Quantitative and Uniform in time Chaos Convergence for N-particle system to its single-typical-particle statistical limit

S. Mischler

(CEREMADE Paris-Dauphine & IUF)
Joint work with C. Mouhot (Cambridge) & B. Wennberg (Goteborg)

Paris 6, Barcelona, Lyon 1, X-EDP - 2010

Outlines of the talk

Plan

Physics and Historic question: derivation of kinetic/fluid equations from particles motion

Maxwell and Boltzmann XIX century
Hilbert's sixth problem (ICM 1900 Paris):
Mathematical Treatment of the Axioms of Physics ?

That means : how to derive rigorously the equations of continuum fluid mechanics (as well as and before, kinetic equations) from the first principle (Newton first law of motion)
(for many-particle dynamics) ?

In other words: can we rigorously derive the mesoscopic ($=$ statistic) dynamics and the macroscopic ($=$ fluid) dynamics from the description of miscroscopic dynamics ?

Boltzmann-Grad limit to Boltzmann equation

- "Boltzmann-Grad" limit (Grad 50s) formally derive the Boltzmann equation from particle systems
- Best (and astonishing result) to this date: Lanford 1974 proves rigorously the limit for very short time (shorter than the free mean path). Idea: use Bogoliubov (or BBGKY) hierarchy
- Why is it hard? Reversible dynamics on the particle system, irreversible dynamics at the Boltzmann level ("*H*-theorem")!
- Related to the mysterious and still poorly understood "stosszahlansatz" (molecular chaos assumption)
- Making the Boltzmann-Grad limit rigorous for arbitrary large time remains a major open problem 1
- From Boltzmann to Navier-Stokes by Bardos, Golse, Levermore, Lions, Masmoudi, Saint-Raymond (90'-2001)

mean-field limit to Vlasov equation

- Derive the Vlasov-Poisson equation from Newton first principle (N particles evolve according to Hamiltonian dynamic associated to Coulombian potential) in the "mean-field" limit
- Neunzert, Braun & Hepp 70s derive the Vlasov equation for smooth and bounded potential, improved by Hauray, Jabin 2007 allowing (too) soft singularity.
- idea: use empirical measures

$$\forall\, Y = (y_1, ..., y_N) \in E^N \quad \text{define} \quad \mu_Y^N(dy) := \frac{1}{N} \sum_{i=1}^{N} \delta_{y_i}(dy)$$

$Y(t)$ solution to an edo $\Rightarrow \mu^N_{Y(t)}$ solution to the associated transport equation. One has to prove (Lipschitz) stability in the probability space $P(E)$ (for a weak distance)

- However, making that mean field limit rigorous for the true singular potential remains a major open problem 2

Less ambitious Kac program (1956)

Derive the (space homogeneous) Boltzmann equation from a jump (collisional) process. First rigorous mathematic treatment of the deduction of Boltzmann equation from microscopic dynamics.

Kac introduced the notion of chaos

Kac stressed two open questions

- Hard spheres model: "The above proof suffers from the defect that it works only if the restriction on time is independent of the initial distribution. It is therefore inapplicable to the physically significant case of hard spheres because in this case our simple estimates yield a time restriction which depends on the initial distribution. A general proof that Boltzmann's property propagates in time is still lacking"

$\rightarrow$ proved by Sznitman 1984 (nonlinear martingale approach, compactness and uniqueness arguments)

- Uniform spectral gap: Deduce spectral gap/exponential trend to equilibrium for the nonlinear Boltzmann eq from the spectral gap for the family of Master eqs

$\rightarrow$ proved by a direct way by Mouhot 2006 (using : linearized L^2 spectral gap Grad 63; L^1 moments Povzner 1965, quantitative H-theorem: Carlen, Carvalho 1992)

Bibliography

- Equations: Maxwell 1867, Boltzmann 1871, Vlasov 1938
- Hierarchy: Bogoliubov 1946, Kirkwood, Born, Green
- Kac program 1951-1975: Wild, Kac, McKean, Tanaka, Grünbaum (1971)
- Chaos to Maxwell function: Mehler 1866, Borel 1925, Sznitman 1989
- Grad limit: Grad 1958, Lanford, King, Illner, Pulvirenti, Cercignani 1994
- Mean field Vlasov limit: Neunzert, Wick 1971, Braun, Hepp 1977, Dobrushin 1979, Spohn, 1991, Hauray, Jabin 2007
- Probability approach: Sznitman 1989, Méléard, Graham, Fournier, Guérin, Malrieu, Villani, Bolley, Guillin
- Uniform spectral gap: Kac, Janvresse 2001, Carlen, Carvalho, Loss, Maslen, Villani, Lieb, Gernimo, Le Roux

More Bibliography on Boltzmann

Propagation of chaos for Maxwell molecules

- M. Kac, *Foundation of kinetic theory* (1956), *Some probabilistic aspects of the Boltzmann equation (1973)*
- H. Tanaka, *Propagation of chaos for certain Markov processes of jump type with nonlinear generators*, Proc. Japan Acad (1969)
- R. Peyré, *A few ideas about quantitative convergence of collison models to the mean field limit*, JSP 2009 or 2010

Propagation of chaos for hard spheres cross-section

- A.F. Grunbaum, *Propagation of chaos for the Boltzmann equation*, ARMA (1971)
- A.-S. Sznitman, *Equations de type de Boltzmann, spatialement homogènes*, ZWVG (1984)

Plan

- Mean field limit for a system of particles:
  - N undistinguishable particles, any of them undergoes the action of order $\frac{1}{N}$ from the $N-1$ other particles
  - N particles evolve according to a Markov (collisional) process
  - prove the LLN for the N particles when $N \to \infty$
- propagation of chaos (= “weak” independence of coordinates of a stochastic vector)
- chaos quantification: rate of convergence depending on the number N of particles
- uniform in time mean field limit

Deterministic N-particle system

N-**particle system.** We consider a states or admissible configurations space E. We take $E \subset \mathbb{R}^d$.

The configuration of a deterministic N-particle system is described by a point in the phases state, and thus by

- its state variable $Y = (y_1, ..., y_N) \in E^N$.

A physical system is then described as

$$\begin{aligned} \bar{Y} &= \text{optimal solution to a stationary problem} \in E^N, \\ Y(t) &= \text{solution to a evolution problem} \in E^N. \end{aligned}$$

When particles are undistinguishable the phase space becomes $E^N/\mathfrak{S}_N$ and the system is equivalently described by

- the associated empirical $\mu^N_Y \in \mathbf{P}(E)$,

with

$$\mu^N_Y(dy) := \frac{1}{N} \sum_{i=1}^N \delta_{y_i}(dy) \in \mathbf{P}(E).$$

Stochastic N-particle system

When the system is stochastic the above state variable is a random variable. One can choose to describe the system thanks to a probability density (= law of the above mentioned random variable). In that case, the system is described by

- the law $f^N \in \mathbf{P}(E^N)/\mathbf{P}_{sym}(E^N)$ of the random variable Y,

and physical system as

$$\begin{aligned} \bar{f}^N &= \text{optimal solution to a stationary problem } \in \mathbf{P}(E^N)/\mathbf{P}_{sym}(E^N), \\ f^N(t) &= \text{solution to a evolution problem} \in \mathbf{P}(E^N)/\mathbf{P}_{sym}(E^N), \end{aligned}$$

where $\mathbf{P}_{sym}(E^N)$ stands for the space of symmetric probabilities on E^N, i.e. invariant under coordinates permutations. The fact that we deal with $\mathbf{P}_{sym}(E^N)$, instead of $\mathbf{P}(E^N)$, comes from that we assume the particles undistinguishable. In that case, we can also describe the system by

- the law $\hat{f}^N = \pi^N_P f^N \in \mathbf{P}(\mathbf{P}(E))$ of the random varibale μ^N_Y.

Notice that randomness may be the result of:
- the evolution dynamic itself;
- the initial datum which is not known with certainty, the evolution dynamic being deterministic

Law of the typical particle: that is a probability measure $f(t) \in \mathbf{P}(E)$ defined as the solution of a nonlinear PDE equation (or a stochastic process $Y(t)$ defined as the solution of a nonlinear martingale problem, and then $f(t) := \mathcal{L}(Y(t))$.

N-particle system is described by

- $X(t) \in E^N \leftrightarrow \mu^N_X \in P(E)$
- $f^N(t) \in P(E^N)$
- $\hat f^N(t) \in P(P(E)) \leftrightarrow f^N_k(t) \in P(E^k) \forall\, k \le N$

where f^N_k is the k-th marginal of f^N defined by

$$f^N_k := \int_{E^{N-k}} f^N(., dx_{k+1}...dx_N) \in P(E^k).$$

How to deduce the behavior of the typical particle from the behavior of the N-particle system ?

Pb 1: Law of large numbers: $\mu^N_{X(t)} \rightharpoonup f(t)$ or $f^N_1 \rightharpoonup f(t)$ when $N \to \infty$

Pb 2: propagation of chaos: $f^N_k \rightharpoonup f(t)^{\otimes k}$ when $N \to \infty$

Example 1: ODE / Vlasov

In $E = \mathbb{R}^d$, we consider a system of N undistinguishable deterministic particles in interaction which dynamic is given by

$$\dot{x}_i = A_i(X), \quad x_i(0) = \text{initial datum}, \quad 1 \le i \le N,$$

with a interaction term of "mean field" type

$$\begin{aligned} A_i(X) &= \frac{1}{N} \sum_{j \neq i} a(x_j - x_i) = \frac{N-1}{N} (a \star \mu^{N-1}_{\hat{X}^i})(x_i) \\ &= \frac{1}{N} \sum_{j=1}^{N} a(x_j - x_i) = (a \star \mu^N_X)(x_i) =: A(x_i, \mu^N_X), \end{aligned}$$

if $a : E \to E$ is smooth, and we can then choose $a(0) = 0$.

The system is therefore described by the trajectory

- $X^N(t) = (x_1^N(t), ..., x_N^N(t)) =: T_t(X) \in E^N$

obtained as the solution of a ODE.

Alternatively; one can introduce the empirical measure

- $f^N(t,dx) := \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i^N(t)}(dx) \in \mathbf{P}(E)$,

which solves the Vlasov equation (the associated mean field equation)

$$\partial_t f = Q(f,f) := -\mathrm{div}(A(x,f)\,f) \quad (0,\infty)\times E.$$

Other possibility consist in introducing

- $F^N(t,dY) := \delta_{X^N(t)}(dY) \in \mathbf{P}(E^N)$,

which solves the Master (Liouville) equation

$$\partial_t F = A^N F := -\sum_{i=1}^{N} \mathrm{div}_i(A_i(X)\,F) \quad (0,\infty)\times E^N.$$

That last equation can also be solved in $\mathbf{P}_{sym}(E^N)$ for any initial law $F_0^N(dY)$ which correspond to deterministic or random initial datum.
In that case $F^N(t,.) := T_t\sharp F_0^N$ with (mass transport formula)

$$\int_{E^N} \varphi(V)\,(T_t\sharp F_0^N)(dV) := \int_{E^N} \varphi(T_t(V))\,F_0^N(dV)$$

Method 1: empirical measure

f^N solves the Vlasov equation

$$\partial_t f^N = -\mathrm{div}(A(x, f^N)\, f^N), \qquad f_0^N = \mu^N_{X_0^N}.$$

If D_1 is a distance on$P(E)$, for instance $D_1 = W_p$ is the Monge-Kantorovich-Wasserstein distance, if $D_1(f_0^N, f_0) \to 0$ and if f solves the Vlasov equation

$$\partial_t f = -\mathrm{div}(A(x, f)\, f), \qquad f(0, .) = f_0,$$

then

$$\sup_{[0,T]} D_1(f^N, f) \le C_T\, D_1(f_0^N, f_0) \to 0.$$

Remark. That is not the chaos propagation, but we may deduce of it a quantified chaos propagation

$$\sup_{[0,T]} D_\infty(F^N, \delta_f) \le C_T\, D_\infty(F^N, \delta_f) \to 0.$$

Example 2: SDE / McKean-Vlasov

In $E = \mathbb{R}^d$, we consider a system of N undistinguishable particles

- $X^N(t) = (x_1^N(t), ..., x_N^N(t)) \in E^N$

in stochastic interaction

$$dx_i = (a \star \mu_X^N)(x_i)\, dt + dB_t^i$$

for a family B_t^1, ... B_t^N of independent Brownian motions.
Ito formula implies

- $F^N(t, dY) =$ law of $X^N(t)$

salsifies the Master (Kolomogorov) equation

$$\partial_t F = A^N F := -\sum_i^N \Delta_i F - \sum_{i=1}^N \mathrm{div}_i(A_i(X)\, F) \quad (0,\infty) \times E^N$$

and the associated mean field equation is the McKean-Vlasov equation

$$\partial_t f = Q(f,f) := -\frac{1}{2}\,\Delta f + \mathrm{div}(A(x,f)\, f) = 0 \quad (0,\infty) \times E$$

Method 2: coupling

We introduce Y the solution to a subsidiary problem such that the coordinates Y_1, ... Y_N are independent stochastic processes and we prove

$$\sup_{[0,T]} \mathbf{E}\Big(\underbrace{\frac{1}{N}\sum_{j=1}^{N} |X_j(t) - Y_j(t)|}_{=:distance\ in\ E^N} \Big) \le \frac{C_T}{\sqrt{N}}$$

That implies

$$\sup_{[0,T]} \mathbf{E} \left| \int_E \varphi(z)\, \mu^N_{X(t)}(dz) - \int_E \varphi(z)\, \mu^N_{Y(t)}(dz) \right| \le \frac{C_T\, \|\varphi\|_{Lip}}{\sqrt{N}}$$

The Y_j are built in such a way that $\mathcal{L}(Y_j)$ is the solution to the McKean-Vlasov equation. That implies $\mu^N_Y \rightharpoonup f$ if $\mu^N_{Y_0} \rightharpoonup f_0$ with f solution to the McKean-Vlasov equation associated to the initial datum f_0, and moreover when $Y_{j,0} \sim f_0$ the above convergence may be quantified and is of order $1/N$ (LLN)
Thanks to a triangular inequality, we conclude with

$$\sup_{[0,T]} \mathbf{E} \left| \int_E \varphi(z)\, \mu^N_{X(t)}(dz) - \int_E \varphi(z)\, f_t(dz) \right| \le \frac{C_T}{\sqrt{N}}.$$

Example 3: N-particle Boltzmann-Markov process

N-particle system $V = (v_1, ..., v_N)$, $v_i \in E = \mathbb{R}^3$ undergoing random Boltzmann type collisions.
Markov process $(V_t)_{t\geq 0}$ defined step by step as follows:
(i) draw randomly $\forall$ $(v_{i'}, v_{j'})$ collision time $T_{i',j'} \;\sim\; Exp(\Phi(|v_{i'} - v_{j'}|))$; then select the post-collisional velocity (v_i, v_j) such that

$$T_{i,j} = \min_{(i',j')} T_{i',j'}.$$

(ii) draw randomly $\sigma \in S^2$ according to the density law $b(\cos\theta)$ with $\cos\theta = \sigma \cdot (v_i - v_j)/|v_i - v_j|$ and define the post-collisional velocities (v_i^*, v_j^*) thanks to

$$v_i^* = \frac{v_i + v_j}{2} + \frac{|v_j - v_i|}{2}\,\sigma, \qquad v_j^* = \frac{v_i + v_j}{2} - \frac{|v_j - v_i|}{2}\,\sigma.$$

Observe that momentum and energy are conserved

$$v_i^* + v_j^* = v_i + v_j, \qquad |v_i^*|^2 + |v_j^*|^2 = |v_i|^2 + |v_j|^2.$$

Finally, this two bodies collisions jump process satisfies

$$\sum_{i=1} v_i(t) = \text{cst}, \quad \sum_{i=1} |v_i(t)|^2 = \text{cst}.$$

Example 3: Master equation for the N-particle system

Equivalently, after time rescaling, the motion of the N-particle system is given through the master equation on the law $F_t^N \in \mathbf{P}(E^N)$ which in dual form reads

$$\partial_t \langle F_t, \varphi \rangle = \langle F_t^N, G^N \varphi \rangle \qquad \forall\, \varphi \in C_b(E^N)$$

with $G^N = (A^N)^*$ given by

$$(G^N \varphi)(V) = \frac{1}{N} \sum_{i,j=1}^{N} \Phi(|v_i - v_j|) \int_{S^2} b(\cos\theta_{ij})\, [\varphi^*_{ij} - \varphi]\, d\sigma,$$

where $\varphi = \varphi(V)$, $\varphi^*_{ij} = \varphi(V^*_{ij})$, $V^*_{ij} = (v_1, .., v_i^*, .., v_j^*, .., v_N)$.

- Maxwell interactions with cut-off: $\Phi = 1$, $b = 1$;
- Maxwell interactions without cut-off: $\Phi = 1$, $b \notin L^1$;
- Hard spheres interactions: $\Phi(z) = |z|$, $b = 1$.

The nonlinear Boltzmann equation

Nonlinear homogeneous Boltzmann equation on $P(\mathbb{R}^3)$ defined by

$$\partial_t f_t = Q(f_t, f_t), \quad f_0 \in P_2(\mathbb{R}^3)$$

with

$$\langle Q(f,f), \varphi \rangle := \int_{\mathbb{R}^6 \times S^2} |w_1 - w_2| \, b(\theta) \, (\phi(w_2') - \phi(w_2)) \, d\sigma \, f(dw_1) \, f(dw_2)$$

where

$$w_2' = \frac{w_1 + w_2}{2} + \frac{|w_2 - w_1|}{2} \, \sigma.$$

The equation generate a nonlinear semigroup

$$\forall \, f_0 \in P_2(\mathbb{R}^3) \qquad S_t f_0 := f_t.$$

Naive idea: 1-marginal

$$\partial_t f^N = A_N\, f^N$$

implies

$$\partial_t f_1^N = (A_N\, f^N)_1 = A_{N,2} f_2^N \quad \rightarrow \quad \partial_t \pi_1 = A_2 \pi_2 \quad \text{and ? ...}$$

We carry on the idea by taking ℓ-th marginal

- Start from a N-particle system

$$\partial_t f^N = A_N\, f^N \quad \text{or} \quad f^N(t) = e^{t\, A_N}\, f_0^N = \sum_{k=0}^{\infty} \frac{t^k}{k!}\, A_N^k\, f_0^{\otimes N}$$

- Write the equation for the ℓ-th marginal distribution

$$\partial_t f_\ell^N = A_{N,\ell+1}\, f_{\ell+1}^N \quad \text{or} \quad f_\ell^N(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\, (A_N^k\, f_0^{\otimes N})_\ell$$

Unclosed equation when $\ell < N$.

- Kac's argument: take $\varphi \in C_b(E^\ell)$ and write the dual identity

$$\langle f_\ell^N(t), \varphi \rangle = \sum_{k=0}^{\infty} \frac{t^k}{k!} \langle f_0^{\otimes N}, A_N^k(\varphi \otimes 1^{\otimes N-\ell}) \rangle$$

- Pass now to the limit $N \to \infty$

$$\langle \pi_\ell(t), \varphi \rangle = \sum_{k=0}^{\infty} \frac{t^k}{k!} \langle f_0^{\otimes k+\ell}, \varphi_k \rangle, \quad \varphi_k \in C(E^{k+\ell}).$$

For $\varphi, \psi \in C(E)$ Kac proves

$$(\varphi \otimes \psi)_k = \sum_{i=0}^{k} \frac{k!}{i!\,(k-i)!} \varphi_i \otimes \psi_{k-i}$$

so that we may recognize

$$\langle \pi_2(t), \varphi \otimes \psi \rangle = \sum_{i \le k} \frac{t^k}{i!\,(k-i)!} \langle f_0^{\otimes k+2}, \varphi_i \otimes \psi_{k-i} \rangle = \langle \pi_1(t), \varphi \rangle \, \langle \pi_1(t), \psi \rangle.$$

- We come back to the family of equations on the marginals of order ℓ

$$\partial_t f_\ell^N = A_{N,\ell+1}\, f_{\ell+1}^N,$$

which (again) are unclosed equations if $\ell < N$.

- We pass to the limit $N \to \infty$

$$(*) \qquad \partial_t \pi_\ell = A_{\ell+1}\, \pi_{\ell+1}.$$

We obtain a family of solutions $(\pi_\ell)_{\ell \geq 1}$ to an infinite hierarchy of equations

- We remark that $\bar\pi_\ell(t) = f(t)^{\otimes \ell}$ is a solution of $(*)$. we also remark that if we are able to prove that $(*)$ admits a unique solution then $\pi_\ell = \bar\pi_\ell$. We conclude by establishing the uniqueness of the solution of $(*)$.

The following result is a "quantified version" of the above method

Plan

Main result 1 - vague version

Theorem (Uniform in time chaos convergence (in the sense of convergence in law of any k-marginals))

$$\sup_{t \in [0,T)} \left| \int_{E^N} \left(f_t^N - f_t^{\otimes N} \right) \varphi \, dV \right| \le \theta(N) \underset{N \to \infty}{\longrightarrow} 0.$$

- $T \in (0, +\infty]$,
- $E = \mathbb{R}^d$, $d = 3$, $V = (v_1, ..., v_N) \in E^N$
- $f_0 = f_{in} \in P(E)$ with enough moments bounded,
  f_t = evolution of one typical particle in the mean-field limit,
  $f_t^{\otimes N}(V) = f_t(v_1) \dots f_t(v_N)$,
- $f_0^N = f_{in}^{\otimes N}$, f_t^N = evolution of N-particle system $\in P_{sym}(E^N)$,
- $\varphi = \varphi_1 \otimes \dots \otimes \varphi_k$, $\varphi_j \in \mathcal{F} \subset C_b(E)$, ex: $\mathcal{F} = W^{1,\infty}$ or H^s,
- $N \ge 2\,k$.

Main result 2 - for Boltzmann model

Theorem (Chaoticity of the N-particle steady states)

$$\left| \int_{E^N} \left(\gamma^N - \gamma^{\otimes N} \right) \varphi \, dV \right| \le \theta(N) \underset{N\to\infty}{\longrightarrow} 0.$$

- $E = \mathbb{R}^d$, $d = 3$, $V = (v_1, ..., v_N) \in E^N$
- $\gamma^N :=$ steady state for the N-particle system
  . $= \text{meas}(S^{dN-1}(\sqrt{N}))^{-1} \, \delta_{S^{dN-1}(\sqrt{N})} \in \mathbf{P}(E^N)$,
- $\gamma(v) := (2\pi)^{-d/2} \exp(-|v|^2/2)$,
- $\varphi = \varphi_1 \otimes ... \otimes \varphi_k$, $\varphi_j \in H^s$,
- $N \ge 2\,k$,
- $f_0^N = [f_{in}^{\otimes N}]_{S^{dN-1}(\sqrt{N})} =$ conditioned product measure.

In other words,

$$\gamma^N \text{ is } \gamma\text{-chaotic}$$

Theorem (Uniform in N spectral gap (in the sense of convergence in law of any k-marginals))

$$\sup_{t\geq T}\sup_{N}\left|\int_{E^N}\left(f_t^N-\gamma^N\right)\varphi\,dV\right|\leq\varepsilon_k(T)\xrightarrow[T\to\infty]{}0,$$

$$\approx\quad f_t^N\xrightarrow[t\to\infty]{}\gamma^N\quad\textit{uniformly in } N.$$

- $E=\mathbb{R}^d,\ d=3,\ V=(v_1,....,v_N)\in E^N$
- $f_0^N=[f_{in}^{\otimes N}]_{S^{dN-1}(\sqrt{N})}$,
- $f_t^N=$ evolution of N-particle system $\in P_{sym}(E^N)$,
- $\varphi=\varphi_1\otimes...\otimes\varphi_k,\ \ \varphi_j\in\mathcal{F}\subset C_b(E)$, ex: $\mathcal{F}=W^{1,\infty}$ or H^s,
- $k\in\mathbb{N}$

Main features

- We prove propagation of chaos with quantitative rates
- Most importantly and new: estimates are uniform in time
  $\Rightarrow N \to \infty$ limit and $t \to \infty$ limit commute!
- We may deal with mixtures of Vlasov, McKean and Boltzmann models.
- Two important physical collision models: hard spheres and Maxwell molecules without cutoff $\Rightarrow$ Give quantitative estimates of previous non-constructive convergence result (Sznitman 1984)
- The method is "almost" new. It is strongly inspired by Grünbaum work (1971) where he claimed he proved convergence result for the hard spheres model. But his proof is definitively wrong ! He essentially recovered the non-constructive convergence result for the Maxwell cut-off model by Kac & McKean.
- The underlining philosophy is a numerical analyst intuition: identity (1) consistency estimates and (2) stability estimates on the limit PDE and refuse any compactness and probability arguments

Main features again

- We follow, complete and improve Grunbaum's program;
- analysis argument (no probability!). The cornerstone of the proof: to prove accurate "stability estimates" on the nonlinear 1-typical particle flow;
- consistency of order $\mathcal{O}(1/N^{1-\varepsilon})\ \forall\, \varepsilon \in (0,1)$;
- error of order $\mathcal{O}(1/N^{1/2})$, $\sim \mathcal{O}(1/N^{1/d})$ or worst because we write the equation in $P(P(\mathbb{R}^3))$ and we use some results from the theory of the concentration of measure (at time $t = 0$): the worse error is made at time $t = 0$ (and then it is not deteriorated by the flow);

Main result - Remarks

(a) For $k = 1$ we deduce that

$$f_1^N(t) := \int_{E^{N-1}} f_t^N dv_2 \dots dv_N \underset{N\to\infty}{\longrightarrow} f(t).$$

The density $f_1^N(t)$ of one typical particle of the N-particle system behaves as $f(t)$ the solution of the mean-field (nonlinear) equation. That proves the mean-field convergence $\approx$ law of large numbers.

(b) The assumption $f_0^N = f_{in}^{\otimes N}$ (or $\approx f_{in}^{\otimes N}$ in the sense that f_0^N is f_{in}-chaotic) is strong. The mean-field limit only holds when molecular chaos holds at the initial time.

Main result - Remarks

(c) The conclusion is also stronger than a mere law of large number because for any $k \geq 1$

$$f_k^N(t) \underset{N\to\infty}{\longrightarrow} f(t)^{\otimes k}.$$

That is the Kac's definition of chaos.

Definition (Chaos - Kac 1956)

A sequence $f^N \in P_{sym}(E^N)$ is f-chaotic, $f \in P(E)$, if for any $k \geq 1$

$$f_k^N \underset{N\to\infty}{\rightharpoonup} f^{\otimes k} \quad \text{weakly in } P(E^k),$$

where the k-th marginal f_k^N of f^N is defined by

$$f_k^N := \int_{E^{N-k}} f^N dv_{k+1} \dots dv_N,$$

and the convergence is the the weak $* \ \sigma(M^1(E^k), C_b(E^k))$ convergence.

Main result - Remarks

(d) In other words, f_0^N is f_0-chaotic implies f_t^N is f_t-chaotic.

(e) The notion of chaos is closed (wider) to the notion of independence in probability theory. If V is a stochastic variable in E^N such that the coordinates are independent variables and have same law $f \in P(E)$ then $V \sim f^{\otimes N}$. In the case of chaos the tensorization structure is required only asymptotically when $N \to \infty$.

(f) Be careful that even when $f_0^N = f_0^{\otimes N}$ we never have $f_t^N = g_t^{\otimes N}$ for a given N (except when there is no interaction between the particles of the N-particle system!). The "independence" (in a weak sense) is recovered only in the limit

Main result - Remarks

(g) The θ function splits into

$$\theta(N) = \theta(k, N) = \theta_1(\varphi, N) + \theta_2(\varphi, T, N) + \theta_3(\varphi, T; f_0^N, f_0),$$

- with $\theta_1 \approx C_0/N$, $\theta_2 \approx C_\varepsilon/N^{1-\varepsilon}$ for any $\varepsilon > 0$, and never better than $\theta_3 \leq C\, N^{-1/2}$, so that θ_3 is the worst term;

- θ_3 is the only term depending on the initial data;

- $\theta_3(\varphi, T; \cdot)$ behaves like a distance between f_0^N and its (possibly) chaos limit f_0;

- Our error estimate $\theta(N)$ makes sense for any given initial data f_0 and f_0^N. We do not ask the initial datum f_0^N to be f_0-chaotic. Of course, more f_0^N "is closed to chaoticity" more f_t^N will be.

Main result - Remarks

(h) We can remove the function φ and the integer k by defining

$$D(g^N; g) := \sup_{k\geq 1} \sup_{\|\varphi_i\|_{\mathcal{F}}\leq 1} \frac{1}{k^3} \left| \langle g^N - g^{\otimes N}, \varphi \otimes 1^{\otimes N-k} \rangle \right|,$$

and proving

$$\sup_{[0,T)} D(f_t^N; f_t) \leq \tilde{\theta}(N) \to 0.$$

(i) We are not able to prove that

$$\sup_{[0,T]} D(f_t^N; f_t) \leq C \left(\frac{1}{N^\alpha} + D(f_0^N, f_0) \right)$$

for some “distance” D which measures how close to a chaos state "$g \in P(E)$" is a probability $g^N \in P_{sym}(E^N)$ and $C, \alpha > 0$.

Main result - Remarks

(j) If we accept a bad estimate rate, we may deduce

$$\sup_{[0,T)} W_1(f_k^N,(t), f(t)^{\otimes k}) \leq \theta_k(N)$$

(k) Our theorem applies to the Vlasov model and the McKean-Vlasov at least for smooth and bounded coefficients, and for the space homogeneous Boltzmann equation in the following cases:
- true Maxwell molecules (without cut-off) cross-section
- hard spheres cross-section
- hard potential with Grad's cut-off cross-section

(l) For the Boltzmann equation and the McKean-Vlasov equation we can chose $T = +\infty$.

Main result - Remarks

(m) It is worth mentioning that the result for $k = 2$ implies that f_t^N is f_t-chaotic, that means the result for any $k \geq 2$. However, if one proceeds along this line the distance estimate $\theta(k, N)$ will be very bad (the decay with respect to N will be very low).

(n) A stronger estimate (stronger topology and taking into account all the N-particle distribution) should be

$$\sup_{[0,T]} H_N(f_t^N; f_t) \leq \theta_H(N)$$

with

$$H_N(g^N; g) := \frac{1}{N} \int_{E^N} g^N \log \frac{g^N}{g^{\otimes N}} \, dV.$$

Main result - Remarks

(o) Open problems:

- $T = +\infty$ with optimal rate $\theta(N) = \mathcal{O}(N^{-1/2})$;

- more general cross-section (true hard or soft potential) and Landau equation;

- Vlasov equation and McKean-Vlasov equation with singular interactions;

- the entropy convergence $\sup_{[0,T]} H_N(f_t^N; f_t) \le \theta_H(N)$;

- quantification of the chaos for the equilibrium state (elastic or inelastic Boltzmann model)

- rate of convergence to equilibrium for the nonlinear PDE from the analysis of the N-particle system dynamic

- for the inelastic Boltzmann equation + diffuse excitation can we deduce from the $N \to \infty$ limit

$$\frac{d}{dt} H(f(t)|g) \le 0$$

where g stands for the unique steady state?

Plan

Proof of Theorem 3

• (1) Theorem 1 and 2 write (for $N \geq 2\,k$)

$$\sup_{[0,\infty)} \left| \int_{E^k} \left(f_k^N(t) - f(t)^{\otimes k}\right) \varphi \, dV \right| + \left| \int_{E^k} \left(\gamma_k^N - \gamma^{\otimes k}\right) \varphi \, dV \right| \leq \theta(N) \underset{N\to\infty}{\longrightarrow} 0.$$

• (2) We know (from Carlen, Carvalho, and then Villani, Mouhot 90'-2006) that

$$\|f_t - \gamma\|_{L^1} \leq C_{f_0}\, e^{-\lambda\, t}.$$

• (3) Gathering these estimates, we get

$$\forall\, N \geq 2k \qquad \left| \int (f_k^N(t) - \gamma_k^N)\, \varphi \right| \leq 2\,\theta(N) + k\, C_{f_0}\, e^{-\lambda\, t}$$

• (4) On the other hand, we know (from Kac, and then Carlen, Loss) that

$$\forall\, N \geq k \qquad \left| \int (f_k^N(t) - \gamma_k^N)\, \varphi \right| \leq C_{N, f_0^N}\, e^{-\lambda_N\, t}.$$

• (5) As a consequence of (3) and (4) we obtain the uniform (with respect to N) convergence:

$$\left| \int (f_k^N(t) - \gamma_k^N)\, \varphi \right| \leq \min\left(2\,\theta(N) + k\, C_{f_0}\, e^{-\lambda\, t}, C_{N, f_0^N}\, e^{-\lambda_N\, t}\right) \underset{t\to\infty}{\longrightarrow} 0$$

We split

$$\left\langle f_t^N - f_t^{\otimes N}, \varphi \otimes 1^{\otimes N-k} \right\rangle =$$
$$= \left\langle f_t^N, \varphi \otimes 1^{\otimes N-k} - R_\varphi \circ \mu_V^N \right\rangle \qquad (= T_1)$$
$$+ \left\langle f_t^N, R_\varphi(\mu_V^N) \right\rangle - \left\langle f_0^N, R_\varphi(S_t^{NL} \mu_V^N) \right\rangle \qquad (= T_2)$$
$$+ \left\langle f_0^N, R_\varphi(S_t^{NL} \mu_V^N) \right\rangle - \left\langle (S_t^{NL} f_0)^{\otimes k}, \varphi \right\rangle \qquad (= T_3)$$

where R_φ is the "polynomial function" on $P(\mathbb{R}^3)$ defined by

$$R_\varphi(\rho) = \int_{E^k} \varphi \, \rho(dv_1) \dots \rho(dv_k)$$

and S_t^{NL} is the nonlinear semigroup associated to the nonlinear mean-field limit by $g_0 \mapsto S_t^{NL} g_0 := g_t$.

T_1 : A combinatory trick.

Define the symmetrical function associated to $\varphi \otimes 1^{\otimes(N-k)}$ by

$$\varphi \otimes \widetilde{1^{\otimes(N-k)}}(V) = \frac{1}{\sharp \mathfrak{S}_N} \sum_{\sigma \in \mathfrak{S}_N} \varphi \otimes 1^{\otimes(N-k)}(V_\sigma).$$

Lemma (A.F. Grunbaum)

$$N \geq 2k \qquad \sup_{V \in \mathbb{R}^{3N}} \left| \varphi \otimes \widetilde{1^{\otimes(N-k)}}(V) - R_\varphi(\mu^N_V) \right| \leq \frac{2\,k^2\,\|\varphi\|_{C(E^k)}}{N}$$

Because f^N is symmetric and a probability we get

$$|T_1| \leq \theta_1(N) := \frac{2\,k^2\,\|\varphi\|_{C(E^k)}}{N}.$$

With $A_{N,k} := \{(i_1, ..., i_k); i_\ell \neq i_{\ell'} \ \forall \, \ell \neq \ell'\}$, $B_{N,k} := A_{N,k}^c$

$$\begin{aligned}
R_\varphi(\hat\mu_X^N) &= \frac{1}{N^k} \sum_{i_1,...,i_k=1}^{N} \varphi(x_{i_1}, ..., x_{i_k}) \\
&= \frac{1}{N^k} \sum_{(i_1,...,i_k)\in A_{N,k}} \varphi(x_{i_1}, ..., x_{i_k}) + \frac{1}{N^k} \sum_{(i_1,...,i_k)\in B_{N,k}} \varphi(x_{i_1}, ..., x_{i_k}) \\
&= \frac{1}{N^k} \frac{1}{(N-k)!} \sum_{\sigma\in\mathfrak{S}_N} \varphi(x_{\sigma(1)}, ..., x_{\sigma(k)}) + \mathcal{O}\left(\frac{k^2}{N} \|\varphi\|_\infty\right) \\
&= \frac{1}{N!} \sum_{\sigma\in\mathfrak{S}_N} \varphi(x_{\sigma(1)}, ..., x_{\sigma(k)}) + \mathcal{O}\left(2\,\frac{k^2}{N} \|\varphi\|_\infty\right)
\end{aligned}$$

T_3 : Assume that the nonlinear flow satisfies

$$(A5) \qquad W_1(f_t, g_t) \le C_T \, W_1(f_0, g_0) \quad \forall \, f_0, g_0 \in P(E)$$

Then

$$\begin{aligned} |T_3| &= \left| \left\langle f_0^N, R_\varphi(S_t^{NL} \mu_V^N) \right\rangle - \left\langle (S_t^{NL} f_0)^{\otimes k}, \varphi \right\rangle \right| \\ &= \left| \left\langle f_0^N, R_\varphi(S_t^{NL} \mu_V^N) - R_\varphi(S_t^{NL} f_0) \right\rangle \right| \\ &\le [R_\varphi]_{C^{0,1}} \left\langle f_0^N, W_1(S_t^{NL} \mu_V^N, S_t^{NL} f_0) \right\rangle \\ &\le [R_\varphi]_{C^{0,1}} \, C_T \left\langle f_0^N, W_1(\mu_V^N, f_0) \right\rangle \end{aligned}$$

and we have to estimate

$$\Omega_N(f_0) := \left\langle f_0^{\otimes N}, W_1(\mu_V^N, f_0) \right\rangle .$$

Rachev and Rüschendorf establish (functional LLN)

$$\left\langle f_0^{\otimes N}, W_2^2(\mu_V^N, f_0)\right\rangle \le \frac{C(f_0)}{N^{2/(d+4)}}$$

We also establish

$$\left\langle f_0^{\otimes N}, \|\mu_V^N - f_0\|^2_{H^{-d/2-1/2}}\right\rangle \le \frac{C(f_0)}{N}$$

More generally we may recognize

$$\begin{aligned}\left\langle f_0^N, W_1(\mu_V^N, f_0)\right\rangle &= \int_{P(E)\times P(E)} W_1(g,h)\,\hat f_0^N(dg)\,\delta_{f_0}(dh)\\ &= \inf_{\pi\in\Pi(\hat f_0^N,\delta_{f_0})}\int_{P(E)\times P(E)} W_1(g,h)\,\pi(dg,dh) = W_1(\hat f_0^N,\delta_{f_0})\end{aligned}$$

where $\hat f_0^N, \delta_{f_0} \in P(P(E))$ and $\hat f_0^N(\Phi) = \langle f_0^N, \Phi(\mu^N)\rangle$.

We conclude thanks to : (f_0^N is f_0-chaotic) $\Leftrightarrow$ ($\hat f_0^N \to \delta_{f_0}$) (Hewitt-Savage theorem)

Sketch of the proof of theorem 1 (T_2) - VI -

T_2 : We write

$$T_2 \quad = \quad \left\langle f_t^N, R_\varphi(\mu_V^N) \right\rangle - \left\langle f_0^N, R_\varphi(S_t^{NL} \mu_V^N) \right\rangle$$

T_2 : We write

$$\begin{aligned} T_2 &= \left\langle f_t^N, R_\varphi(\mu_V^N) \right\rangle - \left\langle f_0^N, R_\varphi(S_t^{NL}\mu_V^N) \right\rangle \\ &= \left\langle f_0^N, T_t^N(R_\varphi \circ \mu_V^N) - (T_t^\infty R_\varphi)(\mu_V^N) \right\rangle \end{aligned}$$

with

- $T_t^N =$ dual semigroup (acting on $C_b(E^N)$) of the N-particle flow $f_0^N \mapsto f_t^N$;
- $T_t^\infty =$ pushforward semigroup (acting on $C_b(P(E))$) of the nonlinear semigroup S_t^{NL} defined by $(T^\infty \Phi)(\rho) := \Phi(S_t^{NL}\rho)$;

T_2 : We write

$$
\begin{aligned}
T_2 &= \left\langle f_t^N, R_\varphi(\mu_V^N) \right\rangle - \left\langle f_0^N, R_\varphi(S_t^{NL}\mu_V^N) \right\rangle \\
&= \left\langle f_0^N, T_t^N(R_\varphi \circ \mu_V^N) - (T_t^\infty R_\varphi)(\mu_V^N) \right\rangle \\
&= \left\langle f_0^N, (T_t^N \pi_N - \pi_N T_t^\infty)\, R_\varphi \right\rangle
\end{aligned}
$$

with

- $T_t^N =$ dual semigroup (acting on $C_b(E^N)$) of the N-particle flow $f_0^N \mapsto f_t^N$;
- $T_t^\infty =$ pushforward semigroup (acting on $C_b(P(E))$) of the nonlinear semigroup S_t^{NL} defined by $(T^\infty \Phi)(\rho) := \Phi(S_t^{NL}\rho)$;
- $\pi_N =$ projection $C(P(E)) \to C(E^N)$ defined by $(\pi_N \Phi)(V) = \Phi(\mu_V^N)$.

Sketch of the proof VI

$$\begin{aligned} T_2 &= \left\langle f_0^N, (T_t^N \pi_N - \pi_N T_t^\infty)\, R_\varphi \right\rangle \\ &= \left\langle f_0^N, \int_0^T T_{t-s}^N\, (G^N \pi_N - \pi_N G^\infty)\, (T_s^\infty\, R_\varphi)\, ds \right\rangle \\ &= \int_0^T \left\langle f_{t-s}^N, (G^N \pi_N - \pi_N G^\infty)\, (T_s^\infty\, R_\varphi) \right\rangle\, ds \end{aligned}$$

where

- G^N is the generator associated to T_t^N and G^∞ is the generator associated to T_t^∞.

Now we have to make some assumptions

- (A1) f_t^N has enough bounded moments;
- (A2) $G^\infty \Phi(\rho) = \langle Q(\rho), D\Phi(\rho)\rangle$;
- (A3) $(G^N \pi^N \Phi)(V) = \langle Q(\mu_V^N), D\Phi(\mu_V^N)\rangle + \mathcal{O}([\Phi]_{C^{1,\eta}}/N)$
- (A4) $S_t^{NL} \in C^{1,\eta}(P(E); P(E))$.

Plan

Semigroups S_t^N, T_t^N and T_t^∞

The master equation generates a linear semigroup $(S_t^N)_{t\geq 0}$ on $P_{2,sym}(\mathbb{R}^{3N})$: $S_t^N\, p_0 = p_t$ for any $p_0 \in P_{2,sym}(\mathbb{R}^{3N})$, where p_t is the solution of the master equation with initial datum p_0.

The unbounded operator G^N is the generator of the dual semigroup (T_t^N) on $C_{0,sym}(\mathbb{R}^{3N})$.

$$\forall\, \varphi \in C_{0,sym}(\mathbb{R}^{3N}), \forall\, p_0 \in P_{2,sym}(\mathbb{R}^{3N}) \quad \langle p_0, T_t^N \varphi\rangle = \langle p_t, \varphi\rangle.$$

We define the pull-back semigroup T_t^∞ on $C(P(\mathbb{R}^3))$ by setting

$$\forall\, \Phi \in C(P(\mathbb{R}^3)),\ \forall\, p \in P(\mathbb{R}^3) \quad (T_t^\infty \Phi)(p) = \Phi(S_t p)$$

where S_t is the nonlinear semigroup.

Remark. A polynomial belongs to $C(P(\mathbb{R}^3))$

The $C^{1,a}$ space, $a \in (0,1]$

$\Phi \in C^{1,a}(P(\mathbb{R}^3))$ if $\Phi \in C(P(\mathbb{R}^3))$ and $\exists D\Phi : P(\mathbb{R}^3) \to C(\mathbb{R}^3)$

$$\forall \mu,\ \nu \in P(\mathbb{R}^3) \quad \Big| \Phi(\nu) - \Phi(\mu) - \langle \nu - \mu, D\Phi[\mu] \rangle \Big| \le C \, \|\nu - \mu\|_{TV}^{1+a}.$$

We define

$$[\Phi]_a = \sup_{\mu,\nu \in P(\mathbb{R}^3)} \frac{\Big| \Phi(\nu) - \Phi(\mu) - \langle \nu - \mu, D\Phi[\mu] \rangle \Big|}{\|\nu - \mu\|_{TV}^{1+a}}.$$

Remark. For any $\varphi \in C(\mathbb{R}^{3k})$, $R_\varphi \in C^{1,1}(P(\mathbb{R}^3))$ and

$$[R_\varphi]_1 \le k^2 \, \|\varphi\|_{C(\mathbb{R}^{3k})}.$$

The pull-back semigroup and its generator

The generator G^∞ is defined on $C^{1,a}(P(\mathbb{R}^3))$ by

$$\forall\, \Phi \in C^{1,a}(P(\mathbb{R}^3))\ \forall\, p_0 \in P_2(\mathbb{R}^3) \quad (G^\infty \Phi)(p_0) := \langle Q(p_0,p_0), D\Phi(p_0)\rangle$$

since

$$\begin{aligned}
(G^\infty \Phi)(p_0) &= \frac{d}{dt}(T^\infty_t \Phi)(p_0)|_{t=0} = \frac{d}{dt}\Phi(p_t)|_{t=0} = \lim_{t\to 0} \frac{\Phi(p_t) - \Phi(p_0)}{t} \\
&= \lim_{t\to 0} \left\{ \left\langle \frac{p_t - p_0}{t}, D\Phi[p_0] \right\rangle + \mathcal{O}\left(\frac{\|p_t - p_0\|^{1+a}_{TV}}{t} \right) \right\} \\
&= \left\langle \frac{dp_t}{dt}|_{t=0}, D\Phi[p_0] \right\rangle = \langle Q(p_0,p_0), D\Phi[p_0]\rangle\,.
\end{aligned}$$

Boltzmann model: Stability / expansion of T_t^∞ in total variation norm

Lemma (The Boltzmann flow is $C^{1,1/2}$)

$\forall\, \rho \in P_2(\mathbb{R}^3)$, $\forall\, t \geq 0$ *there exists* $\mathcal{L}_t[\rho] \in C(\mathbb{R}^3)$ $\forall\, \rho' \in P_2(R^3)$

$$\sup_{[0,T]} \left\| S_t(\rho') - S_t(\rho) - \mathcal{L}_t[\rho](\rho' - \rho) \right\|_{TV} \leq C_T \, \|\rho' - \rho\|_{TV}^{3/2}$$

$\Phi \in C^{1,1/2}(P(\mathbb{R}^3))$ implies $T_t^\infty(\Phi) \in C^{1,1/2}(P(\mathbb{R}^3))$ for $t \in [0, T]$ and $[T_t^\infty(\Phi)]_{1/2} \leq C_T \, [\Phi]_{1/2}$ since :

$$\begin{aligned}
(T_t^\infty \Phi)(\rho') &= \Phi(S_t(\rho')) \\
&= \Phi\Big(S_t(\rho) + \mathcal{L}_t[\rho](\rho' - \rho) + \mathcal{O}(\|\rho' - \rho\|_{TV}^{3/2})\Big) \\
&= \Phi(S_t(\rho)) + D\Phi[S_t(\rho)]\big(\mathcal{L}_t(\rho)(\rho' - \rho)\big) + \mathcal{O}(\|\rho' - \rho\|_{TV}^{3/2}) \\
&= (T_t^\infty \Phi)(\rho) + D\Phi[S_t(\rho)]\big(\mathcal{L}_t(\rho)(\rho' - \rho)\big) + \mathcal{O}(\|\rho' - \rho\|_{TV}^{3/2})
\end{aligned}$$

Proof of the lemma

Denote by f_t, g_t, h_t the unique solutions

$$\partial_t f_t = Q(f_t, f_t), \qquad f_0 = \rho \in L^1(e^{a\,|v|}),$$

$$\partial_t g_t = Q(g_t, g_t), \qquad g_0 = \rho' \in L^1(e^{a\,|v|}),$$

$$\partial_t h_t = \tilde{Q}(f_t, h_t) := Q(f_t, h_t) + Q(h_t, f_t), \quad h_0 = g_0 - f_0 = \rho' - \rho.$$

Classically, the following bounds hold

$$\|h_t\|_{L^1_2} \le C_T \, \|\rho' - \rho\|_{L^1_2}$$

$$\|g_t - f_t\|_{L^1_2} \le C_T \, \|\rho' - \rho\|_{L^1_2},$$

with $L^1_2 := L^1(\mathbb{R}^3, (1 + |v|^2)\, dv)$.

Introduce $\phi_t := g_t - f_t - h_t$ which satisfy

$$\partial_t \phi_t = \tilde{Q}(f_t + g_t, \phi_t) + \tilde{Q}(g_t - f_t, h_t), \qquad \phi_0 = 0.$$

The L^1 norm $y_t := \|\phi_t\|_{L^1_2}$ satisfies the differential inequality

$$y_t' \le C\, y_t + C\, \|g_t - f_t\|_{L^1_3} \|h_t\|_{L^1_3}, \qquad y_0 = 0.$$

By interpolation using higher moments, and the previous estimates:

$$\|g_t - f_t\|_{L^1_3} \|h_t\|_{L^1_3} \le C\, \|\rho' - \rho\|_{L^1}^{3/2}.$$

We deduce from the resolution of differential inequality on y_t that

$$\sup_{[0,T]} y_t \le C_T\, \|\rho' - \rho\|_{L^1}^{3/2}$$

When ρ, ρ' are true measures we make a regularization argument.

Boltzmann model: Consistency

Take $\Phi \in C^{1,1/2}(P(\mathbb{R}^3))$, set $\phi = D\Phi[\hat\mu^N_V]$ and compute

$$\begin{aligned} G^N(\Phi\circ\hat\mu^N_V) &= \frac{1}{2N}\sum_{i,j=1}^N |v_i - v_j| \int_{S^2} [\Phi(\hat\mu^N_{V^*_{ij}}) - \Phi(\hat\mu^N_V)]\, d\sigma \\ &= \frac{1}{2N}\sum_{i,j=1}^N |v_i - v_j| \int_{S^2} \langle \hat\mu^N_{V^*_{ij}} - \hat\mu^N_V, \phi\rangle\, d\sigma \qquad (= I_1) \\ &+ \frac{1}{2N}\sum_{i,j=1}^N |v_i - v_j| \int_{S^2} \mathcal{O}(\|\hat\mu^N_{V^*_{ij}} - \hat\mu^N_V\|^{3/2}_{TV})\, d\sigma \quad (= I_2). \end{aligned}$$

On the one hand, we have

$$I_2 = \frac{1}{2N}\sum_{i,j=1}^{N}|v_i - v_j|\int_{S^2}\mathcal{O}\Big(\Big(\frac{4}{N}\Big)^{3/2}\Big)\,d\sigma$$

$$\leq \frac{[\Phi]_{1/2}}{N}\left(\frac{1}{N^2}\sum_{i,j=1}^{N}(1+|v_i|^2+|v_j|^2)\right) \leq C\,\frac{[\Phi]_{1/2}\,\|\hat\mu_V^N\|_{M_2^1}}{N^{1/2}}.$$

On the other hand, we have

$$I_1 = \frac{1}{2N^2}\sum_{i,j=1}^{N}|v_i - v_j|\int_{S^2}[\phi(v_i^*)+\phi(v_j^*)-\phi(v_i)-\phi(v_j)]\,d\sigma$$

$$= \langle Q(\hat\mu_V^N,\hat\mu_V^N),\phi\rangle = (G^\infty\Phi)(\hat\mu_V^N).$$

Vlasov and McKean-Vlasov model: Consistency

To make the presentation simpler we assume $d = 1$.
The generator G^N writes for any $\varphi \in C^2(\mathbb{R}^N;\mathbb{R})$, $V \in \mathbb{R}^N$

$$(G^N\varphi)(V) = \sum_{i=1}^N \Delta_i \varphi + \sum_{i=1}^N F^N \left(v_i, \mu^{N-1}_{\hat{V}_i}\right) \cdot \partial_{v_i}\varphi,$$

while the nonlinear meanfield McKean-Vlasov operator on $P(\mathbb{R})$ is

$$Q(f) = \Delta f - \partial \left(F(v,f)\, f\right).$$

First, the map $\mathbb{R}^N \to H^{-s_1}(\mathbb{R}^d)$, $s_1 > 5/2 := d/2 + 2$, $V \mapsto \mu^N_V$ is of class C^2 with

$$\partial_{v_i}\mu^N_V = \frac{1}{N}\,\partial\delta_{v_i}, \qquad \Delta_{v_i}\mu^N_V = \frac{1}{N^2}\,\partial^2\delta_{v_i}.$$

We say that $\Phi : P(\mathbb{R}) \to \mathbb{R}$ is of class $C^{2,1}$ if

$$\Phi(\nu) = \Phi(\mu) + \langle \nu - \mu, D\Phi[\mu]\rangle + D^2\Phi[\mu](\nu-\mu,\nu-\mu) + R$$

with

$$\|R\|_{H^{-s_2}} \le C_\Phi\, \|\nu - \mu\|^3_{H^{-s_1}}, \quad s_2 := s_1 + 2.$$

Take $\Phi \in C^{2,1}_b$. The map $\mathbb{R}^N \to \mathbb{R}$, $V \mapsto \Phi(\mu^N_V)$ is C^2 and, denoting $\phi = \phi_V(\cdot) = D\Phi\left[\mu^N_V\right] \in (H^{-s_1}(\mathbb{R}^d))' = H^{s_1}(\mathbb{R}^d)$, we can write

$$
\begin{aligned}
\partial_{v_i}\Phi\left(\mu^N_V\right) &= \left\langle D\Phi\left[\mu^N_V\right], \frac{1}{N}\,\partial\delta_{v_i}\right\rangle = \frac{1}{N}\,\partial\phi_V(v_i) \\
\Delta_{v_i}\Phi\left(\mu^N_V\right) &= \frac{1}{N}\,\Delta\phi_V(v_i) + \frac{1}{N^2}\,D^2\Phi\left[\mu^N_V\right](\partial\delta_{v_i}, \partial\delta_{v_i}).
\end{aligned}
$$

Finally, we compute

$$
\begin{aligned}
G^N\Phi(\mu^N_V) &= \sum_{i=1}^N \Delta_i\left(\Phi(\mu^N_V)\right) + \sum_{i=1}^N F^N\left(v_i, \mu^N_V\right)\cdot\partial_i\left(\Phi(\mu^N_V)\right) \\
&= \sum_{i=1}^N \frac{1}{N}\left(\Delta\phi_V(v_i) + F^N()\,\partial\phi_V(v_i)\right) + \sum_{i=1}^N \frac{1}{N^2}\,D^2\Phi(..) \\
&= \underbrace{\langle\mu^N_V, \Delta\phi + F^N(v,\mu^N_V)\,\partial\phi\rangle}_{=\langle Q(\mu^N_v),\phi\rangle =: G^\infty\Phi(\mu^N_V)} + \mathcal{O}(1/N)
\end{aligned}
$$

Plan

Conclusion

We have proved a quantified version of chaos propagation which is furthermore uniform in time (for the Boltzmann model)

The key point is to estimate the convergence of $T_t^N \pi^N$ to $\pi^N \, T_t^\infty$ as operators acting from $C(P(E))$ with values in $C(E^N)$ which a consequence of
- a stability result (expansion of order > 1) for he nonlinear semigroup
- consistency result on the associated generators

That requires to develop a "differential calculus" on $P(E)$ seen as an embedded manifold of $\mathcal{F}'$, $\mathcal{F} \subset UC_b(E)$

Open problems

- $T = +\infty$ with optimal rate $\theta(N) = \mathcal{O}(N^{-1/2})$;

- more general cross-section (true hard or soft potential) and Landau equation;

- Vlasov equation and McKean-Vlasov equation with singular interactions;

- the entropy convergence $\sup_{[0,T]} H_N(f_t^N; f_t) \le \theta_H(N)$;

- quantification of the chaos for the equilibrium state (elastic or inelastic Boltzmann model)

- rate of convergence to equilibrium for the nonlinear PDE from the analysis of the N-particle system dynamic

- for the inelastic Boltzmann equation + diffuse excitation can we deduce from the $N \to \infty$ limit

$$\frac{d}{dt} H(f(t)|g) \le 0$$

where g stands for the unique steady state?