Exit times of diffusions with incompressible drifts

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Drift-enhanced diffusions

We study the effects of incompressible drifts u on diffusion.

- Without incompressibility assumption $\nabla \cdot u = 0$, effects may be arbitrary
- Typically, an incompressible drift enhances diffusion due to extra mixing

We will show that the last statement is not always true!
(At least in some sense.)

Drifts and principal eigenvalues

Long time behavior of solutions of

$$\phi_t + u \cdot \nabla \phi = \Delta \phi$$

on a bounded domain $\Omega \subseteq \mathbb{R}^n$ with Dirichlet boundary conditions is governed by the principal eigenvalue λ^u of $L^u = -\Delta + u \cdot \nabla$. Namely, $\|\phi(t,\cdot)\| \sim e^{-\lambda^u t}$.

If ψ^u is the (normalized) principal eigenfunction, then

$$\lambda^u = \int_\Omega \psi^u L^u \psi^u dx = \int_\Omega |\nabla \psi^u|^2 dx \geq \inf_{w \in H_0^1(\Omega)} \frac{\|\nabla w\|^2}{\|w\|^2} = \lambda^0$$

by incompressibility of u. So addition of any incompressible $u \neq 0$ increases the rate of decay of ϕ (diffusion is always enhanced in this sense).

Drifts and explosions

So we have that

$$-\Delta\psi + u \cdot \nabla\psi = \lambda\psi$$

has a positive solution if and only if $\lambda = \lambda^u$, with $\lambda^u \geq \lambda^0$.

Now consider positive solutions of the explosion problem

$$-\Delta\psi + u \cdot \nabla\psi = \lambda e^{\psi}$$

on Ω, with Dirichlet boundary conditions (cold boundary).

- There is and explosion threshold λ_*^u such that positive solutions exist when $\lambda < \lambda_*^u$ and do not exists when $\lambda > \lambda_*^u$
- For $u = 0$: Joseph-Lundgren, Keener-Keller, Crandall-Rabinowitz (1970s)
- For general u: Berestycki-Novikov-Kiselev-Ryzhik (2009)

Drifts and explosions

Is it true that $\lambda_*^u \geq \lambda_*^0$ for all *u* (and all Ω)?

- Intuition says ‘yes’ because mixing should increase interaction between the hot spots inside Ω and the cold boundary $\partial\Omega$.
- Berestycki-Kagan-Joulin-Sivashinsky (1997) say ’no’, based on a numerical treatment of the case of Ω being a long rectangle.
- What is going on with the intuitive picture of drift-enhanced diffusion?

Drifts and exit times

Consider a diffusing particle, starting at $x \in \Omega$ and subject to the drift u. That is, consider the stochastic process

$$dX_t^x = u(X_t^x)dt + \sqrt{2}dW_t \qquad \text{and} \qquad X_0^x = x$$

where W_t is the Brownian motion (factor $\sqrt{2}$ can be scaled out).

If $\tau^u(x) \geq 0$ is the expected exit time of X_t^x from Ω, then

$$\begin{aligned} -\Delta\tau^u + u \cdot \nabla\tau^u &= 1 && \text{on } \Omega, \\ \tau^u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Main problem: Which incompressible drift (if any) maximizes τ^u in some sense? When is it $u = 0$?

- We will look at $\|\tau^u\|_\infty$ as the measure of size of τ^u.
- We can answer the second question in $\mathbb{R}^2$ in this sense.

Results

Theorem

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded simply connected domain with a C^1 boundary. Then $u = 0$ maximizes $\|\tau^u\|_\infty$ within the class of incompressible drifts if and only if Ω is a disc.

A stronger version of one direction of this result extends to $\mathbb{R}^n$:

Theorem

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with a C^1 boundary and u an incompressible drift on Ω. Then for any $p \in [1, \infty]$,

$$\|\tau^u\|_p \leq \|\tau^{0,\Omega^*}\|_p$$

where $\Omega^* \subseteq \mathbb{R}^n$ is a ball with $|\Omega^*| = |\Omega|$ and τ^{0,Ω^*} is the expected exit time for zero drift on Ω^*.

Proof of Theorem 2

Let $\tau = \tau^u$ and consider its symmetric rearrangement τ^* on Ω^*.

- If $\Omega_h = \{x \in \Omega \,|\, \tau(x) > h\}$ and $\Omega_h^* = \{x \in \Omega^* \,|\, \tau^*(x) > h\}$, then Ω_h^* is the ball (same center as Ω^*) with $|\Omega_h| = |\Omega_h^*|$.

Isoperimetric inequality gives

$$\int_{\partial\Omega_h^*} |\nabla\tau^*| d\sigma \int_{\partial\Omega_h^*} \frac{d\sigma}{|\nabla\tau^*|} = |\partial\Omega_h^*|^2 \leq |\partial\Omega_h^*|^2 \leq \int_{\partial\Omega_h} |\nabla\tau| d\sigma \int_{\partial\Omega_h} \frac{d\sigma}{|\nabla\tau|}$$

Co-area formula gives

$$\int_{\partial\Omega_h^*} \frac{1}{|\nabla\tau^*|} d\sigma = -\frac{d}{dh}|\Omega_h^*| = -\frac{d}{dh}|\Omega_h| = \int_{\partial\Omega_h} \frac{1}{|\nabla\tau|} d\sigma$$

So $\int_{\partial\Omega_h^*} |\nabla\tau^*| d\sigma \leq \int_{\partial\Omega_h} |\nabla\tau| d\sigma = |\Omega_h| = |\Omega_h^*|$ and hence

$$\frac{d\tau^*}{dr}\Big((\tau^*)^{-1}(h)\Big) = \frac{1}{|\partial\Omega_h^*|} \int_{\partial\Omega_h^*} |\nabla\tau^*| d\sigma \leq \frac{|\Omega_h^*|}{|\partial\Omega_h^*|} = \frac{d\tau^{0,\Omega^*}}{dr}\Big((\tau^*)^{-1}(h)\Big)$$

Therefore $\tau^* \leq \tau^{0,\Omega^*}$, and the result follows.

Proof of Theorem 1

Assume that $\Omega \subseteq \mathbb{R}^2$ is not a disc.

- Opposite direction follows from Theorem 2.
- We will show $\|\tau^u\|_\infty > \|\tau^0\|_\infty$ for some incompressible u.
- We will only consider drifts with $u \cdot \nu = 0$ on $\partial\Omega$ (tangential to the boundary).

Main idea: Look at infinite amplitude drift limit $\bar{\tau}^u = \lim_{A\to\infty} \tau^{Au}$:

$$\begin{aligned} -\Delta\tau^{Au} + Au \cdot \nabla\tau^{Au} &= 1 && \text{on } \Omega, \\ \tau^{Au} &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Show that there is u such that $\|\bar{\tau}^u\|_\infty > \|\bar{\tau}^0\|_\infty = \|\tau^0\|_\infty$. Then

$$\|\tau^{Au}\|_\infty > \|\tau^0\|_\infty$$

for $A \gg 1$.

Infinite amplitude drifts

Stream function of u tangential to $\partial\Omega$ is ψ such that $u = \nabla^\perp \psi = (-\psi_{x_2}, \psi_{x_1})$ and $\psi = 0$ on $\partial\Omega$.

- So level sets of ψ are streamlines of u.
- It follows from results of Freidlin-Wentzell (1993) that the limit $\bar{\tau}^u = \lim_{A\to\infty} \tau^{Au}$ exists and is uniform in Ω if the stream function ψ of u has a single critical point in Ω (i.e., ψ is a 'hill' or a 'valley').
- Then $\bar{\tau}^u = 0$ on $\partial\Omega$ and $\bar{\tau}^u$ satisfies the Freidlin problem

$$\bar{\tau}^u(y) = -\int_0^{\psi(y)} \frac{|\Omega_{\psi,h}|}{\int_{\Omega_{\psi,h}} \Delta\psi \, dx} \, dh$$

  with $\Omega_{\psi,h} = \{x \in \Omega \,|\, \psi(x) > h\}$. So $\bar{\tau}^u$ is constant on $\partial\Omega_{\psi,h}$.
- Let x_0 be the critical point of ψ and define

$$I(\psi) = \|\bar{\tau}^{\nabla^\perp\psi}\|_\infty = \bar{\tau}^{\nabla^\perp\psi}(x_0) = -\int_0^{\|\psi\|_\infty} \frac{|\Omega_{\psi,h}|}{\int_{\Omega_{\psi,h}} \Delta\psi \, dx} \, dh$$

  If $\nabla^\perp\psi$ maximizes $\|\bar{\tau}^u\|_\infty$, then ψ is a critical point of I.

Variation of the drift

For any smooth vector field w supported inside Ω, let

$$\frac{d}{dt} Y_t^x = w(Y_t^x) \qquad \text{and} \qquad Y_0^x = x,$$

and define $\psi_\varepsilon^w(x) = \psi(Y_\varepsilon^x)$. If ψ is a critical point of I, then

$$\frac{d}{d\varepsilon} I(\psi_\varepsilon^w)\Big|_{\varepsilon=0} = 0$$

for each w. This can be showed to be equivalent to the PDE

$$-2\Delta\phi(x) = 1 + |\nabla\phi(x)|^2 \left(\int_{\partial\Omega_{\phi,\phi(x)}} \frac{d\sigma}{|\nabla\phi|}\right)\left(\int_{\partial\Omega_{\phi,\phi(x)}} |\nabla\phi|\, d\sigma\right)^{-1}$$

with $\phi = \bar{\tau}^{\nabla^\perp\psi}$ a reparametrization of ψ.

Domains with τ^0 having a single critical point

Now assume τ^0 maximizes I and has a single critical point. Then $-\Delta\tau^0 = 1$ and $\nabla^\perp\tau^0 \cdot \nabla\tau^0 = 0$ give

$$-\Delta\tau^0 + A\nabla^\perp\tau^0 \cdot \nabla\tau^0 = 1.$$

So $\phi = \bar{\tau}^{\nabla^\perp\tau^0} = \tau^{A\nabla^\perp\tau^0} = \tau^0$ and

$$-2\Delta\phi(x) = 1+|\nabla\phi(x)|^2 \left(\int_{\partial\Omega_{\phi,\phi(x)}} \frac{d\sigma}{|\nabla\phi|}\right)\left(\int_{\partial\Omega_{\phi,\phi(x)}} |\nabla\phi|\, d\sigma\right)^{-1}$$

becomes

$$2 = 1 + |\nabla\tau^0|^2 F(\tau^0).$$

Then $\theta(x) = G(\tau^0(x))$ with $G(s) = \int_0^s F(r)^{-1/2} dr$ solves

$$1 = |\nabla\theta|^2 \qquad (\text{and } \theta = 0 \text{ on } \partial\Omega)$$

If Ω is a not a disc, solutions have more than one interior singularity. Since τ^0 is analytic, θ can only be singular at x_0.

General domains

Lemma

For any Ω, the set of maxima (minima) of τ^0 is discrete.

- This uses analyticity of τ^0 and simple connectedness of Ω.

Pick one maximum x_0, then h close to $\tau^0(x_0)$, and let Ω' be the component of $\Omega_{\tau^0,h}$ containing x_0 (and no other critical points).

Lemma

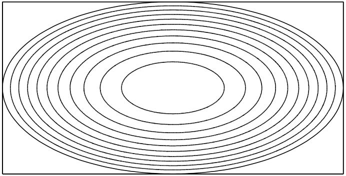
If $u = 0$ maximizes $\|\tau^u\|_\infty$ on Ω, then it maximizes $\|\tau^u\|_\infty$ on Ω'.

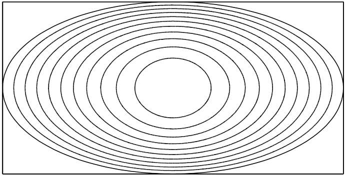
- If not, take v supported in Ω' with $\|\bar{\tau}^v_{\Omega'}\|_{L^\infty(\Omega')} > \|\tau^0_{\Omega'}\|_{L^\infty(\Omega')}$. Let $u_0 = \nabla^\perp \tau^0$, and $u = u_0$ in $\Omega \setminus \Omega'$ and $u = v$ in Ω'.
- Then $\tau^{Au} - \tau_0 = \tau^{Au} - \tau^{Au_0} \to 0$ on $\Omega \setminus \Omega'$ as $A \to \infty$.
- So $\tau^{Au} \to h$ on $\partial\Omega'$ and hence $\tau^{Au} \to h + \bar{\tau}^v_{\Omega'}$ on Ω'. Then

$$\|\tau^{Au}\|_{L^\infty(\Omega)} \to h + \|\bar{\tau}^v_{\Omega'}\|_{L^\infty(\Omega')} > h + \|\tau^0_{\Omega'}\|_{L^\infty(\Omega')} = \|\tau^0_\Omega\|_{L^\infty(\Omega)}.$$

So if $u = 0$ maximizes $\|\tau^u\|_\infty$, then Ω' must be a disc and τ^0 radial on it. Analyticity of τ^0 shows that Ω must be a disc.

An example





- τ^0 and the maximizer $\bar{\tau}^u$ for Ω an ellipse (maximizer is obtained by solving the level set PDE numerically).
- Level sets of the maximizer near its maximum are circular. This is the case for general solutions of the level set PDE.

Open problems

- Existence/uniqueness of the solutions of the PDE

$$-2\Delta\phi(x) = 1 + |\nabla\phi(x)|^2 \left(\int_{\partial\Omega_{\phi,\phi(x)}} \frac{d\sigma}{|\nabla\phi|}\right)\left(\int_{\partial\Omega_{\phi,\phi(x)}} |\nabla\phi|\, d\sigma\right)^{-1}$$

- $\bar{\tau}^u = \lim_{A\to\infty} \tau^{Au}$ is well defined for general u satisfying some non-degeneracy assumptions (not only those whose stream function has a single critical point). If general u maximizes $\bar{\tau}^u$, does $\phi = \bar{\tau}^u$ satisfy the PDE (with integration over connected components of level sets of ϕ)?
- Are there maximizers of $\|\tau^u\|_\infty$ and $\|\bar{\tau}^u\|_\infty$ for general Ω?
- How about other norms, e.g., $\|\tau^u\|_p$? One direction true.
- How about more dimensions (no stream functions there)? One direction true.