dynamics of topological defects in semilinear hyperbolic PDEs

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our main concern

Goal: theorems describing dynamics of topological defects in solutions of equations of the form

$$\boxed{\varepsilon \Box u + \frac{1}{\varepsilon} f(u) = 0} \qquad \text{(NLW)}$$

for suitable f, possibly with non-zero right-hand side and/or coupled to other equations (eg for electromagnetic fields).

We will mainly consider $f : \mathbb{R}^k \to \mathbb{R}^k$ given by $\boxed{f(u) = (|u|^2 - 1)u}$, $k = 1$ or 2.

Motivations:

1. *many* parallel results about elliptic and parabolic PDE.
2. cosmology

basic equations of cosmology

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = 8\pi T_{\mu\nu}(g, U)$$

equation for U evolving in metric g

Here, g is metric of spacetime, U represents matter fields.

“Sie gleicht aber einem Gebäude, dessen einer Flügel aus vorzüglichem Marmor (linke Seite der Gleichung), dessen anderer Flügel aus minderwertigem Holze gebaut ist (rechte Seite der Gleichung)”

Einstein, *Journal of the Franklin Institute* 1936.

Equation for U is considered to arise as the classical limit of conjectured grand unified (quantum field) theory (eg, some form of Yang-Mills-Higgs).

For realistic models, this is a large semilinear hyperbolic system with a complicated potential.

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For matter equations with a fixed metric, some mathematical questions:

- small-time behavior of solutions for well-prepared initial data?

  many conjectures, some theorems: solutions exhibit energy concentration sets of a topological character, evolving via timelike minimal surface equation or related equations.

- how does data become well-prepared?

  general conjecture: initially random data is regularized by rapidly expanding universe

- global or large-time results.....

  issues: collisions of defects, dynamics of non-smooth defects, radiation.....

- any of the above, coupled to Einstein's system.

semilinear elliptic PDE $\leftrightsquigarrow$ minimal surfaces

for $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^k$, $n > k \in \{1, 2\}$ solving

$$-\Delta u + \frac{1}{\varepsilon^2}(u^2 - 1)u = \alpha,$$

(or related gauge theories), energy concentrates around surfaces of codimension k, with mean curvature proportional to α.

(in particular, minimal surfaces if $\alpha = 0$.)

Much more precise descriptions available if $k = 1$.

$k = 1$: Mortola, Sternberg, Sternberg-Zumbrun, Hutchinson-Tonegawa, Kowalczyk, Pacard-Ritoré, del Pino-Kowalczyk-Wei....

$k = 2$: Bethuel-Rivière Rivière, Lin-Rivière, J.- Soner, Alberti-Baldo-Orlandi, Bethuel-Brezis-Orlandi, Bethuel-Orlandi-Smets, J.-Sternberg....

parabolic PDE $\leftrightsquigarrow$ curvature-driven surfaces

for $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^k$, $n > k \in \{1, 2\}$ solving

$$u_t - \Delta u + \frac{1}{\varepsilon^2}(u^2 - 1)u = 0,$$

energy concentrates around surfaces of codimension k, with velocity proportional to mean curvature.

Much more precise descriptions available if $k = 1$. Results with nonzero right-hand side also known.

$k = 1$: Bronsard-Kohn, de Mottoni-Schatzmann, X. Chen, Evans-Soner-Souganidis, Ilmanen, Soner, Barles-Soner-Souganidis.....

also $k = 1$, different scaling and different phenomena: Fisher, Kolmogorov-Petrovskii-Piskunov, Freidlin, Berestycki and collaborators....

$k = 2$: Lin, J.- Soner, Ambrosio-Soner, Lin-Rivière, Bethuel-Orlandi-Smets....

related mathematical work: wave equations

- J, Lin : dynamics of point vortices in solutions of

$$u_{tt} - \Delta u + \frac{1}{\varepsilon^2}(|u|^2 - 1)u = 0 \quad \text{for } u : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2 \qquad (1)$$

  Also Gustafson and Sigal for magnetic vortices. In all these results,
  - vortices move at sub-relativistic speeds
  - point defects rather than submanifolds
- soliton dynamics for somewhat different equation: *relativistic* dynamics of point particles Stuart.
- asymptotic stability of flat kink: for single equation when $N = 3$, small perturbations (in $H^{24} \times H^{23}$) of flat kink converge to 0 in L^∞, with control over rate. Cuccagna 2008.
- Conditional results on behavior of (1), for $u : \mathbb{R}^n \to \mathbb{R}^k$, $n > k$, when $\varepsilon \to 0$. Bellettini, Novaga, Orlandi 2008.

results, scalar/hypersurface case

Theorem (J, 2010)

Let $\Gamma \subset (-T,T)\times\mathbb{R}^N$ be a **smooth** *timelike hypersurface of constant Minkowski curvature α.*
Then exists a solution $u : (-T,T)\times\mathbb{R}^N \to \mathbb{R}$ of

$$u_{tt} - \Delta u + \frac{1}{\varepsilon^2}(u^2-1)u = \alpha$$

whose energy concentrates around Γ, and such that

$$\|u - U_\varepsilon\|_{L^2(\mathcal{N})} \le C\sqrt{\varepsilon}$$

for an explicitly constructed function U_ε exhibiting a Lorenz-contracted transition across Γ.

generalizations to α smooth also established.
The case $\alpha = 0$ was proved earlier.

results, vector/string case

Theorem (J, 2009)

Let $\Gamma \subset (-T,T) \times \mathbb{R}^N$ be a **smooth** *timelike hypersurface of constant Minkowski curvature α.*
Then exists a solution $u : (-T,T) \times \mathbb{R}^N \to \mathbb{R}^2$ of $\varepsilon \Box u + \frac{1}{\varepsilon}(|u|^2 - 1)u = 0$ whose energy concentrates around Γ. Γ.

In addition, u exhibits a "string" near Γ; the precise statement is slightly complicated.

Theorem (Czubak and J, 2010)

parallel results for the self-dual Abelian Higgs model, with estimate of $\|u - U_\varepsilon\|_{L^2/gauge}$ for explicitly constructed U_ε

underway: solutions of self-dual $su(2)$ Yang-Mills Higgs with energy concentrating around codimension 3 minimal surface. (modulo well-posedness, joint with Czubak.)

remarks

- conditions on initial data include quantitative versions of:
  $\star$ small energy away from Γ_0,
  $\star\star$ a defect (interface if $k = 1$, or "string" if $k = 2$) near Γ_0, and
  $\star\star\star$ small energy near Γ_0, given the presence of the defect, *in a frame that moves with* Γ.
- Analogous results are certainly false without well-prepared data.
- There is no reason to believe any similar results hold globally in t.
- In many cases we also establish estimates on energy-momentum tensors (EMT).
- EMT convergence in limit $\varepsilon \to 0$ may actually fail, unlike elliptic and parabolic cases.
- For vector case, convergence (in good cases) away from Γ to wave map into S^1 as $\varepsilon \to 0$.

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