renormalization group

\[ P^\beta \left[z(1) = 0 \mid z(0.5) = z' \right] = P^\beta \left[z(1) = 0 \mid z(0) = z' \sqrt{2} \right] \]

\[ P^\beta \left[z(1) = 0 \mid z(0) = 0 \right] = \int \pi(z', 0.5) P^\beta \left[z(1) = 0 \mid z(0) = z' \sqrt{2} \right] dz' < \left( \int \pi(z', 0.5) dz' \right) P^\beta \left[z(1) = 0 \mid z(0) = 0 \right] \]

\[ = P^\beta \left[z(1) = 0 \mid z(0) = 0 \right] \]

This gives a contradiction. The pinning probability must be zero.

Note: we used monotonicity in z (maximum principle).
Vanishing of Pinning Probability at $p=0.5$ (log scale)
Log probability vs. $1/(p-1/2)$
Sketch of proof of $\ln P \sim - \frac{C}{(2p-1)}$
(G. Kasyan, 2007)
Estimating the probability of remaining inside the parabola

\[ \tau_n = 1/4^n, \quad n = 0,1,2... \]

\[ P^\beta [z(1) = 0] \geq P^\beta \bigg\{ \left| z(0) \right| < 1; \left| z(1 - \tau_1) \right| < 1/2; \ldots \left| z(1 - \tau_n) \right| < 1/2^n; \ldots \bigg\} \]

\[ = P^\beta \bigg\{ \left| z(1 - \tau_n) \right| < 1/2^n; n = 0,1,2\bigg\} \]

\[ \geq \prod_{n=1}^{\infty} \inf_{|z| < 2^{-n+1}} P^\beta \bigg\{ \left| z(1 - \tau_n) \right| < 1/2^n \bigg| \left| z(1 - \tau_{n-1}) \right| = z \bigg\} \]

\[ = \prod_{n=1}^{\infty} \inf_{|z| < 1} P^\beta \cdot 2^{n(2p-1)} \bigg\{ \left| z(3/4) \right| < 1/2 \bigg| z(0) = z \bigg\} \]

\[ \equiv \prod_{n=1}^{\infty} p \left( \beta \cdot 2^{n(2p-1)} \right) \]

Probabilities of transitioning between time 0 and 3/4 with different values of the coupling coefficient.
Large Deviations Estimate

Exit from parabolic region

\[
P^\beta \left\{ \left| z(3/4) \right| > 1/2 \mid \left| z(0) \right| < 1 \right\} \approx e^{-C \beta} , \quad \beta >> 1
\]
Most likely exit path

Follow the flow (to $z=0$) for time $\tau - 1/\beta$

Go against (large) flow for time $1/\beta$
Large Deviations Estimate

\[ P(\gamma) \approx \exp \left\{ - \int_0^\tau \left| \frac{d\gamma}{ds} - \nu(\gamma, s) \right|^2 ds \right\} \]

\( \approx \exp \left\{ - \int_{\tau-1/\beta}^{\tau} \left| \frac{d\gamma}{ds} + \beta \frac{\gamma^p}{(1-s)^{3p/2}} e^{-\frac{p\gamma^2}{2(1-s)}} \right|^2 ds \right\} \]

\( \approx \exp \left\{ - C \int_{\tau-1/\beta}^{\tau} \beta^2 ds \right\} \quad \text{because} \quad \frac{d\gamma}{ds} \approx \beta \)

\( \approx \exp \left\{ - C\beta \right\} \quad \text{for} \quad \beta \gg 1 \)
Lower bound for pinning probability

\[
P[z(1) = 0] \geq \prod_{n=1}^{\infty} P^\beta_2^{n(2^{p-1})} \left\{ z(3/4) < 1/2 \| z(0) < 1 \right\} \\
\geq \prod_{n=1}^{\infty} \left( 1 - C_1 e^{-C_2 \beta_2^{n(2^{p-1})}} \right) \\
\sum_{n=1}^{\infty} \ln \left( 1 - C_1 e^{-C_2 \beta_2^{n(2^{p-1})}} \right) = e^{n=1} \\
\approx \exp \left( - C_1 \sum_{n=1}^{\infty} e^{-C_2 \beta_2^{n(2^{p-1})}} \right) \quad :.
\]

\[
\ln P[z(1) = 0] \geq -C_1 \sum_{n=1}^{\infty} e^{-C_2 \beta_2^{n(2^{p-1})}} \approx -C_1 \sum_{n=1}^{\infty} e^{-C_2 \beta(1+n \ln 2 \cdot (2^{p-1}))} \\
\approx - \frac{C_3}{2p-1} \quad \text{(}2p-1 \ll 1\text{)}
\]
Conclusions

- Pinning of stock prices at strikes near expiration dates can be explained in terms of price-impact due to demand for deltas.

- Linear model is exactly solvable. Agrees very well with data.

- Power-law price-impact functions such as those used in Econophysics give rise to SDEs that generalize the linear model.

- The price impact function \( \frac{dP}{P} \propto Q^p \)

  produces pinning at the strike if and only if \( p > 0.5 \).

- Models of price-impact with \( p > 0.5 \) are therefore more likely to be observed in practice, since they are consistent with the observable phenomenon of pinning.
<table>
<thead>
<tr>
<th>References</th>
</tr>
</thead>
</table>
| Krishnan, Hari I. Nelken  
The effect of stock pinning on option prices, *RISK*, December 2001 |
| Avellaneda, M. and M.D. Lipkin  
| Ni, S.X., N. Pearson and A. M. Poteshman  
| Jeannin, Marc, Iori, Giulia and Samuel, David  
| Kasyan, Gennady, Avellaneda, M. and Lipkin M., forthcoming |
| Kasyan, Gennady, Ph. D. Thesis, NYU, forthcoming |