Reduced basis method for parametrized optimal control problems governed by PDEs

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École Polytechnique Fédérale de Lausanne

J.-L. Lions and E. Magenes Memorial Days Workshop
University Pierre et Marie Curie (UPMC Paris VI) Laboratoire Jacques-Louis Lions

Acknowledgements: A.T. Patera (MIT)

Sponsors: European Research Council - Mathcard Project, Progetto Roberto Rocca Politecnico di Milano-MIT, Swiss National Science Foundation
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2. Parametrized linear-quadratic Optimal Control Problems
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   - An (idealized) application for blood flows
Reduction strategies for Parametrized Optimal Control Problems

The characterization of a system in terms of physical quantities and/or geometrical configuration usually depends on a set of parameters.

The system response will be parameter dependent as well, and so will be the optimal control.

**Parametrized optimal control problems:** the prediction of optimal control inputs and the optimization of given output of interests is required for each different value of the parameters.

The computational effort may be unacceptably high and, often, unaffordable when

- performing the optimization process for many different parameter values (*many-query context*)
- for a given new configuration, we want to compute the solution in a rapid way (*real-time context*)
Reduction strategies for Parametrized Optimal Control Problems

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- performing the optimization process for many different parameter values (many-query context)
- for a given new configuration, we want to compute the solution in a rapid way (real-time context)

**Goal**: to achieve the **accuracy** and **reliability** of a high fidelity approximation but at greatly reduced cost of a **low order model**.
Previous works and improvements on parametrized optimal control

- early works by Ito and Ravindran [1998 and 2001]: optimal control of Navier-Stokes equations

- the RB method has been applied to parametrized linear-quadratic advection diffusion optimal control problems in different contexts: [Quarteroni, R., Quaini, 07], [Tonn, Urban, Volkwein, 11], [Grepl, Karcher, 11] considering low-dimensional control variable.

- we aim at developing a reduced framework that enables to handle with general control functions, i.e. **infinite dimensional** distributed and/or boundary control functions.

- an **efficient and rigorous** a posteriori error estimation, necessary both for constructing the reduced order model and measuring its accuracy, is still partially missing for a large class of optimal control problems. Previous preliminary works by [Dedé, 2010] [Tonn, Urban, Volkwein, 11] [Grepl, Karcher, 11]
Optimal control problems [Lions, 1971]

In general, an optimal control problem (OCP) consists of:

- a control function $u$, which can be seen as an input for the system,
- a controlled system, i.e. an input-output process: $\mathcal{E}(y, u) = 0$, beign $y$ the state variable
- an objective functional to be minimized: $\mathcal{J}(y, u)$
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- an objective functional to be minimized: $\mathcal{J}(y, u)$

\[
\text{find the optimal control } u^* \text{ and the state } y(u^*) \text{ such that the cost functional } \\
\mathcal{J}(y, u) \text{ is minimized subject to } \mathcal{E}(y, u) = 0
\]
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- an objective functional to be minimized: $J(y, u)$

\[
\begin{align*}
\text{STATE PROBLEM} & \quad \text{Output} \\
\quad \text{Optimization: update control } u & \quad J(y, u)
\end{align*}
\]

\textit{find the optimal control $u^*$ and the state $y(u^*)$ such that the cost functional $J(y, u)$ is minimized subject to $E(y, u) = 0$}

We restrict attention to:

- quadratic cost functionals, e.g. $J(y, u) = \frac{1}{2} ||y - y_d||^2 + \frac{\alpha}{2} ||u||^2$
- stationary, linear, elliptic state equations: in particular (scalar coercive) advection-diffusion equations and then (vectorial noncoercive) Stokes equations
Parametrized optimal control problems

A parametrized optimal control problem (OCP\(\mu\)) consists of:

- a control function \(u(\mu)\), which can be seen as an input for the system,
- a controlled system, i.e. an input-output process:
  \(\mathcal{E}(y(\mu), u(\mu); \mu) = 0\),
- an objective functional to be minimized: \(\mathcal{J}(y(\mu), u(\mu); \mu)\)

Given \(\mu \in \mathcal{D}\), find the optimal control \(u^*(\mu)\) and the state \(y^*(\mu)\) such that the cost functional \(\mathcal{J}(y(\mu), u(\mu); \mu)\) is minimized subject to \(\mathcal{E}(y(\mu), u(\mu); \mu) = 0\)

where \(\mu \in \mathcal{D} \subset \mathbb{R}^p\) denotes a \(p\)-vector whose components can represent:

- coefficients in boundary conditions
- physical parametrization
- geometrical configurations
- data (observation)
Saddle-point formulation for optimal control problems
The abstract optimization problem

**Functional setting:** \( Y \) state space, \( U \) control space, \( Q(\equiv Y) \) adjoint space, \( \mathcal{Z} \) observation (data) space s.t. \( Y \subset \mathcal{Z} \),

In the following: \( y, z \in Y \quad u, v \in U, \quad p, q \in Q \).
The abstract optimization problem: **quadratic cost functional**

**Functional setting:** \( Y \) state space, \( U \) control space, \( Q(\equiv Y) \) adjoint space, \( Z \) observation (data) space s.t. \( Y \subset Z \),

In the following: \( y, z \in Y \), \( u, v \in U \), \( p, q \in Q \).

- we consider the following **quadratic cost functional** to be minimized

\[
J(y, u; \mu) = \frac{1}{2} m(y - y_d(\mu), y - y_d(\mu); \mu) + \frac{\alpha}{2} n(u, u; \mu)
\]

where \( y_d(\mu) \in Z \) is a given function, \( \alpha > 0 \) regularization parameter.

We make the following assumptions: for any \( \mu \in D \)

- \( m(\cdot, \cdot; \mu) : Z \times Z \to \mathbb{R} \) is a symmetric, continuous and positive bilinear form
- \( n(\cdot, \cdot; \mu) : U \times U \to \mathbb{R} \) is a symmetric, continuous and coercive bilinear form
The abstract optimization problem: **linear constraint**

**Functional setting:** $Y$ state space, $U$ control space, $Q(\equiv Y)$ adjoint space, $Z$ observation (data) space s.t. $Y \subset Z$,

In the following: $y, z \in Y$, $u, v \in U$, $p, q \in Q$.

- the **linear constraint** is expressed by the following state equation (elliptic PDE in weak form)

$$a(y, q; \mu) = c(u, q; \mu) + \langle G(\mu), q \rangle \quad \forall q \in Q$$

where $G(\mu) \in Q'$ for all $\mu \in \mathcal{D}$.

We make the following assumptions: for any $\mu \in \mathcal{D}$

$a(\cdot, \cdot; \mu) : Y \times Q \to \mathbb{R}$ is a continuous and (weakly) coercive bilinear form

$c(\cdot, \cdot; \mu) : U \times Q \to \mathbb{R}$ is a continuous bilinear form

Holding these assumptions, given $u \in U$, the state equation is uniquely solvable for any $\mu \in \mathcal{D}$.
The abstract optimization problem

**Parametrized optimal control problem**

\[
\begin{align*}
\text{minimize} & \quad J(y, u; \mu) = \frac{1}{2} m(y - y_d(\mu), y - y_d(\mu); \mu) + \frac{\alpha}{2} n(u, u; \mu) \\
\text{s.t.} & \quad a(y, q; \mu) = c(u, q; \mu) + \langle G(\mu), q \rangle \quad \forall q \in Q.
\end{align*}
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The abstract optimization problem

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\text{s.t.} & \quad a(y, q; \mu) = c(u, q; \mu) + \langle G(\mu), q \rangle \quad \forall q \in Q.
\end{align*}
\]

In order to develop the Reduced Basis (RB) method

- we firstly recast the problem in the framework of saddle-point problem [Gunzburger, Bochev, 2004]
- we then apply the well-known Brezzi theory [Brezzi, Fortin, 1991]
  \( \leftrightarrow \) existence, uniqueness, stability and optimality conditions

This way we can exploit the analogies with the already developed theory of RB method for Stokes-type problems [R., Veroy, 2007; R., Huynh, Manzoni 2010]
Saddle-point formulation

- Let \( X \equiv Y \times U \) the state and control space
- \( Q \) the adjoint (Lagrange multiplier) space as before

and let \( x = (y, u) \in X \), \( w = (z, v) \in X \), with \( \|x\|_X^2 = \|y\|_Y^2 + \|u\|_U^2 \)
Saddle-point formulation

Let \( X \equiv Y \times U \) the state and control space
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and let \( x = (y, u) \in X, \quad w = (z, v) \in X \), with \( \|x\|_X^2 = \|y\|_Y^2 + \|u\|_U^2 \)

Let
\[
\begin{align*}
A(x, w; \mu) &= m(y, z; \mu) + \alpha n(u, v; \mu) \\
\langle F(\mu), w \rangle &= m(y_d(\mu), z; \mu) \\
t(\mu) &= \frac{1}{2} m(y_d(\mu), y_d(\mu); \mu)
\end{align*}
\]

the cost functional can be expressed as
\[
J(x; \mu) = \frac{1}{2} A(x, x; \mu) - \langle F(\mu), x \rangle + t(\mu) = J(x; \mu) + t(\mu)
\]
Saddle-point formulation

Let \( X \equiv Y \times U \) the state and control space

\( Q \) the adjoint (Lagrange multiplier) space as before

and let \( x = (y, u) \in X \), \( \overline{w} = (z, v) \in X \), with \( \| x \|^2_X = \| y \|^2_Y + \| u \|^2_U \)

Let

\[
\begin{align*}
\mathcal{A}(x, \overline{w}; \mu) &= m(y, z; \mu) + \alpha n(u, v; \mu) \\
\langle F(\mu), \overline{w} \rangle &= m(y_d(\mu), z; \mu) \\
t(\mu) &= \frac{1}{2} m(y_d(\mu), y_d(\mu); \mu)
\end{align*}
\]

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\[
J(x; \mu) = \frac{1}{2} \mathcal{A}(x, x; \mu) - \langle F(\mu), x \rangle + t(\mu) = J(x; \mu) + t(\mu)
\]

Let

\[
\mathcal{B}(\overline{w}, q; \mu) = a(z, q; \mu) - c(v, q; \mu),
\]

the constraint equation can be expressed as

\[
\mathcal{B}(x, q; \mu) = \langle G(\mu), q \rangle \quad \forall q \in Q
\]
Saddle-point formulation: applying Brezzi theory

The optimal control problem can be recast in the form: given $\mu \in \mathcal{D}$,

$$\begin{cases} 
\min J(x; \mu) = \frac{1}{2} A(x, x; \mu) - \langle F(\mu), x \rangle, \\
B(x, q; \mu) = \langle G(\mu), q \rangle \quad \forall q \in Q.
\end{cases}$$
Saddle-point formulation: applying Brezzi theory

The optimal control problem can be recast in the form: given $\mu \in \mathcal{D}$,

$$
\begin{align*}
\min_{x} \mathcal{J}(x; \mu) &= \frac{1}{2} A(x, x; \mu) - \langle F(\mu), x \rangle, \\
\mathcal{B}(x, q; \mu) &= \langle G(\mu), q \rangle \quad \forall q \in Q.
\end{align*}
$$

Let $X_0 = \{ w \in X : \mathcal{B}(w, q; \mu) = 0 \, \forall q \in Q \} \subset X$, the assumptions we have made imply that (e.g. [Gunzburger, Bochev, 2004]):

- the bilinear form $A(\cdot, \cdot; \mu)$ is continuous over $X \times X$ and coercive over $X_0$
- the bilinear form $\mathcal{B}(\cdot, \cdot; \mu)$ is continuous over $X \times Q$ and satisfies the following inf-sup condition
  $$
  \exists \beta_0 > 0 : \quad \beta(\mu) = \inf_{q \in Q} \sup_{w \in X} \frac{\mathcal{B}(w, q; \mu)}{\|w\|_X \|q\|_Q} \geq \beta_0, \quad \forall \mu \in \mathcal{D}
  $$
- the bilinear form $A(\cdot, \cdot; \mu)$ is symmetric and non-negative over $X$
Saddle-point formulation: applying Brezzi theory

- the optimal control problem

\[
\min_{\bar{x} \in X} \mathcal{J}(\bar{x}; \mu) \quad \text{subject to} \quad B(\bar{x}, q; \mu) = \langle G(\mu), q \rangle \quad \forall q \in Q.
\]

has a unique solution \( \bar{x} = (y, u) \in X \) for any \( \mu \in \mathcal{D} \)
Saddle-point formulation: applying Brezzi theory

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\]

has a unique solution \( x = (y, u) \in X \) for any \( \mu \in \mathcal{D} \)

- that solution can be determined by solving the optimality system

\[
\begin{cases}
A(x(\mu), w; \mu) + B(w, p(\mu); \mu) = \langle F(\mu), w \rangle & \forall w \in X, \\
B(x(\mu), q; \mu) = \langle G(\mu), q \rangle & \forall q \in Q,
\end{cases}
\]

Compact form

Given \( \mu \in \mathcal{D} \), find \( U(\mu) \in \mathcal{X} \) s.t:

\[
\mathcal{X} = X \times Q, \quad U = (x, p), \quad W = (w, q)
\]

\[
B(U, W; \mu) = A(x, w; \mu) + B(w, p; \mu) + B(x, q; \mu)
\]

\[
F(W; \mu) = \langle F(\mu), w \rangle + \langle G(\mu), q \rangle
\]

\[
B(U(\mu), W; \mu) = F(W; \mu) \quad \forall W \in \mathcal{X}.
\]
Saddle-point formulation: applying Brezzi theory

- The optimal control problem

\[
\min_{x \in X} J(x; \mu) \quad \text{subject to} \quad B(x, q; \mu) = \langle G(\mu), q \rangle \quad \forall q \in Q.
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has a unique solution \(x = (y, u) \in X\) for any \(\mu \in \mathcal{D}\)

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\end{align*}
\]

Compact form

given \(\mu \in \mathcal{D}\), find \(U(\mu) \in \mathcal{X}\) s.t:

\[
\begin{align*}
\mathcal{X} &= X \times Q, & U &= (x, p), & W &= (w, q) \\
B(U, W; \mu) &= A(x, w; \mu) + B(w, p; \mu) + B(x, q; \mu) \\
F(W; \mu) &= \langle F(\mu), w \rangle + \langle G(\mu), q \rangle \\
B(U(\mu), W; \mu) &= F(W; \mu) \quad \forall W \in \mathcal{X}.
\end{align*}
\]

- At this point we may apply the Galerkin-FE approximation
Reduced Basis method for parametrized optimal control problems
RB method: Construction

$\mu$-OCP, optimality system

$\text{Pb}(\mu; U(\mu))$

$U(\mu) \in X : B(U(\mu), W; \mu) = F(W) \quad \forall W \in X$
RB method: Construction

\[ \text{\( \mu \)-OCP, optimality system} \]

\[ \text{Pb}(\mu; U(\mu)) \]

\[ U(\mu) \in X : B(U(\mu), W; \mu) = F(W) \quad \forall W \in X \]

\[ \text{Truth approximation (FEM)} \]

\[ \text{Pb}_N(\mu; U_N(\mu)) \]

\[ U_N(\mu) \in X_N : B(U_N(\mu), W; \mu) = F(W) \quad \forall W \in X_N \]

- **Truth Hypothesis**: \( U_N(\mu) \) “indistinguishable” from \( U(\mu) \).
- **RB Motivation**: \( \mu \rightarrow U_N(\mu) \), \( J_N(\mu) \) too expensive and slow in many-query and real-time contexts.
RB method: Construction

**μ-OCP, optimality system**

\[
P_b(\mu; U(\mu))
\]

\[
\mu \in \mathcal{X} : \quad B(U(\mu), W; \mu) = F(W) \quad \forall W \in \mathcal{X}
\]

**Truth approximation (FEM)**

\[
P_b^N(\mu; U^N(\mu))
\]

\[
U^N(\mu) \in \mathcal{X}^N : \quad B(U^N(\mu), W; \mu) = F(W) \quad \forall W \in \mathcal{X}^N
\]

**Sampling (Greedy)**

**Space Construction**

(Hierarchical Lagrange basis)

**OFFLINE**

\[
S_N = \{\mu^i, \quad i = 1, \ldots, N\}
\]

\[
\mathcal{X}_N = \text{span}\{U^N(\mu^i), \quad i = 1, \ldots, N\}
\]

\[
\text{dim}(\mathcal{X}_N) = N \ll N = \text{dim}(\mathcal{X}^N)
\]
RB method: Construction

\( \mu \)-OCP, optimality system

\[
Pb(\mu; U(\mu)) \quad U(\mu) \in \mathcal{X} : \quad B(U(\mu), W; \mu) = F(W) \quad \forall W \in \mathcal{X}
\]

Truth approximation (FEM)

\[
Pb_N(\mu; U_N(\mu)) \quad U_N(\mu) \in \mathcal{X}_N : \quad B(U_N(\mu), W; \mu) = F(W) \quad \forall W \in \mathcal{X}_N
\]

Sampling (Greedy)
Space Construction
(Hierarchical Lagrange basis)

OFFLINE

\[
S_N = \{ \mu^i, \ i = 1, \ldots, N \}
\]
\[
\mathcal{X}_N = \text{span}\{ U_N(\mu^i), \ i = 1, \ldots, N \}
\]
\[
\text{dim}(\mathcal{X}_N) = N \ll \mathcal{N} = \text{dim}(\mathcal{X}_N)
\]

Reduced Basis (RB) approximation

\[
Pb_N(\mu; U_N(\mu)) \quad U_N(\mu) \in \mathcal{X}_N : \quad B(U_N(\mu), W; \mu) = F(W) \quad \forall W \in \mathcal{X}_N
\]

[\text{R., Huynh, Patera 2008, Talk by A. Manzoni}]
RB method: smooth parametric dependency

- Guess: $\mathcal{M}^N$ is low dimensional - smooth dependence on $\mu$ (Lipschitz continuity)
- (Adaptive) sampling procedure for parameter exploration
- reduced basis made of optimal solutions of the FE “truth” problem
- Evaluation procedure: (optimal) Galerkin projection

[Noor, Peters, Porshing, Lee, ...]
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[Noor, Peters, Porshing, Lee, ...]

How to be rigorous, rapid and reliable?
- depends on the sampling procedure for parameter exploration (greedy algorithm)
- exploits an Offline/Online stratagem based on the affinity assumption:

\[ B(U, W; \mu) = \sum_{q=1}^{Q_b} \Theta_a^q(\mu)B^q(U, W), \quad F(W; \mu) = \sum_{q=1}^{Q_f} \Theta_f^q(\mu)F^q(W) \]

- relies on a posteriori error analysis
Reduced Basis Method: approximation stability

Reduced Basis (RB) approximation: given \( \mu \in D \), find \((x_N(\mu), p_N(\mu)) \in X_N \times Q_N:\)

\[
\begin{align*}
A(x_N(\mu), w; \mu) + B(w, p_N(\mu); \mu) &= \langle F(\mu), w \rangle & \forall w \in X_N \\
B(x_N(\mu), q; \mu) &= \langle G(\mu), q \rangle & \forall q \in Q_N
\end{align*}
\]

How to define the reduced basis spaces?
Reduced Basis Method: approximation stability

Reduced Basis (RB) approximation: given $\mu \in D$, find $(x_N(\mu), p_N(\mu)) \in X_N \times Q_N$:

\[
\begin{aligned}
&A(x_N(\mu), w; \mu) + B(w, p_N(\mu); \mu) = \langle F(\mu), w \rangle & \forall w \in X_N \\
&B(x_N(\mu), q; \mu) = \langle G(\mu), q \rangle & \forall q \in Q_N
\end{aligned}
\]

(*)

How to define the reduced basis spaces? we have to provide a spaces pair $\{X_N, Q_N\}$ that guarantee the fulfillment of an equivalent parametrized Brezzi \textit{inf-sup} condition

\[
\beta_N(\mu) = \inf_{q \in Q_N} \sup_{w \in X_N} \frac{B(w, q; \mu)}{\|w\|_X \|q\|_Q} \geq \beta_0, \quad \forall \mu \in D.
\]

For the state and adjoint variables: aggregated spaces

\[
Y_N \equiv Q_N = \text{span}\{y^N(\mu^n), p^N(\mu^n)\}_{n=1}^N
\]

For the control variable:

\[
U_N = \text{span}\{u^N(\mu^n)\}_{n=1}^N
\]
Reduced Basis Method: approximation stability

**Reduced Basis (RB) approximation:** given \( \mu \in D \), find \((x_N(\mu), p_N(\mu)) \in X_N \times Q_N\):

\[
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\end{align*}
\]

\((*)\)

How to define the reduced basis spaces? We have to provide a spaces pair \(\{X_N, Q_N\}\) that guarantee the fulfillment of an equivalent parametrized Brezzi *inf-sup* condition

\[
\beta_N(\mu) = \inf_{q \in Q_N} \sup_{w \in X_N} \frac{B(w, q; \mu)}{||w||_{X} ||q||_{Q}} \geq \beta_0, \quad \forall \mu \in D.
\]

For the state and adjoint variables: aggregated spaces

\[
Y_N \equiv Q_N = \text{span}\{y^N(\mu^n), p^N(\mu^n)\}_{n=1}^N
\]

For the control variable:

\[
U_N = \text{span}\{u^N(\mu^n)\}_{n=1}^N
\]

Let \(X_N = Y_N \times U_N\), we can prove that

- \(\beta_N(\mu) \geq \alpha^N(\mu) > 0\) being \(\alpha^N(\mu)\) the coercivity constant associated to the FE approximation of the PDE operator
- Brezzi theorem \(\implies\) for any \(\mu \in D\), the RB approximation \((*)\) has a unique solution depending continuously on the data
RB method: algebraic formulation

We can express the RB state, control and adjoint solutions as

\[ x_N(\mu) = \sum_{j=1}^{3N} x_{Nj}(\mu) \sigma_j, \quad \text{denoting } X_N = \text{span}\{\sigma_j\}_{j=1}^{3N} \]

\[ p_N(\mu) = \sum_{j=1}^{2N} p_{Nj}(\mu) \tau_j, \quad \text{denoting } Q_N = \text{span}\{\tau_j\}_{j=1}^{2N} \]

recalling the assumption of affine parametric dependence (also for the RHS)

\[
\begin{align*}
A(x, w; \mu) &= \sum_{q=1}^{Q_a} \Theta_a^q(\mu) A^q(x, w), \\
B(w, p; \mu) &= \sum_{q=1}^{Q_b} \Theta_b^q(\mu) B^q(w, p)
\end{align*}
\]

for a new parameter \( \mu \), the RB solution can be written as a combination of basis functions with weights given by the following reduced basis linear system:

\[
\begin{cases}
\sum_{j=1}^{3N} \sum_{q=1}^{Q_a} \Theta_a^q(\mu) A^q_{ij} x_{Nj}(\mu) + \sum_{l=1}^{2N} \sum_{q=1}^{Q_b} \Theta_b^q(\mu) B^q_{li} p_{Nl}(\mu) = \sum_{q=1}^{Q_f} \Theta_f^q(\mu) F^q_i, & 1 \leq i \leq 3N \\
\sum_{j=1}^{3N} \sum_{q=1}^{Q_b} \Theta_b^q(\mu) B^q_{lj} x_{Nj}(\mu) = \sum_{q=1}^{Q_g} \Theta_g^q(\mu) G^q_l, & 1 \leq l \leq 2N
\end{cases}
\]

size: \( 5N \times 5N \)
RB method: Offline/Online decomposition

\[
\begin{aligned}
&\sum_{j=1}^{3N} \sum_{q=1}^{Q_a} \Theta_a^q(\mu) A_{ij}^q x_{Nj}(\mu) + \sum_{l=1}^{2N} \sum_{q=1}^{Q_b} \Theta_b^q(\mu) B_{li}^q p_{Nl}(\mu) = \sum_{q=1}^{Q_f} \Theta_f^q(\mu) F_i^q, \quad 1 \leq i \leq 3N, \\
&\sum_{j=1}^{3N} \sum_{q=1}^{Q_b} \Theta_b^q(\mu) B_{ij}^q x_{Nj}(\mu) = \sum_{q=1}^{Q_g} \Theta_g^q(\mu) G_i^q, \quad 1 \leq l \leq 2N,
\end{aligned}
\]
RB method: Offline/Online decomposition

$$
\begin{aligned}
&\sum_{j=1}^{3N} \sum_{q=1}^{Q_a} \Theta^q_a(\mu) A^q_{ij} x_{Nj}(\mu) + \sum_{l=1}^{2N} \sum_{q=1}^{Q_b} \Theta^q_b(\mu) B^q_{li} p_{Nl}(\mu) = \sum_{q=1}^{Q_f} \Theta^q_f(\mu) F^q_i, \quad 1 \leq i \leq 3N, \\
&\sum_{j=1}^{3N} \sum_{q=1}^{Q_b} \Theta^q_b(\mu) B^q_{ij} x_{Nj}(\mu) = \sum_{q=1}^{Q_g} \Theta^q_g(\mu) G^q_i, \quad 1 \leq l \leq 2N,
\end{aligned}
$$

- **Offline** pre-processing: compute and store the basis function \(\{\sigma_i\}_i\) and \(\{\tau_j\}_j\)
- store the matrices \(A^q\) and \(B^q\) given by
  \[
  (A_N)^q_{ij} = A^q(\sigma_j, \sigma_i), \quad (B_N)^q_{li} = B^q(\sigma_i, \tau_l), \quad 1 \leq i, j \leq 3N, \quad 1 \leq l \leq 2N,
  \]
- as well as the RHS \(F^q_i, G^q_l\). **Operation count**: depends on \(N, Q_a, Q_b, Q_f, Q_g\) and \(N\).
RB method: Offline/Online decomposition

\[
\begin{align*}
&\left\{ \sum_{j=1}^{3N} \sum_{q=1}^{Q_a} \Theta^q_a(\mu) A^q_{ij} \times N_j(\mu) + \sum_{l=1}^{2N} \sum_{q=1}^{Q_b} \Theta^q_b(\mu) B^q_{li} p_{Nl}(\mu) = \sum_{q=1}^{Q_f} \Theta^q_f(\mu) F^q_i, \quad 1 \leq i \leq 3N, \\
&\sum_{j=1}^{3N} \sum_{q=1}^{Q_b} \Theta^q_b(\mu) B^q_{ij} \times N_j(\mu) = \sum_{q=1}^{Q_g} \Theta^q_g(\mu) G^q_i, \quad 1 \leq l \leq 2N, \right. \\
\end{align*}
\]

- **Offline** pre-processing: compute and store the basis function \( \{\sigma_i\}_i \) and \( \{\tau_j\}_j \) store the matrices \( A^q \) and \( B^q \) given by

\[
(A_N)^q_{ij} = A^q(\sigma_j, \sigma_i), \quad (B_N)^q_{li} = B^q(\sigma_i, \tau_l), \quad 1 \leq i,j \leq 3N, \quad 1 \leq l \leq 2N,
\]
as well as the RHS \( F^q_i, G^q_l \). **Operation count:** depends on \( N, Q_a, Q_b, Q_f, Q_g \) and \( N \).

- **Online** evaluations: evaluate coefficients \( \Theta^q(\mu) \), assemble RB matrices and vectors

\[
\begin{align*}
A_N(\mu) &= \sum_{q=1}^{Q_a} \Theta^q_a(\mu) A^q_N, & B_N(\mu) &= \sum_{q=1}^{Q_b} \Theta^q_b(\mu) B^q_N, \\
F_N(\mu) &= \sum_{q=1}^{Q_f} \Theta^q_f(\mu) F^q_N, & G_N(\mu) &= \sum_{q=1}^{Q_g} \Theta^q_g(\mu) G^q_N;
\end{align*}
\]
and solve

\[
\begin{pmatrix}
A_N(\mu) & B_N^T(\mu) \\
B_N(\mu) & 0
\end{pmatrix}
\begin{pmatrix}
x_N(\mu) \\
p_{N}(\mu)
\end{pmatrix}
= \begin{pmatrix}
F_N(\mu) \\
G_N(\mu)
\end{pmatrix}
\]

**Operation count:** \( O((5N)^3 + (Q_a + Q_b)N^2 + (Q_f + Q_g)N) \) independent of \( N \), \( N \ll N \).
RB Method: a posteriori error estimation

**Goal**: provide **rigorous**, **sharp** and **inexpensive** estimators for the **error on the solution** variables and for the **error on the cost functional**
RB Method: a posteriori error estimation

**Goal:** provide **rigorous**, **sharp** and **inexpensive** estimators for the **error on the solution** variables and for the **error on the cost functional**

**General noncoercive (Babuška) framework** (as Stokes problems, e.g. [Rozza et al., 2010]):

- Recall the compact (weakly coercive) formulation: given $\mu \in D$,
  $$\text{find } U(\mu) \in \mathcal{X} \text{ s.t: } B(U(\mu), W; \mu) = F(W; \mu) \quad \forall W \in \mathcal{X}.$$  

Note that $\exists \hat{\beta}_0 > 0 : \hat{\beta}(\mu) = \inf_{W \in \mathcal{X}} \sup_{U \in \mathcal{X}} \frac{B(U, W; \mu)}{\|U\|_{\mathcal{X}} \|W\|_{\mathcal{X}}} \geq \hat{\beta}_0, \quad \forall \mu \in D.$

- the FE and RB approximations satisfy an equivalent inf-sup condition

- define the residual $r(W; \mu) = F(W; \mu) - B(U_N, W; \mu)$ and apply Babuška theorem
RB Method: a posteriori error estimation

**Goal:** provide **rigorous**, **sharp** and **inexpensive** estimators for the **error on the solution** variables and for the **error on the cost functional**

**General noncoercive (Babuška) framework** (as Stokes problems, e.g. [Rozza et al., 2010]):

- Recall the compact (weakly coercive) formulation: given \( \mu \in \mathcal{D} \),

  \[
  \text{find } U(\mu) \in \mathcal{X} \text{ s.t.: } B(U(\mu), W; \mu) = F(W; \mu) \quad \forall W \in \mathcal{X}.
  \]

  Note that \( \exists \beta_0 > 0 : \beta(\mu) = \inf_{W \in \mathcal{X}} \sup_{U \in \mathcal{X}} \frac{B(U, W; \mu)}{\|U\|_{\mathcal{X}} \|W\|_{\mathcal{X}}} \geq \beta_0, \quad \forall \mu \in \mathcal{D} \).

  the FE and RB approximations satisfy an equivalent inf-sup condition

- define the residual \( r(W; \mu) = F(W; \mu) - B(U_N, W; \mu) \) and apply Babuška theorem

**A posteriori error estimation on the solution variables**

\[
(\|x^N(\mu) - x_N(\mu)\|_{\mathcal{X}}^2 + \|p^N(\mu) - p_N(\mu)\|_{Q}^2)^{1/2} \leq \frac{\|r(\cdot; \mu)\|_{\mathcal{X}'} + \Delta_N(\mu)}{\beta_{LB}(\mu)} := \Delta_N(\mu)
\]

- \( 0 < \beta_{LB}(\mu) \leq \beta^N(\mu) \) is a constructible lower bound given by the successive constraint method (SCM) [R., Huynh, Manzoni 2010]

- thanks to the affinity assumption, we can provide the standard Offline/Online stratagem for the efficient computation of \( \|r(\cdot; \mu)\|_{\mathcal{X}'} \)
RB Method: a posteriori error estimation

Goal: provide rigorous, sharp and inexpensive estimators for the error on the solution variables and for the error on the cost functional.
RB Method: a posteriori error estimation

**Goal:** provide rigorous, sharp and inexpensive estimators for the error on the solution variables and for the error on the cost functional

Define the error on the cost functional as

\[ J^N(\mu) - J_N(\mu) = J(y^N(\mu), u^N(\mu); \mu) - J(y_N(\mu), u_N(\mu); \mu) \]

with standard arguments (e.g. [Becker et al., 2000; Dedé, 2010]) we can easily show that

\[ J^N(\mu) - J_N(\mu) = \frac{1}{2} r(U^N(\mu) - U_N(\mu); \mu) \]
Goal: provide rigorous, sharp and inexpensive estimators for the error on the solution variables and for the error on the cost functional

Define the error on the cost functional as

\[ J^N(\mu) - J_N(\mu) = J(y^N(\mu), u^N(\mu); \mu) - J(y_N(\mu), u_N(\mu); \mu) \]

with standard arguments (e.g. [Becker et al., 2000; Dedé, 2010]) we can easily show that

\[ J^N(\mu) - J_N(\mu) = \frac{1}{2} r(U^N(\mu) - U_N(\mu); \mu) \]

The error estimator \( \Delta_{J_N}(\mu) \) does not need any additional ingredients than those already discussed:

- the efficient computation of \( \| r(\cdot, \mu) \|_{X'} \)
- the calculation of the lower bound \( \hat{\beta}_{LB}(\mu) \)
RB Method: the “complete game”

- **Offline stage** involves precomputation of FE structures required for the RB space construction and the certified error estimates.

- **Online stage** has complexity only depending on $N$ and allows resolution of the Optimal Control Problem for any $\mu \in D$ with a certified error bound.
Numerical results - I

Optimal control of parametrized advection-diffusion equations
Boundary control for a Graetz convection-diffusion problem

the desired function is parameter dependent:

\[ y_d(\mu) = \mu_3 \chi_{\hat{\Omega}_o} \]

parameter domain: \( D = [6, 20] \times [1, 3] \times [0.5, 3] \)

We consider the following optimal control problem:

\[
\begin{align*}
\text{minimize} & \quad J(y_o(\mu), u_o(\mu); \mu) = \frac{1}{2} \| y_o(\mu) - y_d(\mu) \|^2_{L^2(\hat{\Omega}_o)} + \frac{\alpha}{2} \| u_o(\mu) \|^2_{L^2(\Gamma^o_C)} \\
\text{subject to} & \quad \begin{cases}
- \frac{1}{\mu_1} \Delta y_o(\mu) + x_{o2}(1 - x_{o2}) \frac{\partial y_o(\mu)}{\partial x_{o1}} = 0 & \text{in } \Omega^o(\mu) \\
y_o(\mu) = 1 & \text{on } \Gamma^o_D \\
\frac{1}{\mu_1} \nabla y_o(\mu) \cdot n = u_o(\mu) & \text{on } \Gamma^o_C(\mu) \\
\frac{1}{\mu_1} \nabla y_o(\mu) \cdot n = 0 & \text{on } \Gamma^o_N(\mu),
\end{cases}
\end{align*}
\]

- \( \mu_1 \) Péclet number, \( \mu_2 \) heigth of the channel
- the flow has an imposed temperature at the inlet and a parabolic convection field
- observation in the thermal layer \( \hat{\Omega}_o \)
Boundary control for a Graetz convection-diffusion problem

\[
\min_J(y_o(\mu), u_o(\mu); \mu) = \frac{1}{2} \|y_o(\mu) - y_d(\mu)\|^2 + \frac{\alpha}{2} \|u_o(\mu)\|^2
\]

s.t.

\[
- \frac{1}{\mu_1} \Delta y_o(\mu) + x_{o2}(1 - x_{o2}) \frac{\partial y_o(\mu)}{\partial x_{o1}} = 0 \quad \text{in } \Omega_o(\mu)
\]

\[
\frac{1}{\mu_1} \nabla y_o(\mu) \cdot n = u_o(\mu) \quad \text{on } \Gamma^o_C(\mu)
\]

\[
\text{+ BCs}
\]

Functional spaces:

\[
Y_o = H^1_{\Gamma_D}(\Omega_o) \quad U_o = L^2(\Gamma^o_C) \quad Q_o \equiv Y_o
\]

observation space:

\[
\mathcal{Z}_o = L^2(\hat{\Omega}_o)
\]

- the problem is mapped to a reference domain \( \Omega = \Omega_o(\mu_{\text{ref}}) \) with \( \mu_{\text{ref}} = (\cdot, 1, \cdot) \)
- we obtain an affine decomposition with \( Q_a = 1 \), \( Q_b = 5 \), \( Q_f = 1 \), \( Q_g = 4 \)
- we fixed \( \alpha = 0.07 \) and we used \( P^1 \) finite elements for the FE approximation of the state, adjoint and control variables
Boundary control for a Graetz convection-diffusion problem

Comparison of Brezzi inf-sup constant $\beta_N(\mu)$ and coercivity constant of the state equation $\alpha^N(\mu)$ as a function of $\mu_2$.

<table>
<thead>
<tr>
<th>Number of FE dof $N$</th>
<th>8915</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of parameters $P$</td>
<td>3</td>
</tr>
<tr>
<td>Number of RB functions $N$</td>
<td>39</td>
</tr>
<tr>
<td>Dimension of RB linear system</td>
<td>$39 \cdot 5$</td>
</tr>
<tr>
<td>Affine operator components $Q$</td>
<td>6</td>
</tr>
</tbody>
</table>

Linear system dimension reduction: 50:1

FE evaluation $t_{FE}(s)$: 14.5

RB evaluation $t_{RB}^{\text{online}}(s)$: 0.1

RB evaluation $t_{RB}^{\text{offline}}(s)$: 3970

Representative solution for $\mu = (12, 2, 2.5)$. 

State $y_N$ and adjoint $y_N$.
Boundary control for a Graetz convection-diffusion problem

Lower bound $\hat{\beta}_{LB}(\mu)$ for the Babuška inf-sup constant $\hat{\beta}(\mu)$ as a function of the parameter $\mu_1$.

<table>
<thead>
<tr>
<th>Number of FE dof $N$</th>
<th>8915</th>
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<tbody>
<tr>
<td>Number of parameters $P$</td>
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</tr>
<tr>
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<td></td>
</tr>
<tr>
<td>Affine operator components $Q$</td>
<td>6</td>
</tr>
</tbody>
</table>

Linear system dimension reduction $50:1$
FE evaluation $t_{FE}$ (s) $14.5$
RB evaluation $t_{RB}^{\text{online}}$ (s) $0.1$
RB evaluation $t_{RB}^{\text{offline}}$ (s) $3970$

Error estimation (•) and true error (○) for the solution (left) and the cost functional (right).
Numerical results - II

Optimal control of parametrized Stokes problems
How to extend the method

\[
\begin{aligned}
\text{minimize } J(v, p, u; \mu) &= \frac{1}{2} m(v - v_d(\mu), v - v_d(\mu); \mu) + \frac{\alpha}{2} n(u, u; \mu) \\
\text{subject to } \begin{cases} 
    a(v, \xi; \mu) + b(\xi, p; \mu) = \langle F(\mu), \xi \rangle + c(u, \xi; \mu) & \forall \xi \in V, \\
    b(v, \tau; \mu) = \langle G(\mu), \tau \rangle & \forall \tau \in M,
\end{cases}
\end{aligned}
\]

Functional setting:

\[
V = [H^1(\Omega)]^2 \quad M = L^2(\Omega) \quad \text{velocity and pressure spaces}
\]

\[
Y = V \times M \quad \text{state space}, \quad Q \equiv Y \quad \text{adjoint space}, \quad U \quad \text{control space}
\]

- two nested saddle-point
  - outer: optimal control
  - inner: Stokes constraint

- reduced basis functions computed by solving \( N \) times the FE approximation (with stable spaces pair for velocity and pressure variables)

- stability of the RB approximation of the Stokes constraint fulfilled by introducing suitable \textit{supremizer operators} [R., Veroy, 2007; R., Huynh, Manzoni 2010]

- stability of the RB approximation of the whole optimal control problem fulfilled by defining suitable \textit{aggregated spaces} for the state and adjoint variables
Test 1: confirming theoretical results (Couette flow)

\[\begin{align*}
\Omega_0(\mu) & = (0, \mu_1) \times (1, 0) \\
\Gamma_D^o & = (0, 0) \times (1, 0) \\
\Gamma_N^o & = (1, \mu_1) \times (0, 0) \\
\end{align*}\]

\[\mu_1: \text{height of the pipe}\]

forcing term state equation: \( f_0(\mu) = (0, -\mu_2) \)

parameter domain: \( \mathcal{D} = [0.5, 2] \times [0.5, 1.5] \)

We consider the following optimal control problem

\[
\begin{align*}
\text{minimize} \quad & J(v_0(\mu), p_0(\mu), u_0(\mu)) = \frac{1}{2} \| v_{o1}(\mu) - x_{o2} \|^2_{L^2(\Omega_0)} + \frac{\alpha}{2} \| u_0(\mu) \|^2_{L^2(\Omega_0)} \\
\text{s.t.} \quad & \begin{cases}
-\nu \Delta v_0 + \nabla p_0 = f_0(\mu) + u_0 & \text{in } \Omega_0(\mu) \\
\text{div } v_0 = 0 & \text{in } \Omega_0(\mu) \\
v_{o1} = x_{o2}, \quad v_{o2} = 0 & \text{on } \Gamma_D^o(\mu) \\
- p_0 n_{o1} + \nu \frac{\partial v_{o1}}{\partial n_{o1}} = 0, \quad v_{o2} = 0 & \text{on } \Gamma_N^o(\mu),
\end{cases}
\end{align*}
\]

Example of uncontrolled flow
velocity on the left
pressure on the right
Test 1: confirming theoretical results (Couette flow)

\[
\min \ J(v_o(\mu), p_o(\mu), u_o(\mu)) = \frac{1}{2} \|v_{o1}(\mu) - x_{o2}\|^2 + \frac{\alpha}{2} \|u_o(\mu)\|^2
\]

\[
\begin{cases}
-\nu \Delta v_o + \nabla p_o = f_o(\mu) + u_o & \text{in } \Omega_o(\mu) \\
\text{div} \ v_o = 0 & \text{in } \Omega_o(\mu) \\
v_{o1} = x_{o2}, \ v_{o2} = 0 & \text{on } \Gamma_D^o(\mu) \\
-\rho_o n_{o1} + \nu \frac{\partial v_{o1}}{\partial n_{o1}} = 0, \ v_{o2} = 0 & \text{on } \Gamma_N^o(\mu),
\end{cases}
\]

Functional spaces: velocity \( V_o = [H_1^1(\Omega_o)]^2 \) and pressure \( M_o = L^2(\Omega_o) \)

\[
\text{state } Y_o = V_o \times M_o \quad \text{adjoint } Q_o \equiv Y_o \quad \text{control } U_o = [L^2(\Omega_o)]^2
\]

- the problem is mapped to a reference domain \( \Omega = \Omega_o(\mu_{\text{ref}}) \) with \( \mu_{\text{ref}} = (1, \cdot) \)
- we obtain an affine decomposition with \( Q_a = 1, \ Q_b = 4, \ Q_f = 1, \ Q_g = 5 \)
- we fixed \( \alpha = 0.008, \ \nu = 0.1 \) and used \( P^2-P^1 \) Taylor-Hood finite elements for the FE approximation of the velocity and pressure variables (for the control variable we used \( P^2 \) finite elements)
Test 1: confirming theoretical results (Couette flow)

comparison of Brezzi inf-sup constant $\beta_N(\mu)$ and Babuška in-sup constant $\alpha^N(\mu)$ of the Stokes operator.

Representative solutions for $\mu = (1.7, 1.5)$: (state) pressure on the left, (state) velocity in the middle, control on the right.
Test 1: confirming theoretical results (Couette flow)

Lower bound \( \hat{\beta}_{LB}(\mu) \) for the Babuška inf-sup constant \( \hat{\beta}(\mu) \) as a function of the parameter \( \mu \).

<table>
<thead>
<tr>
<th>Number of FE dof ( N' )</th>
<th>17439</th>
</tr>
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<tbody>
<tr>
<td>Number of parameters ( P )</td>
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<tr>
<td>Number of RB functions ( N )</td>
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<td>Dimension of RB linear system</td>
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<td>Affine operator components ( Q )</td>
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<td>Linear system dimension reduction</td>
<td>80:1</td>
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<tr>
<td>FE evaluation ( t_{FE} ) (s)</td>
<td>16.1</td>
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<tr>
<td>RB evaluation ( t_{RB}^{\text{online}} ) (s)</td>
<td>0.1</td>
</tr>
<tr>
<td>RB evaluation ( t_{RB}^{\text{offline}} ) (s)</td>
<td>7820</td>
</tr>
</tbody>
</table>

Estimation (●) and true error (○) for the solution (left) and the cost functional (right).
An (idealized) application in haemodynamics: a data assimilation problem

- we consider an inverse boundary problem in hemodynamics, inspired by the recent works [D’Elia et al., 2011; Perego et al., 2011]
- simplified model of an arterial bifurcation
- we suppose to have a measured velocity profile on the red section, but not the Neumann flux on $\Gamma_C$ that will be our control variable
- starting from the velocity measures we want to find the control variable in order to retrieve the velocity and pressure fields in the whole domain.

given new geometrical configuration ($\mu_{geom}$) and parametrized measurements $\mu_{meas}$ on the red section

find the unknown Neumann boundary condition on $\Gamma_C$ and retrieve the whole velocity and pressure fields
An (idealized) application in haemodynamics: a data assimilation problem

The state velocity and pressure variables \( \{v_o, p_o\} \) satisfy the following Stokes problem in the original domain \( \Omega_o(\mu) \):

\[
\begin{align*}
-\nu \Delta v_o + \nabla p_o &= 0 & \text{in } \Omega_o(\mu), \\
\text{div } v_o &= 0 & \text{in } \Omega_o(\mu), \\
v_o &= 0 & \text{on } \Gamma_D(\mu), \\
v_o &= g(\mu_{in}) & \text{on } \Gamma_{in}(\mu), \\
-p_o n_o + \nu \frac{\partial v_o}{\partial n_o} &= u_o & \text{on } \Gamma_C(\mu),
\end{align*}
\]

Then we consider the following (parametrized) cost functional to be minimized

\[
J(v_o(\mu), p_o(\mu), u_o(\mu); \mu) = \frac{1}{2} \int_{\Gamma_{obs}} |v_o(\mu) - v_d(\mu)|^2 \, d\Gamma_o + \frac{\alpha_1}{2} \int_{\Gamma_C} |\nabla u_o(\mu)|^2 \, d\Gamma_o + \frac{\alpha_2}{2} \int_{\Gamma_C} |u_o(\mu)|^2 \, d\Gamma_o
\]
An (idealized) application in haemodynamics: a data assimilation problem

Average computed error between the truth FE solution and the RB approximation.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of FE dof $\mathcal{N}$</td>
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<td>Number of parameters $P$</td>
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<td>Dimension of RB linear system</td>
<td>$43 \cdot 13$</td>
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<tr>
<td>Affine operator components $Q$</td>
<td>71</td>
</tr>
</tbody>
</table>

FE evaluation $t_{FE}$ (s) 22  
RB evaluation $t_{RB}^{\text{online}}$ (s) 0.15

$\mu = (1, \pi/5, \pi/6, 1, 1.7, 2.2, 0.8, 1)$.  
$\mu = (1.2, \pi/6, \pi/6, 0.8, 2.5, 2.1, 0.3, 1)$.  

(EPFL) École Polytechnique Fédérale de Lausanne
we provided a **certified RB method** for the solution of parametrized optimal control problems with **high dimensional** control variable

- reduction to low-dimensionality of the whole control problem and not just of the state equation
- **key ingredient**: saddle-point formulation of the optimal control problem
- full Offline/Online decomposition strategy

...
References


For more information see

http://sma.epfl.ch/~rozza

http://cmcs.epfl.ch/

http://augustine.mit.edu