Non-conforming high order approximations of the elastodynamics equation

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Journées Lions-Magenes
Plan of the talk

1. Introduction and motivations
2. Non-conforming high order methods
3. Theoretical analysis
4. Some applications
Introduction: what is a seismic wave?

- The **wiggles** in a seismogram indicate that the ground is being (or was) **vibrated** by seismic waves.
- Seismic waves are **propagating vibrations** that carry **energy** from the source of the shaking outward in all directions (e.g., stone is thrown into the water).
- Seismic waves that are set up during an earthquake are more complex than those on the pond.

ChristChurch (NZ), 2011-02-22 at 8.06.48 AM (magnitude 6.3)

Circles on water

European earthquake potential hazard
The are many different seismic waves. The most important ones are

- Compressional or P (primary)
- Transverse or S (secondary)
- Love
- Rayleigh

An earthquake radiates P and S waves in all directions.

The interaction of the P and S waves with Earth’s surface → surface waves.

P and S waves travel at different speeds (used to locate earthquakes)
Introduction: what is a seismic wave? (cont’d)

Prof. Ammon’s course “An Introduction to Earthquakes”, PSU (http://eqseis.geosc.psu.edu/~cammon/)

P-Waves (Compressional)

- The ground is vibrated in the direction the wave is propagating.
- Travel through all types of media.
- \( C_P = \sqrt{\frac{\lambda + 2\mu}{\rho}} \) (P-wave velocity)
- Typical speed: \( \sim 1 \rightarrow 14 \text{ km/sec} \)

S-Waves (Transverse)

- The ground is vibrated in the perpendicular direction to that the wave is propagating.
- Travel only through solid media.
- \( C_S = \sqrt{\frac{\mu}{\rho}} \) (S-wave velocity)
- Typical speed: \( \sim 1 \rightarrow 8 \text{ km/sec} \)

Much of the damage close to an earthquake is the result of strong shaking caused by S-waves.
The mathematical problem

Equilibrium equations for an elastic medium subjected to an external force $\mathbf{f}^{\text{ext}}$ read

$$\rho \partial_{tt} \mathbf{u} - \nabla \cdot \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{f}^{\text{ext}} \quad \text{in } \Omega \times (0, T)$$

- $\mathbf{u}$ displacement of the medium
- $\rho$ material density
- $\epsilon(\mathbf{u}) = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^\top)$ strain tensor
- $\boldsymbol{\sigma}(\mathbf{u}) = \lambda \nabla \cdot \mathbf{u} \mathbf{I} + 2\mu \epsilon(\mathbf{u}) = \mathbf{D} : \epsilon(\mathbf{u})$ stress tensor (Hooke’s Law)
- $\lambda$, $\mu$ Lamé elastic coefficients
- suitable boundary conditions on $\partial \Omega$
- suitable initial conditions

Variational Formulation

For any time $t \in (0, T]$ find $\mathbf{u} = \mathbf{u}(t) \in \mathcal{V} = H^1_D(\Omega)$ such that

$$(\rho \partial_{tt} \mathbf{u}, \mathbf{v})_\Omega + (\boldsymbol{\sigma}(\mathbf{u}), \epsilon(\mathbf{v}))_\Omega = \mathcal{L}(\mathbf{v}) \quad \forall \mathbf{v} \in \mathcal{V}$$

In applications, $\mathbf{f}^{\text{visc}} = -2\rho \zeta \partial_t \mathbf{u} - \rho \zeta^2 \mathbf{u}$ is usually added ($\zeta$ decay factor).
Numerical solution of (seismic) wave propagation problems

- Waves are typically propagated over many periods
  \( \Rightarrow \) control the numerical dispersion and dissipation errors
- Accurate description of the behaviour at interfaces
  \( \Rightarrow \) high-order accuracy
- The size of the bodies excited is large relative to the wavelengths of interest
  \( \Rightarrow \) efficient and scalable implementation
- Complex geometries and strong contrasts in the media
  \( \Rightarrow \) methods that accommodate non-conformities

**Non-Conforming Spectral Element Methods**

- Discontinuous Galerkin Spectral Element Method (DGSEM)
  [Reed & Hill, 1973], [LeSaint & Raviart, 1974]
- Mortar Spectral Element Method (MSEM)
  [Bernardi, Maday & Patera, 1994]
  \( \Rightarrow \) Weak continuity across interfaces
  \( \Rightarrow \) Non-conforming meshes
  \( \Rightarrow \) Non-uniform polynomial approximation degrees
DGSEMs: decompositions

1 - Macro Level (this is usually provided by Engineers)

- Subdivide $\Omega$ into $R$ non-overlapping macro-regions $\Omega_k$: i.e., $\overline{\Omega} = \bigcup_{k=1}^{R} \Omega_k$.
- For each $\Omega_k$, define a polynomial approximation order $N_k$.

2 - Meso Level

- For each $\Omega_k$, introduce a conforming partition $\mathcal{T}_{h_k}$, i.e., $\overline{\Omega}_k = \bigcup_{j=1}^{J_k} \overline{\Omega}_k^j$.
- Subdivide the skeleton in $M$ elementary components $\gamma_m$ (edges).
DGSEMs: decompositions

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DGSEMs: approximation space

For each $\Omega_k$, and for each $\Omega^j_k \subset \Omega_k$

- $F^j_k : \hat{\Omega} = (-1, 1)^d \rightarrow \Omega^j_k$.
- $\{\hat{x}_i\} : \text{set of of } n = (N_k + 1)^d \text{ Gauss–Legendre-Lobatto (GLL) points.}$
- $\{L_i\} : \text{set of the Lagrangian polynomials based on the approximation points } \hat{x}_i.$

**Approximation space**

$$X_\delta(\Omega_k) = \{v_\delta \in C^0(\hat{\Omega}_k) : v_\delta|_{\Omega^j_k} \circ F^j_k \in Q_{N_k}(\hat{\Omega}) \ \forall \ \Omega^j_k\}.$$  

$$V_\delta = \{v_\delta \in \left[L^2(\Omega)\right]^d : v_\delta|_{\Omega_k} \in [X_\delta(\Omega_k)]^d \ \forall \ Omega_k \ \text{and} \ v_\delta|_{\Gamma_D} = 0\},$$
Semi-discrete DG formulation

\( \forall t \in (0, T] \) find \( u_\delta = u_\delta(t) \in V_\delta \) such that

\[
(r \partial_{tt} u_\delta, v)_\Omega + A_{DG}(u_\delta, v) = L(v) \quad \forall v \in V_\delta
\]

where \( A_{DG}(\cdot, \cdot) = A(\cdot, \cdot) + B(\cdot, \cdot) \)

\[
A(u, v) = (\sigma(u), \varepsilon(v))_\Omega \quad \forall u, v \in V_\delta
\]

\( B(\cdot, \cdot) \) controls the jumps of the discrete solution, and

\[
L(v) = (f^{ext}, v)_\Omega + BCs \quad \forall v \in V_\delta
\]
Trace operators and definition of $\mathcal{B}(\cdot, \cdot)$

On each edge $\gamma$, we define the average and jump operators for $\mathbf{v}$ and $\sigma$

$$
\{ \mathbf{v} \} = \frac{1}{2} (\mathbf{v}_i + \mathbf{v}_j), \quad \llbracket \mathbf{v} \rrbracket = \mathbf{v}_i \otimes \mathbf{n}_i + \mathbf{v}_j \otimes \mathbf{n}_j,
$$

$$
\{ \sigma \} = \frac{1}{2} (\sigma_i + \sigma_j), \quad \llbracket \sigma \rrbracket = \sigma_i \cdot \mathbf{n}_i + \sigma_j \cdot \mathbf{n}_j,
$$

$\mathcal{B}(\mathbf{u}, \mathbf{v}) = \sum_{\gamma} - (\{ \sigma(\mathbf{u}) \}, \llbracket \mathbf{v} \rrbracket)_\gamma + \theta (\llbracket \mathbf{u} \rrbracket, \{ \sigma(\mathbf{v}) \})_\gamma + \eta_\gamma (\llbracket \mathbf{u} \rrbracket, \llbracket \mathbf{v} \rrbracket)_\gamma$

$\theta = -1, 1, 0 \rightarrow$ SIPG, NIPG, IIPG [Arnold, Brezzi, Cockburn, Marini, SINUM 2001/02].

$$
\eta_\gamma = \alpha \{ \lambda + 2\mu \} A \frac{N^2_\gamma}{h_\gamma}, \quad \text{with } \alpha > 0 \text{ at our disposal}
$$

where $\{ q \}_A = \text{harmonic average}, \ N_\gamma = \max \{ N_i, N_j \}, \ h_\gamma = \min \{ h_i, h_j \}$
Boundedness and coercivity of $A_{DG}(\cdot, \cdot)$

Let $V(\delta) = V_\delta \oplus (H^2(\Omega) \cap H^1_{\Gamma_D}(\Omega))$, and define

\[
\|v_\delta\|_{DG}^2 = \|D^{1/2} \varepsilon(v_\delta)\|_{0,\Omega}^2 + \sum_\gamma \eta_\gamma \|v_\delta\|_{0,\gamma}^2 \quad \forall v_\delta \in V_\delta,
\]

\[
\|v\|_{DG}^2 = \|v\|_{DG}^2 + \sum_{k=1}^K \left( \frac{h_k}{N_k^2} \right)^2 \|v\|_{2,\Omega_k}^2 \quad \forall v \in V(\delta).
\]

There exist $M, \kappa > 0$ such that:

\[
A_{DG}(u, v) \leq M \|u\|_{DG} \|v\|_{DG} \quad \forall u, v \in V(\delta)
\]

\[
A_{DG}(u_\delta, u_\delta) \geq \kappa \|u_\delta\|_{DG}^2 \quad \forall u_\delta \in V_\delta.
\]

For SIPG/IIPG methods coercivity holds provided the stability parameter $\alpha$ is chosen sufficiently large.
Semi-discrete error estimates

Provided that, for any \( t \in [0, T] \), the exact solution \( u = u(t) \in H^s(\Omega), \ s \geq 2 \)

\[
\sup_{t \in [0, T]} \| \| (u - u_\delta)(t) \| \|_{DG} \lesssim \frac{h^{r-1}}{N^{s-3/2}} \quad r = \min\{N + 1, s\}
\]

Remarks:

- Optimal in \( h \), suboptimal in \( N \) by a factor of \( N^{1/2} \)
  
  Know for DG methods (cf. [Houston, Schwab, Süli, 2000], Perugia and Schötzau, 2002].)

- A similar method was introduced and analyzed by [Riviè re, Wheeler, 2003] and [Riviè re, Shaw, Whiteman, 2007] but the bilinear form contains the additional term

\[
\sum_\gamma \eta_\gamma (\text{\textbf{[} } \partial_t u_\delta \text{\textbf{]} } , \text{\textbf{[} } v_\delta \text{\textbf{]} } )_\gamma
\]

- See also [Grote et al., 2006]

* Antonietti, Mazzieri, Quarteroni, Rapetti, CMAME, to appear
Time integration (leap-frog)

\[
\begin{aligned}
\begin{cases}
M \ddot{U}_\delta + A^{DG} U_\delta = F \\
U_\delta(0) = u_{0,\delta} \\
\dot{U}_\delta(0) = u_{1,\delta}
\end{cases}
\end{aligned}
\]

- Set \( \Delta t = T/N \), and \( t_n = n\Delta t \), for \( n = 0, \ldots, N \)

- leap-frog scheme:

\[
\begin{aligned}
M U^n_\delta &= [2M - (\Delta t)^2 A^{DG}] U^{n-1}_\delta - M U^{n-2}_\delta + (\Delta t)^2 F^{n-1} \quad n \geq 2 \\
M U^1_\delta &= [M - \frac{(\Delta t)^2}{2} A^{DG}] u_{0,\delta} - \Delta t M u_{1,\delta} + \frac{(\Delta t)^2}{2} F^0
\end{aligned}
\]
Fully-discrete error estimates

It holds

\[ \| u(t^n) - u^n_s \|_{DG} \lesssim \Delta t^2 + \frac{h^{r-1}}{N^{s-3/2}} \]

where \( r = \min(N + 1, s) \).

Remarks:

- Optimal in \( h \), suboptimal in \( N \) by a factor of \( N^{1/2} \)
- Proof: follows as in [Rivière, Shaw, Whiteman, 2003].
- Cf. also [Grote, Schötzau 2009]
Numerical results (toy problem):

- $\Omega = (0, 1)^2$, $\Gamma_D \equiv \partial \Omega$, $\rho = \lambda = \mu = 1$
- $u(t, x, y) = \sin(\sqrt{2}\pi t) \left[ -\sin^2(\pi x) \sin(2\pi y); \sin(2\pi x) \sin^2(\pi y) \right]$

The error is computed at the observation time $t^* = 2 \ (\Delta t = 1 \cdot 10^{-4})$. 

$N_1 = N_2 = 2$
Numerical results: elastic scattering by a circle

- $\Omega = (0, 2000)m \times (0, 2000)m$
- Circular inclusion
  - Radius $R = 250m$
  - Center $C = (1000, 1000)m$
- Plane wave ($\omega = 15Hz$).

**Case 1 - STIFF INCLUSION**

- Square: $C_S = 1155m/s$, $C_P = 2000m/s$, $\rho = 1Kg/m^3$
- Circle: $C_S = 1732m/s$, $C_P = 3000m/s$, $\rho = 2Kg/m^3$
- $h_{Square} \approx 70m$
- $h_{Circle} \approx 100m$

**Case 2 - SOFT INCLUSION**

- Square: $C_S = 2310m/s$, $C_P = 4000m/s$, $\rho = 2Kg/m^3$
- Circle: $C_S = 600m/s$, $C_P = 1400m/s$, $\rho = 1Kg/m^3$
- $h_{Square} \approx 70m$
- $h_{Circle} \approx 35m$
Displacements along the circular inclusion

Conforming SEM, DGSEM (N=4)

STIFF vs SOFT inclusion

SOFT inclusion: snapshots $t = 1.5, 2.25$ s.
Geophysical application I: Gubbio (PG) alluvial basin

- It's a strong seismicity region (between Eurasian and African plates).
- Seismic events characterized by magnitude up to 7.
- Umbria earthquake (26.09.97, h11:40, $M_W$ 6.0).
- Alluvium is loose, unconsolidated (not cemented together into a solid rock) soil.

Courtesy of R. Paolucci, DIS, PoliMi
Geophysical application I: Gubbio alluvial basin

NON CONFORMING (48091 spectral nodes)

CONFORMING (61385 spectral nodes)

Deep Structure: Mechanical Parameters

<table>
<thead>
<tr>
<th>Layer</th>
<th>$V_p$ [m/s]</th>
<th>$V_s$ [m/s]</th>
<th>$\rho$ [Kg/m$^3$]</th>
<th>$Q_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basin</td>
<td>700</td>
<td>350</td>
<td>1900</td>
<td>100</td>
</tr>
<tr>
<td>Bedrock</td>
<td>3500</td>
<td>1800</td>
<td>2200</td>
<td>100</td>
</tr>
</tbody>
</table>

- Polynomial degree $N = 4$.
- External force: Ricker-type function at $x_s$
Geophysical application II: Time histories

High order DG methods for elastodynamics

Journées Lions-Magenes 20 / 26
The Acquasanta bridge on the Genova-Ovada railway.

Remarkable structure both for the site features and the local geological and geomorphological conditions.

The foundations of several piers are located on weak rock.

Some instability problems have been detected in the past on the valley slope towards Ovada.
Geophysical application II: Acquasanta bridge, Liguria Italy

Discretization parameters

<table>
<thead>
<tr>
<th>Layer</th>
<th>(c_P) [(m/s)]</th>
<th>(c_S) [(m/s)]</th>
<th>(\rho) [(Kg/m^3)]</th>
<th>(Q(\zeta))</th>
<th>(N) (SEM)</th>
<th>(N) (DGSEM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bridge</td>
<td>1218</td>
<td>716.7</td>
<td>1750</td>
<td>100</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1100</td>
<td>635</td>
<td>2400</td>
<td>100</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1100</td>
<td>635</td>
<td>2400</td>
<td>250</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>1700</td>
<td>982.5</td>
<td>2600</td>
<td>250</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>2300</td>
<td>1330</td>
<td>2800</td>
<td>250</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>2300</td>
<td>1330</td>
<td>2800</td>
<td>250</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

Seismic moment is given at the point \(S\).
Geophysical application II: Acquasanta bridge, Liguria Italy

DGSEM ($\approx 40000$ d.o.f.)

SEM ($\approx 100000$ d.o.f.)

R1 valley (Displ. vertical)

R2 bridge (Displ. Horizontal)
Geophysical application III: Christchurch earthquake, 22.02.2011, $M_W$ 6.2

- Geological map of Christchurch provided by GNS (New Zealand research institute) [Forsyth et al. 2008]
- Geological cross-section: numbers refer to the stations within a 40 km radius from the epicenter

Courtesy of R. Paolucci, DIS, PoliMi
Geophysical application III: Christchurch earthquake, 22.02.2011, $M_W$ 6.2

- **DGSEM approach** for the Central Business District (CBD)
- **Not-Honoring** approach for the contact of alluvial and soft sediments [Guidotti et al., 2011]
- Soft-soil $h \approx 25m$, $N = 2$
- Central Business District $h \approx 5m$, $N = 1$

<table>
<thead>
<tr>
<th>Layer</th>
<th>$c_S [m/s]$</th>
<th>$c_P [m/s]$</th>
<th>$\rho [Kg/m^3]$</th>
<th>$Q(\zeta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Foundation</td>
<td>400</td>
<td>650</td>
<td>2400</td>
<td>200</td>
</tr>
<tr>
<td>Building</td>
<td>100</td>
<td>163</td>
<td>2400</td>
<td>200</td>
</tr>
</tbody>
</table>

Mechanical parameters [Bielak et al. 2010]
Conclusions

Non-Conforming Spectral Element Methods

- preserve the spectral accuracy typical of high order methods
- have a high accuracy and extremely low dispersion error
- improve the geometrical flexibility of spectral discretizations
- can naturally handle non-conforming grids and complex geometries
  - independent local meshes
  - different spectral approximation degrees
- allow the study of combined effect of strong lateral variations of soil properties and topographic amplification in heterogeneous media
Ongoing work

- Simulation of 3D complex earthquake scenarios
- Extension to tetrahedral meshes
- Implicit/high order time-integration schemes
- Effective preconditioners for the resulting system of equations
- Release of the new code SPEED
- SPectral Elements in Elastodynamics with Discontinuous Galerkin
  jointly developed at MOX and DIS - PoliMI

Acknowledgements

We are grateful to R. Paolucci, C. Smerzini (DIS, PoliMi), and M. Stupazzini (Munich RE) for providing input for the geophysical applications and their help in the analysis of numerical results.

THANK YOU FOR YOUR ATTENTION!
Richter scale: quantify the “size” of an earthquake

- $M_L$ (local magnitude scale): is approximately determined from the logarithm of the amplitude of waves recorded by seismographs.
- The energy released ((closely correlated to its destructive power)) scales with the $3/2$ power of the shaking amplitude
  
  **Example:** An earthquake that measures 5.0 on the Richter scale has a shaking amplitude 10 times larger than one that measures 4.0 (and corresponds to an energy release of $31.6 = (10^1)^{3/2}$ times greater)
- In the 1970s $M_L$ scale was replaced by the moment magnitude scale (MMS)
- MMS ($M_W$) measures the size of earthquakes in terms of the energy released:
  
  $$M_W = \frac{2}{3} \log_{10} M_0 - 10.7$$

- Seismic moment $M_0 = \mu AD$
  
  - $\mu$ is the shear modulus of the rocks involved in the earthquake
  - $A$ is the area of the rupture along the geologic fault where the earthquake occurred
  - $D$ is the average displacement on $A$
Stability of the leap-frog integration scheme

**Aim**

Establish the largest time step $\Delta t$ such that the solution remains bounded with respect to problems’s data, i.e.,

$$
\Delta t \leq C_{CFL} \frac{h}{C_P}
$$

where

- $h =$ mesh-size
- $C_P = \sqrt{(\lambda + 2\mu)/\rho}$ (=P-wave velocity)
- $C_{CFL} =$ Courant number (depends on $N$)

- Generalized eigenvalue problem on $\hat{\Omega}$ to estimate $C_{CFL}$

The CFL condition is satisfied if there exists a positive constant $c^*(\lambda, \mu, \alpha)$ such that

$$
C_{CFL} \leq \frac{c^*(\lambda, \mu, \alpha)}{N^2}.
$$

For the NIPG $c^*(\lambda, \mu, \alpha) = c^*(\lambda, \mu)$. For the SIPG $c^*(\lambda, \mu, \alpha) \sim \alpha^{-1/2}$
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Numerical results: Sampling ratio $\delta = 0.2$ and $C_P/C_S = 2$

Computed upper bound for $C_{\text{CFL}}$ as a function of $N$

<table>
<thead>
<tr>
<th>$N$</th>
<th>SEM</th>
<th>MSEM</th>
<th>SIPG</th>
<th>NIPG</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.3376</td>
<td>0.3333</td>
<td>0.2621</td>
<td>0.2163</td>
</tr>
<tr>
<td>3</td>
<td>0.1967</td>
<td>0.1770</td>
<td>0.1368</td>
<td>0.1045</td>
</tr>
<tr>
<td>4</td>
<td>0.1206</td>
<td>0.1118</td>
<td>0.0795</td>
<td>0.0607</td>
</tr>
<tr>
<td>5</td>
<td>0.0827</td>
<td>0.0776</td>
<td>0.0530</td>
<td>0.0400</td>
</tr>
<tr>
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<td>0.0570</td>
<td>0.0374</td>
<td>0.0281</td>
</tr>
<tr>
<td>7</td>
<td>0.0449</td>
<td>0.0434</td>
<td>0.0280</td>
<td>0.0210</td>
</tr>
<tr>
<td>8</td>
<td>0.0351</td>
<td>0.0342</td>
<td>0.0216</td>
<td>0.0162</td>
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<tr>
<td>9</td>
<td>0.0281</td>
<td>0.0277</td>
<td>0.0172</td>
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</tr>
<tr>
<td>10</td>
<td>0.0231</td>
<td>0.0227</td>
<td>0.0140</td>
<td>0.0105</td>
</tr>
</tbody>
</table>

$N$-rate: -1.8463, -1.8253, -1.9247, -1.9360

Remarks

- The constants for the **SIPG** are around 70% with respect the SEM, while for the **MSEM** are around 95%
- The **NIPG** has constants always more restrictive than those of **SIPG**

High-order (possibly implicit) time integration schemes (as Runge-Kutta methods) are under investigation
Gubbio (PG) alluvial basin: external force

\[ f(x, t) = g(x)h(t) \]

with

\[ g(x) = \delta(x - x_s)\hat{w} \quad h(t) = h_0[1 - 2\beta(t - t_0)^2]\exp(-\beta(t - t_0)^2) \]

Ricker-type function

where

- \( \delta = \) Dirac distribution
- \( x_s = \) Source location
- Direction of applied load [cf. Casadei et al, 2002]
- \( h_0 = \) scaling factor
- \( t_0 = 2 \text{ s} \) (time shift)
- \( \beta = \pi^2\nu_{max}^2s^{-1} \)
- \( \nu_{max} = \) (maximum frequency) = 3 Hz