Problèmes de contrôle liés aux mouvements de foules

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Motivation: Evacuation / egress in panic situation

<table>
<thead>
<tr>
<th>Date</th>
<th>Place</th>
<th>Venue</th>
<th>Deaths</th>
<th>Injured</th>
<th>Reason</th>
</tr>
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<tbody>
<tr>
<td>1964</td>
<td>Lima, Peru</td>
<td>Stadium</td>
<td>318</td>
<td>500</td>
<td>Goal disallowed</td>
</tr>
<tr>
<td>1992</td>
<td>Bastia, Corsica</td>
<td>Stadium</td>
<td>17</td>
<td>1900</td>
<td></td>
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<tr>
<td>1996</td>
<td>Lusaka, Zambia</td>
<td>Stadium</td>
<td>9</td>
<td>78</td>
<td>Stampede after Zambia’s victory over Sudan</td>
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<tr>
<td>1996</td>
<td>Guatemala City,</td>
<td>Stadium</td>
<td>80</td>
<td>180</td>
<td>Fans trying to force their way into the stadium</td>
</tr>
<tr>
<td></td>
<td>Guatemala</td>
<td></td>
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<tr>
<td>1999</td>
<td>Minsk, Belarus</td>
<td>Subway</td>
<td>51</td>
<td>150</td>
<td>Heavy rain at rock concert</td>
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<td>2000</td>
<td>Durban, South Africa</td>
<td>Disco</td>
<td>13</td>
<td>44</td>
<td>Tear gas</td>
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<td>2000</td>
<td>Roskilde, Denmark</td>
<td>Stadium</td>
<td>8</td>
<td>25</td>
<td>Failure of loud speakers</td>
</tr>
</tbody>
</table>

Framework: Model

Velocity

Initial crowd

Target crowd

Control region
(Fixed)
Macroscopic model

Search $u$ such that

\[ \left\{ \begin{array}{l}
\partial_t \mu + \nabla \cdot ((v + 1_\omega u) \mu) = 0 \\
\mu(0) = \mu^0, \; \mu(T) \approx \mu^1
\end{array} \right. \]

with

- $v :$ population velocity
- $1_\omega(x)u(x, t) :$ control
Problematic
We search \( u \), called control, such that the solution \( \mu \) to system
\[
\begin{cases}
\partial_t \mu + \nabla \cdot ((v + \mathbb{1}_\omega u)\mu) = 0 \\
\mu(0) = \mu^0
\end{cases}
\]
satisfies:

- \( \mu \) near a given target at time \( T \)

\[
\forall \varepsilon > 0, \mu^0, \mu^1, \exists u \text{ s.t. } d(\mu(T), \mu^1) \leq \varepsilon.
\]

- **Approximate controllability**

- \( \bar{y} \) reach a target at time \( T \)

\[
\forall \mu^0, \mu^1, \exists u \text{ s.t. } \mu(T) = \mu^1.
\]

- **Exact controllability**

We call **minimal time** the infimum of \( T \) for which the approximate/exact controllability holds.
Framework: Distance?

Distance between two continuous crowds
If we represent the population by a **density compactly supported**:

\[
\| \mu_0 - \mu_1 \|_{L^p} = \| \mu_0 - \mu_2 \|_{L^p}
\]

**The** \(L^p\) **distance is not a good distance for the crowds!!!**
Monge problem (1781)

**Distance**: minimal cost to send a mass on another.

Continuous case

Discrete case
Framework: Wasserstein distance

We denote by:

- $\Gamma := \{ \gamma : \mathbb{R}^d \to \mathbb{R}^d \text{ Borel} \}$.
- $\mathcal{P}_{ac}^c(\mathbb{R}^d) := \{ \mu \in \mathcal{P}_c(\mathbb{R}^d) : \text{abs. cont. w.r.t. the Lebesgue measure} \}$.

**Definition**

Let $\gamma \in \Gamma$ and $\mu \in \mathcal{P}_{ac}^c(\mathbb{R}^d)$. **Push-forward** of $\mu$ with $\gamma$:

$$(\gamma \# \mu)(E) := \mu(\gamma^{-1}(E)),$$

for all $E \subset \mathbb{R}^d$ such that $\gamma^{-1}(E)$ is $\mu$-measurable.

**Definition (Monge Problem)**

Let $p \in [1, \infty)$ and $\mu, \nu \in \mathcal{P}_{ac}^c(\mathbb{R}^d)$. **Wasserstein distance** between $\mu$ and $\nu$:

$$W_p(\mu, \nu) = \min_{\gamma \in \Gamma} \left\{ \left( \int_{\mathbb{R}^d} |\gamma(x) - x|^p d\mu \right)^{1/p} : \gamma \# \mu = \nu \right\}.$$
Framework: Wasserstein distance

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**Definition**

Let \( \gamma \in \Gamma \) and \( \mu \in \mathcal{P}_{ac}^c(\mathbb{R}^d) \). **Push-forward** of \( \mu \) with \( \gamma \):

\[
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**Definition (Monge Problem)**

Let \( p \in [1, \infty) \) and \( \mu, \nu \in \mathcal{P}_{ac}^c(\mathbb{R}^d) \). **Wasserstein distance** between \( \mu \) and \( \nu \):

\[
W_p(\mu, \nu) = \min_{\gamma \in \Gamma} \left\{ \left( \int_{\mathbb{R}^d} |\gamma(x) - x|^p \, d\mu(x) \right)^{1/p} : \gamma \# \mu = \nu \right\}.
\]
Definition

We define the flow (or the characteristic) associated to \( w : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d \) as \((x^0, t) \mapsto \Phi^w_t(x^0)\) such that for all \( x^0 \in \mathbb{R}^d \), \( t \mapsto \Phi^w_t(x^0) \) is solution to

\[
\begin{aligned}
\dot{x}(t) &= w(x(t), t), \quad t \geq 0, \\
x(0) &= x^0.
\end{aligned}
\]

Theorem (Method of Characteristics)

Let \( T > 0 \), \( \mu^0 \in \mathcal{P}_c(\mathbb{R}^d) \) and \( w \) a velocity field uniformly bounded, \textbf{Lipschitz} in space and measurable in time. Then

\[
\begin{aligned}
\partial_t \mu + \nabla \cdot (w \mu) &= 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\
\mu(0) &= \mu^0 & \text{in } \mathbb{R}^d,
\end{aligned}
\]

admits a unique solution \( \mu \) in \( C^0([0, T]; \mathcal{P}_c(\mathbb{R}^d)) \).
Controllability: Geometric condition

Geometric condition

(i) \( \forall x^0 \in \text{supp}(\mu^0), \exists t^0 \in (0, T) : \Phi_{t^0}^v(x^0) \in \omega. \)

(ii) \( \forall x^1 \in \text{supp}(\mu^1), \exists t^1 \in (0, T) : \Phi_{-t^1}^v(x^1) \in \omega. \)

Theorem (D.-Morancey-Rossi 2017)

Let \( \mu^0, \mu^1 \in \mathcal{P}^{ac}_c(\mathbb{R}^d) \). Assume the Geometric Condition.
System (1) is **approximately controllable** with a Lipschitz control.
Controllability: Sketch of proof for the approx. contr.

Global strategy

(i) Step 1: We send $\mu^0$ to $\nu^0$ supported in a square $S \subset \omega$. We send $\mu^1$ to $\nu^1$ supported in a square $S \subset \omega$.

(ii) Step 2: We send approximately $\nu^0$ to $\nu^1$.

Final computation:

$\mu^0 u^1 u^3 \rightarrow \nu^0 u^3 \rightarrow \nu^1 \leftarrow u^2 \rightarrow \mu(T)$
Discretization following the mass of $\mu^0$ and $\mu^1$

Assume that supp($\mu^0$), supp($\mu^1$) $\subset$ Square $\subset \Omega$.

\[
\int_{A^0_{i,j}} d\mu^0 = \int_{A^1_{i,j}} d\mu^1 = \frac{1}{n^2}
\]
Controllability: Sketch of proof for the approx. contr.

Center of the cells

\[ \int_{B_{ij}^0} d\mu^0(x) = \int_{B_{ij}^1} d\mu^1(x) = \frac{1}{n^2} - \frac{1}{n^3} \]

\[ B_{ij}^0 = (b_{ij}^0 - b_{ij}^{0+}) \times (b_{ij}^{0-}, b_{ij}^{0+}) \quad B_{ij}^1 = (b_{ij}^1 - b_{ij}^{1+}) \times (b_{ij}^{1-}, b_{ij}^{1+}) \]

Remark: We do not control the mass outside \( B_{ij}^0 \).
Construction of the flow

We send linearly $\mu^0_{|B^0_{ij}}$ on $B^1_{ij}$:

Remark: $|B^0_{ij}| \xrightarrow{n \to \infty} 0$. 
Construction of the flow

For all \( x^0 = (x_1^0, x_2^0) \in A_{ij} \), we build the flow

\[
\Phi^u_t(x^0) := \begin{pmatrix}
a_i^+ - x_1^0 c_i^- (t) + \frac{x_1^0 - a_i^-}{a_i^+ - a_i^-} c_i^+ (t) \\
a_i^- - a_i^+ \\
a_{ij}^+ - x_2^0 \\
a_{ij}^- - a_{ij}^+
\end{pmatrix},
\]

where

\[
\begin{align*}
c_i^- (t) &= (b_i^- - a_i^-)t + a_i^- , \\
c_i^+ (t) &= (b_i^+ - a_i^+)t + a_i^+ , \\
c_{ij}^- (t) &= (b_{ij}^- - a_{ij}^-)t + a_{ij}^- , \\
c_{ij}^+ (t) &= (b_{ij}^+ - a_{ij}^+)t + a_{ij}^+ .
\end{align*}
\]

Thus

\[
\Phi^u_T(A_{ij}) = B_{ij} .
\]

**Remark:** We take a \( C^\infty \) extension outside \( A_{ij} \).
Construction of the control

The corresponding velocity is given by

\[
\begin{align*}
  u_1(x, t) &= \alpha_i(t)x_1 + \beta_i(t), \\
  u_2(x, t) &= \alpha_{ij}(t)x_2 + \beta_{ij}(t),
\end{align*}
\]

where

\[
\begin{align*}
  \alpha_i(t) &= \frac{b_i^+ - b_i^- + a_i^- - a_i^+}{c_i^+(t) - c_i^-(t)}, \\
  \beta_i(t) &= \frac{a_i^+ b_i^- - a_i^- b_i^+}{c_i^+(t) - c_i^-(t)}, \\
  \alpha_{ij}(t) &= \frac{b_{ij}^+ - b_{ij}^- + a_{ij}^- - a_{ij}^+}{c_{ij}^+(t) - c_{ij}^-(t)}, \\
  \beta_{ij}(t) &= \frac{a_{ij}^+ b_{ij}^- - a_{ij}^- b_{ij}^+}{c_{ij}^+(t) - c_{ij}^-(t)}.
\end{align*}
\]
Estimation of the distance

Define

\[ R := (0, 1)^2 \setminus \bigcup_{i,j} B_{ij}^1, \]

We have

\[
W_1(\mu^1, \mu(T)) \leq \sum_{i,j=1}^{n} W_1(\mu^1 \times 1_{B_{ij}^1}, \mu(T) \times 1_{B_{ij}^1}) + W_1(\mu^1 \times 1_R, \mu(T) \times 1_R).
\]

Included in \( B_{ij}^1 \)

Small mass

No control
There exist measurable maps $\gamma_{ij} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $\overline{\gamma} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$\gamma_{ij} \#(\mu^1 \times 1_{B_{ij}^1}) = \mu(T) \times 1_{B_{ij}^1} \quad \text{and} \quad \overline{\gamma} \#(\mu^1 \times 1_R) = \mu(T) \times 1_R.$$ 

We have

$$W_1(\mu^1 \times 1_{B_{ij}^1}, \mu(T) \times 1_{B_{ij}^1}) = \int_{B_{ij}^1} |x - \gamma_{ij}(x)| d\mu^1(x)$$

$$\leq [(b_{i+}^1 - b_{i-}^1) + (b_{ij}^{1+} - b_{ij}^{1-})] \int_{B_{ij}^1} d\mu^1(x)$$

$$\leq (b_{i+}^1 - b_{i-}^1 + b_{ij}^{1+} - b_{ij}^{1-}) \left( \frac{1}{n^2} - \frac{1}{n^3} \right)$$

and

$$W_1(\mu^1 \times 1_R, \mu(T) \times 1_R) \leq \int_R |x - \overline{\gamma}(x)| d\mu^1(x)$$

$$\leq \text{diam}(S) \frac{1}{n}.$$ 

Thus

$$W_1(\mu^1, \mu(T)) \xrightarrow{n \to \infty} 0.$$
Minimal time: Microscopic model

If we take \( \mu^0 = \frac{1}{n} \sum_{i=1}^{n} \delta_{X^0_i} \) (with \( X^0_i \neq X^0_j \) for all \( i \neq j \)) as initial data in

\[
\begin{aligned}
\{ & \partial_t \mu + \nabla \cdot ((v + 1_\omega u) \mu) = 0 \\
\mu(0) = \mu^0, 
\end{aligned}
\]

then the solution is given by

\[
\mu(t) = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i(t)}
\]

where \( X_i \) is solution to

\[
\begin{aligned}
\{ & \dot{X}_i(t) = v(X_i(t)) + 1_\omega(X_i(t))u(X_i(t), t) \\
X_i(0) = X^0_i, 
\end{aligned}
\]

where \( X^0 := \{X^0_1, \ldots, X^0_n\} \).

We will call this system the microscopic model.
Minimal time : Microscopic model

Computation of the optimal time

**Theorem**

Assume the Geometric Condition and $\omega$ convex.
Assume the $\{t^0_i\}_{i \in \{1,...,n\}}$ et $\{t^1_i\}_{i \in \{1,...,n\}}$ are increasingly and decreasingly ordered respectively, then the **minimal time** to exactly steer $X^0$ to $X^1$ is equal to:

$$T_0 := \max_{i \in \{1,...,n\}} \{t^0_i + t^1_i\}.$$ 

where

$$\begin{align*}
t^0_i &:= \inf\{t \geq 0 : \Phi^\nu_t(X^0_i) \in \omega\} \text{ arrival time of } X^0_i \text{ in } \omega \\
t^1_i &:= \inf\{t \geq 0 : \Phi^\nu_t(X^1_i) \in \omega\} \text{ arrival time of } X^1_i \text{ in } \omega
\end{align*}$$
Minimal time : Macroscopic model

We define for all $m \in [0, 1]$

\[
\begin{align*}
\mathcal{F}_0^{-1}(m) &:= \inf \{ t \geq 0 : \mathcal{F}_0(t) \geq m \}, \\
\mathcal{F}_1^{-1}(m) &:= \inf \{ t \geq 0 : \mathcal{F}_1(t) \geq m \},
\end{align*}
\]

with for all $t \geq 0$

\[
\begin{align*}
\mathcal{F}_0(t) &:= \mu^0(\{ x^0 \in \text{Supp}(\mu^0) : t^0(x^0) \leq t \}), \\
\mathcal{F}_1(t) &:= \mu^1(\{ x^1 \in \text{Supp}(\mu^0) : t^1(x^1) \leq t \}).
\end{align*}
\]

Theorem (D.-Morancey-Rossi 17’)

Let $\mu^0, \mu^1 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$. Assume the Geometric Condition and $\omega$ convex.

\[
T_0 := \max_{m \in [0,1]} \{ \mathcal{F}_0^{-1}(m) + \mathcal{F}_1^{-1}(1 - m) \}.
\]

Then

(i) For all $T > T_0$, System (1) is **approximately controllable** from $\mu^0$ to $\mu^1$ at time $T$ with a control $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ uniformly bounded, Lipschitz in space and measurable in time.

(ii) For all $T \in (T_2^*, T_0)$, System (1) is **not approximately controllable** from $\mu^0$ to $\mu^1$, where $T_2^*$ is the time at which each agent has crossed the control region.
Algorithm 1

**Step 1**: Discretisation of $\mu^0$ and $\mu^1$
(i) Construction of the uniform mesh
(ii) Computation of the cells $B_{ij}^0$ and $B_{ij}^1$ following the mass

**Step 2**: Computation of the discrete minimal time

$$T_0 := \max_{1 \leq i \leq N-R} \{ t_{i+R}^0 + t_i^1 \}$$

where $(t_i^0)_i$ increasing and $(t_i^1)_i$ decreasing.

**Step 4**: Computation of the optimal discrete permutation

**Step 5**: Concentration of the masses (if necessary)

**Step 6**: Final computation
Consider the initial data $\mu^0$ and the target $\mu^1$ defined by
\[
\mu^0 := \begin{cases} 
1/8 & \text{if } (x, y) \in (0, 4) \times (1, 3), \\
0 & \text{otherwise}
\end{cases}
\]
and
\[
\mu^1 := \begin{cases} 
1/16 & \text{if } (x, y) \in (8, 14) \times (0, 4) \setminus (9, 13) \times (2, 3), \\
0 & \text{otherwise}.
\end{cases}
\]

We fix the velocity field $v := (1, 0)$ and the control region $\omega := (5, 7) \times (0, 4)$. The **minimal time** is equal to : 8s.
Numerical simulation: Macroscopic model
images.math.cnrs.fr/modelisation-de-mouvements-de.html
B. Maury, J. Venel, 2011.
Thank you for your attention!
Controllability

Exact controllability

Remark

- With a Lipschitz velocity field, the flow is a homeomorphism, then $\text{supp}(\mu^0)$ and $\text{supp}(\mu^1)$ have to be homeomorph. In particular, we cannot separate a mass in two parts or bring together to different masses.

- Even with a BV velocity field we cannot bring together to different masses.

- For a Borel velocity field, the solution is not guaranteed unique.

Theorem (D.-Morancey-Rossi 2017)

Let $\mu^0, \mu^1 \in \mathcal{P}_c(\mathbb{R}^d)$. Assume the Geometric Condition.

- System (1) is not always exactly contr. with a Lipschitz control (or BV).

- There exists a couple $(\mu, u)$ solution of system (1) such that $\mu(T) = \mu^1$ with a Borel control.
**Exact controllability**

**Remark**
- With a **Lipschitz velocity** field, the flow is a homeomorphism, then \(\text{supp}(\mu^0)\) and \(\text{supp}(\mu^1)\) have to be homeomorph. In particular, we cannot separate a mass in two parts or bring together to different masses.
- Even with a BV velocity field we cannot bring together to different masses.
- For a **Borel velocity** field, the solution is not guaranteed unique.

**Theorem (D.-Morancey-Rossi 2017)**

Let \(\mu^0, \mu^1 \in \mathcal{P}_c(\mathbb{R}^d)\). Assume the Geometric Condition.
- System (1) is **not always exactly contr.** with a Lipschitz control (or BV).
- There exists a couple \((\mu, u)\) solution of system (1) such that \(\mu(T) = \mu^1\) with a Borel control.
Framework: Wasserstein distance

For $\mu, \nu \in \mathcal{P}_c(\mathbb{R}^d)$, we denote by $\Pi(\mu, \nu)$ the set of transference plans from $\mu$ to $\nu$, i.e. the probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ which have marginals $\mu$ and $\nu$:

$$\int_{y \in \mathbb{R}^d} d\pi(x, y) = d\mu(x) \quad \text{and} \quad \int_{x \in \mathbb{R}^d} d\pi(x, y) = d\nu(y).$$

Definition (Kantorovich problem)

Let $p \in [1, \infty)$ and $\mu, \nu \in \mathcal{P}_c(\mathbb{R}^d)$. Wasserstein distance between $\mu$ and $\nu$:

$$W_p(\mu, \nu) = \min_{\pi \in \Pi(\mu, \nu)} \left\{ \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d\pi(x, y) \right)^{1/p} \right\}$$
Minimal time : Macroscopic model, sketch of proof

Step 1 : Uniform discretization of $\text{supp}(\mu^0)$ and $\text{supp}(\mu^1)$

We take $h$ small enough such that the cells $K_h$ satisfies for a $t^* > 0$

$$\Phi_{t^*}(K_h) \subset\subset \omega,$$

Step 2 : Discretization following the mass of the cells $K_h$

- Each cell will the same mass $1/n^2$.
- The rest will be negligible.
Step 3 : Association of the masses

We use the results of the discrete case to associate the masses

- We approximate the measure by a sum of Dirac (centers of the cells).
- We control this discrete approximation.
- We follow the trajectory of the Dirac masses, up to a concentration of the mass.

Difficulty : Compare \( T_0(X^0, X^1) \) and \( T_0(\mu^0, \mu^1) \).
We recall that

\[
\begin{cases}
    \dot{X}_i(t) = v(X_i(t)) + \mathbf{1}_{\omega}(X_i(t))u(X_i(t), t) \\
    X_i(0) = X_i^0
\end{cases}
\]

**Theorem (D.-Morancey-Rossi 2017)**

Assume the Geometric Condition and \( \omega \) convex.

System (1) is **exactly controllable** with a Lipschitz control.

Moreover the **minimal time** to exactly steer \( X^0 \) to \( X^1 \) is equal to:

\[
T_0 = \min_{\sigma \in S_n} \max_{i \in \{1, \ldots, n\}} |t^0_i + t^1_{\sigma(i)}|
\]

where

\[
\begin{cases}
    t^0_i := \inf\{t \geq 0 : \Phi_t^v(X_i^0) \in \omega\} \\
    t^1_i := \inf\{t \geq 0 : \Phi_t^v(X_i^1) \in \omega\}
\end{cases}
\]
Define

\[ K_{ij} := \begin{cases} \| (Y^0_i, t^0_i) - (Y^1_j, T - t^1_j) \|_{\mathbb{R}^{d+1}} & \text{if } t^0_i < T - t^1_j, \\ \infty & \text{otherwise}, \end{cases} \]

where

\[ \begin{cases} t^0_i := \inf \{ t \geq 0 : \Phi^v_i (X^0_i) \in \omega \} \\ t^1_i := \inf \{ t \geq 0 : \Phi^v_i (X^1_i) \in \omega \} \end{cases} \]
Computation of the optimal permutation

Consider the minimisation problem

$$\inf_{\pi \in \mathcal{B}_n} \left\{ \frac{1}{n} \sum_{i,j=1}^{n} K_{ij} \pi_{ij} \right\},$$

where $\mathcal{B}_n$ is the set of bistochastic matrices $\pi := (\pi_{ij})_{1 \leq i, j \leq n}$, i.e.

$$\sum_{i=1}^{n} \pi_{ij} = 1, \sum_{j=1}^{n} \pi_{ij} = 1, \pi_{ij} \geq 0.$$  

The infimum is reached.  
Since $\mathcal{B}_n$ is convex, there exists a minimum which is a permutation matrix.
No intersection of the trajectories

By contradiction: no intersection of the trajectories

\[ (Y_{\sigma(j)}, T - t_{\sigma(j)}) \]

\[ (Y_{\sigma(i)}, T - t_{\sigma(i)}) \]

\[ (Y_{i}^{0}, t_{i}^{0}) \]

\[ (Y_{j}^{0}, t_{j}^{0}) \]
Algorithm 2 Minimal time problem for exact contr. : Discrete case

Step 1 : Computation of the minimal time.

\[ T_0 := \max_{1 \leq i \leq N} \{ t^0_i + t^1_i \} \]

where \( (t^0_i)_i \) increasing and \( (t^1_i)_i \) decreasing.

Step 2 : Computation of an optimal permutation to steer \( X^0 \) to \( X^1 \)

\[
\inf_{\pi \in \mathcal{B}_n} \left\{ \frac{1}{n} \sum_{i,j=1}^{n} K_{i,j} \pi_{ij} \right\}
\]

Step 3 : Computation of the control \( u \) and the solution \( X \)
Numerical simulation: Microscopic model

Initial configuration $X^0$

Final configuration $X^1$

$v := (1, 0)$.
Numerical simulation : Microscopic model